Workshop on

Stochastic Partial Differential Equations

Statistical Issues and Applications

Centre for Mathematical Physics and Stochastics - MaPhySto

Organized by

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1 Introduction

The MaPhySto–workshop "Stochastic Partial Differential Equations – Statistical Issues and Applications" was held 4 – 6 January 2001 at the Department of Statistics and Operations Research, University of Copenhagen.

The aim of the workshop was to identify promising research directions concerning statistical issues, including ill-posed problems, and to discuss where stochastic partial differential equations can fruitfully be applied. The focus was on applications in finance and hydrology.

This booklet contains extended abstracts of the talks given at the workshop followed by the list of participants.

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Bernstein - von Mises Theorem and Bayes Estimation for Parabolic SPDEs

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Recently, infinite dimensional stochastic differential equations (SDEs), like the stochastic partial differential equations (SPDEs), are being paid a lot of attention in view of their modeling applications in neurophysiology, turbulence, oceonography and finance, see Itô (1984), Walsh (1986) and Kallianpur and Xiong (1995), Holden *et al.* (1996) and Carmona and Rozovskii (1999). In view of this it becomes necessary to estimate the unknown parameters in SPDEs.

Various methods of estimation in finite dimensional SDEs has been extensively studied during the last three decades as the observation time tends to infinity (see, Liptser and Shiryayev (1978), Basawa and Prakasa Rao (1980), Kuchler and Sørensen (1997), Prakasa Rao (1999) and Kutoyants (1999)) or as the intensity of noise tends to zero (see, Ibragimov and Has'minskii (1981), Kutoyants (1984, 1994)). On the other hand, this problem for infinite dimensional SDEs is young.

Consider an SPDE:

$$du(t,x) = \theta Au(t,x)dt + dW(t,x),\ 0 \le t \le T,\ x \in G$$

where G is a suitable Euclidean domain, A is a partial differential operator, W is a cylindrical Brownian motion and θ is an unknown parameter to be estimated on the basis of observations of the diffusion field u.

Loges (1984) initiated the study of parameter estimation in such models. When the length of the observation time becomes large ($T \to \infty$), he obtained consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a real valued drift parameter in a Hilbert space valued SDE. Koski and Loges (1986) extended the work of Loges (1984) to minimum contrast estimators. Koski and Loges (1985) applied the work to a stochastic heat flow problem. Kim (1996) also studied the properties of MLE in a similar set up. Mohapl (1992) studied the asymptotics of MLE in a in a nuclear space valued SDE. For partially observed SPDE systems of both parabolic and hyperbolic type, parameter estimation is studied by Aihara (1992, 1994, 1995), Aihara and Bagchi (1988, 1989, 1991), Bagchi and Borkar (1984). Nonparametric estimation where θ is an infinite dimensional parameter was studied by Ibragimov and Khasminskii (1998, 1999).

Huebner, Khasminskii and Rozovskii (1992) introduced spectral method and obtained consistency, asymptotic normality and asymptotic efficiency of MLE of a parameter in the drift coefficient of an SPDE. Spectral approach allows one to obtain asymptotics of estimators under conditions which guarantee the singularity of the measures generated by the corresponding diffusion field for different parameters. Unlike in the finite dimensional cases, where the total observation time was assumed to be long or intensity of the noise was assumed to be small ($\epsilon \to 0$), here both are kept fixed. Here the asymptotics are obtained when the number of Fourier coefficients (n) of the solution of SPDE becomes large.

Huebner, Khasminskii and Rozovskii (1992) gave two contrast examples in one of which they obtained consistency, asymptotic normality and asymptotic efficiency of the MLE as noise intensity decreases to zero under the condition of absolute continuity of measures generated by the process for different parameter values (the situation is similar to the classical case) and in the other they obtained these properties as the finite dimensional projection becomes large under the condition of singularity of the measures generated by the process. The study of small noise asymptotics of the MLE for more general parabolic SPDEs in the absolutely continuous case was studied by Huebner (1999).

The spectral asymptotics for MLE was extended by Huebner and Rozovskii (1995) to more general parabolic SPDEs where the partial differential operators commute and satisfy some order conditions. Piterberg and Rozovskii (1995) studied the properties MLE of a parameter in SPDE which are used to model the upper ocean variability in physical oceanography. Piterbarg and Rozovskii (1996) studied the properties of MLE based on discrete observations of the corresponding diffusion field. Huebner (1997) extended the problem to the ML estimation of multidimensional parameter. Lototsky and Rozovskii (1999) studied the same problem without the commutativity condition.

The Bernstein-von Mises theorem, concerning the convergence of suitably normalised and centered posterior distribution to normal distribution, was first proved by Bernstein (1917) and von Mises (1931) in specific i.i.d. cases. It plays a fundamental role in asymptotic Bayesian inference, see Le Cam and Yang (1990). A complete proof in the i.i.d. caes was first given by Le Cam (1953). Since then the theorem has been extended to many dependent cases. Borwanker *et al.* (1972) obtained the theorem for discrete time Markov processes. For the linear homogeneous diffusion processes, the Bernstein - von Mises theorem was proved by Prakasa Rao (1980). Prakasa Rao (1981) extended the theorem to a two parameter diffusion field. Bose (1983) extended the theorem to the homogeneous nonlinear diffusions and Mishra (1989) to the nonhomogeneous diffusions. As a further refinement in Bernstein-von Mises theorem, Bishwal (2000a) obtained sharp rates of convergence to normality of the posterior distribution and the Bayes estimators in the Ornstein-Uhlenbeck process.

All these above work on Bernstein-von Mises theorem are concerned with finite dimensional SDEs. Recently Bishwal (1999) proved the Bernstein-von Mises theorem and obtained asymptotic properties of regular Bayes estimator of the drift parameter in a Hilbert space valued SDE when the corresponding diffusion process is observed continuously over a time interval $[0, T]$. The asymptotics are studied as $T \to \infty$ under the condition of absolute continuity of measures generated by the process. Results are illustrated for the example of an SPDE. The situation is analogous to the finite dimensional SDEs, where the measures are absolutely continuous.

Here we prove the Bernstein-von Mises theorem for a certain class of linearly parametrized parabolic SPDEs when the corresponding random field is observed continuously in time. We will consider two types of asymptotics: 1) spectral asymptotics when the measures generated by the corresponding diffusion field for different parameter values are singular and 2) small noise asymptotics when the above measures are absolutely continuous with respect to each other. As a consequence of the above theorem, it will be shown that regular Bayes estimators, for smooth priors and loss functions, are asymptotically equivalent to the maximum likelihood estimators, strongly consistent, asymptotically normally distributed and asymptotically efficient in the Hajek-Le Cam sense. An example of heat equation is given where the assumptions made are verified. First part is from Bishwal (1998) and the second part from (2000b). As further refinement of these results, rates of normal approximation of the posteriors and Bayes estimators are shown. The rate is interesting which differs from the classical i.i.d. case. For the heat equation, it is of the order $n^{-3/2}$ where n is the number of observable Fourier coefficients. This part is from Bishwal (2001). Similar rate was shown by Bishwal, Markussen and Sørensen (2000) for the MLE.

Sequential estimators are known to have better properties than the MLE in finite dimensional SDEs (see Küchler and Sørensen (1997)). Sequential estimation in SPDEs is studied in Bishwal and Sørensen (2000).

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Stochastic PDEs, infinite dimensional diffusions and interest rate dynamics

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Interest rates admit a natural description in terms of a random field indexed by time and maturity: for example, one may represent the structure of interest rates at date t by the forward rate $f(t,T)$ for the maturity $T = t + x$, x units of time ahead from the current date. Here x is the time to maturity of the corresponding forward contract. Forward rates are linked to bond prices by

$$B(t,T) = \exp - \int_t^T f(t,u)du$$

The forward rates $f(t,T)$ are modeled in the Heath Jarrow Morton (HJM) framework as a family of diffusion processes indexed by the maturity date T: for each $T \geq t$, $f(t,T)$ is the solution of

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T).dW_t \tag{1}$$

where W is a finite dimensional Wiener process on a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$. Driving a infinite (continuum) of diffusion processes with the same finite dimensional noise sources creates considerable amount of correlations among different forward rates, leading to risk-free arbitrage strategies in the model, which is undesirable from a financial point of view. A sufficient condition for avoiding such arbitrage opportunities is the existence of an *equivalent martingale measure* $\mathcal{Q} \sim \mathcal{P}$ such that for each maturity T, the bond price $B(t,T)$ is a $\mathcal{Q}$-martingale. In the diffusion context, one can parametrize $\mathcal{Q}$ by its Girsanov kernel with respect to $\mathcal{P}$. As shown by Heath, Jarrow &Morton [9] this leads to a restriction on the form of the drift in Eq.1:

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,u)du + \sigma(t,T)\gamma(t) \tag{2}$$

where $\gamma(t)$ is a $\mathcal{F}_t$ predictable process, depending only on t and not on T and verifying a Novikov condition allowing to link $\mathcal{P}$ to $\mathcal{Q}$ via a Girsanov transformation:

$$E \exp[- \int_0^T \gamma(t)dW_t^i - \int_0^T \gamma(t)^2 dt] < +\infty \tag{3}$$

However absence of arbitrage gives no more restrictions on the process γ, leaving a large degree of freedom for the modeling of interest rate dynamics. Most of the literature on this topic [1, 9, 8, 10] deals with the case $\gamma = 0$ which is the case when $\mathcal{P} = \mathcal{Q}$ describing the "risk-neutral" dynamics of interest rates. This corresponds to the dynamics one

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would use in a Monte Carlo simulation for pricing interest rate option. In this case, if one denotes by $r(t,x) = f(t, t+x)$ the forward rate curve as a function of time to maturity, then $r(t,.)$ is a function defined on $[0,\infty[$, solution of a stochastic evolution equation the random field $r(t,x)$ is the solution to a hyperbolic first-order stochastic PDE with finite dimensional Wiener process as noise source [10]:

$$\frac{\partial r(t,x)}{\partial t} = \frac{\partial r}{\partial x} + \sigma(t,x)\int_0^x \sigma(t,u)du + \sigma(t,x).\eta(t,x) \tag{4}$$

where η is a white noise in t, eventually correlated in x.

However, statistical analysis of observations reveals interesting properties of this random field which seem to be at odds with such a specification [4, 3, 5]: the mean-reverting character of interest rates, the influence of the curvature of the yield curve on the dynamics and the increasingly oscillating behavior of higher order principal components of forward rates as well as their as the quick decay of their eigenvalues are empirical facts which have no generic theoretical counterparts in HJM models, which have to be fine tuned to obtain these effects. This often leads to postulating complicated time dependent volatility structures to reproduces empirical stylized facts. Also, the fact that Eq. 4 is a first order SPDE means that there is no smoothing effect: an anomaly / non-smoothness in the initial condition will not be "arbitraged out" as in real markets by the evolution described in Eq.4. As illustrated in [5], these empirical facts can be easily accomodated by introducing a curvature dependent (second order derivative) term in Eq.4, leading to a parabolic SPDE for r.

Considering the forward rate curve $r_t = r(t,.)$ as the state variable, Eq. 4 can also be regarded as an infinite dimensional diffusion taking values in some functional (Hilbert) space H to be specified [10]:

$$dr_t = A.r_t dt + \Sigma.dB_t \tag{5}$$

$$A.r = \frac{\partial r}{\partial x} + \sigma(t,.)\int_0^{.} \sigma(t,u)du \tag{6}$$

where the operator A is now considered as an operator in H and B_t is an infinite dimensional cylindrical Brownian motion on H. Σ is a densely defined linear operator in H which, in the case of a finite factor model, reduces to an operator of finite rank. The Hilbert space H should contain all possible forward rate curves; the choice of the space is discussed in detail in [8], see also [1]. However from a statistical perspective many joint specifications of (Σ, H) lead to the same (weak) solution of the SDE; reciprocally, given certain statistical properties of the yield curve, the choice of H influences that of Σ [5]. The main point here is that it is the nature of A which determines the properties of the evolution semigroup of the forward rates, and the HJM operator in Eq.5 does not generically possess the properties pointed out above.

The statistical properties described above –mean reversion, structure of principal components, fast decay of PC variances, inreasingly oscillating PCs– can be easily reproduced by choosing A as a Sturm Liouville operator:

$$dr_t = A.r_t dt + \Sigma.dB_t \tag{7}$$

$$A.f = \frac{1}{s(x)}\frac{\partial}{\partial x}[a(x)\frac{\partial f}{\partial x}]. \tag{8}$$

This means that $r(t, x)$ is now the solution of a parabolic stochastic PDE. We study in [5] a simple example of this model where

$$A = \frac{\partial}{\partial x} + \kappa \frac{\partial^2}{\partial x^2} \tag{9}$$

and show how all the statistical properties observed in empirical data can be recovered with a *single* additional parameter κ, without introducing a complex time dependent volatility structure.

Considered as a diffusion in Hilbert space, the model has even a simpler structure: it is a Hilbert space valued Ornstein Uhlenbeck, the infinite dimensional analog of the Vasicek model, in which the short rate is an Ornstein Uhlenbeck process. In fact, the projections of r_t onto the eigenvectors of A are scalar Ornstein Uhlenbeck processes, independent if the process is covariance stationary [5]. This model can then also be considered as an infinite factor model with factors which are Ornstein Uhlenbeck processes.

The remaining question is whether the parabolic SPDE model is compatible with the HJM arbitrage condition (2). This is where the infinite dimensional nature of the driving noise plays a role. The infinite dimensional nature of the noise is shown to guarantee the absence of arbitrage in this framework: the non-degeneracy of the volatility operator Σ allows a parabolic SPDE of the type (7) to be expressed in the form 5; see [6]. Thus infinite factor HJM models appear not simply as technical generalization of finite factor models but as an elegant way to conciliate arbitrage restrictions and econometric observations.

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Tracking Volatility

Jakša Cvitanić, Robert Liptser, and Boris Rozovskii

Abstract: This paper is concerned with nonlinear filtering of the volatility coefficient in a Black-Scholes type model that allows stochastic volatility. More specifically we assume that the asset price process $S = (S_t)_{t\geq 0}$ is given by

$$dS_t = rS_t dt + \sqrt{v_t} S_t dB_t,$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion and v_t is the (stochastic) volatility process. Moreover, it is assumed that $v_t = v(\theta_t)$ where v is a nonnegative function and $\theta = (\theta_t)_{t\geq 0}$ is a homogeneous Markov jump process, taking values in the finite alphabet $\{a_1, \ldots, a_M\}$, with the intensity matrix $\Lambda = ||\lambda_{ij}||$ and the initial distribution $p_q = P(\theta_0 = a_q),\ q = 1, \ldots, M$.

The random process θ is unobservable. Following to Frey and Runggaldier [4], we assume also that the asset price S_t is measured only at random times $0 < \tau_1 < \tau_2 < \ldots,$. This assumption is designed to reflect the discrete nature of high frequency financial data (e.g. tick-by-tick stock prices). The random time moments τ_k "represent instances at which a large trade occurs or at which a market maker updates his quotes in reaction to new definition."

In the above setting the problem of volatility estimation is reduced naturally to a special nonlinear filtering problem. We remark that while quite natural, the latter problem does not fit into the "standard" framework and requires new technical tools.

In this paper, we derive a mean-square optimal recursive Bayesian filter for θ_t based on the observations of $S_{\tau_1}, S_{\tau_2}, \ldots$ for all $\tau_k \leq t$. In addition we derive Duncan-Mortensen-Zakai and Wonham-Kushner type equations for posterior distributions of θ_t and prove uniqueness of their solutions.

1. Introduction

In the classical Black-Scholes model for financial markets, the stock price S_t is modeled as a Geometric Brownian motion, namely with diffusion coefficient equal to σS_t, where "volatility" σ is assumed to be constant. The volatility parameter is the most important one when it comes to option pricing, so, naturally, many researchers generalized the constant volatility model to so-called stochastic volatility models, where σ_t is itself random and time dependent. There are two basic classes of models - complete and incomplete. In complete models the volatility is assumed to be a functional of the stock price, while in incomplete models it is driven by some other source of noise, possibly correlated with the original Brownian motion. In this paper we study a particular

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incomplete model in which the volatility process is independent of the driving Brownian motion process. This has the economic interpretation of the volatility being influenced by market, political, financial and other factors which are independent of the systematic risk (the Brownian motion process) associated to the particular stock price under study. It is also close in spirit to the way traders think about volatility - as a parameter that changes with time, and whose future value in a given period of interest has to be estimated/predicted. They need the estimate of the volatility to decide how they will trade in financial markets, especially derivatives markets. In fact, the notion of volatility is so important to traders that they even quote option prices in volatility units rather than in dollars (or some other currency). It is also important for investment banks who need to know the model for the future volatility in order to be able to price custom-made financial products, whose payoff depends on the future path of the underlying stock price. Very recently new contracts have been developed, which directly trade the volatility itself (volatility swaps, for example). We plan to address the issue of pricing options within the framework of our model in future research.

Estimation of volatility from observed stock prices is not a trivial task in either complete or incomplete models, in part because the prices are observed at discrete, possibly random time points. Since volatility itself is not observed, it is natural to apply filtering methods to estimate the volatility process from the historical stock price observations. Nevertheless, this has only recently been investigated in continuous-time models, in particular by Frey and Runggaldier [4]. Most of the earlier research was concentrated on either the time series approach (ARCH-GARCH models) or on calibration methods. The latter do not use historical data to estimate the volatility, but try instead to find the volatility process that matches best (in some appropriately defined sense) the observed present prices of frequently traded option contracts. In other words, methods have been developed to solve the "inverse problem" of finding the volatility process such that the corresponding theoretical prices of options become close, in some sense, to the observed market prices of options. In this paper we adopt the approach of filtering the volatility from the observed historical stock prices. In the future work we plan to combine the two approaches, thereby estimating the model by taking into account both the historical behavior of volatility (as indicated through past stock prices), as well as the market opinion about its future behavior (as indicated through present option prices).

More precisely, we are concerned with nonlinear filtering of the volatility coefficient in a Black-Scholes type model in which the asset price process $S = (S_t)_{t\geq 0}$ is given by

$$dS_t = rS_t dt + \sqrt{v_t} S_t dB_t, \tag{1}$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion and v_t is the (stochastic) volatility process (see [3]). Moreover, it is assumed that $v_t = v(\theta_t)$, $v(x)$ is bounded positive (known) function, and $\theta = (\theta_t)_{t\geq 0}$ is a homogeneous Markov jump process taking values in the finite alphabet $\{a_1, \dots, a_M\}$ with the intensity matrix $\Lambda = ||\lambda_{ij}(t)||$ and the initial distribution $p_q = P(\theta_0 = a_q)$, $q = 1, \dots, M$.

The random process θ is unobservable. Following Frey and Runggaldier [4], we assume also that the asset price S_t is observed only at random times $0 := \tau_0 < \tau_1 < \tau_2 < \dots$,. This assumption is designed to reflect the discrete nature of high frequency financial data (e.g. tick-by-tick stock prices). The random time moments τ_k "represent instances at which a large trade occurs or at which a market maker updates his quotes in reaction to new information" (see [3]).

In the above setting the problem of volatility estimation is reduced naturally to a special nonlinear filtering problem. We remark that while quite natural, the latter problem does not fit into the "standard" framework and requires new technical tools.

Frey and Runggaldier [4] derived a Kallianpur-Striebel type formula for the optimal mean-square filter for θ_t based on the observations of $S_{\tau_1}, S_{\tau_2}, \dots$ for all $\tau_k \le t$ and investigated Markov Chain approximations for this formula. In this paper, we extend their result in that we derive exact Duncan-Mortensen-Zakai and Wonham-Kushner type filters for θ_t.

2. Preliminaries.

In this Section we introduce additional notation and further specialize the mathematical model.

To begin with let us consider the observation process. As in Frey and Runggaldier [4], we assume that the price process is observed only at random time moments $0 < \tau_1 < \tau_2 < \dots$. More specifically, observation process is the discrete-time stochastic process $(\tau_k, S_{\tau_k})_{k\ge 1}$. By technical reasons it is more convenient to deal with the Gaussian process $U_t = \log S_t$ ruther then the original price process S_t. Obviously,

$$U_t - U_{\tau_{k-1}} = \int_{\tau_{k-1}}^t \Big(r - \frac{1}{2}v(\theta_s)\Big)ds + \int_{\tau_{k-1}}^t v^{1/2}(\theta_s)dB_s.$$

Write $U_{k-1}(t) := U_t - U_{\tau_{k-1}}$ and set

$$Y_t = I_{\{\tau_{k-1}<t\le\tau_k\}} U_{k-1}(t)$$

where $\tau_0 = 0$. Of course, the sequences $(\tau_k, S_{\tau_k})_{k\ge1}$and $((\tau_k, Y_{\tau_k})_{k\ge1})$ provide the same information regarding the volatility process θ_t. Thus, without loss of generality, we can assume that the observation is given by the process

$$G_k = \{(\tau_k, Y_{\tau_k}); (\tau_{k-1}, Y_{\tau_{k-1}}); \dots ; (\tau_1, Y_{\tau_1})\}.$$

The counting process

$$N_t = \sum_{k=1}^{\infty} I(\tau_k \ge t)$$

and the random integer-valued measure

$$\mu\big([0,t]\times\Gamma\big) = \int_0^t I(Y_s \in \Gamma)dN_s$$

will play an important role in the future.

We assume that the triple (B_t, θ_t, N_t) are defined on a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ subject to standard conditions. Let $(\mathcal{G}_t)$ be the right continuous complete filtration generated by μ, that is by the processes $\int_0^t \int_{\mathbb{R}} f(y)\mu(ds,dy)$ with bounded (measurable) functions f. Obviously, $(\mathcal{G}_t) \subset (\mathcal{F}_t)$.

The following assumptions will be in force everywhere below:

A The Brownian motion (B_t) is independent of (θ_t, N_t).

B The counting process (N_t) is a double Poisson (Cox) process with the stochastic intensity $n(\theta_t)$, where n is bounded and strictly positive function. The jumps of the processes (θ_t) and (N_t) are disjoint.

The first part of **B** means that $N_t - \int_0^t n(\theta_s)ds$ is a martingale with respect to $(\mathcal{F}_t)$ or, equivalently, that $\widehat{N}_t = \int_0^t n(\theta_s)ds$ is an $(\mathcal{F}_t)$ −adapted compensator of N_t

Note also, that assumption **A** yields that the conditional distribution

$$P\left(U_{k-1}(t) \le y | \mathcal{G}_{t-}; \theta_{[0,\tau_k]}\right)$$

is Gaussian with the density

$$p_{(k-1),\theta}(y) = \frac{1}{\sqrt{2\pi\sigma^2_{k-1}(\theta,t)}} e^{-\frac{(y-m_{k-1}(\theta,t))^2}{2\sigma^2_{k-1}(\theta,t)}}$$

where

$$m_{k-1}(\theta,t) = \int_{\tau_{k-1}}^{t\wedge\tau_k} \big(r - \frac{1}{2}v(\theta_s)\big)ds$$
$$\sigma^2_{k-1}(\theta,t) = \int_{\tau_{k-1}}^{t\wedge\tau_k} v(\theta_s)ds.$$

It is also convinient to introduce the vector process $\mathbf{I}_t$ with the components

$$I(\theta_t = a_1), I(\theta_t = a_2), \dots, I(\theta_t = a_M).$$

Obviously, $\mathbf{I}_t$ is just another representation of θ_t.

Set

$$\mathfrak{v} = v(a_1), v(a_2), \dots, v(a_M)$$
$$\mathfrak{n} = n(a_1), n(a_2), \dots, n(a_M)$$

It is readily checked that $v(\theta_t) = \mathfrak{v}\mathbf{I}_t$, $n(\theta_t) = \mathfrak{n}\mathbf{I}_t$, and the process $\mathbf{I}_t$ is a semimartingale (with respect to $(\mathcal{F}_t)$) given by the Itô equation

$$\mathbf{I}_t = \mathbf{I}_0 + \int_0^t \Lambda\mathbf{I}_s ds + \mathfrak{I}_t \qquad (2)$$

where $(\mathfrak{I}_t)$ is a purely discontinuous vector martingale. The paths of every component of $(\mathfrak{I}_t)$ is right continuous with unit size jumps, and admits left-hand limits (see Lemma 9.2 in [8]).

Note that the disjointness of jumps of (θ_t) and (N_t) is equivalent to the disjointness of jumps of $(\mathfrak{I}_t)$ and (N_t).

Write

$$\varrho_{k-1}(t,y) = \frac{E\big(\mathsf{diag}(\mathbf{I}_t)\mathfrak{n}^* p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}{E\big(\mathfrak{n}\mathbf{I}_t p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}$$

and

$$\varrho(t,y)=\sum_{k=1}^{\infty} I_{\{\tau_{k-1}<t\le\tau_k\}}\varrho_{k-1}(t,y)$$

where $p_{(k-1),\theta}(y)$ is the Gaussian density introduced above, * is the transposition symbol, and $\mathsf{diag}(\mathbf{I}_t)$ is the scalar matrix with the diagonal $\mathbf{I}_t$.

Note that the integrand in $\varrho_k(t,y)$ is a functional of $(\mathbf{I}_t)$. Indeed, the parameters of the density $p_{(k-1),\theta}(y)$ can be rewritten as follows:

$$m_{k-1}(\theta,t)=\int_{\tau_{k-1}}^{t\wedge\tau_k}\big(r-\frac{1}{2}\mathfrak{v}\mathbf{I}_s\big)ds$$
$$\sigma^2_{k-1}(\theta,t)=\int_{\tau_{k-1}}^{t\wedge\tau_k}\mathfrak{v}\mathbf{I}_s ds.$$

3.Filters

3.1 The Wonham-Kushner filter

Set $\pi_t(\mathbf{I})=E(\mathbf{I}_t|\mathcal{G}_t)$. The process $\pi_t(\mathbf{I})$ describes the dynamics of posterior conditional distributions

$$P(\theta_t=a_1|\mathcal{G}_t), P(\theta_t=a_2|\mathcal{G}_t),\dots, P(\theta_t=a_M|\mathcal{G}_t)$$

Of course, this posterior distribution fully defines the mean-square optimal nonlinear filter.

The main result of this paper is formulated in the following Theorem.

Theorem. *The posterior distribution $\pi_t(\mathbf{I})$ verifies the following equation*

$$\begin{aligned}\pi_t(\mathbf{I}) &= \pi_0(\mathbf{I})+\textstyle\int_0^t\Lambda\pi_s(\mathbf{I})ds\\ &+\textstyle\sum_{k:\tau_k\le t}\big(\varrho_{k-1}(\tau_k,Y_{\tau_k})-\pi_{\tau_k-}(\mathbf{I})\big)\\ &-\textstyle\int_0^t\Big(\mathsf{diag}(\pi_s(\mathbf{I}))-\pi_s(\mathbf{I})\pi^*_s(\mathbf{I})\Big)\mathfrak{n}^*ds.\end{aligned} \tag{3}$$

Equations for posterior distributions of jump processes is often referred to as Wonham-Kushner equations and we will follow this tradition. It is redily checked that Wonham-Kushner equation can be rewritten as follows:

$$\begin{aligned}\pi_t(\mathbf{I}) &= \pi_0(\mathbf{I})+\int_0^t\Lambda\pi_s(\mathbf{I})ds\\ &\quad+\int_0^t\int_{\mathbb{R}}\big(\varrho(s,y)-\pi_{s-}(\mathbf{I})\big)\big(\mu-\widetilde{\nu}\big)(ds,dy))\end{aligned}$$

where

$$\widetilde{\nu}(dt,dy)=\sum_{k=1}^{\infty} I_{\{\tau_{k-1}<t\le\tau_k\}}E\Big(\mathfrak{n}\mathbf{I}p_{(k-1),\theta}(y)|\mathcal{G}_t\Big)dtdy$$

is the compensator of μ with respect to the "observation". Moreover

$$\int_0^t \int_{\mathbb{R}} \big(\mu - \widetilde{\nu}\big)(ds, dy)$$

is a martingale and "$\mu - \widetilde{\nu}$" plays the role of the innovation process. The latter form of the Kushner-Wonham equation is quite similar to its well known version developed for diffusion type observation process (see [8], [12]).

The proof of the Theorem is based on a general filtering result for semimartingales - Theorem 4.10.1 in Liptser and Shiryaev [9].

Continuous time filtering equation with discontinuous observation were addressesed by many authors (see e.g. Grigelionis [5], [6]; Elliott, Aggoun and Moore [2]; Krylov and Zatezalo [7] etc.) Unfortunately, these works do not cover our setting.

3.2 Duncan-Mortensen-Zakai equation

If $n(x) \equiv 1$, i.e. N_t is the Poisson process with the unit intensity, the jump moments $\tau_1, \tau_2, \dots$, do not carry any "information" regarding the volatility and the observation process $\mathcal{G}$ can be reduced to the sequence $Y_{\tau_1}, Y_{\tau_2}, \dots$ It is readily checked that in this case the filtering equation takes a much simpler form:

$$\pi_t(\mathbf{I}) = \pi_0(\mathbf{I}) + \int_0^t \Lambda \pi_s(\mathbf{I}) ds + \sum_{k:\tau_k \le t} \varrho_{k-1}(\tau_k, Y_{\tau_k})$$

where

$$\varrho_{k-1}(t, y) = \frac{E\big(\mathbf{I}_t p_{(k-1),\theta}(y) | \mathcal{G}_{t-}\big)}{E\big(p_{(k-1),\theta}(y) | \mathcal{G}_{t-}\big)}$$

This fact inspires the idea to find a new measure P' that is absolutely continuous with respect to the original measure P and such that (N_t, P') is the Poisson process with the unit intensity. Such change of the measure is possible, if the filtering problem is treated on a finite time interval $[0, T]$. Specifically, the new measure is defined by

$$dP' = \mathfrak{z}_T dP_T$$

where P_T is the restriction of P to $\mathcal{F}_T$, and $t \le T$,

$$\mathfrak{z}_t = \exp\Big(\int_0^t - \log n(\theta_{s-}) dN_s - \int_0^t \frac{1 - n(\theta_s)}{n(\theta_s)} d\widehat{N} s\Big) \tag{4}$$

(see e.g. Section 19.4 in Ch. 19 [8]).

In the future, an expectation with respect to the measure P' will be denoted E'. Write $\pi'_t(\mathfrak{z}\mathbf{I}) = E'\big(\mathfrak{z}_t^{-1} \mathbf{I}_t | \mathbf{G}_t\big)$, and $\pi'_t(\mathfrak{z}) = E'\big(\mathfrak{z}_t^{-1} | \mathbf{G}_t\big)$). By the Kallianpur-Striebel formula, we have

$$\pi_t(\mathbf{I}_t) = \frac{\pi'_t(\mathfrak{z}^{-1}\mathbf{I})}{\pi'_t(\mathfrak{z}^{-1})}.$$

The equation for $\pi'_t(\mathfrak{z}^{-1}\mathbf{I})$ in the case of diffusion type observation (usually referred to as the Duncan-Mortensen-Zakai equation) is well known (see e.g. [12]). To derive an analog of this equation in the present setting, we start with the Kushner type equations for the time evolution of $\pi'_t(\mathfrak{z}^{-1}\mathbf{I})$ and $\pi'_t(\mathfrak{z}^{-1})$. To this end, let us discuss briefly some important features of our model with respect to the new measure P'. Since (θ_t) and (N_t) have disjoint jumps, the distribution of the process θ remains the same under the new measure. Similarly, since the predictable covariance of (B_t) and $(N_t - \widehat{N}_t)$ is zero, it ie readily checked that the process (B_t) is a standard Brownian motion independent of (θ_t) and (N_t). Further, since (N_t) is the Poisson process with the unit intensity, we have

$$\varrho'_{k-1}(t,y) = \frac{E'\big(\mathbf{I}_t p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}{E'\big(p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}$$

and $\varrho'(t,y) = \sum\limits_{k=1}^{\infty} \varrho'_{k-1}(t,y)$. The compensator $\widetilde{\nu}$ is is given now by the formula

$$\widetilde{\nu}'(dt,dy) = \sum_{k=1}^{\infty} I_{\{\tau_{k-1}<t\le\tau_k\}} E'\big(p_{(k-1),\theta}(y)|\mathcal{G}_t\big)dtdy.$$

Making use of Theorem 4.10.1 from [9], one can show that the following representation for $\pi'_t(\mathfrak{z}^{-1})$ holds true:

$$\pi'_t(\mathfrak{z}^{-1}) = 1 + \int_0^t \int_{\mathbb{R}} \varrho'(s,y)(\mu - \widetilde{\nu}')(ds,dy)$$

Write

$$w(s,y) = \frac{E(\mathfrak{n}\mathbf{I}p_{(k-1),\theta}(y)|G_{s-})}{E(p_{(k-1),\theta}(y)|G_{s-})}. \tag{5}$$

It is readily checked that

$$\frac{\varrho'(s,y)}{\pi'_{s-}(\mathfrak{z}^{-1})} = w(s,y) - 1 \tag{6}$$

Theorem 4.10.1 in [9], (5) , and (6) yield the following linear equation for $\pi'_t(\mathfrak{z}^{-1})$:

$$\pi'_t(\mathfrak{z}^{-1}) = 1 + \int_0^t \int_{\mathbb{R}} \pi'_{s-}(\mathfrak{z}^{-1})\big(w(s,y) - 1\big)(\mu - \widetilde{\nu}')(ds,dy).$$

Obviously the Dolean-Dade exponent is the unique solution of this equation. Thus we have

$$\begin{aligned}\pi'_t(\mathfrak{z}^{-1}) &= \exp\Big(\int_0^t \int_{\mathbb{R}} \log w(s,y)\mu(ds,dy) - \\ &\quad - \int_0^t \int_{\mathbb{R}} (w(s,y) - 1)\widetilde{(\nu)}(ds,dy)\Big)\end{aligned}$$

To derive the Duncan-Mortensen-Zakai filter for $\pi'_t(\mathfrak{z}^{-1}\mathbf{I})$, we begin with deriving the semimartingale decomposition for $\mathfrak{z}_t^{-1}\mathbf{I}_t$. From (4) by the Itô formula, we have

$$\mathfrak{z}_t^{-1} = 1 + \int_0^t \mathfrak{z}_{s-}^{-1}(n(\theta_{s-}) - 1)(dN_s - ds).$$

Now, taking into account (1) and applying Itô's formula to $\mathfrak{z}_t\mathbf{I}_t$ we get

$$\begin{aligned}\mathfrak{z}_t\mathbf{I}_t &= \mathbf{I}_0 + \int_0^t \Lambda\mathfrak{z}_s\mathbf{I}_s ds + \int_0^t \mathfrak{z}_{s-}\mathbf{I}_{s-}d\mathfrak{I}_s \\ &\quad + \int_0^t \mathfrak{z}_{s-}\mathbf{I}_{s-}(n(\theta_{s-})-1)(dN_s - ds).\end{aligned}$$

Now, applying Theorem 4.10.1 in [9], we find

$$\begin{aligned}&\pi'_t(\mathfrak{z}^{-1}\mathbf{I}) \\ =\ & \pi'_0(\mathbf{I}) + \int_0^t \Lambda\pi'_s(\mathfrak{z}^{-1}\mathbf{I})ds \\ +\ & \int_0^t\int_{\mathbb{R}} \big(\varphi'(s,y) - \pi'_{s-}(\mathfrak{z}^{-1}\mathbf{I})\big)(\mu - \widetilde{\nu}')(ds,dy),\end{aligned}$$

where $\varphi'(t,y) = \sum\limits_{k=1}^{\infty} I_{\{\tau_{k-1}<t\le\tau_k\}}\varphi'_{k-1}(t,y)$ and

$$\varphi'_{k-1}(t,y) = \frac{E'\big(\mathsf{diag}(\mathfrak{z}_t^{-1}\mathbf{I}_t)\mathfrak{n}^* p_{(k-1),\theta}(y)|\mathcal{G}'_{t-}\big)}{E'\big(p_{(k-1),\theta}(y)|\mathcal{G}'_{t-}\big)}.$$

The above results can be reformulated as follows:

Theorem. *The Duncan-Mortensen-Zakai filter is given by the equations*

$$\begin{aligned}\pi'_t(\mathfrak{z}^{-1}) &= 1 + \sum_{k:\tau_k\le t}\pi'_{\tau_k-}(\mathfrak{z}^{-1})\big(w(\tau_k,Y_{\tau_k})-1\big) \\ &\quad - \int_0^t \pi'_s(\mathfrak{z}^{-1})\big(\pi_s(\mathfrak{n}\mathbf{I})-1\big)ds\end{aligned}$$

and

$$\begin{aligned}&\pi'_t(\mathfrak{z}^{-1}\mathbf{I}) \\ =\ & \pi'_0(\mathbf{I}) + \int_0^t \Lambda\pi'_s(\mathfrak{z}^{-1}\mathbf{I})ds \\ &+ \sum_{k:\tau_k\le t}\big(\varphi'(\tau_k,Y_{\tau_k}) - \pi'_{\tau_k-}(\mathfrak{z}^{-1}\mathbf{I})\big) \\ &- \int_0^t \Big(\mathsf{diag}(\pi'_s(\mathfrak{z}^{-1}\mathbf{I})\mathfrak{n}^* - \pi'_s(\mathfrak{z}^{-1}\mathbf{I})\Big)ds.\end{aligned}$$

The proof follows from the following relations:

$$\begin{aligned}&\int_{\mathbf{R}}\widetilde{\nu}'(ds,dy) = ds \\ &\int_{\mathbf{R}} w(s,y)\widetilde{\nu}'(ds,dy) = \pi_s(\mathfrak{n}\mathbf{I})ds \\ &\int_{\mathbf{R}} \varphi'(s,y)\widetilde{\nu}'(ds,dy) = \mathsf{diag}\big(\pi'_s(\mathfrak{z}^{-1})\big)ds.\end{aligned}$$

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Inverse problems for stochastic partial differential equations

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1. Motivation and examples

Many physical models are represented by linear and nonlinear stochastic partial differential equations. The linear SPDEs are of the form

$$du(t,x) = \left(A_0 u(t,x) + \sum_{k=1}^{r} \theta_k(t,x) A_k u(t,x)\right) dt + dW(t,x), \qquad (1)$$
$$0 \le t \le T, x \in G \subset \mathbb{R}^d,$$

where $A_0 + \sum_{k=1}^{r} \theta_k A_k$ is an elliptic differential operator with unknown coefficients $\theta_k = \theta_k(t,x)$, and W is a cylindrical Brownian motion. The cylindrical Brownian motion is used only to simplify the presentation. A more general noise term, $BdW(t,x)$, with some correlation operator B can be considered without difficulty.

Example 1: **Sea surface temperature anomalies.** The study of the evolution of the sea surface temperature anomalies $u(t,x)$ (that is, u is the deviation of the sea surface temperature from some average value), where t denotes time and x, the space variable, is of interest for environmental research as well as for the development of modern technological devices sensitive to water temperature. The evolution of the temperature anomalies is described by a linear stochastic partial differential equation. The stochastic nature of the equation is due to short-term atmospheric forcing. Frankignoul and Reynolds (1983) suggested adding a noise term such as cylindrical Brownian motion $W(t,x)$, because typical weather conditions do not have an influence on the sea temperature for more than three days. This assumption was supported by experimental data (see also Ostrovskii and Piterbarg (1997)). The evolution of the sea surface temperature anomalies has the following form

$$du(t,x) = (\kappa \Delta u(t,x) - (v, \nabla)\, u(t,x) - \lambda u(t,x))\ dt + dW(t,x).$$

Model parameters are the heat conductivity κ, Newton's cooling coefficient λ, and the velocity vector v of the top layer of the ocean. It is desirable to estimate these parameters to get a more accurate model for the temperature evolution.

Example 2: **Groundwater flow.** The groundwater level fluctuates in response to the combined action of deep percolation, evaporation, pumping, and groundwater depletion. A model for a one-dimensional groundwater flow is (see Serrano and Unny

(1987):

$$\begin{aligned}
du(t,x) &= K\Delta u(t,x)\,dt + dW(t,x), \quad x \in (0,L), t > 0\\
u(t,0) &= C,\\
\frac{\partial u}{\partial x}(t,L) &= 0,\\
\frac{\partial u}{\partial x}(0,x) &= v_0(x)
\end{aligned}$$

where u is the deviation of the groundwater level from the steady state. The physical parameter K describes water percolation and is the model parameter to be estimated from the measurements of u. The initial condition v_0 is the initial hydraulic gradient and is a random variable due to inaccurate measurements. There are a number of other interesting models in hydrology that can be reduced to (1) (see e.g. Serrano and Adomin (1996)).

2. Estimation

Being a relatively new area of stochastic analysis, stochastic partial differential equations have only recently been introduced as a modelling tool in applied sciences, and the approach to estimation has been mostly intuitive and experimental. Therefore, identification in many models is an open problem.

Rigourous mathematical study of the estimation problems for model (1) involves asymptotical analysis of the corresponding estimator. Just as in other statistical problems, small noise or long time asymptotics can be used. In the small noise asymptotics, estimation problems for (1) were studied by Ibragimov and Khasminski (1997 and later), and in the long time asymptotics, by Aihara (1992), Bagchi and Borkar (1984) and others.

Still, under certain conditions, a consistent estimation in (1) is possible even if the amplitude of the noise and the observation time are fixed. The solution is projected on a special orthonormal basis in $L_2(G)$, and the dimension of the projection becomes the new asymptotic parameter. The following *commutativity assumption* is often made to simplify the analysis: the operators $A_0, \ldots, A_r$ have a common system of eigenfunctions $\{\phi_i,\ i \geq 1\}$ that is an orthonormal basis in $L_2(G)$. The solution is then projected on this basis, and the estimator is based on the first N Fourier coefficients of the solution.

The first examples of such estimators for the model with $r = 1$ and the corresponding coefficient θ_1 independent of t, x were studied by Huebner et al. (1992). This parametric model was further analysed, with and without the commutativity assumption, by Huebner and Rozovskii (1995), Huebner (1997), Piterbarg and Rozovskii (1997), Lototsky (1997), Lototsky and Rozovskii (1999).

In the non-parametric setting, the following particular case of (1) was studied under the commutativity assumption:

$$\begin{aligned}
du(t,x) &= (A_0 + \theta_0(t)A_1)\,u(t,x)\,dt + dW(t,x), \quad t \in (0,T],\ x \in G\\
u(0,x) &= u_0(x), \qquad u|_{\partial G} = 0.
\end{aligned}$$

The set Θ of admissible functions θ_0 is such that $A_0 + \theta_0(t)A_1$ is a strongly elliptic operator for all $t \in [0,T]$ and all $\theta_0 \in \Theta$. Possible estimators for θ_0 in this model are sieve estimator (Huebner and Lototsky (2000a)) or kernel estimator (Huebner and Lototsky (2000b)).

If the initial condition u_0 is not random, then the Fourier coefficients $u_1(t), \dots, u_N(t)$ are independent Ornstein–Uhlenbeck process, and the drift of each process contains the unknown function $\theta_0(t)$ and the eigenvalues of the operators A_0, A_1. As a result, we get the drift estimation problem when the observations are independent but not identically distributed diffusions. In the i.i.d. setting, this problem was studied by Nguyen and Pham (1982) using sieves and by Kutoyants (1984) using kernel estimators.

Sieve estimator

The sieve estimator $\hat{\theta}^N$ of $\theta_0(t)$ is obtained by maximizing the likelihood function based on the N Fourier coefficients of the solution. The maximization is carried out over a sieve Θ_N, that is, a finite dimensional subspace of Θ. The family of spaces $\{\Theta_N,\ N \geq 1\}$ is chosen so that the approximation error decreases to zero as the the number N of observations increases. We assume that every function $\theta \in \Theta$ can be represented as an infinite linear combination of known functions $\{h_j, j \geq 1\}$:

$$\theta(t) = \sum_{j=1}^{\infty} \theta_j h_j(t)$$

and the functions $\{h_j\}$ are orthonormal on $[0,T]$. If we choose the sieve Θ_N to be the span of $h_1(t), \dots, h_{d_N}(t)$, then the *sieve maximum likelihood estimate* will be of the form

$$\hat{\theta}^N = \sum_{j=1}^{d_N} \hat{\theta}_j h_j(t).$$

The numbers $\hat{\theta}_j$ are computed by solving a linear system of equations; the matrix and the right hand side of the system are determined by the observations.

The choice of the number d_N, the dimension of Θ_N, in relation to the number N of Fourier coefficients of the solution, is an essential part of designing the estimator. In the paper Huebner and Lototsky (2000a) we describe the procedure for choosing d_N and prove consistency and asymptotic normality of the estimator.

Kernel estimator

We consider the following estimate $\hat{\theta}^N$ of θ_0:

$$\hat{\theta}^N(t) = \int_0^T R_{h_N}(s-t)dX^N(s),$$

where R is a kernel function, $R_{h_N}(s) = R(s/h_N)/h_N$ with $h_N \to 0, N \to \infty$, and X^N is a certain process constructed from $u_1, \dots, u_N$. In the apper Huebner and Lototsky (2000b) we prove the mean-square convergence of the type

$$\lim_{N\to\infty} \sup_{\theta_0 \in \Theta} \sup_{t \in [t_1,t_2]} N^{\gamma} E|\hat{\theta}^N(t) - \theta_0(t)|^2 < \infty,$$

and explicitly compute the rate $\gamma > 0$ which is determined by the parameter class Θ and the orders of the operators A_0, A_1.

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An estimation problem for partial stochastic differential equations

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Introduction. Let G be a bounded region in R^d with smooth boundary $\partial G, Q_T$ means cylinder $[0,T] \times G$ and $\Gamma_T = [0,T] \times \partial G$. Let

$$L = \sum_{|k| \le 2p} a_k(t,x) D_x^k$$

be a strongly elliptic differential operator with the coefficients a_k being smooth functions defined on Q_T. Here $k = (k_1, \dots, k_d), |k| = \sum_{i=1}^d k_i, D_x^k = D_{x_1}^{k_1} \dots D_{x_d}^{k_d}$.

Below we consider the problem of estimation of a functional parameter $\theta(x), x \in G$, on the base of observations $u_\epsilon(t,x), (t,x) \in Q_T$, and u_ϵ is the solution to the problem

$$du_\epsilon(t) = Lu_\epsilon dt + \theta(x) g(u_\epsilon) dt + \epsilon dw(t), \quad u_\epsilon(0,x) = \phi(x), \quad B(x, D_x) u|_{\Gamma_T} = h. \qquad (1)$$

A general estimation problem has been set up in [1]. Different estimation problems for the linear case, $g(u) = u$, has been considered in [3]. Below we continue these investigations.

Denote u_0 the solution to the nondisturbed problem

$$\frac{\partial u}{\partial t} = Lu + \theta g(u)$$

with the same boundary and initial conditions as in (1). We suppose that the operator L, the function g, the region G, the initial and boundary conditions satisfy such regularity conditions which ensure the existence and unicity of the sufficiently smooth solution u in some interval $[0,T]$. In particular, we suppose that there exist constants c_1, c_2 such that

$$|g(u)| \le c_1 |u|, |g(u_1) - g(u_2)| \le c_2 |u_2 - u_1|.$$

On such regularity conditions see, for example, [4].

We restrict our consideration by the case when the noise $w(t)$ is the cylindrical Wiener process (see [5]). We assume below that the order $2p$ of the operator L is strongly greater than the dimension d. Under this condition the solution u_ϵ to (1) exists, is unique and with probability one satisfies Hölder conditions of some positive order $\gamma > 0$ (see [5], [1]).

Denote $< .,. >, |.|$ inner product and norm on the Hilbert space $L_2(G)$ of functions of x. Denote $\mathcal{L}_2$ the Hilbert space of $L_2(G)$-valued fuctions $\varphi(t)$ with the scalar product and norm

$$(\varphi, \psi) = \int_0^T < \varphi(t), \psi(t) > dt, \quad ||\varphi|| = \int_0^T |\varphi|^2 dt.$$

We denote $||.||_H$ norm in a normed space H.

Below we suppose that the operator L, the function g and the noise level ϵ are known to the statistician and that the only unknown parameter is the function $\theta(x)$. However we know that $\theta \in \Theta$ where Θ is a known subset of $L_2(G)$. We assume also that all functions in Θ are continuous. Our statistical problem looks as follows. We are given a known function $\Phi : L_2(G) \longrightarrow H$ where H is a Hilbert or an Euclidean space. The problem is to estimate the value $\Phi(\theta)$ of the known function Φ at the unknown point θ.

Here are a few examples:

1. $\Phi(\theta) = \int_G \varphi(x)\theta(x)dx$ where φ is a known function from $L_2(G)$ (a bounded linear functional of θ).

2. $\Phi(\theta) = \int_G |\theta(x)|^2 dx$ (a quadratic functional of θ).

3.$\Phi(\theta) = \int_G K(x,y)\theta(y)dy, \quad \int_G \int_G K^2(x,y)dxdy < \infty$(a linear Hilbert-Schmidt operator of θ).

4.$\Phi(\theta) = \theta(x_0)$ where x_0 is a given point of G (an unbounded linear functional of θ).

5. $\Phi(\theta) = \theta$ (a linear bounded but not Hilbert-Schmidt operator).

2. Asymptotically efficient estimators. Definition. Denote $\mathbf{P}_\theta^{(\epsilon)}$ the distribution of the solution u_ϵ to (1). It follows easely from the results of Kozlov [6] that the likelihood ratio

$$\frac{d\mathbf{P}_{\theta+\mathbf{h}}^{(\epsilon)}}{d\mathbf{P}_\theta^{(\epsilon)}}(u_\epsilon) = \exp\left\{\frac{1}{\epsilon}\int_0^T < hg(u_\epsilon), dw(t) > -\frac{1}{2\epsilon^2}\int_0^T |hg(u_\epsilon)|^2 dt\right\}. \qquad (2)$$

To begin with suppose that the set Θ is one-dimensional and consists of all functions $\theta(x) = \alpha e(x), \quad e(x) \equiv 1, \quad -\alpha_0 \leq \alpha \leq \alpha_0$. Let $\Phi(\theta) = \alpha$. It means that in fact we deal with parametric problem of estimation of the one-dimensional parameter α. It follows easely from (2) that the family of measures $\mathbf{P}_\alpha^{(\epsilon)}$ satisfies Le Cam's LAN condition (see [7], ch.2):

$$\frac{d\mathbf{P}_\alpha^{(\epsilon)}}{d\mathbf{P}_\mathbf{0}^{(\epsilon)}} = \exp\left\{\frac{\alpha}{\epsilon}\int_0^T < g(u_0), dw(t) > -\frac{\alpha^2}{2\epsilon^2}\int_0^T |g(u_0)|^2 dt + o_{pr}(1)\right\}.$$

The classic Hajek-LeCam inequality (see [7], ch.2) implies then that for any $\delta > 0$

$$\liminf_\epsilon \sup_{|\beta|\leq\delta} \epsilon^{-2}\mathbf{E}_\beta|\alpha - \alpha^*|^\mathbf{2} \geq ||\mathbf{g}(\mathbf{u_0})||^{-\mathbf{2}} = \left(\int_\mathbf{0}^\mathbf{T}\int_\mathbf{G}(\mathbf{g}(\mathbf{u_0}(\mathbf{t},\mathbf{x})))^\mathbf{2}\mathbf{dtdx}\right)^{-\mathbf{2}}.$$

Evidently in this case $||g(u_0)||^2$ is Fisher's information of the problem. (We can consider even more general case when $g = g(u;\alpha_1,\ldots,\alpha_k)$ is a differentiable function of parameters α_j. In this case Fisher's information matrix of the problem is equal to $\mathbf{I} = \left(\int_\mathbf{0}^\mathbf{T}\frac{\partial\mathbf{g}}{\partial\alpha_\mathbf{i}}\frac{\partial\mathbf{g}}{\partial\alpha_\mathbf{j}}\mathbf{dt}\right)_{\mathbf{i,j=1,\ldots,k}}$.)

To treat our main problem, the nonparametric case, we notice that because of (2) the family of distributions $\{\mathbf{P}_\theta^{(\epsilon)}\}$ satisfies the LAN conditions in the sense of [8] with the norming operators $A_\epsilon = \epsilon(g(u_0)0^{-1}, (g(u_0))^{-1}$ denoting the operator of multiplication by the function $(g(u_0(t,x)))^{-1}$ (cf [3]). The version of Hajek-Le Cam inequality from [8] gives us the possibility to write some analogues of Hajek-LeCam inequality. For the sake of simplicity we restrict oureself by the case when the set Θ is sufficiently massive. Namely we assume that for any point $\theta \in \Theta$ the intersection of Θ with L_2-balls $O_r(\theta)$ with the center in θ and radius r is dense in $O_r(\theta)$ for sufficiently small r.

Theorem 1. *Let $\Phi(\theta)$ be a Frechét differentiable function. Assume that the derivative $\Phi'(\theta)$ is a Hilbert- Schmidt operator. Then for any estimator Φ_ϵ of $\Phi(\theta)$*

$$\inf_V \liminf_{\epsilon\to 0} \sup_{v\in V} \epsilon^{-2}\mathbf{E_v}||\Phi(\theta)-\Phi_\epsilon||^2_{\mathbf{H}} \geq \mathbf{tr}(\Phi'\mathbf{v_0^2}(\Phi')^*) \tag{3}$$

where inf *is taken over all L_2-vicinities V of θ and v_0^2 denotes the operator of multiplication by the function $v_0^2 = \left(\int_0^T g(u_0(t,x))^2 dt\right)^{-1}$.*

We call estimators $\hat{\Phi}_\epsilon$ asymptotically efficient if for them the equality signe is realized in (3).

Examples.

1. In the case of the example 1.1 the derivative $\Phi'(\theta) = \phi$ and

$$\liminf_{\epsilon} \sup_{v\in V} \epsilon^{-2}\mathbf{E_v}|\Phi(\theta)-\Phi_\epsilon|^2 \geq \int_{\mathbf{G}} \frac{\phi^2(\mathbf{x})}{\int_0^{\mathbf{T}}(\mathbf{g}(\mathbf{u_0}(\mathbf{t},\mathbf{x})))^2\mathbf{dt}}\mathbf{dx}.$$

2. In the case of the example 1.2 the derivative $\Phi'(\theta) = 2\theta$ and

$$\liminf_{\epsilon} \sup_{V} \epsilon^{-2}\mathbf{E_v}|\Phi(\theta)-\Phi_\epsilon|^2 \geq \mathbf{4}\int_{\mathbf{G}} \frac{\theta^2(\mathbf{x})}{\int_0^{\mathbf{T}}(\mathbf{g}(\mathbf{u_0}(\mathbf{t},\mathbf{x})))^2\mathbf{dt}}\mathbf{dx}.$$

3. In the case of the example 1.3 the derivative is the integral operator with the kernel K and

$$\liminf_{\epsilon} \sup_{V} \epsilon^{-2}\mathbf{E_v}|\Phi(\theta)-\Phi_\epsilon|^2 \geq \int_{\mathbf{G}}\int_{\mathbf{G}} \mathbf{K^2(x,y)v_0^2(x)dxdy}.$$

4. In the case of the examples 1.4, 1.5 $tr(\Phi' v_0^2(\Phi')^*) = \infty$ and we can say only that for any estimator Φ_ϵ

$$\liminf_{\epsilon} \sup_{V} \epsilon^{-2}\mathbf{E}|\Phi(\theta)-\Phi_\epsilon|^2 = \infty$$

and to treate these examples we need a different approach which we consider in the next section.

3. Estimation of θ. Lower bounds. Consider a metric space (M, ρ) where $M \subseteq L_2(G)$ and ρ is a metric such that

1. $\rho(x,y) \geq |x-y|, \quad c > 0;$

2. if $|x-y|=0$, then$\rho(x,y)=0$.

Denote $C_\delta(B,M,\rho)=C_\delta(B)$ Kolmogorov's δ- capacity of a set B in (M,ρ) (see, for example, [9]). Introduce now Shannon's capasity $\mathcal{C}_\epsilon(\Theta)$ of the set Θ letting

$$\mathcal{C}_\epsilon(\Theta)=\sup I(u_\epsilon,\theta)$$

where $I(u_\epsilon,\theta)$ denotes Shannon's information in u_ϵ about θ and the upper bound is taken over all random fields $\theta(x)$ on G connected with u_ϵ by the problem (1), independent on w and such that all realizations of θ belong to Θ.

Theorem 2. *Let the parametric set Θ be a subset of (M,ρ). Then for any estimator θ_ϵ of θ*

$$\sup_{\theta\in\Theta}\mathbf{P}_\theta\left\{\rho(\theta_\epsilon,\theta)\geq\delta\right\}\geq\mathbf{1}-\frac{\mathcal{C}_\epsilon(\Theta)+\mathbf{1}}{\mathbf{C_{2\delta}}(\Theta)-\mathbf{1}}.$$

In particular

$$\sup_{\theta}\mathbf{E}_\theta|\theta-\theta_\epsilon|^{\mathbf{2}}\geq\sup_{\delta>\mathbf{0}}\delta^{\mathbf{2}}\left(\mathbf{1}-\frac{\mathcal{C}_\epsilon(\Theta)+\mathbf{1}}{\mathbf{C_{2\delta}}(\Theta)-\mathbf{1}}\right). \tag{4}$$

The proof of the theorem is the same as in [2] and [3].

There exist many results about asymptotic behaviour of C_δ when $\delta\to 0$ (see [9]). Shannon's capacity $\mathcal{C}_\epsilon$ satisfies the following inequality (cf. [3]):

$$\mathcal{C}_\epsilon(\Theta)\leq C\epsilon^{-2}\sup_{\theta\in\Theta}|\theta|^2 \tag{5}$$

where the constant C depends on Θ and the given elements of the problem (1).

Let $a_n\uparrow\infty$. Let $\{\phi_n\}$ be an orthonormal sequence in $L_2(G)$. We say that $\Sigma=\Sigma\{a_n\}$ is an ellipsoidal disk in $L_2(G)$ if

$$\Sigma=\{\theta\in L_2(G):\theta=\sum_n c_n\phi_n,\sum_n c_n^2a_n^2\leq 1\}.$$

Theorem 3 . *Let $\Theta\supset\Sigma\{a_n\}$. Then*

$$\sup_{\theta\in\Theta}\mathbf{E}_\theta||\theta_\epsilon-\theta_\epsilon||\geq\mathbf{c}\mathbf{a}_{\mathbf{N+1}}^{-\mathbf{1}} \tag{6}$$

where $N=\min\{n:n\geq Ca_{n+1}^{-2}\epsilon^{-2}+2\}$ and c,C are constants.

The inequality (6) can be deduced from (4) and (5) in the same way as in [3].

Example. Let $\beta=(\beta_{\mathbf{1}}\ ,\dots,\beta_{\mathbf{d}}),\beta_{\mathbf{j}}>\mathbf{0}$. Denote $H_\beta^{(p)}(G)M$ the set of all functions $f\in L_p(G)$ such that the derivatives $\frac{\partial^{r_j}f}{\partial x_j^{r_j}}$ satisfy in $L_2(G)$ Hölder's condition of order $\alpha_j,\quad 0<\alpha_j\leq 1,\quad r_j+\alpha_j=\beta_j$, and constant M. If the index $p=2$, we omit it.

Theorem 4. *If $\Theta\supset H_\beta(G)M$, then for any estimator θ_ϵ of θ*

$$\sup_{\theta\in\Theta}\mathbf{E}_\theta|\theta_\epsilon-\theta|\geq\mathbf{C}\epsilon^{\frac{\mathbf{2}\beta}{\mathbf{2}\beta+\mathbf{1}}} \tag{7}$$

where β is defined by the relation $\beta^{-1} = \sum \beta_j^{-1}$ and C is a positive constant.

Consider now the estimation problem of the example 1.4.

Theorem 5. *If $\Theta \supset H_\beta^{(\infty)}(G)M$, then for all estimators θ_ϵ of $\theta(x_0)$, $x_0 \in G$,*

$$\sup_{\theta\in\Theta} \mathbf{E}_\theta|\theta(\mathbf{x_0}) - \theta_\epsilon| \geq \mathbf{C}\epsilon^{\frac{\mathbf{2}\beta}{\mathbf{2}\beta+\mathbf{1}}}. \tag{8}$$

The proof of this theorem follows [3], and uses the method initiated by B.Levit and consisting of proper selection of appropriate one-parametric estimation problems related to the initial problem (see details in [6], ch.4).

4. Consistent estimators of θ. A naturally looking estimator for θ

$$\frac{\int_0^T g(u_\epsilon(t))(du_\epsilon(t) - Lu_\epsilon(t)dt)}{\int_0^T (g(u_\epsilon(t))^2 dt} \tag{9}$$

evidently has no sense. To correct this "naive" estimator we argue as follows. Notice at first that under our conditions $\{u_\epsilon, (t,x) \in Q_T\}$ are observable, they are statistics. Let $f(t)$ be an $L_2(G)$- valued step function whose values are infinitely differentiable functions of $x \in G$ with compact support strongly inside G. Then

$$\int_0^T < f(t), du_\epsilon(t) = \sum_j < f(t_{j-1}), u_\epsilon(t_j) - u_\epsilon(t_{j-1}) >$$

and

$$\int_0^T < u_\epsilon(t), L^* f(t) > dt$$

also are statistics. We write the difference of these statistics as

$$\int_0^T < f(t), du_\epsilon(t) - Lu_\epsilon(t)dt > . \tag{10}$$

This integral is a statistic. The set of above defined step functions is dense enough in $L_2(G)$ and standard arguments show that the integrals (10) will be statistics for all $f(t)$ such that $||f(t)||^2 + ||L^*f||^2 < \infty$. Finally, if $a : R^1 \to R^1$ is continuous and $|a(x)| \leq c(1+|x|)$, then the integrals

$$\int_0^T a(u_\epsilon(t)) < f(t), du_\epsilon(t) - Lu_\epsilon(t)dt > \tag{11}$$

also are statistics.

Define now corrected versions of our naive estimators (9) as follows

$$\theta_\epsilon(x) = \theta_\epsilon(x, f_x) = \frac{\int_0^T g(u_\epsilon(t) < f_x, du_\epsilon(t) - Lu_\epsilon(t)dt >}{\int_0^T g^2(u_\epsilon(t))dt}. \tag{12}$$

The function f_x does not depend on t and for all t is equal to $\phi_x(y)$ where $\phi_x(y),\ \ y \in G$ is a continuous function with a sharp maxima at the point x. We may, for example take $\phi_x(y) = (r(\epsilon))^d K(r(\epsilon)(x-y)$ where $\int_{R^d} K(x)dx = 1$ and $r(\epsilon) \to \infty$ when $\epsilon \to 0$ (kernel estimators). These estimators evidently are statistics.

Theorem 6. *Assume that* $\inf_{x\in G} |g(u_0(0,x))| = m > 0$. *Let* $\Theta \subset H_\beta(G)M$, *beta* $> 1/2$. *Then there exist estimators* θ_ϵ *such that*

$$\sup_{\theta\in\Theta} \mathbf{E}_\theta|\theta_\epsilon - \theta|^{\mathbf{2}} \leq \mathbf{c}\epsilon^{\frac{4\beta}{2\beta+1}}$$

where β *is defined as above and* c *is a constant.*

The proof of the theorem is a reprise of the arguments given in [3], though the arguments are more involved. The estimators θ_ϵ are selected from the estimators (12).

Corollary. *Let under the conditions of the Theorem 6 the set* $\Theta = H_\beta(G)M,\ \ \beta > 1/2$. *Then*

$$\inf_{\theta_\epsilon}\sup_{\theta\in\Theta} \mathbf{E}_\theta|\theta_\epsilon - \theta|^{\mathbf{2}} \asymp \epsilon^{\frac{4\beta}{2\beta+1}}.$$

The estimators of the Theorem 6 can be used also to prove the following result.

Theorem 7. *Let* $\Theta = H_\beta^\infty(G)M$. *Let* $x_0 \in \theta$. *Then*

$$\inf_{\theta_\epsilon}\sup_{\theta\in\Theta} \mathbf{E}_\theta|\theta_\epsilon - \theta(\mathbf{x_0})|^{\mathbf{2}} \asymp \epsilon\frac{\mathbf{4}\beta}{\mathbf{2}\beta+\mathbf{1}}.$$

5. Asymptotically efficient estimators. Construction. Let us return to the problem of estimation of the values $\Phi(\theta)$ of a function Φ with a Hilbert-Schmidt derivative Φ'. A natural way to construct estimators for $\Phi(\theta)$ is to put the estimator of the theorem 6 into the function Φ to get the plug-in estimator $\Phi(\theta_\epsilon)$.

Theorem 8. *Assume that* $\inf_x |g(u_0(0,x)| = m > 0$. *Assume that the set* $\Theta = H_\beta(G)M$. *Let* $\Phi(\theta)$ *be a Frecht́ differentiable function and let the derivative* $\Phi'(\theta)$ *as a function of* θ *satisfy Hölder condition of order* α. *If* $\beta > 1/2 + 1/(2\alpha)$, *then there exist estimators* Φ_ϵ *of* $\Phi(\theta)$ *such that the differences* $\epsilon^{-1}(\Phi_\epsilon - \Phi(\theta))$ *are asimtotically normal with mean zero and correlation operator* $\Phi'(\theta)v_0$ *and*

$$\lim_\epsilon \epsilon^{-1}\mathbf{E}_\theta||\Phi_\epsilon - \Phi(\theta)||^{\mathbf{2}}_{\mathbf{H}} = \mathbf{tr}(\Phi'(\theta)\mathbf{v_0}(\Phi'(\theta))^*).$$

Theorem 9. *Let under the conditions of the Theorem 8 the function* $\Phi(\theta) = K\theta,\ \ K$ *be a linear Hilbert-Schmidt operator. If* $\beta > 1/2$, *there exist estimators* K_ϵ *of* $\Phi(\theta)$ *such that the differences* $\epsilon^{-1}(K_\epsilon - K\theta)$ *are asymptotically Gaussian with mean zero and correlation operator* Kv_0 *and*

$$\lim_\epsilon \epsilon^{-1}\mathbf{E}_\theta||\mathbf{K}_\epsilon - \mathbf{K}\theta||^{\mathbf{2}}_{\mathbf{H}} = \mathbf{tr}(\mathbf{K}\mathbf{v_0}\mathbf{K}^*).$$

The proof of both Theorems follows the same lines as in [3]. It follows from the results of Sect. 2 that the estimators of Theorems 7, 8 are asymptotically efficient.

Remark. It follows from the condition $\Theta \subset H_\beta(G)M, \quad \beta > 1/2$ that all functions from Θ are continuous and satisfy Hölder's condition of order $\beta - 1/2$.

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Parameter identification for rescaled solutions of PDE with random data

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We present a systematic study of classical statistical inference problems (parameter estimation and hypothesis testing) for random fields arising as rescaled solutions of the nonlinear diffusion equation (the Burgers' turbulence problem) as well as solutions of fractional diffusion equation with weakly dependent and strongly dependent random initial conditions.

A nonlinear diffusion equation known as the Burgers' equation describes various physical phenomena, from nonlinear acoustic and kinematic waves, to the growth of molecular interfaces and formation of large-scale structures of the universe (see Woyczynski (1998), Leonenko and Woyczynski (1998a), Leonenko (1999) and the references therein).

One-dimensional Burgers' equation have also emerged in models of financial markets (see He and Lelard (1993), Hodges and Carverhill (1993, 1997).

We introduce a new statistical model of random fields arising in Burgers' turbulence and provide statistical inference tools for them, both in the space and in the frequency domain.

We begin with a review of results on parabolically rescaled solutions of Burgers' equation (with viscosity parameter $\mu > 0$)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}$$

with weakly dependent and strongly dependent random initial data (see Leonenko and Woyczynski(1998a, 2001)). The statistical inference for rescaled solutions of Burgers' equation with weakly dependent initial conditions is reduced to the statistical analysis for stationary Gaussian processes with the covariance function of the form

$$B(x) = c\left(1 - \frac{x^2}{4\mu t}\right)e^{-\frac{x^2}{8\mu t}}, \quad x \in R$$

and the spectral density of the form

$$g(\lambda) = q\lambda^2 e^{-2\mu t\lambda^2}, \quad \lambda \in R,$$

where c, q and μ are unknown parameters.

On the other hand, statistical inference for the rescaled solutions of Burgers' equation with strongly dependent initial data can be reduced to analysis of stationary Gaussian process with the spectral density of the form

$$f(\lambda) = p\left|\lambda\right|^{1+\alpha} e^{-2\mu t\lambda^2}, \quad \lambda \in R,\ 0 < \alpha < 1,$$

where p and μ are unknown parameters. The parameter α, called here the fractional exponent is also unknown. This parameter characterizes the decay at infinity of the correlation function of the initial data.

We discuss estimation of several important physical parameters of the equation itself (such as kinematic viscosity μ) and parameters of the initial data (such as fractional exponent). We use information on parameters containes in the limiting covariance structure of parabolically rescaled solution of Burgers' equation with random data (estimation in space domain). On the other hand our statistical analysis in frequency domain is based on asymptotic theory of minimum contrast estimators. These results are obtained jointly with W.A.Woyczynski (Case Western Reserve University, Cleveland).

Similar results for multidimensional Burgers' equation can be found in Leonenko Woyczynski (1999).

Fractional diffusion equations were introduces to describe physical phenomena such as diffusion in porous media with fractal geometry, kinematics in viscoelastic media, relaxation processes in complex systems, propagation of seismic waves and turbulence(see Leonenko Woyczynski (1998), Anh and Leonenko (1999, 2000, 2001), Anh et al. (1999), Angulo et al.(2000), Beghin et al.(2000) and references therein). These equations are obtained from the classical diffusion equation by replacing the first and/or second-order derivative by a fractional derivative.

We consider the following fractional diffusion equation

$$\frac{\partial^\beta u}{\partial t^\beta} = -\mu(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u, \quad \mu > 0$$

with random data, where the time derivative of order $\beta \in (0,1\}$ is defined in Caputo-Djrbashian sense, and the operators $-(I-\Delta)^{\gamma/2}$, $\gamma \geq 0$, and $(-\Delta)^{\alpha/2}$, $\alpha > 0$, are interpreted as inverses of the Bessel and Riesz potentials respectively and Δ is the n-dimansional Laplacian.

We present a spectral representation of the mean-square solution of the fractional diffusion equation with random data. Gaussian and non-Gaussian limiting distributions of the renormalized solution are described in terms of multiple stochastic integral representations.

In particular, the second order spectral density of rescaled solution

$$g(\lambda) = \frac{const}{|\lambda|^{n-\kappa}} E_\beta^2(-\mu t^\beta\, |\lambda|^\alpha), \lambda \in R^n$$

where $\kappa \in (0,n)$ is the parameter of the strongly dependent initial condition, and

$$E_\beta(-x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{\Gamma(\beta j+1)}, x \geq 0$$

is the Mittag-Leffler function.

A Gaussian random field with the spectral density $g(\lambda)$ can be used as model of physical phenomena with important features such as long-range dependence and intermittency simultaneously.These results are obtained jointly with V.Anh (Queensland University of Technology, Brisbane)

The minimum contrast estimates of unknown parameters of this spectral density can be used (see Leonenko and Moldavska(1999)).

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Likelihood inference for a linear SPDE observed at discrete points in time and space

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1. Introduction

The purpose of the present paper is to propose an approximate likelihood and study the asymptotic properties of the associated maximum likelihood estimator for the parameter $\theta = (\eta_1, \eta_2, \xi_0, \xi_1, \xi_2)$ given observations in discrete points in time and space of the stationary solution of the parabolic ($\eta_2 = 0$) or hyperbolic ($\eta_2 > 0$) stochastic partial differential equation

$$\eta_2 \frac{\partial^2}{\partial t^2} V(t,x) + \eta_1 \frac{\partial}{\partial t} V(t,x) = \xi_0 V(t,x) + \xi_1 \frac{\partial}{\partial x} V(t,x) \\ + \xi_2 \frac{\partial}{\partial x^2} V(t,x) + W_\xi(t,x), \quad t \in \mathbb{R}, 0 < x < 1 \quad \text{(1a)}$$

with Dirichlet boundary conditions

$$V(t,0) = V(t,1) = 0, \quad t \in \mathbb{R}. \tag{1b}$$

Here the parameters satisfies $\eta_1, \xi_2 > 0$, $\eta_2 \geq 0$ and the stochastic disturbance term $W_\xi(t,x)$ is related to Brownian white noise $W(t,x)$ via the equation

$$W_\xi(t,x) = e^{-\frac{\xi_1}{2\xi_2}x} W(t,x). \tag{2}$$

The proposed approximate likelihood would have to be optimized using numerical methods. The motivation for studying this statistical problem is applications in mathematical finance to the modeling of the term structure for bonds of different maturity times, see Cont (1998) and Santa-Clara & Sornette (1999). In these models the spatial component represents time to maturity. In Cont (1998) it is argued that the short rate ($x = 0$) and the long rate ($x = 1$) can be modeled independently of the profile from the short rate to the long rate. Moreover it is argued that the deviation from the average profile can be modeled by the solution $V(t,x)$ to the stochastic partial differential equation. Realistic data thus consist of observations at discrete points in time and space organized in a lattice. The spatial resolution is usually fairly low consisting of e.g. 20 maturity times. Calculating the discrete Fourier transforms and using the Galerkin approximation would thus be inadequate and result in biased estimates. The solutions to the parabolic and hyperbolic equations have different properties, see the discussion in Cont (1998), whence it is of interest to test the hypothesis of a parabolic equation against a hyperbolic equation.

The paper is organized as follows. In section 2 we describe for which parameters there exists a stationary solution to (1), give a representation of the stationary solution, and describe the sample path properties of the solution. In section 3 we give a time series representation for a observation of $V(t,x)$ in discrete points in time and space organized in a lattice. In section 4 we propose an approximate likelihood and give conditions under which this approximate likelihood has the same first order asymptotic properties as the exact likelihood. Moreover we derive the likelihood ratio test for a parabolic equation against a hyperbolic equation.

This extended abstract is a shortened version of Markussen (2001b).

2. The stationary solution and its properties

The stochastic partial differential equation (1) contains a second order derivative w.r.t. time and is thus most easily formulated in terms of white noise calculus in the sense of e.g. Holden et al. (1996). But since the equation (1) lives in one dimensional space there exists an ordinary solution, which coincides with the solution of the corresponding equation in the calculus of Walsh (1986), see also Kallianpur & Xiong (1995). The following theorem is well-known.

Theorem 1. *If the parameter $\theta = (\eta_1, \eta_2, \xi_0, \xi_1, \xi_2)$ lies in the parameter space $\Theta \subset \mathbb{R}^5$ given by*

$$\eta_1, \xi_2 > 0, \qquad \eta_2 \geq 0, \qquad \xi_0, \xi_1 \in \mathbb{R}, \qquad \frac{\xi_1}{4\xi_2} + \pi^2 \xi_2 > \xi_0$$

then there exists a unique stationary solution $V(t,x) = \sum_{k=1}^{\infty} U_k(t) X_k(x)$ to the stochastic partial differential equation (1). Here the deterministic functions $X_k(x)$ are given by

$$X_k(x) = \sqrt{2} \sin(\pi k x) e^{-\frac{\xi_1}{2\xi_2} x},$$

and the coefficients $U_k(t)$ are independent stochastic processes. Let the independent Brownian motions $B_k(t)$, $k \geq 1$ be given by

$$B_k(t) = \int_0^t \int_0^1 W_\xi(s,y) X_k(y) e^{\frac{\xi_1}{\xi_2}} \, dy \, ds,$$

and put

$$\lambda_k = \xi_0 - \frac{\xi_1^2}{4\xi_2} - \pi^2 k^2 \xi_2, \qquad \mu_k = \eta_1^2 + 4\eta_2 \lambda_k.$$

In the parabolic case $U_k(t)$ is a stationary solution to the stochastic differential equation

$$dU_k(t) = \frac{\lambda_k}{\eta_1} U_k(t) \, ds + dB_k(t),$$

and in the hyperbolic case $U_k(t)$ is the first component of a stationary solution $\bar{U}_k(t) = (U_k(t), \tilde{U}_k(t))$ to the stochastic differential equation

$$d\bar{U}_k(t) = \begin{cases} \frac{1}{2\eta_2} \begin{pmatrix} -\eta_1 & \sqrt{\mu_k} \\ \sqrt{\mu_k} & -\eta_1 \end{pmatrix} \bar{U}_k(t) \, dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dB_k(t) & \text{if } \mu_k > 0 \\ \frac{1}{2\eta_2} \begin{pmatrix} -\eta_1 & 2\eta_2 \\ 0 & -\eta_1 \end{pmatrix} \bar{U}_k(t) \, dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dB_k(t) & \text{if } \mu_k = 0 \\ \frac{1}{2\eta_2} \begin{pmatrix} -\eta_1 & \sqrt{-\mu_k} \\ -\sqrt{-\mu_k} & -\eta_1 \end{pmatrix} \bar{U}_k(t) \, dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dB_k(t) & \text{if } \mu_k < 0. \end{cases} \tag{3}$$

In the hyperbolic case the paths of the coefficient processes $U_k(t)$ are continuous differentiable as is seen from (3). This fact suggest that the solution $V(t,x)$ is more smooth in the hyperbolic case than in the parabolic case. Following Walsh (1986) Theorem 3.8 we can prove Theorem 2.

Theorem 2. *The solution $V(t,x)$ to the stochastic partial differential equation* (1) *has a version that is continuous in (t,x). Moreover if for some fixed $t_0 < \infty$,*

$$\omega_t(\delta) = \sup_{x,y\in[0,1]:|x-y|\le\delta} |V(t,x) - V(t,y)|, \quad t \in [0,t_0],$$
$$\omega(\delta) = \sup_{s,t\in[0,t_0],x,y\in[0,1]:((s-t)^2+(x-y)^2)^{\frac{1}{2}}\le\delta} |V(s,x) - V(t,y)|$$

are the moduli of continuity in space at time t respectively in time and space then there exists a constant $\alpha < \infty$ and random variables Y_t, $t \in [0,t_0]$ and Y with exponential moments such that for $0 \le \delta \le 1$,

$$\omega_t(\delta) \le Y_t\delta^{\frac{1}{2}} + \alpha\delta^{\frac{1}{2}}\sqrt{\log(\delta^{-1})},$$
$$\omega(\delta) \le \begin{cases} Y\delta^{\frac{1}{4}} + \alpha\delta^{\frac{1}{4}}\sqrt{\log(\delta^{-1})} & \text{if } \eta_2 = 0 \\ Y\delta^{\frac{1}{2}}\sqrt{\log(\delta^{-1})} + \alpha\delta^{\frac{1}{2}}\log(\delta^{-1}) & \text{if } \eta_2 > 0. \end{cases}$$

Theorem 2 states that the solution $V(t,x)$ to (1) has paths that essentially are Hölder continuous of order $\frac{1}{2}$ in space and $\frac{1}{4}$ in time in the parabolic case, and of order $\frac{1}{2}$ in time and space in the hyperbolic case. The paths are thus substantially more rough in time in the parabolic case, in which case they also by Walsh (1986) Theorem 3.10 have non-vanishing quartic variation.

3. Time Series Representation

Let a fixed time step $\Delta > 0$ and fixed rational spatial coordinates $\frac{a_n}{b} \in (0,1)$, $n = 1,\dots,N$ be given. Let the N-dimensional time series $V^\Delta(t)$, $t \in \mathbb{N}_0$, and the $2b$-dimensional time series $U^\Delta(t)$, $t \in \mathbb{N}_0$ be given by

$$V^\Delta(t) = V\big(t\Delta, \frac{a_n}{b}\big)_{n=1,\dots,N}, \qquad U^\Delta(t) = \big(\textstyle\sum_{j=0}^\infty U_{k+2bj}(t\Delta)\big)_{k=1,\dots,2b},$$

and let the matrices $\Xi \in \mathbb{R}^{N\times N}$ and $\Psi \in \mathbb{R}^{N\times 2b}$ be given by

$$\Xi = \mathrm{diag}\big(e^{-\frac{\xi_1}{2\xi_2}\frac{a_n}{b}}\big)_{n=1,\dots,N}, \qquad \Psi = \big(\sqrt{2}\sin\big(\pi k\tfrac{a_n}{b}\big)\big)_{\substack{n=1,\dots,N \\ k=1,\dots,2b}}.$$

Then the time series $V^\Delta(t)$ has the state space representation

$$V^\Delta(t) = \Xi\Psi U^\Delta(t), \quad t \in \mathbb{N}_0.$$

The components of the time series $U^\Delta(t)$ are independent and given as infinite sums of the independent time series $U_k^\Delta(t) = U_k(t\Delta)$. In the parabolic case, U_k^Δ is a first order autoregressive process

$$U_k^\Delta(t+1) = e^{\frac{\lambda_k}{\eta_1}\Delta}U_k^\Delta(t) + \varepsilon_k^\Delta(t+1) \sim \mathcal{N}\big(0, \frac{-1}{2\eta_1\lambda_k}\big),$$
$$\varepsilon_k^\Delta(t) \text{ i.id. } \mathcal{N}\big(0, \frac{-1}{2\eta_1\lambda_k}(1 - e^{2\frac{\lambda_k}{\eta_1}\Delta})\big).$$

In the hyperbolic case, $U_k^\Delta(t)$ is the first component of the similarly defined two-dimensional first order autoregressive process $\bar U_k^\Delta(t) = (U_k^\Delta(t), \tilde U_k^\Delta(t))$,

$$\bar U_k^\Delta(t+1) = \rho_k^\Delta \bar U_k^\Delta(t) + \bar\varepsilon_k^\Delta(t+1) \sim \mathcal{N}_2(0, \sigma_k^2),$$
$$\bar\varepsilon_k^\Delta \text{ i.id. } \mathcal{N}_2(0, \sigma_k^2 - \rho_k^\Delta \sigma_k^2 \rho_k^{\Delta *}).$$

Here the structure of the autoregression coefficient ρ_k^Δ and the stationary variance σ_k^2 depends on the sign of μ_k. If $\mu_k > 0$ then

$$\rho_k^\Delta = e^{-\frac{\eta_1}{2\eta_2}\Delta} \begin{pmatrix} \frac{\sqrt{\mu_k}}{2\eta_2}\Delta & 0 \\ 0 & -\frac{\sqrt{\mu_k}}{2\eta_2}\Delta \end{pmatrix}, \qquad \sigma_k^2 = \begin{pmatrix} \frac{\eta_2^2}{-2\eta_1\lambda_k} & \frac{\eta_1\eta_2^2}{-2\lambda_k\mu_k} \\ \frac{\eta_1\eta_2^2}{-2\lambda_k\mu_k} & \frac{\eta_2^2}{-2\eta_1\lambda_k} + \frac{4\eta_2^2}{\eta_1\mu_k} \end{pmatrix},$$

if $\mu_k = 0$ then

$$\rho_k^\Delta = e^{-\frac{\eta_1}{2\eta_2}\Delta} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_k^2 = \begin{pmatrix} \frac{2\eta_2}{\eta_1^3} & \frac{1}{\eta_1^2} \\ \frac{1}{\eta_1^2} & \frac{1}{\eta_1\eta_2} \end{pmatrix},$$

and if $\mu_k < 0$ then

$$\rho_k^\Delta = e^{-\frac{\eta_1}{2\eta_2}\Delta} \begin{pmatrix} \cos(\frac{\sqrt{-\mu_k}}{2\eta_2}\Delta) & \sin(\frac{\sqrt{-\mu_k}}{2\eta_2}\Delta) \\ -\sin(\frac{\sqrt{-\mu_k}}{2\eta_2}\Delta) & \cos(\frac{\sqrt{-\mu_k}}{2\eta_2}\Delta) \end{pmatrix},$$
$$\sigma_k^2 = \begin{pmatrix} \frac{-1}{2\eta_1\lambda_k} & \frac{-1}{2\lambda_k\sqrt{-\mu_k}} \\ \frac{-1}{2\lambda_k\sqrt{-\mu_k}} & \frac{-1}{2\eta_1\lambda_k} + \frac{\eta_1}{\lambda_k\mu_k} \end{pmatrix}.$$

4. Approximate Likelihood Inference

Assume observations of $V(t,x)$ in discrete points in time and space at the lattice given by

$$t = \Delta, 2\Delta, \dots, T\Delta, \qquad x = \frac{a_1}{b}, \dots, \frac{a_N}{b} \tag{4}$$

are at our disposal, i.e. that the time series

$$V^\Delta(t) = \Xi\Psi U^\Delta(t), \quad t = 1, \dots, T$$

have be observed. In practical applications it might be necessary to let the spatial sampling points $\frac{a_n}{b}$, $n = 1, \dots, N$ depend on the time point t. If the spatial sampling points e.g. are altered to $\frac{a_n(t)}{b}$ or $x(t) + \frac{a_n}{b}$, $n = 1, \dots, N$ in some appropriate periodic fashion then we conjecture Theorem 4 below to remain true for some other Fisher information matrix.

Let the cutoff point $K(T)$ and the white noise variances $\tau_k^2(T)$ be given by

$$K(T) = \begin{cases} \lceil \frac{1}{\pi}\sqrt{\frac{\eta_1}{2\xi_2\Delta}}\sqrt{\log T}\rceil & \text{if } \eta_2 = 0 \\ \lceil T^{\frac{1}{4}} \rceil & \text{if } \eta_2 > 0 \end{cases},$$
$$\tau_k^2(T) = \sum_{j\in\mathbb{N}_0 : k+2bj \geq K(T)}^{\infty} \frac{-1}{2\eta_1\lambda_{k+2bj}}, \quad k = 1, \dots, 2b. \tag{5}$$

We propose to approximate the distribution of $V^{\Delta}(t)$ by the distribution of $\Xi\Psi\hat{U}^{\Delta}(t)$, where

$$\hat{U}^{\Delta}(t) = \Big(\textstyle\sum_{j\in\mathbb{N}_0:k+2bj<K(T)} U^{\Delta}_{k+2bj}(t) + \hat{\varepsilon}_k(t)\Big)_{k=1,\dots,2b},$$
$$\hat{\varepsilon}_k(t) \text{ i.id. } \mathcal{N}\big(0,\tau_k^2(T)\big),$$

Since we have an explicit description of the finite dimensional state space model $\Xi\Psi\hat{U}^{\Delta}(t)$ the approximate likelihood can easily be calculated via the Kalman-Bucy filter, and thereafter optimized via numerical methods.

Let $\varphi^{\Delta}_k(\omega)$, $\omega \in (-\frac{1}{2},\frac{1}{2}]$ be the spectral density of the time series $U^{\Delta}_k(t)$ under the parameter θ. If θ_0 is the true value of the parameter then the time series $V^{\Delta}(t) = \Xi\Psi U^{\Delta}(t)$ has spectral density $\Phi^{\Delta}_{\theta_0}(\omega)$ given by

$$\Phi^{\Delta}_{\theta_0}(\omega) = \Xi\Psi\mathrm{diag}\,\Big(\textstyle\sum_{j=0}^{\infty}\varphi^{\Delta}_{k+2bj}(\omega)\Big)_{k=1,\dots,2b}\Psi^*\Xi^*,$$

and the approximation $\Xi\Psi\hat{U}^{\Delta}(t)$ has spectral density $\Phi^{\Delta}_{T,\theta_0}(\omega)$ given by

$$\Phi^{\Delta}_{T,\theta_0}(\omega) = \Xi\Psi\mathrm{diag}\,\Big(\textstyle\sum_{j\in\mathbb{N}_0:k+2bj<K(T)}\varphi^{\Delta}_{k+2bj}(\omega) + \tau_k^2(T)\Big)_{k=1,\dots,2b}\Psi^*\Xi^*.$$

We measure the quality of the approximation via the L_2-distance between the spectral density $\Phi^{\Delta}_{\theta_0}(\omega)$ and the approximation $\Phi^{\Delta}_{T,\theta_0}(\omega)$ in the L_2-space of $\mathbb{C}^{N\times N}$-valued functions on $(-\frac{1}{2},\frac{1}{2}]$. For a matrix $A\in\mathbb{C}^{N\times N}$, the Schatten p-norm $\|A\|_p$, $p\in[1,\infty]$ is defined as the l_p-norm of the eigenvalues of the positive semi definite matrix $|A| = (A^*A)^{\frac{1}{2}}$. For a matrix valued function $\Phi : (-\frac{1}{2},\frac{1}{2}] \to \mathbb{C}^{N\times N}$, the L_p-norm $\|\Phi\|_p$ is defined as the usual L_p-norm of the real function $\|\varphi(\cdot)\|_p$. These L_p-norms behaves much like the usual L_p-norms and especially satisfies the Hölder inequality.

Lemma 1. *If the cutoff point $K(T)$ and white noise variances $\tau_k^2(T)$ are given by* (5) *then $T^{\frac{1}{2}}\|\Phi^{\Delta}_{\theta} - \Phi^{\Delta}_{T,\theta}\|_2$ is bounded as $T\to\infty$.*

The analysis now relies on the following theorem proved in Markussen (2001a).

Theorem 3. *Let $V(t)$, $t\in\mathbb{N}$ be a N-dimensional Gaussian time series, i.e.*

$$V_T = \big(V(1),\dots,V(T)\big)^* \sim \mathcal{N}_{T\times N}\big(0,\Sigma_T(\varphi)\big),$$

where $\Sigma_T(\varphi)$ is the Toeplitz matrix associated to the spectral density $\Phi(\omega)$. For each $T\in\mathbb{N}$ let $\Phi_{T,\theta}(\omega)$, $\theta\in\Theta\subseteq\mathbb{R}^d$ be a family of spectral densities and let $l_T(\theta)$ be the corresponding log likelihood ratio,

$$l_T(\theta) = -\frac{1}{2}\log\det\Sigma_T(\Phi_{T,\theta}) - \frac{1}{2}tr\big(\Sigma_T(\Phi_{T,\theta})^{-1}V_TV_T^*\big). \tag{6}$$

If $T^{\frac{1}{2}}\|\Phi - \Phi_{T,\theta_0}\|_2$ is bounded as $T\to\infty$ then under additional mild regularity conditions, see Markussen (2001a), the maximum likelihood estimator $\hat{\theta}_T = argmax_{\theta\in\Theta}l_T(\theta)$ is $\sqrt{T}$-consistent for θ_0 and the localized log likelihood ratio converges uniformly to a Gaussian shift process, i.e. there exists d-dimensional random variables G_T such that

$$E\Big(\sup_{u\in\mathbb{R}^d:|u|\le r,\theta_0+T^{-\frac{1}{2}}u\in\Theta}|l_T(\theta_0+T^{-\frac{1}{2}}u) - l_T(\theta_0) - \big(u^*G_T - \frac{1}{2}u^*J_{\theta_0}u\big)|\Big) \tag{7}$$

vanishes for every $r > 0$ and G_T converges in distribution to $\mathcal{N}_d(0, J_{\theta_0})$ as $T \to \infty$. Here the Fisher information matrix J_{θ_0} is given by

$$J_{\theta_0} = \lim_{T\to\infty} \left(\tfrac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} tr\big(\Phi_{T,\theta_0}^{-1}(\omega)\partial_i \Phi_{T,\theta_0}(\omega)\Phi_{T,\theta_0}^{-1}(\omega)\partial_j \Phi_{T,\theta_0}(\omega)\big)\, d\omega \right)_{i,j=1,\dots,d}.$$

The additional regularity conditions are satisfied for the stochastic partial differential equation model under consideration, see Markussen (2001a), whence Theorem 3 applies by lemma 1. Using the LAN-property (7) we get the following theorem.

Theorem 4. *Let $\theta_0 = (\eta_1, \eta_2, \xi_0, \xi_1, \xi_2)$ be the true parameter and let the cutoff point $K(T)$ and the white noise variances $\tau_k^2(T)$ be given by (5). Then the approximate log likelihood $l_T(\theta)$ given in (6) can be calculated via the Kalman-Bucy filter, and the approximate maximum likelihood estimator $\hat{\theta}_T = argmax_{\theta\in\Theta} l_T(\theta)$ is normal $\sqrt{T}$-consistent for θ_0 with Fisher information*

$$\left(\tfrac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} tr\big(\Phi_{\theta_0}^{\Delta}(\omega)^{-1}\partial_i \Phi_{\theta_0}^{\Delta}(\omega)\Phi_{\theta_0}^{\Delta}(\omega)^{-1}\partial_j \Phi_{\theta_0}^{\Delta}(\omega)\big)\, d\omega \right)_{i,j=1,\dots,5}$$

and asymptotically efficient as $T \to \infty$ in the sense of Hajek-LeCam, see LeCam & Yang (2000). If $\eta_2 = 0$ then the likelihood ratio test statistic $\chi(T) = 2\sup_{\theta\in\Theta} l_T(\theta) - 2\sup_{\theta\in\Theta:\eta_2=0} l_T(\theta)$ for a parabolic against a hyperbolic equation converges in distribution to $\frac{1}{2}\varepsilon_0 + \frac{1}{2}\chi_1^2$ as $T \to \infty$. Here ε_0 is the point measure in 0 and χ_1^2 is the chi-square distribution with one degree of freedom.

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What is the Time Value of an Option?

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Plotted against time, the price of a common American option traded on a Canadian or US market exhibits changes as random as an ordinary stock. From an in-the-money option, such a behavior is anticipated due to the presence of the intrinsic value. An out-of-the money option has no intrinsic value; the random behavior is thus a feature of the time value, or premium, paid for the option on the market. A closer look at summary statistics published by exchanges after the bell shows that absolute changes in the option price are usually smaller then those of the underlying equity. This is caused by the cushioning effect of the option's time value. Though the premium is known to decline with time and has the cushioning effect, the option price and its underlying security price arise from independent negotiations between traders on the stock and option markets, respectively. It is thus worthwhile to ask about its volatility and how the volatility affects the predictability of the American option price. The argument below shows that under fairly general assumptions about the equity price and the time value, the conditional expectation one would naturally use for prediction of the option price is random and satisfies a stochastic partial differential equation (SPDE). The assumptions arise from analysis of the current option and equity market prices posted daily by the Montreal and Toronto Stock Exchanges, respectively. The equation can be used for option pricing in a similar manner as the deterministic counterpart with minor modifications and shall be useful for those who need to include the expected premium prices and their volatility into their calculations.

The definition of an option and basic risk management strategies are described in [19]. The market price of an American option consists of two quantities: the time-value and the intrinsic value. The instantaneous price, say, of a call option with an exercise price Q Canadian dollars is thus described by the formula

$$C(t, S_t, Q) = R(t, S_t, Q) + \max(S_t - Q, 0), \tag{1}$$

where $R(t, S_t, Q)$ is a random function describing the time value at time t and $\max(S_t - Q, 0)$ is the intrinsic value of the option. We consider a contract that gives the right to buy a single share of a common stock. Literature on financial derivatives [18] commonly assumes that S_t, the underlying equity price at time t, is a random quantity generated from a stochastic process satisfying for $s \leq t$ an Itô's equation

$$dS_t = \alpha(t, S_t) + \beta(t, S_t)dw_t, \quad S_s = y. \tag{2}$$

Functions $\alpha(t, y)$ and $\beta(t, y)$ are deterministic, w_t is a Wiener process and $S_s = y$ is the initial condition. This model is often acceptable even if the "ideal conditions" in the market considered by [2] are not fulfilled. The particular shape of $\alpha(t, y)$ and $\beta(t, y)$ can be determined by analysis of historical data, fundamentals of the company issuing the underlying stock, market conditions etc. Definition, meaning and assumptions assuring that a unique solution of (2) exists are e.g. in [13]. Investigation of the historical equity market prices etc. was done using methods described, for example, in [4], [9], and [6].

Statistical inference on continuous-time stochastic processes is discussed in [16]. The time value is of the form

$$R(t, S_t, Q) = r(t, S_t, Q) + \epsilon(t, S_t, Q), \tag{3}$$

where $r(t, y, Q)$ is a deterministic function and $\epsilon(t, y, Q)$ is a Gaussian random field. The choice is determined by the data and their spatio-temporal analysis. Theoretically, $R(t, S_t, Q)$ should be always non-negative before the time of expiry, a condition the Gaussian field cannot satisfy. However, if the last trade on an option took place early in the day and the stock price changed substantially by the end of the session, which is a usual situation caused by the low volume of options traded on the market compared to the volume of the underlying stock, the time value calculated from closing data using (1) may well be negative. Use of non-negative time values only causes an undesired loss of information. Common transformations for variance stabilization that might result in a model with non-negative values did not contribute to fit improvement either. Validation of the model (3) and related problems are discussed e.g. in [14], [8] and [10].

Let us suppose that (2) determines a (unique) conditional probability density function $p(s, y, t, x)$. Following [3], [5] and others it is natural to predict the value $f(S_t)$ of an arbitrary integrable function of the equity price S_t using the conditional expectation based on equity price observed at the present time s:

$$E[f(S_t)|S_s = y] = \int_{-\infty}^{\infty} f(x)p(s, y, t, x)dx. \tag{4}$$

Hence, introducing the quantity

$$\eta(s, y, Q) = E[\epsilon(t, S_t, Q)|S_s = y] \tag{5}$$

we can describe the expected market price of the option at time t prior to the date of expiry conditioned on the current equity value observed at time s as

$$V(s, y, Q) = E[C(t, S_t, Q)|S_s = y] \tag{6}$$

$$= \int_{-\infty}^{\infty} (r(t, x, Q) + \max(x - Q, 0))p(s, y, t, x)dx + \eta(s, y, Q). \tag{7}$$

The second summand $\eta(s, y, Q)$ is random and since integration is a linear operation, it is a Gaussian random variable. It describes the uncertainty arising from negotiations on the option market, which cannot be explained by the information contained in the underlying equity price.

In most applications, $p(s, y, t, x)$ is likely to satisfy the backwards Kolmogorov equation [11]. An expression similar to (6) is thus obtained by solving the SPDE

$$\frac{\partial}{\partial_s}V(s, y, Q) = -\alpha(t, y)\frac{\partial}{\partial_y}V(s, y, Q) - \frac{\beta(t, y)^2}{2}\frac{\partial^2}{\partial_y^2}V(s, y, Q) + \dot{W}(s, y, Q) \tag{8}$$

with terminal condition

$$V(t, y, Q) = r(t, Q, y) + \max(y - Q, 0) \tag{9}$$

and Gaussian white noise term $\dot{W}(s,y,Q)$. Notice that Q is treated as a parameter. Without the noise, (8) is the backwards Kolmogorov equation. Solution of (8) has form

$$V(s,y,Q)=\int_{-\infty}^{\infty}V(t,x,Q)p(s,y,t,x)dx+\int_{s}^{t}\int_{-\infty}^{\infty}p(s,y,\tau,x)W(d\tau,dx,Q). \quad (10)$$

where $W(s,y,Q)$ is a Brownian sheet and the last integral is in Itô's sense. A general theory introducing SPDE's and methods of solution is given, for example, in [7] and [17]. The expected value of (10) serves as the option value predictor, whereas the variance of (10) is necessary to assess the error of the prediction.

The equation (8) can be modified to incorporate further factors, such as the interest rate. Let us take a writer of a covered call option, for example. On American markets, options are issued and expire the third Friday of the month. Their lifetime is about three months and more. In anticipation of an important event such as company's quarterly report or announcement of the federal reserve within the option's lifetime, the writer may plan to change position after the announcement and if possible, cover his options by equivalent option contracts and sell the underlying equity afterwards. Buying equivalents of its options, the writing institution is loosing not only the intrinsic value (if present) and a part of the premium but also the interest associated with assets fixed in stocks the writer must hold to cover his options. Due to the presence of the time value, the likelihood a market participant who bought the option on the market will exercise it is nearly zero. It is always more advantageous to cash in the intrinsic value on the option market and add some of the premium back into the seller's account. The assumption that the writer has no need to cover the exercise expenses prior to the expiry date is thus plausible and allows us to calculate the lost interest easily. This interest loss can be incorporated in the equation (8) by subtracting an extra term from the right side:

$$\frac{\partial}{\partial_s}V(s,y,Q)=-\rho(s)V(s,y,Q) \quad (11)$$

$$-\alpha(t,y)\frac{\partial}{\partial_y}V(s,y,Q)-\frac{\beta(t,y)^2}{2}\frac{\partial^2}{\partial_y^2}V(s,y,Q)+\dot{W}(s,y,Q).$$

The function $\rho(s)$ is positive and describes the possibly time dependent interest rate. The solution $V(s,y,Q)$ of (11) with terminal condition (9) is interpreted as the expected expense associated with buying the option back on the market at time t before expiry conditioned on the equity price at time s. Solution of the equation has form

$$V(s,y,Q)=\int_{-\infty}^{\infty}V(t,x,Q)e^{\int_s^t\rho(u)du}p(s,y,t,x)dx+ \quad (12)$$

$$\int_{s}^{t}\int_{-\infty}^{\infty}e^{\int_\tau^t\rho(u)du}p(s,y,\tau,x)W(d\tau,dx,Q).$$

The Gaussian distribution of $V(s,y,Q)$ in (12) allows us to specify a confidence region for the predicted expense. The distribution of $V(s,y,Q)$ is likely to contain parameters estimable by methods described in [12]. Investigation of variability due to the use of estimated instead of exact parameter values may be thus of interest. The function

$e^{\int_s^t \rho(u)du}$ in (12) describes the appreciation of the stock over time and in this case it increases the writer's loss. Comparison of the outlined option valuation procedure and the familiar Black-Scholes formula can be done by addressing the theoretical background leading to the formulas and the corresponding empirical valuation methods. Both approaches take an Itô's process as a model of the stock market price. Here, the ideal market assumptions required by [2] for elimination of the equity price randomness and resulting in their popular equation are replaced by the assumption that the optimal predictor of the option price is described by the conditional expectation calculated from the equity's probability distribution; see [3], [15] and [5]. Consequently, the option does not have to be European unless interest should be included in calculations. In addition, the method introduced above involves the premium charged to the intrinsic value on the option market. [1] assess their model by testing "market efficiency" using historical data and different hedging strategies. If their valuation formula was wrong then one of the strategies would lead to a systematic profit or loss for the user. Diagnostics of results presented here relies on a common analysis of historical data and model residuals.

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Stochastic partial differential equations driven by fractional Brownian motion

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We use fractional white noise theory to study stochastic partial differential equations driven by multiparameter fractional Brownian motion with all its Hurst coefficients in the interval (1/2, 1). In particular, we give explicit solutions of the fractional versions of the following SPDEs: The linear heat equation, the Laplace-Poisson equation and the quasi-linear heat equation.

Compared to the corresponding SPDEs driven by the classic multi-parameter Brownian motion (when all the Hurst coefficients are being equal to 1/2), the solutions in the fractional case tend be be smoother. In particular, in the fractional case we typically get genuine L^2–solutions (and not just distribution valued solutions) in higher dimensions than in the classical case. This feature makes the fractional white noise more tractable than the classical white noise when we apply SPDEs in mathematical modelling.

The talk is partially based on joint works with Yaozhong Hu and Tusheng Zhang.

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Parameter estimation in equations of stochastic fluid mechanics under Lagrangian data

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Let $u(t,x;\theta) \in R^d$, be a random velocity field, $t \geq 0$, $x \in R^d$ depending on an unknown multi dimensional parameter θ. Denote by $X(t,r)$ the displacement of a particle released at moment $t=0$ at the point r, driven by the given velocity field, i.e

$$\dot{X}(t,r) = u(t, X(t,r)), \quad X(0,r) = r, \tag{1}$$

where the dot means time derivative. The sample

$$X_1(t),\ X_2(t), \dots,\ X_N(t), \quad t \in (0,T) \tag{2}$$

corresponding to different initial values

$$X_n(t) = X(t, r_n), \quad X(0, r_n) = r_n, \quad n = 1, \dots, N$$

is called Lagrangian data with observation time T. We stress that the sample (2) is not independent because all $X_n(t)$ come from the same equation (1). Our purpose is to study asymptotic behavior of the Maximum Likelihood estimator $\hat{\theta}$ of θ based on Lagrangian data as $T \to \infty$. Originally this problem has come from oceanography where measurements by drifters (current following devices) have been increasingly growing during the last two decades (Davis, 1991). We consider two models of a turbulent flow with infinitely small and finite velocity correlation time respectively. The asymptotic behavior of $\hat{\theta}$ crucially depends on the sign of the Lyapunov exponent, λ, for the corresponding flow of diffeomorphisms. In the case of the delta-correlated velocity field λ has been found in (Le Jan, 1985, Baxendale and Harris, 1986). An explicit expression for the Lyapunov exponent for the turbulent flow with memory is given here. In the first model we assume that

$$u(t,x) = \bar{u}(x;\theta) + u'(t,x),$$

where $\bar{u}(x;\theta)$ is a deterministic vector field and $u'(t,x)$ is a Gaussian white noise in time, i.e.

$$Eu'(t,x) = 0, \quad Eu'(t,x)u'(s,y)^* = \delta(t-s)B(x,y).,$$

Define (Nd)-dimensional vectors $y = (x_1, \dots, x_N)$, $U(y) = (\bar{u}(x_1), \dots, \bar{u}(x_N))$, $Y(t) = (X_1(t), \dots, X_N(t))$, and $(Nd \times Nd)$-matrix $D(y) = (B(x_m, x_n))$. The process $Z(t)$ is a diffusion process in R^{Nd} with the drift $U(y)$ and the diffusion matrix $D(y)$, (Kunita, 1990), i.e.

$$dY(t) = U(Y(t))dt + D^{1/2}(Y(t))dW(t).$$

Thus, the problem is reduced to estimation in a finite dimension diffusion with the diffusion matrix degenerating on some hyperplanes. In the case of non-degenerate diffusion this problem was comprehensively considered before (e.g. Citovich, 1977 and Kutoyants, 1993). The incremental observed information can be found using the general maximum likelihood ideology (e.g. O.E.Barndorf-Nielsen and M.Sørensen, 1994)

$$I_{kl}(\theta) = D^{-1}(Y(t))\frac{\partial U(Y(t);\theta)}{\partial \theta_k} \cdot \frac{\partial U(Y(t);\theta)}{\partial \theta_l},$$

where the dot means the dot product in R^{Nd}. Now let $\bar{u}(x;\theta) \equiv \theta$ and $u'(t,x)$ be isotropic. Isotropy implies that the correlation tensor is expressed through two functions $b_L(r), b_N(r) : R^1_+ :\to R^1$ called the longitudional and transversal correlation functions respectively (Monin and Yaglom, 1971, 1975),

$$b_{ij}(x) = b_N(r)\delta_{ij} + \frac{y_i y_j}{r^2}(b_L(r) - b_N(r)),$$

where $r = |x|$. We suppose that $b_{ij}(x)$ are twice differentiable

$$b_L(r) = b_0 - \frac{1}{2}\beta_L r^2 + O(r^4), \quad b_N(r) = b_0 - \frac{1}{2}\beta_N r^2 + O(r^4), \tag{3}$$

and

$$b_L(r), \quad b_N(r) \to 0$$

as $r \to \infty$, where $b_0, \beta_L, \beta_N > 0$. The top Lyapunov exponent for the flow of diffeomorphisms generated by the considered velocity field is given by

$$\lambda = \frac{d-1}{2}\beta_N - \frac{\beta_L}{2}.$$

The following statement addresses the asymptotic of the mean square error for the first model.

Theorem 1 (L.Piterbarg, 1998) *Relation*

$$\lim_{T\to\infty} TE(\hat{\theta} - \theta)(\hat{\theta} - \theta)^* = \frac{B(0)}{N}$$

holds in the following cases

$$d \geq 4, \quad or \quad d = 3,\ \lambda \geq 0, \quad or \quad d = 2,\ \lambda > 0,$$

and relation

$$\lim_{T\to\infty} TE(\hat{\theta} - \theta)(\hat{\theta} - \theta)^* = B(0)$$

holds if one from the following is true

$$d = 2,\ \lambda < 0, \quad or \quad d = 1.$$

Now consider another model corresponding to the turbulence with memory

$$u(t,x) = \bar{u}(x;\theta) + u'(t,x;\theta),$$

where the fluctuation velocity satisfies an Euler equation with a Gaussian white noise forcing

$$\frac{\partial u'}{\partial t} + u' \cdot \nabla u' = a(u',x;\theta) + \xi(t,x), \tag{4}$$

where $a(u',x;\theta)$ is a function presenting dissipation and deterministic forcing and known up to the unknown parameter,

$$E\xi(t,x) = 0, \quad E\xi(t,x)\xi(s,y)^* = \delta(t-s)B(x,y). \tag{5}$$

In this case the vector $Z(t) = (\dot{X}_1(t), X_1(t), \dots, \dot{X}_N(t), X_N(t))$ is a diffusion in R^{2Nd} with a degenerated diffusion matrix. Introduce

$$A(Z(t)) = (a(\dot{X}_1(t) - \bar{u}(X_1(t)), X_1(t)), ..., a(\dot{X}_N(t) - \bar{u}(X_N(t)), X_N(t))).$$

Then the incremental observed information is

$$I_{kl}(\theta) = D^{-1}(Y(t))\frac{\partial A(Z(t);\theta)}{\partial\theta_k} \cdot \frac{\partial A(Z(t);\theta)}{\partial\theta_l}.$$

Assume isotropy of $\xi(t,x)$ such that (3) holds for $B(x,y)$ appearing in (5), and

$$\bar{u}(x;\theta) = \theta, \quad a(u',x;\theta) = -au', \tag{6}$$

where a is unknown parameter and $\tau = 1/a$ is so called the Lagrangian correlation time. Set $\beta = \beta_N - \beta_L$, where β_N, β_L are given in (3) .

Theorem 2 *If* $d = 2$ and $\beta > 0$, *then the Lyapunov exponent for the flow defined by (4,6) is positive and given by*

$$\lambda = \frac{1}{2\tau}\left(\frac{K_{-2/3}(1/6\beta\tau^3)}{K_{1/3}(1/6\beta\tau^3)} - 1\right),$$

where $K_\nu(x)$ is the modified Bessel function of the second kind and order ν.

Proposition 3 *Under conditions of Theorem 2*

$$\lim_{T\to\infty} TE(\hat{\theta}-\theta)(\hat{\theta}-\theta)^* = \frac{B(0)}{a^2N}, \quad \lim_{T\to\infty} TE(\widehat{a}-a)^2 = \frac{2a}{N}.$$

The last statement is not completely proven yet.

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Stochastic Allen Cahn: Analysis and Numerics

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1. Introduction

I will present an analysis of the Allen-Cahn equation perturbed by a space-time noise process. The equation has the following form

$$du = \Big[\epsilon^2 \Delta u + f(u)\Big] dt + \sigma dW(t), \tag{1}$$

where Δ is the Laplacian, ϵ and σ are small parameters, $f(u) = \frac{1}{2}(u - u^3)$, the initial data $u(0)$ should be specified, and homogeneous Neumann boundary conditions are specified on the domain $[0, 1]$. The purpose of this work is see how the many results in the deterministic case $\sigma = 0$ extend to the stochastic case.

2. Allen-Cahn Equation

Background references on this equation include [1], [3], [20], [12], [13], [15], [21], [6], [2].

The dynamics of this equation in the case $\sigma = 0$ divides into regimes. Very quickly, the solution u will converge to $u(t, x) \approx \pm 1$. The leads to solutions consisting of phases $u \approx \pm 1$ separated by transition layers of width ϵ. In intermediate time, the phase boundaries evolve and it can be shown the contours $\{u = 0\}$ obeys an ODE of the following form: let h_i be a well ordered set representing the positions of the contours $\{u = 0\}$, then

$$\frac{dh_i}{dt} = \frac{\epsilon}{\|U'\|^2} \Big[\mu_{i+1} e^{-\sigma_{i+1}(1+\delta_{i,N})\epsilon^{-1}\ell_{i+1}} - \mu_i e^{-\sigma_i(1+\delta_{i,1})\epsilon^{-1}\ell_i}\Big] \tag{2}$$

where $\ell_i = h_i - h_{i-1}$ (for $f(u) = \frac{1}{2}(u - u^3)$, $\mu_i = 4$, $\|U'\|^2 = 2/3$, $\sigma_i = 1$). These equations hold in the case $\sigma = 0$ on an exponentially long time scale e^{1/ϵ^2}. Finally, the solution reaches a stable equilibrium at one of the homogeneous phases $u(x) = \pm 1$ for $0 \le x \le 1$.

Analysis in [17] in the case $\sigma \neq 0$ shows that a solution $u_{\epsilon,\sigma}$ of (1) with noise intensity σ converges to the corresponding deterministic problem u_ϵ in the limit $\sigma^2/\epsilon \to 0$ in the mean square sense. In the limit $\sigma^2/\epsilon \to \infty$, the mean square norm of $u_{\epsilon,\sigma}$ blows up because of the ill posedness of the space-time process $W(t)$ in L_2.

The second part of the analysis is a formal derivation of the SDEs corresponding to (2). A formal derivation in [17] shows that

$$\begin{aligned} dh_i =& \frac{\epsilon}{\|U'\|^2} \Big[\mu_{i+1} e^{-\sigma_{i+1}(1+\delta_{i,N})\epsilon^{-1}\ell_{i+1}} - \mu_i e^{-\sigma_i(1+\delta_{i,1})\epsilon^{-1}\ell_i}\Big] dt \\ &+ \frac{\sigma \epsilon^{1/2}}{\|U'\|} d\beta_i(t). \end{aligned} \tag{3}$$

The derivation is based on asymptotics done by [20] and is not rigorous (in contrast to the results for $\sigma = 0$ derived in [19]). The result is however supported by numerical simulations that suggest convergence in a weak sense.

3. Numerics

To verify the formal derivation of (3) numerically, I chose to simulate the numerical solution of the SDE (3) and compare it to the numerical solution of the stochastic PDE (1).

The numerical solution of SDEs is well understood and the subject of several excellent monographs, for example [11] and [14]. There are no special difficulties in simulating (3).

In contrast however, the numerical solution of parabolic stochastic PDEs is a relevantly recent topic. Research in this area includes Gyongy [10, 9], Shardlow [16], and Davie-Gaines [4]. By far the most general results are those of Gyongy who succeeds in proving convergence of numerical methods for very general types of equations. Shardlow writes for equations of the form (1) and is motivated by work on the approximation of ergodicity [18]. The results consider finite difference schemes, taking the θ method in time and the standard three point approximation to the Laplacian in space, together with a spectral approximation to the noise term. For a time step Δt and grid spacing Δx, the rate of convergence is proven to be $\Delta x^{1/2}$ subject to a stability condition $(1-2\theta)\Delta t \leq \Delta x^2/4$. Davie-Gaines give an interesting result about how rates of convergence will never be better than $\Delta x^{1/2}$ unless more sophisticated approaches are used to evaluate the noise. Other references in this general area include [5], [7], [8].

I implemented schemes for the SDE and Allen-Cahn equation and computed solutions for a simple test case. An initial condition was chosen with a single interface, and the mean and variance computed for the change of the location of the interface by simulating the equations for many realisations of the noise. The computations support the derivation of (3) and are presented in [17].

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On the small time large deviation principle for solutions of stochastic partial differential equations

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The aim of this talk is to present some results on the small time asymptotics of diffusion processes and heat semigroups on Hilbert spaces, which includes solutions of some stochastic evolution equations. First let us recall the basic results in finite dimensions. Let $L = \frac{1}{2}\Delta$ be the one half of the Laplacian operator on R^d. Then we know that the heat kernel is the transition density of the Brownian motion given by

$$P_t(x,y) = (2\pi t)^{-\frac{d}{2}} \exp\{-\frac{d^2(x,y)}{2t}\},$$

where $d(x,y)$ stands for the usual distance on R^d.
It is clear that the following small time asymptotics holds

$$\lim_{t\to 0} 2t \log P_t(x,y) = -d^2(x,y). \tag{1.1}$$

Much work has been done to extend the above asymptotics to general situations where the Laplacian is replaced by general elliptic operators , R^d is replaced by some finite dimensional Riemannian manifolds and $d(x,y)$ is the corresponding Riemannian distance. The results are quite satisfactory, see [6], [23] and references therein.

We are here concerned with the above asymptotics in infinite dimensional cases where L will be the generator of a symmetric diffusion process $X_t, t \geq 0$ on some Hilbert space E. Because of the lack of the transition density , the natural replacement for $P_t(x,y)$ in the equation (1.1) is $P(X_0 \in B, X_t \in C)$, where $C, \quad B$ are two Borel subsets. The distance $d(x,y)$ between two points x, y is replaced by the distance of the two sets C and B. Specifically, we obtained the following small time asymptotics

$$\lim_{t\to 0} 2t \log P(X_0 \in B, X_t \in C) = -d^2(B,C), \tag{1.2}$$

where d is the appropriate Riemannian distance associated with the diffusion. The upper bound and the lower bound are proved separately. The upper bound is proved for any two Borel subsets $B, \quad C$ with positive measures and quite general diffusions with continous diffusion operators. For the lower bound, we assume that the diffusion is a solution of a stochastic differential equation or a stochastic evolution equation on the Hilbert space . We first establish a small time large deviation principle for solutions of stochastic evolution equations of the type:

$$u_t = x - \int_0^t Au_s ds + \int_0^t b(u_s) ds + \int_0^t \sigma(u_s) dW_s \tag{1.3}$$

Then the lower bound follows from the large deviation principle.

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