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Workshop on

Finance and Turbulence

Centre for Mathematical Physics and Stochastics — MaPhySto

and

Centre for Analytical Finance — CAF

Aarhus School of Business and University of Aarhus

Contents

1 Introduction

The Workshop on 'Finance and Turbulence' was held 5-7 May 1999 at University of Aarhus and was organised jointly by CAF (Centre for Analytical Finance) and MaPhySto (Centre for Mathematical Physics and Stochastics), the organizing committee consisting of Ole E. Barndorff-Nielsen, Bent Jesper Christensen, Henning Bunzel (Aarhus) and Michael Sørensen (Copenhagen).

The aim of the Workshop was to discuss the striking similarities, as well as the differences, between key empirical features observed in the financial markets on the one hand and in studies of the turbulence of fluids on the other. Particular emphasis was given to questions relating to realistic stochastic modelling of the phenomena concerned.

The participants came from the fields of Physics, Stochastics, and Mathematical Finance/Econometrics, and among the topics treated were: *Burgers' Equation, Cascades, Extremal Behaviour, Long Range Dependence, Scaling, Selfsimilarity, and Volatility and Intermittency.*

The present booklet contains extended abstracts of (most of) the talks given at the workshop. The programme and the list of participants are also included.

2 Workshop Program

Wednesday May 5 (in Auditorium D1, building 531)

09.00-10.00 Registration and coffee/tea

Chair: Bent Jesper Christensen

10.00-10.50 **Stewart Hodges:**
The Risk Premium In Trading Equilibria Which Support Black-Scholes Option Pricing.

Coffee/tea

11.20-12.10 **Rama Cont:**
Multi-resolution analysis of financial time series.

12.30-14.00 Lunch

Chair: Claudia Klüppelberg

14.40-15.10 **Rudolf Friedrich:**
A new stochastic concept.

15.20-15.50 **Joachim Peinke:**
Turbulence and Finance.

Coffee/tea

16.10-16.40 **Ralf Hendrych:**
Self-similarity and Wavelets.

16.50-17.30 **Albert Shiryaev:**
Kolmogorov and the Turbulence.

17.30-18.30 Poster Session and Small Reception

Thursday May 6 (in Auditorium G1, building 532)

Chair: Hanspeter Schmidli

9.00-9.50 **Roberto Baviera:**
Weak efficiency and information in foreign exchange markets.

10.00-10.50 **Neil Shephard (joint work with Ole E. Barndorff-Nielsen):**
Non-Gaussian OU based models and some of their uses in financial economics.

Coffee/tea

11.20-12.10 **Claudia Klüppelberg:**
Analysing Extremal Behaviour of Financial Time Series.

12.30-14.00 Lunch

Chair: Michael Sørensen

14.00-14.30 **Rosario Delgado (joint work with M. Jolis):**
On a Ogawa-type integral with application to the Fractional Brownian Motion.

14.40-15.10 **Francesco Mainardi:**
Non local transport effects in skewed turbulence via fractional diffusion and Lévy statistics.

15.20-15.50 **Ole E. Barndorff-Nielsen (joint work with Preben Blæsild):**
A case study in turbulence.

Coffee/tea

16.10-16.40 **Patrick Cheridito:**
Long-Range Dependence and Option Pricing.

16.50-17.20 **Nils Svanstedt:**
Two-scale limits and mean fields for the Navier-Stokes equation for oscillatory fluids.

17.30-18.00 Discussion

19.00-22.00 Conference Dinner

Friday May 7 (in Auditorium D2, building 531)

Chair: Goran Peskir

10.10-11.00 **Mykola Leonenko:**
Non-Gaussian scenarios for fractional diffusion-wave equation with singular data.

Coffee/tea

11.20-12.10 **Martin Greiner:**
What can we learn from one-dimensional observables in fully developed Navier-Stokes turbulence?.

12.30-14.00 Lunch

14.00-14.50 **Rimas Norvaiša:**
p-variation, integration and stock price modelling.

3 Abstracts of talks and posters

(The abstracts/papers are ordered alphabetically after the lastname of the author who presented the work.)

Ole E. Barndorff-Nielsen (MaPhySto) and Preben Blæsild (Aarhus):

A case study in turbulence.

ABSTRACT: A turbulence data set* consisting of 100 time series, each comprising 125.000 consecutive and equidistantly spaced observations, is analysed. The time series are regarded as i.i.d.

Indications of systematic deviations from Kolmogorov's theory of homogeneous and isotropic turbulence are noted and some preliminary modelling of key features of the data is proposed.

Key words: continuous time AR(3) behaviour; Lamperti transformation; normal inverse Gaussian law; selfsimilarity; vague stationarity.

*Data kindly put at our disposal by Rudolf Friedrich and Joachim Peinke.

Weak efficiency and information in foreign exchange markets

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Weak efficiency and information in foreign exchange markets

Abstract

In this paper we test the efficiency hypothesis in financial market. A market is called efficient if the price variations "fully reflect" relevant information, i.e. a speculator cannot make a profit out of it. A currency exchange market is a natural candidate to check efficiency because of its high liquidity. We perform a statistical study of *weak* efficiency in Deutschemark/US dollar exchange rates using high frequency data. In the *weak* form of efficiency the information can only come from historical prices.

The presence of correlations in the returns sequence implies the possibility of a statistical prevision of market behavior. We show the existence of correlations by means two statistical tools. A first analysis has been performed using structure functions. This approach gives an indication on the returns distributions at different lags τ. We have also computed the generalized correlation functions of the return absolute values; roughly speaking this is a test of the independence of the fluctuations of fixed size. In both cases we have obtained a clear evidence of long term return anomalies. This implies a failure of the usual "random walk" model of the returns; nevertheless the presence of long term correlations does not directly imply the fault of the *weak* efficiency hypothesis : it is not obvious how to use time correlation to make a profit in a realistic investment.

Then we show how this information is relevant for a speculator. First we introduce a measure of the *available* information relevant from a financial point of view, with a technique which reminds the Kolmogorov ϵ-entropy. Second in the case of no transaction costs, we propose a simple investment strategy which leads to an exponential growth rate of the capital related to the *available* information.

We have performed two kind of information analysis in the return series. We show that the *available* information is practically zero if the speculator wants to change his portfolio *systematically* after a fixed lag τ : for him the market is efficient. Instead, a finite *available* information is observed by a *patient* investor who cares only of fluctuation of given size Δ. This is the first case, as far as we know, in which the *available* information obtained by a suitable data analysis is directly linked to the possible earnings of a speculator who follows a particular trading rule.

1 Introduction

A large amount of research suggests that prices are related with information, and in particular it focuses on efficiency in financial markets. A market is inefficient if a speculator can make a profit out of information present in the market. Since the celebrated work of Fama [1] a big effort has been done to test empirically and to understand theoretically the efficiency of financial markets.

A market is said to be efficient if prices "fully reflect" all available information, i.e. such information is completely exploited in order to determine the price, after having taken into account the costs to use this information and a transient time, due to costs, to reach equilibrium. The idea is that the investor destroys information while using it and as a consequence he contributes to produce equilibrium.
In the last years long term correlations have been observed in financial markets. We shall not review in details the contributions to the field. We stress that long term return anomalies are usually revealed via test of efficiency in a *semi-strong* form, i.e. not only considering the asset prices but also some other publicly known news. The interest is generally focused on the market reactions to an event occurred a fixed period time before (three to five typically) such as divested firms [2], mergers [3] or initial public offerings [4, 5]. Recent research [6, 7, 8, 9, 10] has pointed out the existence of long range correlations also in the *weak* form. However only low frequency data are considered and implications on efficiency are not completely understood.
In this paper we focus on efficiency in the *weak* form, i.e we consider only the information coming from historical prices. We are interested on a time scale longer than the typical correlation returns time (few minutes) but lower than the characteristic time after which we do not have statistical relevance of the results: in this sense we deal with *long term* return anomalies. Currency exchange seems to be the natural subject for an efficiency test. We expect that such markets are very efficient as a consequence of the large liquidity. For these reasons we have decided to analyze a one year high frequency dataset of the Deutschemark/US dollar exchange, the most liquid market. Our data, made available by Olsen and Associated, contains all worldwide $1,472,241$ bid–ask Deutschemark/US dollar exchange rate quotes registered by the inter-bank Reuters network over the period October 1, 1992 to September 30, 1993.
One of the main problem in tick data analysis, is the irregular spacing of quotes. In this paper we consider *business* time, i.e. the time of the transaction given by its rank in the sequence of quotes. This seems to be a reasonable way to consider time in a worldwide time series, where time delays and lags of no transaction are often due to geographical reasons.
In this paper we test the independence hypothesis of returns and define and measure an *available* information. In section **2** we check the independence with two different techniques. The first one, called structure functions analysis, shows whether it is possible to rescale properly the distribution functions at different lags [11]. The second one is a direct independence test. The independence of two random variables x, y implies that $f(x)$ and $g(y)$ are uncorrelated for every f and g. We check it for $f(\cdot) = g(\cdot) = |\cdot|^q$. We interpret these quantities as an estimate of the correlation between returns of given size. We want to quantify the *available* information and discuss its financial relevance. In section **3** we consider a speculator with a given resolution, i.e. he is concerned only about fluctuations at least of size Δ. This reminds the ϵ entropy introduced by Kolmogorov [12] in the context of information theory. A similar filter has been first introduced by

Alexander [13, 14]. To show the inefficiency of the market he proposed the following trading rule : if the return moves up of Δ, buy and hold until it goes down of Δ from a subsequent high, then sell and maintain the short position till the return rises again of Δ above a subsequent low.

Here we divide the problem in two parts. First we define the *available* information for any fixed resolution Δ of the speculator. Second, as suggested by Fama [1], we relate the *available* information with the profitability by means of a particular trading rule. We show how this information is related to the optimal growth rate portfolio using a simple approximation in terms of Markov process.

In section **4** we summarize and discuss the results.

2 Long term correlations

After the seminal work of Bachelier [15], it was widely believed that the price variations follow an independent, zero mean, gaussian process. The main implications of the "fundamental principle" of Bachelier are that the price variation is a martingale and it is an independent random process.

Bachelier considers the market a "fair game" : a speculator cannot exploit previous information to make better predictions of forthcoming events. Information can come only from correlations and in absence of them from the shape of the probability distribution of the returns.

For about sixty years this contribution was practically forgotten, and quantitative analysis on financial data started again with advent of computers.

Following Fama [1], we shall call hereafter "random walk" the financial models where the returns

$$r_t \equiv \ln \frac{S_{t+1}}{S_t} \tag{1}$$

are independent variables. In this paper we define S_t as the average between bid and ask price. We do not want to enter here in a detailed analysis of the huge literature about "random walk" models. We just mention that, before the contribution of Mandelbrot [16], the return r_t was considered well approximated by an independent gaussian process. Mandelbrot proposed that the returns were distributed according a Levy-stable, still remaining independent random variables.

At present, it is commonly accepted that the variables

$$r_t^{(\tau)} \equiv \sum_{t'=t+1}^{t+\tau} r_{t'} = \ln \frac{S_{t+\tau}}{S_t} \tag{2}$$

do not behave according a gaussian at small τ, while the gaussian behavior is recovered for large τ. Of course a return r_t distributed according to a Levy, as suggested by Mandelbrot, is stable under composition and then also $r_t^{(\tau)}$ would follow the same distribution for every

τ. A recent proposal is the truncated Levy distribution model introduced by Mantegna and Stanley [17] which fits well the data and reproduces the transition from small to large τ.

Let us focus our attention on independence tests. We remark once again that an influence of the return r_t at time t on the return $r_{t+\tau}$ at time $t+\tau$ implies a not fully efficient market in a *weak* form. The relevance of the question is clear in the case of an investor analyzing historical data to a make market forecast and a profit out of it.

As a test of independence it is generally considered the correlation functions on time intervals τ

$$C(\tau) \equiv \langle r_t r_{t+\tau} \rangle - \langle r_t \rangle \langle r_{t+\tau} \rangle \ , \tag{3}$$

where $\langle \cdot \rangle$ denotes the temporal average

$$\langle A \rangle \equiv \frac{1}{T} \sum_{t=1}^{T} A_t$$

and T is the size of the sample.

The presence of correlations in Deutschemark/US dollar exchange returns before the nineties is a well known fact. For example in [18], where it is considered the same dataset we use, it is shown that the returns are negatively correlated for about three minutes.

We remind that in general uncorrelation does not imply independence. A sort of long term memory can be revealed with appropriate tools, see for example the seminal works in the field of Alexander [13, 14] and Niederhoffer and Osborne [19], and the most recent literature [6, 7, 8, 9, 20, 21], where it is shown that absolute returns or powers of returns exhibit a long range correlation. It is a common belief that it is not possible to exploit this kind of information because of transaction costs.

We shall show in next section that dependent (even if uncorrelated) returns have a clear financial meaning because they imply the existence of *available* information.

In subsection **2.1** we show the persistence of a long range memory for the Deutschemark/US dollar exchange rate by means of the analysis of structure functions. In subsection **2.2**, we test directly the independence of returns with a generalization of the correlation analysis.

2.1 Structure functions

There is some evidence that the process $r_t^{(\tau)}$ cannot be described in terms of a unique scaling exponent [22, 23], i.e. it is not possible to find a real number h such that the statistical properties of the new random variable $r_t^{(\tau)}/\tau^h$ do not depend on τ.

The scaling exponent h gives us information on the features of the underlying process. In the case of independent gaussian behavior of r_t the scaling exponent is 1/2.

On the contrary, the data show that the probability distribution function of $r_t^{(\tau)}/\sqrt{Var[r_t^{(\tau)}]}$ changes with τ [22, 23]. This is an indication that r_t is a dependent stochastic process and it implies the presence of wild fluctuations.
A way to show these features, which is standard for the fully developed turbulence theory [24], is to study the structure functions :

$$F_q(\tau) \equiv \langle |r_t^{(\tau)}|^q \rangle \ . \tag{4}$$

In the simple case where r_t is an independent random process, one has (for a certain range of τ)

$$F_q(\tau) \sim \tau^{hq} \ , \tag{5}$$

where $h > 1/2$ in the Levy-stable case while the gaussian behavior is recovered for $h = 1/2$. The truncated Levy distribution corresponds to $h > 1/2$ for τ sufficiently small and to $h = 1/2$ at large τ. "Random walk" models present always a unique scaling exponent. If the structure function has the behavior in (5) we call the process self-affine (sometimes called uni-fractal).

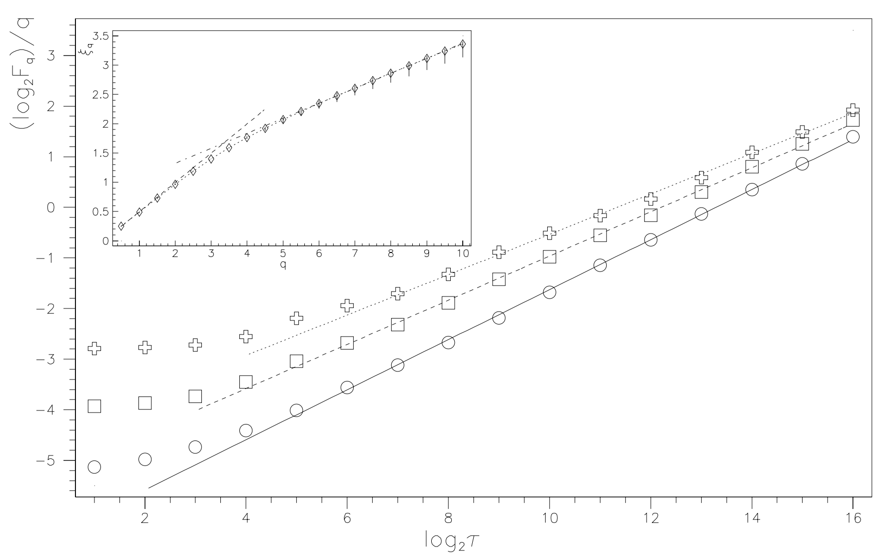

Figure 1: Structure functions $\frac{1}{q}\log_2 F_q(\tau)$ *versus* $\log_2 \tau$ for Deutschemark/US dollar exchange rate quotes. The three plots correspond to different value of q : $q = 2.0$ ($\circ$), $q = 4.0$ ($\square$) and $q = 6.0$ ($+$). In the insert we show ξ_q versus q. We estimate with linear regression two different regions in this graph. The first one is a line of slope 0.5 (dashed line), and the second has a slope 0.256 (dash dotted line).

As previously mentioned a description in terms of a unique scaling exponent h, does not work. Therefore instead of (5) one has

$$F_q(\tau) \sim \tau^{\xi_q} \ , \tag{6}$$

where ξ_q are called scaling exponents of order q. If ξ_q is not linear, the process is called multi-affine (sometimes multi-fractal). Using simple arguments it is possible to see that ξ_q has to be a convex function of q [25]. The larger is the difference of ξ_q from the linear behavior in q the wilder are the fluctuations and the correlations of returns. In this sense the deviation from a linear shape for ξ_q gives an indication of the relevance of correlations. In figure 1 we plot, the $F_q(\tau)$ for three different values of q. A multi-affine behavior is exhibited by different slopes of $\frac{1}{q}\log_2(F_q)$ *vs.* $\log_2(\tau)$, at least for τ between 2^4 and 2^{15}. For larger *business* lags a spurious behavior can arise because of the finite size of the dataset considered. In the insert we plot the ξ_q estimated by standard linear regression of $\log_2 F_q(\tau)$ *vs.* $\log_2(\tau)$ for the values of τ mentioned before. To give an estimation of errors, the most natural way turns out to be a division of the year dataset in two semesters. This is natural in the financial context, since it is a measure of reliability of the second semester forecast based on the first one. We observe that the traditional stock market theory (brownian motion for returns), gives a reasonable agreement with $\xi_q \simeq q/2$ only for $q < 3$, while for $q > 6$ one as $\xi_q \simeq \tilde{h}q + b$ with $\tilde{h} = 0.256$ and $c = 0.811$. We stress once again that such a behavior cannot be explained by a "random walk" model (or other self-affine models) and this effect is a clear evidence of correlations present in the signal.

2.2 Long term correlations analysis

Let us consider the absolute returns series $\{|r_t|\}$, which is often shown to be long range correlated in recent literature [6, 7, 8, 9, 10, 20, 21]. Absolute values mean that we are interested only in the size of fluctuations.
Let us introduce the generalized correlations $C_q(\tau)$:

$$C_q(\tau) \equiv \langle |r_t|^q |r_{t+\tau}|^q \rangle - \langle |r_t|^q \rangle \langle |r_{t+\tau}|^q \rangle \ . \tag{7}$$

We shall see that the above functions will be a powerful tool to study correlations of returns with comparable size: small returns are more relevant at small q, while $C_q(\tau)$ is dominated by large returns at large q (the usual definition of correlation for absolute returns is recovered for $q = 1$).
Following the definitions in [26], let us suppose to have a long memory for the absolute returns series, i.e. the correlations $C_q(\tau)$ approaches zero very slowly at increasing τ, i.e. $C_q(\tau)$ is a power-law:

$$C_q(\tau) \sim \tau^{-\beta_q} \ .$$

If $|r_t|^q$ is an uncorrelated process one has $\beta_q = 1$, while β_q less than 1 corresponds to long range memory.
Instead of directly computing correlations $C_q(\tau)$ of single returns we consider rescaled sums of returns. This is a well established way, if one is interested only in long term

analysis, in order to drastically reduce statistical errors that can affect our quantities [27]. Let us introduce the *generalized cumulative absolute returns* [10]

$$\chi_{t,q}(\tau) \equiv \frac{1}{\tau} \sum_{i=0}^{\tau-1} |r_{t+i}|^q \tag{8}$$

and their variance

$$\delta_q(\tau) \equiv \langle \chi_{t,q}(\tau)^2 \rangle - \langle \chi_{t,q}(\tau) \rangle^2 \ . \tag{9}$$

After some algebra (see Appendix), one can show that if $C_q(\tau)$ for large τ is a power-law with exponent β_q, then $\delta_q(\tau)$ is a power-law with the same exponent :

$$C_q(\tau) \sim \tau^{-\beta_q} \quad \Longrightarrow \quad \delta_q(\tau) \sim \tau^{-\beta_q} \ .$$

In other words the hypothesis of long range memory for absolute returns ($\beta_q < 1$), can be checked via the numerical analysis of $\delta_q(\tau)$.

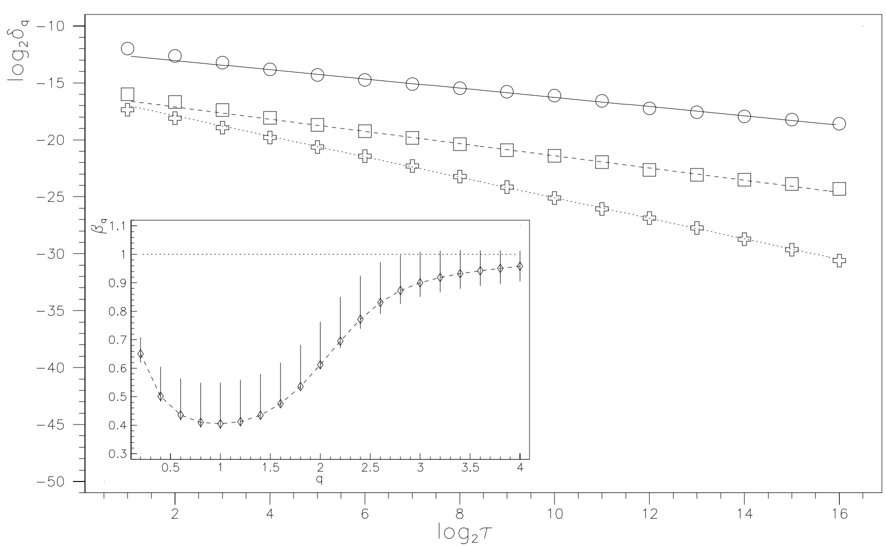

Figure 2: $\log_2 \delta_q$ *versus* $\log_2 \tau$. The three plots correspond to different value of q : $q = 1.0$ ($\circ$), $q = 1.8$ ($\Box$) and $q = 3.0$ ($+$). In the insert we show β_q versus q, the horizontal line shows value $\beta = 1$ corresponding to independent variable.

In figure 2 we plot the δ_q *vs.* τ in log-log scale, for three different values of q. The variance $\delta_q(\tau)$ is affected by small statistical errors, and it confirms the persistence of a long range memory for a τ larger than 2^4 and up to 2^{15}.

The exponent β_q can be profitably estimated by standard linear regression of $\log_2(\delta_q(\tau))$ *versus* $\log_2(\tau)$, and the errors are estimated in the same way of subsection **2.1**.

We notice in the insert that the "random walk" model behavior is remarkably different from the one observed in the Deutschemark/US dollar exchange for $q < 3$. This implies the presence of strong correlations, while one has $\beta_q = 1$ for large values of q, i.e. big fluctuations are practically independent.
An intuitive meaning of the previous results is the following. Using different q one selects different sizes of the fluctuations. Therefore the non trivial shape of β_q is an indication of the existence of long term anomalies.

3 Available information

Let us focus our attention on information analysis of the return r_t. We must treat the dataset in such a way that methods of information theory can be applied.
The usual approach is the codification of the original data in a symbolic sequence. There are several ways to build up such a sequence: one should make sure that this treatment does not change the structure of the process underlying the evolution of the financial data.
In order to construct a symbolic sequence from a time series, at least two steps are needed :

- A *filtering* procedure to remove most of the noise in the dataset.
- A *coarse graining* procedure to partition the range of variability of the filtered data, in order to assign a conventional symbol to each element of the partition.

The codification is then straightforward: a symbol corresponds unambiguously to the data stored in each element of the partition.
From the original signal r_t we obtain a discrete symbolic sequence :

$$c_1, c_2, \ldots, c_i, \ldots$$

where each c_i takes only a finite number, say m, of values. In such a way we reduce ourself to the study of a discrete stochastic process.
A simple way to obtain a symbolic sequence is to consider only a two-valued symbol and define a discrete random variable without performing any filtering operation :

$$c_i = \begin{cases} -1 & \text{if} \quad r_i < 0 \\ +1 & \text{if} \quad r_i \geq 0 \end{cases} \quad . \tag{10}$$

The financial meaning of this codification is rather evident: the symbol -1 occurs if the stock price decreases, otherwise the symbol is 1.
Let us now remind some basic concepts of information theory. Consider a sequence of n symbols $C_n = \{c_1, c_2, \ldots, c_n\}$ and its probability $p(C_n)$. The block entropy H_n is defined by

$$H_n \equiv - \sum_{C_n} p(C_n) \ln p(C_n) \ . \tag{11}$$

The difference

$$h_n \equiv H_{n+1} - H_n \tag{12}$$

represents the average information needed to specify the symbol c_{n+1} given the previous knowledge of the sequence $\{c_1, c_2, \ldots, c_n\}$.
The series of h_n is monotonically not increasing and for an *ergodic* process one has

$$h = \lim_{n\to\infty} h_n \tag{13}$$

where h is the Shannon entropy [28].
It is easy to show that if the stochastic process $\{c_1, c_2, \ldots\}$ is markovian of order k (*i.e.* the probability to have c_n at time n depends only on the previous k steps $n-1, n-2, \ldots, n-k$), then $h_n = h$ for $n \geq k$. In other cases, or h_n goes to zero for increasing n which means that either for n sufficiently large the $(n+1)$th-symbol is predictable knowing the sequence C_n or it tends to a positive finite value. The maximum value of h is $\ln(m)$. It occurs if the process has no memory at all and the m symbols have the same probability.
The difference between $\ln(m)$ and h is intuitively the quantity of information we may use to predict the next result of the phenomenon we observe, i.e. the market behavior. We define *available* information :

$$I \equiv \ln(m) - h = R\ln(m) \tag{14}$$

where $R = 1 - h/\ln(m)$ is called the *redundancy* of the process [28].
Hereafter we limit the discrete process to take only two values, -1 and 1 which have an evident financial meaning. We expect that the high frequency details are not relevant and cannot be easily used by financial analysts. It seems rather reasonable to study the process r_t with a finite lag τ (see subsection **3.1**) or a finite resolution Δ on the values of r_t (see subsection **3.2**).
In the following we shall show that different discretization procedures lead to completely different results. This corresponds to two different kind of investment, one *systematic* and the other *patient*. The *systematic* investor modifies his portfolio every τ steps, where the lag τ is measured in the usual *business* time (but the same results hold also for the *calendar* time). The *patient* investor, instead, waits to update his strategy until a certain behavior of the market is achieved, for example, a fluctuation of size Δ.
In the last part of this section we shall show that this kind of investment seems to be the most suitable for financial aims.

3.1 A naive approach: fixed lag analysis

In a recently proposed time series model [29] the price variation is considered as a result of a true underlying process plus an uncorrelated white noise. If we think the observed return as the sum of these two components, it is natural to try to eliminate the additional noise taking the average of the signal over a given lag.
More precisely we treat the financial data as follows :

- we group the sequence of the r_t in non overlapping blocks of τ data and we define the sum of the data in the j^{th} block

$$r'_j \equiv \sum_{k=j\tau+1}^{j(\tau+1)} r_k \ .$$

Notice that r'_j is equivalent to $r^{(\tau)}_{j\tau}$ where $r^{(\tau)}_t$ is defined in equation (2).

- we decimate the data, i.e. we take from the sequence r'_j only one data every m :

$$R_k \equiv r'_{mk} \ .$$

This procedure should eliminate the eventual short time correlation of the noise.

- we build up the symbolic sequence, like in equation (10) :

$$c_k = \begin{cases} -1 & \text{if} \quad R_k < 0 \\ +1 & \text{if} \quad R_k \geq 0 \end{cases} \quad . \tag{15}$$

Let us remark that the first two steps have been performed to reduce the noise.
The total number of the data of the symbolic sequence is $N/(m\tau)$, where N is the number of original data. This filter is linear, i.e. if the signal is a linear combination of various contributes, at the end of the filtering procedure we have the sum of the filtered contributes. The theory of linear filter is well developed in literature (see for example [27]), and we use this simple approach to check whether a noise is added to our signal.
At this point we have a binary sequence from which we compute the Shannon entropy.
Figure 3 reports the results of our analysis for the linear filter. We plot h_n vs n for various τ and m, compared with the entropy of Bernoulli trials with probability $p = 0.5$ (this is nothing that the usual coin tossing).
We know that the entropy h_n of a fair binary Bernoulli trial must be $\ln(2)$ for every n. The folding of h_n at large n depends on the finite number of sequence elements. It can be proved [30] that the statistical analysis does not give the proper value of h_n for n larger than :

$$n^* \approx \frac{1}{h} \ln(N)$$

where h is the entropy of the signal and N is the length of the sequence.
It should be now clear that the entropy of the sequence is given by the value of the *plateau*.
The entropy does not differ sensibly from $\ln(2)$, of the coin tossing, and, therefore, we cannot make prevision on the market. In conclusion, the financial data cannot be represented as a white noise added on a true underlying signal.
Nevertheless, because of the long term correlations (see section **2**), there is a clear indication that the present state of the market depends non trivially on the past.

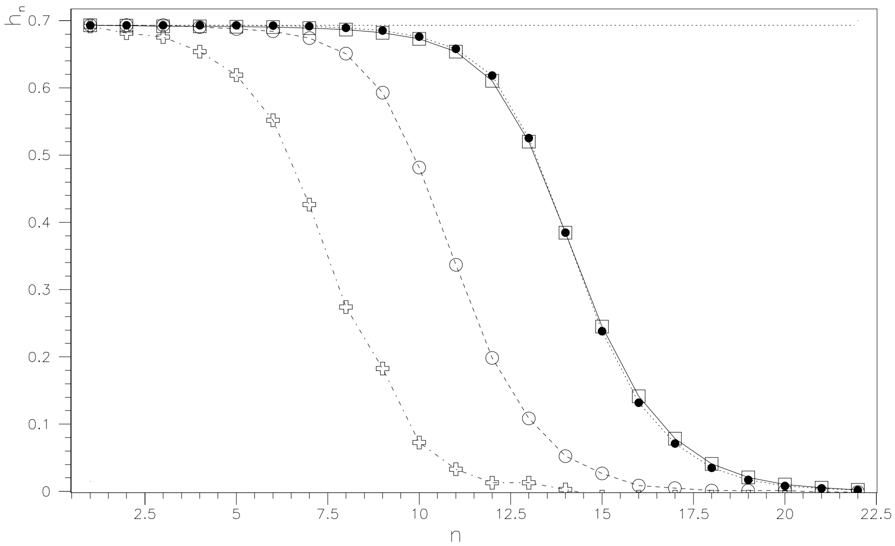

Figure 3: h_n *versus* n. The three plots correspond to different value of τ and m : $\tau = 10,\ m = 10$ ($\square$), $\tau = 10,\ m = 100$ ($\circ$) and $\tau = 100,\ m = 100$ ($+$). We show also the entropy numerically obtained from a coin tossing sequence with the same number of data of the case $\tau = 10,\ m = 10$ ($\bullet$). The dotted line indicates $\ln(2)$.

3.2 A fixed resolution analysis

The failure of the previous analysis lead us to try another approach in order to keep the information present in the financial data, this time we use a non-linear filter with a clear financial meaning.
The procedure to create the symbolic sequence is now :

- we fix a resolution value Δ and we define

$$r_{t,t_0} \equiv \ln \frac{S_t}{S_{t_0}} \ , \tag{16}$$

  where t_0 is the initial *business* time, and $t > t_0$. We wait until an exit time t_1 such as :

$$|r_{t_1,t_0}| \geq \Delta \ .$$

  In this way we consider only market fluctuations of amplitude Δ. Since the distribution of the returns is *almost* symmetric, the threshold Δ has been chosen equal for both positive and negative values. Starting from S_{t_1} we obtain with the same procedure S_{t_2}.

- following the previous prescription we create a sequence of returns

$$\{r_{t_1,t_0}, r_{t_2,t_1}, \ldots, r_{t_k,t_{k-1}}, \ldots\} \ ,$$

from which we obtain the symbolic dynamics :

$$c_k = \begin{cases} -1 & \text{if} \quad r_{t_k,t_{k-1}} < 0 \\ +1 & \text{if} \quad r_{t_k,t_{k-1}} > 0 \end{cases} \quad . \tag{17}$$

We define k as Δ *trading* time, i.e. we enumerate only the transactions at which Δ is reached.

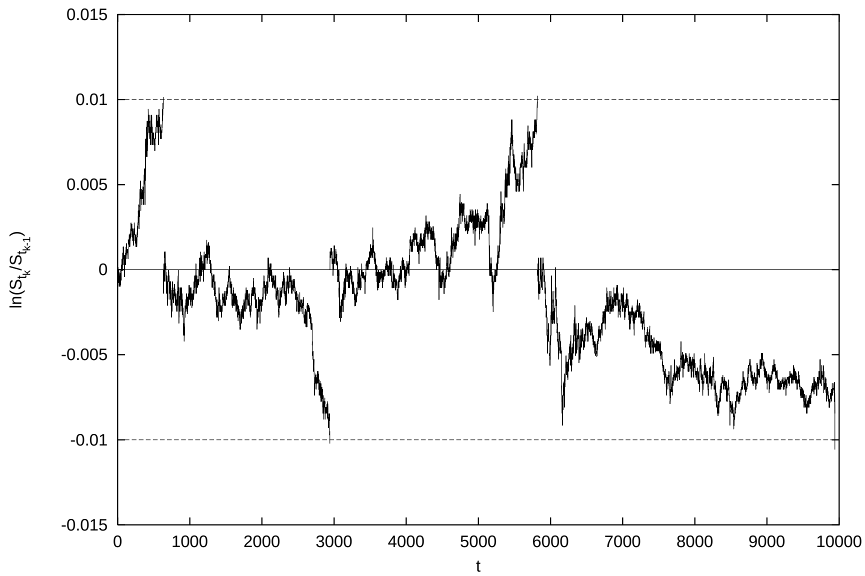

Figure 4: Evolution of $r_{t_k,t_{k-1}}$ with $\Delta = 0.01$. $t_0 = 0$ corresponds at 00:00:14 of October 1, 1992 in calendar time to the , and the $t_4 = 9939$ corresponds at 11:59:28 of October 2, 1992.

Let us notice that the variable $|r_{t_k,t_{k-1}}|$ has a narrow distribution close to the threshold, and for all practical purposes $|r_{t_k,t_{k-1}}|$ can be well approximated with Δ. In figure 4 we show an example of evolution of the $r_{t_k,t_{k-1}}$.
The entropy analysis of the symbolic sequence $\{r_{t_k,t_{k-1}}\}$ gives a completely different result from the one in the previous section. In figure 5 it is shown that the entropy is clearly different from $\ln(2)$ in a wide range of Δ, i.e. there is a set of Δ for which the *available* information (see eq. (14)) is very large.
In figure 6 we plot the *available* information *versus* Δ and the distribution of transaction costs. Because these two quantities do not have similar size they are plotted on different vertical scales but they are superimposed to make easier comparison between them. We

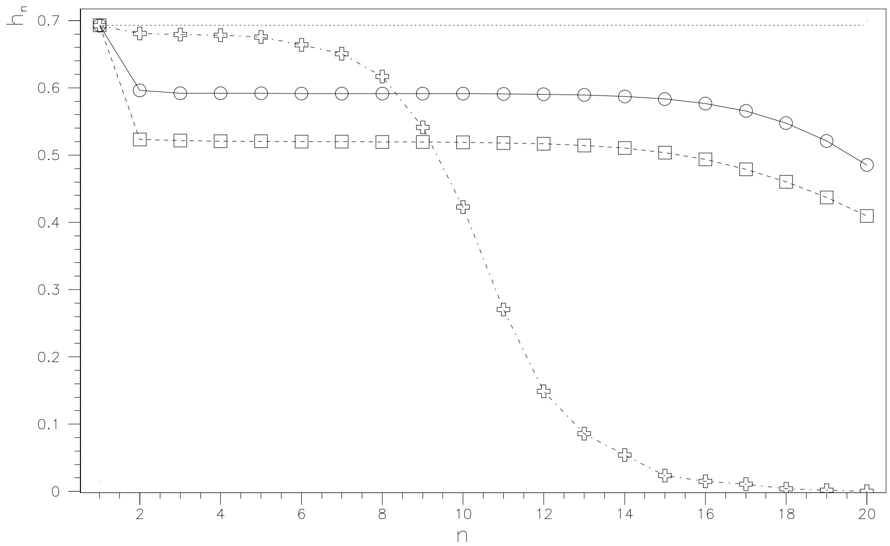

Figure 5: h_n *versus* n. The three plots correspond to different value of Δ : $\Delta = 0.00005$ ($\circ$), $\Delta = 0.0002$ ($\Box$) and $\Delta = 0.004$ ($+$). The dotted line indicates $\ln(2)$.

observe that the maximum of the *available* information is almost in correspondence to the maximum of the distribution of the transaction cost.
We have estimated the transaction costs γ as

$$\gamma_t = \frac{1}{2} \ln \frac{S_t^{(ask)}}{S_t^{(bid)}} \simeq \frac{S_t^{(ask)} - S_t^{(bid)}}{2 S_t^{(bid)}} \ ,$$

of course this is an upper bound for the true transaction cost.
We notice that the *available* information is almost equal zero when we consider very small and very large values of Δ. These limit values cannot be reached for two different reasons; since S_t can assume only discrete values, it is not possible to take the limit $\Delta \to 0$. In addition, we cannot compute I_Δ for large Δ because in the sequence $r_{t_k,t_{k-1}}$ there are not enough data for an efficient statistical analysis.

3.3 Profitable information

We focus our attention on optimal strategies in a financial market with non zero *available* information. We then show the economic relevance of such a quantity in case of *weak* efficiency of the market.
Consider the optimal growth rate strategy for a *patient* speculator. The returns $\{r_{t_k,t_{k-1}}\}$ are almost symmetrically distributed and they can be well approximated by the two threshold values Δ and $-\Delta$.

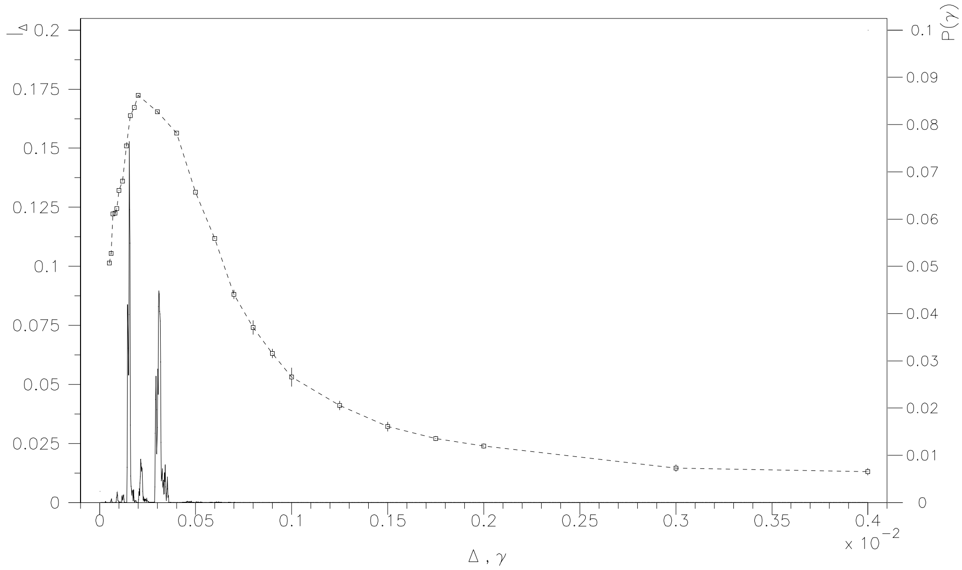

Figure 6: *Available* information I_Δ *versus* Δ (on the left), superimposed to the distribution of transaction costs, $P(\gamma)$ *versus* γ (on the right).

We shall only deal with the markovian case. In fact as suggested by figure 5 and shown in [31], one has that the markovian approximation well reproduces the signal filtered with a fixed resolution Δ. The symmetry of the return distribution and the markovian nature of the process implies that the transition matrix is close to

$$\begin{pmatrix} p_\Delta & 1-p_\Delta \\ 1-p_\Delta & p_\Delta \end{pmatrix} \quad . \tag{18}$$

where p_Δ is the probability to have $+1$ at Δ *trading* time $(t+1)$, knowing that c_t was $+1$ at time t.
In this particular case the *available* information is :

$$I_\Delta = p_\Delta \ln(p_\Delta) + (1-p_\Delta)\ln(1-p_\Delta) + \ln(2) \ . \tag{19}$$

We focus our attention on an investor who decides to diversify his portfolio only in a security asset with a given interest rate return r, and to invest, every Δ *trading* time t, a fraction l_t of his capital in the Deutschemark/US dollar exchange. Our convention is that the fraction l is positive if he exchange dollars into marks, negative vice versa, and we allow the speculator to borrow money from a bank.
We assume a vanishing interest rate return. This hypothesis is reasonable: in the period we are dealing with, the official discount rate fixed by the Federal Reserve is of 3 percent per year and fluctuates between 5.75 and 8.75 percent in the German case. The *patient*

speculator rehedges his portfolio on average every 66 seconds when Δ is equal to the mean transaction cost. The largest Δ corresponds to an average time of 8.6 hours of standby. In the time scales involved the true interest rate return is about one hundred times smaller than Δ: the approximation of a vanishing interest rate appears to be fair.

We deal with the no transaction costs case: this will allow us to understand easily the meaning of *available* information for a *patient* investor. The more general situation with transaction costs is treated in detail in [32].

Let us focus on the investment at time t: the speculator commits a fraction l_t in dollars. At the following time step $t+1$ his capital becomes

$$W_{t+1} = [1 + l_t(Exp(c_{t+1}\Delta) - 1)]W_t \simeq (1 + l_t c_{t+1}\Delta)W_t \ , \tag{20}$$

where the first order approximation in Δ is enough accurate for the Δs considered in this paper (see Figure 6). We notice that in this case, a consequence of the symmetry, is that the optimal l_t can assume only two values $l_t = c_t l$ where l is a real number.

We define the *profitable* information as the exponential rate of the capital of an investor who follows an optimal growth rate strategy. The strict connection between this quantity and the *available* information was first noticed by Kelly [33], who, considering an elementary gambling game, first gave an interpretation of Shannon entropy in the context of optimal investment.

The computation of capital growth rate is a simple application of [33], and for the investment above described is

$$\lambda_{\Delta}(l) \equiv \lim_{T\to\infty} \frac{1}{T} \ln \frac{W_T}{W_0} = p_{\Delta} \ln(1 + l\Delta) + (1 - p_{\Delta}) \ln(1 - l\Delta) \ , \tag{21}$$

where we neglect $O(\Delta^2)$ in (20). It reaches its maximum for

$$l^* = \frac{2p_{\Delta} - 1}{\Delta} \ . \tag{22}$$

An intuitive consequence of equation (22) is that an anti-persistent return ($p_{\Delta} < 1/2$), as in the financial series we have considered, implies that the optimal strategy is to buy marks if the dollar rises, and to do the opposite otherwise. Of course a persistent case ($p_{\Delta} > 1/2$) would imply an l_t greater than zero every time the positive threshold Δ is reached.

From (21) and (22) one has that the optimal growth rate is equal to the *available* information:

$$\lambda^*_{\Delta} = \max_l \lambda_{\Delta}(l) = p_{\Delta} \ln(p_{\Delta}) + (1 - p_{\Delta}) \ln(1 - p_{\Delta}) + \ln(2) = I_{\Delta} \ . \tag{23}$$

We stress that the equivalence between *available* and *profitable* information, if we forget the costs involved in this trading rule, means that a speculator, who follows a particular strategy, has the possibility to obtain a growth rate of his capital exactly equal to this information: this makes clear why we have called it *profitable*.

We underline that we have considered the growth rate measuring the time in Δ *trading* time. To obtain the exponential rate of the capital in the usual *calendar* time we have to normalize (21) with the average exit time for the specified Δ [34]. For example for $\Delta = 0.0002$ corresponding to the maximum *available* information is characterized by $\langle \tau \rangle = 66$ seconds. This means that the average optimal growth rate is equal to 0.27 percent per second.

A naive consequence of previous results could be that an efficient market hypothesis should be rejected.

Unfortunately (for the authors) this is not obvious.

We have previously noticed that the *available* information can be transformed in *profitable*, let us now comment the feasibility of the proposed trading rule.

When Δ is near the value of the maximum *available* information, the speculator changes his position with high frequency, and Δ is comparable with transaction costs: it is not any more possible to neglect them.

Furthermore in equation (22) Δ appears at the denominator, and then the values of l^* can be enormous. For example for Δ corresponding to the maximum of the *available* information, the speculator who follows the optimal growth rate strategy, should borrow 2830 times the capital he has! Even a small fluctuation from the expected average behavior can lead to bankruptcy.

On the other hand if he wants to use reasonable values of l^*, he has to chose a sufficiently large Δ; in this situation the filtered series is almost indistinguishable from a "random walk" and then there is almost no *available* information.

We have now all the ingredients to comment the shape of the *available* information shown in figure 6.

The speculator cannot have a resolution Δ lower than the transaction costs, profits from such an investment would be in fact less than costs. Therefore in this range of Δ the *available* information increases. The discretization of the prize changes does not allow for reaching in a continuous way $\Delta = 0$, where the "random walk" model is practically recovered as shown in the first part of this section.

For Δ larger than the transaction costs the information can be exploited by proper strategies. However, small fluctuations are more difficult to detect and to distinguish from the "noise" and the *profitable* information is almost useless because of the huge values of l^* involved. This fact is even more evident when transaction costs are included.

On the other hand for large Δ, the investors are able to discover the *available* information and to make it profitable with a feasible strategy. As a consequence, the efficient equilibrium is than restored for all practical purposes.

Let us briefly mention what happens instead to the *systematic* investor. We can repeat exactly the above discussion and the only difference is that now he decides to modify his portfolio every τ *business* time. Because there is almost no *available* information (see subsection **3.1**) the optimal growth rate of his capital is vanishing even without considering the costs involved in the transactions.

4 Conclusions

In this paper we have considered the long term anomalies in the Deutschemark/US dollar quotes in the period from October 1, 1992 to September 30, 1993 and we have analyzed the consequences on the *weak* efficiency of this market.

In section **2** we have shown the presence of long term anomalies with two techniques: the structure functions and a generalization of the usual correlation analysis. In particular we have pointed out that "random walk" models (or other self-affine models) cannot describe these features.

Once we have shown the existence of correlations in financial process, we have tested whether they allow for a *profitable* strategy.

With such a goal in mind, in section **3** we have first introduced a direct measure of the *available* information, then we have shown in a particular case that this is equivalent to a *profitable* information. In other words following a suitable trading rule it is possible in absence of transaction costs to have an exponential growth rate of the capital equal to this information.

We have measured the *available* information with a technique which reminds the Kolmogorov ϵ entropy. Two different codifications for financial series (fixed lag τ and finite resolution Δ) lead to completely different results.

The *available* information strongly depends on the kind of investment the speculator has in mind. We show that if he wants to change his position *systematically* at fixed lags τ the *available* information is practically zero: for this investor the market is efficient.

Instead, a *patient* investor, who waits to modify his portfolio till the asset has a fluctuation Δ, observes a finite *available* information.

However, the existence of such a trading rule does not imply that the investment is feasible in practice. Namely we show that when reasonable investments are involved almost no *available* information survives. On the contrary, it is extremely difficult to use it when it is still present.

The technique described here can be considered as a powerful tool to test *weak* efficiency : the speculator contributes to reach efficient equilibria destroying the *available* information that could be exploited in practice. The efficiency hypothesis is then restored for almost all practical purposes.

Appendix

In this appendix we show that if the correlations $C_q(\tau)$ exhibit a long range memory $C_q(\tau) \sim \tau^{-\beta_q}$ then also the variance $\delta_q(\tau)$ of the *generalized cumulative absolute returns* $\{\chi_{t,q}(\tau)\}$ behaves at large τ as $\tau^{-\beta_q}$.

Making explicit expression of $\chi_{t,q}(\tau)$ (see equation (8)) one can write equation (9) as :

$$\delta_q(\tau) = \frac{1}{\tau^2} \sum_{\tau_1=0}^{\tau-1} \sum_{\tau_2=0}^{\tau-1} \langle |r_{t+\tau_1}|^q |r_{t+\tau_2}|^q \rangle - \langle |r_{t+\tau_1}|^q \rangle \langle |r_{t+\tau_2}|^q \rangle \ .$$

Taking into account the fact that r_t is a stationary process, and using the definition of $C_q(\tau)$, one has:

$$\delta_q(\tau) = \frac{1}{\tau} C_q(0) + \frac{2}{\tau^2} \sum_{\tau > \tau_1 > \tau_2 \geq 0} C_q(\tau_1 - \tau_2)$$

where

$$C_q(0) = \langle |r_t|^{2q} \rangle - \langle |r_t|^q \rangle^2 \ .$$

The expression of $\delta_q(\tau)$ can be rewritten as:

$$\delta_q(\tau) = \frac{1}{\tau} C_q(0) + \frac{2}{\tau^2} \sum_{\tau_1=1}^{\tau-1} (\tau - \tau_1) \ C_q(\tau_1) \ .$$

Under the hypothesis $C_q(\tau) \sim \tau^{-\beta_q}$, one has for large τ

$$\frac{2}{\tau^2} \sum_{\tau_1=1}^{\tau-1} (\tau - \tau_1) \ C_q(\tau_1) \sim \tau^{-\beta_q} \ ,$$

which leads to :

$$\delta_q(\tau) = O(\tau^{-1}) + O(\tau^{-\beta_q}) \ .$$

Since $\beta_q \leq 1$, the thesis follows, i.e. :

$$\delta_q(\tau) \sim \tau^{-\beta_q} \ .$$

Acknowledgment

We thank Luca Biferale and Rosario Mantegna for many useful discussions at an early stage of this research. We also thank Paolo Muratore Ginanneschi and Alberto Bigazzi for a careful reading of the manuscript.

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Patrick Cheridito (ETH):

Long-Range Dependence and Option Pricing.

Abstract: Several attempts have been made to remedy some of the shortcomings of the Black-Scholes model by describing the risky asset by a process with correlated increments. Fractional Brownian motion B_t^H exhibits long-range dependence between the increments, but it is not a semimartingale. By a general result of Delbaen and Schachermayer (1994) this guarantees the existence of a sequence of simple predictable integrands yielding a free lunch with vanishing risk. Rogers (1997) and Shiryayev (1998) even constructed arbitrage strategies for fractional models. Hence, it is certainly not reasonable to model the discounted price of a financial asset by

$$S_t = 1 + B_t^H \quad \text{or} \quad S_t = e^{B_t^H}.$$

Rogers (1997) proposed to change the kernel in the Mandelbrot-Van Ness representation of fractional Brownian motion at zero to obtain a Gaussian semimartingale with the same long-range dependence as fractional Brownian motion.

We present an arbitrage strategy which needs to see a smaller filtration than the one of Rogers and exists for fractional Brownian motion with every Hurst parameter $H \in (0,1)$, whereas Shiryayev's strategy can only be defined for $H \in (\frac{1}{2}, 1)$. Further we show that Roger's Gaussian semimartingale R_t is equivalent to Brownian motion. This means that if we model a risky asset by the SDE

$$dS_t = \mu S_t dt + \sigma S_t dR_t, \quad (*)$$

the option pricing formulas are the same as in the Black-Scholes model. However, if we combine such Gaussian semimartingales with different long-range dependence rates

$$R_t = \sum_{i=1}^{n} \alpha_i R_t^i,$$

it might well be that $(*)$ yields a sensible model not equivalent to the Black-Scholes model.

Rama Cont (CMAP, CNRS - Ecole Polytechnique):

Multiresolution analysis of financial time series.

Abstract: While financial applications involve many different time scales, ranging from a few minutes (intraday) to a few months, most techniques used in econometrics focus on modeling the returns (first differences) of price series on a single time scale. By contrast, multiresolution analysis probes the properties of time series at various time scales and focuses on their *scaling* properties i.e. how their statistical features change with time resolution.

We present a "pathwise" approach to the analysis of scaling and regularity properties of financial time series, using the *wavelet transform* as a mathematical microscope for probing the local Hölder regularity of price trajectories. Using three different methods, we estimate the Hölder spectrum of these time series and compare them with properties of various stochastic processes used in financial modeling.

Finally, we compare these results to previous studies on turbulent velocity fields and point out similarites and differences. We will show that, although it is tempting to apply the Kolmogorov cascade approach to model financial data, some serious problems are encountered in such approaches.

Keywords: random cascades, financial time series, high-frequency data, Hölder regularity, multifractal formalism, multiscale stochastic processes, multiresolution analysis, scaling, self-similarity, singularity spectrum, turbulence, wavelet transform.

Rosario Delgado (Barcelona):

On an Ogawa-type integral with application to the Fractional Brownian Motion[†]*.*

EXTENDED ABSTRACT:

Recently, several authors have used different approaches to the construction of a stochastic calculus with respect to the process known as Fractional Brownian motion (FBM for short), introduced in the celebrated paper of Mandelbrot and Van Ness ([2]). The interest on this subject takes root in the recognition of this process as a good model in many applications (in Engineering, Economics, Physics, Biology, ...), due to its long-term dependence and self-similarity character. Since this non-Markovian process is not even a semimartingale, it is not possible to apply the classical theory in order to define a stochastic integral with respect to it. Here we construct a deterministic Ogawa-type integral with respect to a continuous function that, in particular, can be a trajectory of the FBM. This integral is inspired by the (stochastic) Ogawa integral (see for instance Ogawa ([3])), that we also consider in our work when we integrate with respect to the FBM.

1. A deterministic Ogawa-type integral with respect to continuous functions.

Let $t \in [0,\, 1]$, $f \in L^2\left([0,1]\right)$ and $g \in \mathcal{C}\left([0,\, 1]\right)$. For any $n \in \mathbb{N}$ we introduce the sums

$$U_n(f,g,t) = \sum_{i=1}^{n} \big(\int_0^t f(s)\,\phi_i(s)\,ds \big)\, g_i\,,$$

where $\{\phi_i,\, i \geq 1\}$ is the Haar system of $L^2\left([0,\, 1]\right)$ and $\{g_i,\, i \geq 0\}$ are the coefficients of the development of g in terms of the basis of Schauder of $\mathcal{C}\left([0,\, 1]\right)$, $\{\varphi_i,\, i \geq 0\}$. Note that $g_i,\, i \geq 1,$ coincide with the Stieltjes integral of ϕ_i with respect to g on $[0,\, 1]$. The fact that $f\,1_{[0,\,t]} = \sum_{i=1}^{\infty} \big(\int_0^t f(s)\,\phi_i(s)\,ds \big)\,\phi_i$ makes natural the next definition.

Definition 1.1 *We will say that the indefinite (Ogawa-type) integral of f with respect to g exists if $U_n(f,g,t)$ converges for all $t \in [0,1]$, as n goes to infinity, and we will denote its limit by*

$$\int_0^t f(s) dg(s) = \sum_{i=1}^{\infty} \big(\int_0^t f(s)\,\phi_i(s)\,ds \big)\, g_i\,.$$

Equivalently, we also will say that f is Ogawa-type integrable with respect to g.

We point out that this integral coincides with the *smoothed* Stratonovich integral, and under rather general conditions, it also coincides with $\sum_{i=1}^{\infty} \big(\int_0^1 f(s)\,\phi_i(s)\,ds \big)\, I\left(\phi_i, g, t\right)$, where $I\left(\phi_i, g, t\right)$ is the Stieltjes integral of ϕ_i with respect to g on $[0,\, t]$.

A fundamental property that satisfies the integral introduced above is given in the following result.

[†]Joint work with Maria Jolis

Proposition 1.2 *Let f be a simple function of the form $f(s) = \sum_{j=1}^{\ell} a_j \, 1_{[t_j, t_{j+1})}(s)$, with $a_j \in \mathbb{R}$, $0 \leq t_1 < t_2 < \ldots < t_{\ell+1} \leq 1$. Then, for any $g \in \mathcal{C}([0, 1])$, there exists the indefinite Ogawa-type integral of f with respect to g, and equals to their Stieltjes integral, that is,*

$$\int_0^t f(s) \, dg(s) = \sum_{j=1}^{\ell} a_j \left(g(t_{j+1} \wedge t) - g(t_j \wedge t) \right).$$

By other way, Ciesielski et al. ([1]) introduce an integral of any function $f \in L^2([0, 1])$ with respect to any function $g \in \mathcal{C}([0, 1])$, with finite development in the basis of Haar and Schauder, respectively $\{f_i\}_i$ and $\{g_{i'}\}_{i'}$, as $\sum_{i,i' \geq 1} f_i g_{i'} \int_0^t \phi_i(s) \, \phi_{i'}(s) ds$. They prove that their integral can be extended by continuity to any functions f and g in the Besov subspaces of $\mathcal{C}([0, 1])$ $\mathcal{B}_{p,1}^{1-\alpha}$ and $\mathcal{B}_{p,\infty}^{\alpha}$, respectively, with α and p such that $1 \leq p \leq \infty$, and $1/p < \alpha < 1 - 1/p$. For the sake of completeness, we recall that the Besov space $\mathcal{B}_{p,q}^{s}$, with $s > 0$ and $1 \leq p, \, q \leq \infty$, is the Banach space of functions $f : [0, 1] \to \mathbb{R}$ such that $||f||_{s,p,q}$ is finite, endowed with this norm, where

$$||f||_{s,p,q} = \begin{cases} ||f||_p + \left(\int_0^1 \left(\frac{1}{t^s} \, \omega_p(f, t) \right)^q \frac{dt}{t} \right)^{1/q} & \text{if} \quad q < \infty \\ ||f||_p + \sup_{0<t<1} \left(\frac{1}{t^s} \, \omega_p(f, t) \right) & \text{if} \quad q = \infty, \end{cases}$$

and $\omega_p(f, t) = \sup_{0<h\leq t} ||(f(\cdot + h) - f(\cdot)) \, 1_{[0, 1-h]}(\cdot)||_p$.

In the next Proposition we relate the integral of Ciesielski et al. with the Ogawa-type integral that we are considering.

Proposition 1.3 *If $f \in \mathcal{B}_{p,1}^{1-\alpha}$ and $g \in B_{p,\infty}^{\alpha}$ with $1 \leq p \leq \infty$ and $1/p < \alpha < 1 - 1/p$, then f is Ogawa-type integrable with respect to g and this integral coincides with the integral introduced by Ciesielski et altri in [1].*

Our following result gives a sufficient condition of integrability for functions f that are not necessarily continuous. To get this kind of result we have to pay the price of restrict the class of functions g with respect to which we can integrate to a sufficiently good Besov subspace of $\mathcal{C}([0, 1])$.

Theorem 1.4 *Let $H \in (1/2, 1)$ and let $g \in \mathcal{B}_{q,\infty}^{H}$, with $1/q < H - 1/2$. Let f be a function of $L^2([0, 1])$ for which the following condition is satisfied:*

$$(h) \quad f \in \mathcal{B}_{p,\infty}^{r}, \quad \textit{for some } r > 1 - H, \quad \textit{with} \quad p = \frac{q}{q-1}.$$

Then, f is Ogawa-type integrable with respect to g and the indefinite integral is a continuous function.

From the proof of this result we also obtain that there exists a constant $C > 0$ such that

$$||\int_0^{\cdot} f(s)dg(s)||_\infty \leq C\left(||f||_{r,p,\infty} + ||f||_2\right)||g||_{H,q,\infty}\,.$$

Then, a natural question to next consider is that if the indefinite integral belongs to some Besov subspace of $\mathcal{C}([0,\,1])$, and if we can prove some kind of norms inequality for it. The positive answer to this question is given in the next result.

Theorem 1.5 *Under the assumptions of Theorem 1.4, if, in addition, we suppose that the following hypothesis is satisfied:*

$$(h') \quad f \in L^\infty([0,\,1]), \quad \frac{1}{q} \leq 1 - H\,,$$

then, the indefinite Ogawa-type integral belongs to the Besov subspace of $\mathcal{C}([0,\,1])$ $\mathcal{B}^H_{q',\infty}$ *for* $q' = \frac{1}{1-r}$ *and the following norms inequality holds, for some* $C > 0$,

$$||\int_0^{\cdot} f(s)dg(s)||_{H,q',\infty} \leq C\left(||f||_{r,p,\infty} + ||f||_\infty\right)||g||_{H,q,\infty}\,.$$

2. Application to the fractional Brownian motion.

We denote by B^H the FBM of Hurst parameter $H \in (0,\,1)$. This family of processes includes the ordinary Brownian motion, that corresponds to $H = 1/2$. We point out that the case $1/2 < H < 1$ is the most frequently encountered in mathematical modeling. It is known that the trajectories of B^H belong to the Besov subspace of $\mathcal{C}([0,\,1])$ $\mathcal{B}^H_{q,\infty}$, for any $0 < 1/q < H$. In particular, for $H \in (1/2,\,1)$, we can take $0 < 1/q < H - 1/2$. Therefore, we can apply Theorem 1.4, that ensures the integrability in the trajectorial (Ogawa-type) sense of any process whose trajectories live in $L^2([0,\,1]) \cap \mathcal{B}^r_{p,\infty}$, with $r > 1 - H$ and $p = q/(q-1)$. This theorem also gives the continuity of the indefinite integral. If we take $0 < 1/q < (1-H) \wedge (H - 1/2)$ and we suppose, in addition, that the trajectories of the process that we are integrating belong to $L^\infty([0,\,1])$, Theorem 1.5 gives that the indefinite integral belongs to some Besov subspace of $\mathcal{C}([0,\,1])$ and that a norms inequality holds.

Moreover, we can also consider the stochastic Ogawa integral (see [3], for instance) with respect to the FBM of parameter $H \in (0,\,1)$. We denote by b_i^H the coefficients of the development of the trajectories of B^H in the basis of Schauder. Then, we have that for any ω P-a.s, $B^H(\omega) = \sum_{i\geq 1} b_i^H(\omega)\,\varphi_i$. Let $X = (X_t)_{t\in[0,\,1]}$ be a measurable process defined on the same probability space that the FBM, that satisfies that its paths are in $L^2([0,\,1])$ P-a.s. Let us denote by $(x_i)_{i\geq 1}$ the coefficients of the development in the Haar system of the trajectories of X.

Definition 2.1 *For any $t \in [0, 1]$ and $n \in \mathbb{N}$, we define $U_n(X,t)$ to be*

$$\sum_{i=1}^{n} \Big(\int_0^t X_s \, \phi_i(s) \, ds \Big) \Big(\int_0^1 \phi_i(s) \, dB_s^H \Big) \,,$$

where $\int_0^1 \phi_i(s) \, dB_s^H$ is defined in the natural way as b_i^H. Then, we say that X is Ogawa integrable with respect to B^H if there exists the limit in probability, as n goes to infinity, of $U_n(X,t)$, for all $t \in [0, 1]$. We will denote this limit by $\int_0^t X_s \, dB_s^H$. Moreover, if $H \geq 1/2$,

$$\int_0^t X_s \, dB_s^H = \sum_{i=1}^{\infty} \Big(\int_0^1 X_s \, \phi_i(s) \, ds \Big) \Big(\int_0^t \phi_i(s) \, dB_s^H \Big) \,,$$

where $\int_0^t \phi_i(s) \, dB_s^H$ is defined in the natural way.

In the next result we give a sufficient condition for the integrability of a process with respect to B^H, with $H > 1/2$, in the sense of Definition 2.1.

Proposition 2.2 *Let $H \in (1/2, 1)$. Let $X = (X_t)_{t \in [0,1]}$ be a measurable process whose trajectories belong to $L^2([0, 1])$ P-a.s, for which the following condition is satisfied:*

$$(h'') \quad \begin{cases} \text{there exist } p \in (1, \frac{2}{3-2H}) \text{ and } r > 1 - H \text{ such that} \\ \sup_{j \geq 0} \Big(2^{j(r-1/p+1/2)} \big(E(\sum_{k=1}^{2^j} |x_{2^j+k}|^p) \big)^{1/p} \Big) < \infty \,. \end{cases}$$

Then, X is Ogawa integrable with respect to B^H.

Remark 2.3 *It is straightforward to prove that hypothesis (h'') is satisfied for any centered process X starting from zero and with $E(X_s^2) < \infty$ for some $s \in (0, 1]$, whose increments are stationary and with the property of self-similarity of parameter $M > 1-H$. An example of such a process is the FBM of parameter $M > 1 - H$, but there are other examples.*

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[2] B.B. Mandelbrot, J. W. Van Ness (1968), Fractional Brownian Motions, Fractional Noises and Applications, *SIAM Review*, **10 (4)**, 422–437.

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Martin Greiner (MPI Dresden):

What can we learn from one-dimensional observables in fully developed Navier-Stokes turbulence?.

ABSTRACT: Fully developed Navier-Stokes turbulence is a three-dimensional nonlinear process. Standard experimental observations record time series of the velocity field in one point, which according to Taylors frozen flow hypothesis can be interpreted as a one-dimensional spatial cut through the three-dimensional velocity field at a given fixed time. This reduction in dimensions is an important point to keep in mind when analysing and interpreting the 'one-dimensional' data. We discuss its implications on various inertial range observables by employing turbulent cascade models. Such observables are, for example, velocity structure functions and pdfs, and multiplier and wavelet correlations for the energy dissipation field.

Ralf Hendrych (CeVis, Bremen):

Self-similarity and Wavelets.

ABSTRACT: We'll give a historical overview about the concept of self-similar stochastic processes, starting with the work of Lamperti on "semi-stability". This concept of scaling invariance has several generalizations and applications. One problem is the estimation of Hurst-exponents. Different methods are used. Some of them based on the wavelet-transform. The wavelet-transform as a tool for the analysis and synthesis of self-similar stochastic processes will be motivated and discussed.

Stewart Hodges (Warwick):

The Risk Premium In Trading Equilibria Which Support Black-Scholes Option Pricing.

ABSTRACT: This paper provides further analysis of the behaviour of the risk premium on the market portfolio of risky assets. Earlier work by Hodges and Carverhill (1993), and by others, has characterised the evolution of the market risk premium in economies where the variance of the return on the market has constant variance and market index options can be priced using the Black-Scholes model. In such economies the risk premium satisfied a non-linear partial differential, equation called Burgers' equation. This also provides some significant new insights into this analysis. First we describe the nature of the existing results and provide a much simpler and more intuitive derivation. Next, we consider the time homogeneous case. Our original objective was to find a time homogeneous economy which allows the risk premium to vary inversely with the level of the market so that some kind of mean reversion could take place. Sadly, this is impossible. We obtain the interesting, but negative, result that the risk premium must be constant or increasing in the market level for time homogeneous equilibria which rule out arbitrage. Finally, this result is shown to tie in to earlier work asymptotic portfolio selection. the article also illustrates that caution is required in this kind of modelling to avoid writing down models which admit arbitrage. The analysis also shows the limitations of the representative investor paradigm. Current research on the inefficiency of the market portfolio if the form of the risk premium is static will also be described.

Esben Høg (Aarhus School of Business):

A note on a representation and calculation of the long-memory Ornstein-Uhlenbeck process.

Extended Abstract:

Abstract

In this paper we analyze the covariance function for a long memory generalization of Ornstein-Uhlenbeck type processes which are the analogues in continuous time of long memory autoregressions of order 1. A Fractional Brownian Motion with drift is a special case. We find the exact expression for the covariance function of the long memory OUP by using the confluent hypergeometric function.

Introduction

This paper is concerned with the analysis of a long memory version of the Ornstein-Uhlenbeck process.

Behaviour of interest rates and other financial series generally tend to be highly positively autocorrelated with long swings (especially long rates) and sample autocorrelations that die out slowly. Thus they appear to be non–stationary. The concepts of fractional integration or long memory provide a framework for these processes. To incorporate first order autoregressive behaviour as well a fractional Ornstein-Uhlenbeck process is considered. Eventually this model may be used to estimate the long memory (Hurst) index.

The Ornstein-Uhlenbeck process

In continuous time the analogue of autoregressions of order 1 are the Ornstein-Uhlenbeck type processes. Recall a Gaussian Ornstein-Uhlenbeck process of the form

$$\mathrm{d}X = (\mu - \varphi X)\mathrm{d}t + \sigma \mathrm{d}W$$

or

$$X(t) = X(0) + \int_0^t (\mu - \varphi X(u))\mathrm{d}u + \sigma W(t),$$

where $\{W(t) : t \geq 0\}$ is a standard Brownian motion. It has the well known solution

$$X(t) \quad = \quad \mathrm{e}{-}\varphi t X(0) + \frac{\mu}{\varphi}(1 - \mathrm{e}{-}\varphi t) + \sigma \int_0^t \mathrm{e}{-}\varphi(t-u)\mathrm{d}W(u).$$

Now consider the process with the mean subtracted

$$\begin{aligned} \widetilde{X}(t) &= X(t) - \mathbb{E}\,(X(t)) = X(t) - \mathrm{e}{-}\varphi t X(0) + \frac{\mu}{\varphi}(1 - \mathrm{e}{-}\varphi t) \\ &= \sigma \int_0^t \mathrm{e}{-}\varphi(t-s)\mathrm{d}W(s). \end{aligned}$$

Then write

$$
\begin{aligned}
\widetilde{X}(t) &= \sigma\left\{W(t) + \int_0^t [\mathrm{e}{-}\varphi(t-s) - 1]\,\mathrm{d}W(s)\right\} \\
&= \sigma\left\{W(t) + \int_0^t \left[\int_0^{t-s} \frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{e}{-}\varphi u)\mathrm{d}u\right] \mathrm{d}W(s)\right\} \\
&= \sigma\left\{W(t) + \int_0^t \left[\int_s^{t} \frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{e}{-}\varphi(v-s))\mathrm{d}v\right] \mathrm{d}W(s)\right\},
\end{aligned}
$$

so that applying Fubini's theorem for stochastic integrals

$$
\begin{aligned}
\widetilde{X}(t) &= \sigma\left\{W(t) + \int_0^t \left[\int_0^{v} \frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{e}{-}\varphi(v-s))\mathrm{d}W(s)\right] \mathrm{d}v\right\} \qquad (1) \\
&= \sigma\left\{W(t) - \varphi \int_0^t \left[\int_0^{v} \mathrm{e}{-}\varphi(v-s)\mathrm{d}W(s)\right] \mathrm{d}v\right\} \\
&= \sigma W(t) - \sigma\varphi \int_0^t \mathrm{e}{-}\varphi(t-u)W(u)\mathrm{d}u,
\end{aligned}
$$

or in differential representation

$$
d\widetilde{X}(t) = \sigma\left\{\mathrm{d}W(t) - \varphi\left[\int_0^t \mathrm{e}{-}\varphi(t-s)\mathrm{d}W(s)\right]\mathrm{d}t\right\}.
$$

Now the question is, can we make similar calculations with the Brownian Motion replaced by a Fractional Brownian Motion?

Generalization of Fractional Brownian Motion

The answer to the question in the previous section is yes, at least if we define a Fractional Brownian Motion with Hurst coefficient H, where $0 < H < 1$, as

$$
W_d(t) = \frac{1}{\Gamma(d+1)} \int_0^t (t-s)^d \mathrm{d}W(s),
$$

where W is a Brownian Motion, and d is the "difference" parameter $d = H - 1/2$, see for instance Comte & Renault (1996).

Comte & Renault (1996) then define a fractionally integrated process of order d, $-1/2 < d < 1/2$, as

$$
X(t) = \frac{1}{\Gamma(d+1)} \int_0^t (t-s)^d \widetilde{a}(t-s)\mathrm{d}W(s),
$$

or in our notation

$$
X(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \widetilde{a}(t-s)\mathrm{d}W(s) \qquad (2)
$$

where $\widetilde{a}$ is C^1.

The advantage of this definition is that it turns out that it allows to generalize (1) by hiding the singular part t^d inside the Brownian term by replacing W by W_H.

For that purpose stochastic integration w.r.t. W_H is defined in the following. If $W_H(t) = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} \mathrm{d}W(s)$ then under certain regularity conditions,

$$X(t) = \int_0^t c(t-s)\mathrm{d}W_H(s) \tag{3}$$

is defined as

$$X(t) = \int_0^t c(t-s)\mathrm{d}W_H(s) := \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t c(t-s)W_H(s)\mathrm{d}s\right].$$

Thus (2) and (3) are two representations of $X(t)$.

If c is C^1 there is a one–to–one correspondence between $c(\cdot)$ and $\widetilde{a}(\cdot)$.

Moreover it turns out that the generalization of decomposition for ordinary Brownian Motion (cf. (1)) is

$$\widetilde{X}(t) = c(0)W_H(t) + \int_0^t \left[\int_0^v c'(v-s)\mathrm{d}W_H(s)\right]\mathrm{d}v, \tag{4}$$

or in differential representation

$$d\widetilde{X}(t) = c(0)\mathrm{d}W_H(t) + \left[\int_0^t c'(t-s)\mathrm{d}W_H(s)\right]\mathrm{d}t.$$

Example:

The fractionally integrated version of the Ornstein-Uhlenbeck type process is expressed in either of two ways:

$$\begin{aligned}
\widetilde{X}(t) &= \int_0^t a(t-s)\mathrm{d}W(s) = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}\widetilde{a}(t-s)\mathrm{d}W(s), \\
\widetilde{X}(t) &= \int_0^t c(t-s)\mathrm{d}W_H(s),
\end{aligned}$$

where obviously $c(x) = \sigma \mathrm{e}{-}\varphi x$ and it may be shown that

$$\begin{aligned}
a(x) &= \frac{\sigma}{\Gamma(H+1/2)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \mathrm{e}{-}\varphi u(x-u)^{H-1/2}\mathrm{d}u\right] \\
&= \frac{\sigma}{\Gamma(H+1/2)}\left(x^{H-1/2} - \varphi \mathrm{e}{-}\varphi x \int_0^x \mathrm{e}\varphi u u^{H-1/2}\mathrm{d}u\right).
\end{aligned}$$

Also there is a one-to-one relation between the $\widetilde{a}$ and c functions.

To calculate the autocovariances, it is convenient to use the representation with stochastic integration w.r.t. FBM. We use the so-called confluent hypergeometric function (Gradshteyn & Ryzhik 1980, formula (9.210)) to calculate integrals of the form $\int_0^x \mathrm{e}\varphi u u^{\alpha-1}\mathrm{d}u$, also see the Appendix.

Consider the representation

$$\widetilde{X}(t) = \sigma \int_0^t \mathrm{e}{-}\varphi(t-u)\mathrm{d}W_H(u)$$

which by definition is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sigma \mathrm{e}{-}\varphi t \int_0^t \mathrm{e}\varphi u W_H(u)\mathrm{d}u\right]$$

Simple differentiation then shows

$$\begin{aligned}\widetilde{X}(t) &= -\sigma\varphi \mathrm{e}{-}\varphi t \int_0^t \mathrm{e}\varphi u W_H(u)\mathrm{d}u + \sigma \mathrm{e}{-}\varphi t \mathrm{e}\varphi t W_H(t) \\ &= \sigma W_H(t) - \sigma\varphi \int_0^t \mathrm{e}{-}\varphi(t-u) W_H(u)\mathrm{d}u.\end{aligned}$$

The autocovariance function is (for $t \geq s$)

$$\begin{aligned}\mathbb{E}(\widetilde{X}(t)\widetilde{X}(s)) &= \sigma^2 \Big[\mathbb{E}(W_H(t)W_H(s)) - \varphi \int_0^t \mathrm{e}{-}\varphi(t-u)\mathbb{E}(W_H(s)W_H(u))\mathrm{d}u \\ &\quad -\varphi \int_0^s \mathrm{e}{-}\varphi(s-v)\mathbb{E}(W_H(t)W_H(v))\mathrm{d}v \\ &\quad +\varphi^2 \int_0^t \int_0^s \mathrm{e}{-}\varphi(t-u) - \varphi(s-v)\mathbb{E}(W_H(u)W_H(v))\mathrm{d}u\mathrm{d}v\Big].\end{aligned} \tag{5}$$

The exact covariance function for the long memory OUP

We have the following exact expression for the autocovariance function in equation (5):

Proposition 1 *Let $X(t)$ be a long memory OU process starting at time 0 with parameters φ, σ^2, and H. Then the covariance function of $X(\cdot)$ for $t \geq s$ is*

$$\begin{aligned}Cov(X(t), X(s)) &= \frac{\sigma^2}{2}\left[t^{2H}e^{-\varphi s} + s^{2H}e^{-\varphi t} - (t-s)^{2H}\right] \\ &\quad + \frac{\sigma^2\varphi}{4}\Big[-e^{-\varphi(t+s)}\left(M(H,\varphi,s) + M(H,\varphi,t)\right) \\ &\quad\quad + e^{-\varphi(t-s)}\left(M(H,-\varphi,s) + M(H,\varphi,t-s)\right) \\ &\quad\quad + e^{\varphi(t-s)}\left(M(H,-\varphi,t) - M(H,-\varphi,t-s)\right)\Big],\end{aligned} \tag{6}$$

where $M(\cdot,\cdot,\cdot)$ is the following simple "version" of the confluent hypergeometric function ${}_1\mathrm{F}_1$:

$$M(H,\delta,x) = \int_0^x u^{2H} e^{\delta u}\, du \quad = \quad \frac{1}{2H+1} x^{2H+1}\, {}_1\mathrm{F}_1(2H+1, 2H+2, \delta x).$$

Remarks:

- There exist efficient algorithms to calculate $\mathrm{M}(H,\delta,x)$.
- Note that the covariance function for the Fractional Brownian Motion is obtained from (6) when $\varphi = 0$:

$$\varphi = 0: \qquad \mathrm{Cov}(X(t), X(s)) = \frac{\sigma^2}{2}\left[t^{2H} + s^{2H} - (t-s)^{2H}\right], \qquad t \geq s.$$

- Also note that the covariance function for the (ordinary) OU process is obtained from (6) when $H = 1/2$:

$$H = 1/2: \qquad \mathrm{Cov}(X(t), X(s)) = \frac{e^{-\varphi\,(t-s)}\,(1 - e^{-2\,s\varphi})}{2\varphi}, \qquad t \geq s.$$

Proof

The first term in (5) is

$$\frac{\sigma^2}{2}\left[t^{2H} + s^{2H} - (t-s)^{2H}\right].$$

The second term is

$$\begin{aligned}
-\tfrac{\varphi\sigma^2}{2} & \left[\int_0^t \mathrm{e}{-}\varphi(t-u)s^{2H}\mathrm{d}u + \int_0^t \mathrm{e}{-}\varphi(t-u)u^{2H}\mathrm{d}u - \int_0^t \mathrm{e}{-}\varphi(t-u)|s-u|^{2H}\mathrm{d}u\right] \\
= \quad & -\frac{-\varphi\sigma^2}{2}\left[s^{2H}\frac{1}{\varphi}(1-\mathrm{e}{-}\varphi t) + \mathrm{e}{-}\varphi t\int_0^t \mathrm{e}\varphi u u^{2H}\mathrm{d}u\right. \\
& \left. + \int_0^s \mathrm{e}{-}\varphi(t-u)(s-u)^{2H}\mathrm{d}u + \int_s^t \mathrm{e}{-}\varphi(t-s)(u-s)^{2H}\mathrm{d}u\right] \\
= \quad & -\frac{\varphi\sigma^2}{2}\left\{s^{2H}(1-\mathrm{e}{-}\varphi t)/\varphi + \frac{1}{2H+1}\mathrm{e}{-}\varphi t t^{2H+1}\,{}_1\mathrm{F}_1(2H+1, 2H+2, \varphi t)\right. \\
& + \mathrm{e}{-}\varphi(t-s)\frac{1}{2H+1}\left[s^{2H+1}\,{}_1\mathrm{F}_1(2H+1, 2H+2, -\varphi s)\right. \\
& \left.\left. - (t-s)^{2H+1}\,{}_1\mathrm{F}_1(2H+1, 2H+2, \varphi(t-s))\right]\right\}.
\end{aligned}$$

Likewise the third term becomes

$$\begin{aligned}
& -\frac{\varphi\sigma^2}{2}\left\{t^{2H}(1-\mathrm{e}{-}\varphi s)/\varphi + \frac{1}{2H+1}\mathrm{e}{-}\varphi s s^{2H+1}\,{}_1\mathrm{F}_1(2H+1, 2H+2, \varphi s)\right. \\
& + \mathrm{e}\varphi(t-s)\frac{1}{2H+1}\left[t^{2H+1}\,{}_1\mathrm{F}_1(2H+1, 2H+2, -\varphi t)\right. \\
& \left.\left. - (t-s)^{2H+1}\,{}_1\mathrm{F}_1(2H+1, 2H+2, -\varphi(t-s))\right]\right\}.
\end{aligned}$$

Finally, the fourth term reads

$$
\begin{aligned}
& \frac{\varphi^2\sigma^2}{2}\bigg[\int_0^s \mathrm{e}{-\varphi(s-v)} \int_0^t \mathrm{e}{-\varphi(t-u)} u^{2H} \mathrm{d}u\mathrm{d}v \\
& + \int_0^t \mathrm{e}{-\varphi(t-u)} \int_0^s \mathrm{e}{-\varphi(s-v)} v^{2H} \mathrm{d}v\mathrm{d}u \\
& - \int_0^s \int_0^t \mathrm{e}{-\varphi(t-u)-\varphi(s-v)} |v-u|^{2H} \mathrm{d}v\mathrm{d}u\bigg] \\
= \quad & \frac{\varphi^2\sigma^2}{2}\bigg[\int_0^s \mathrm{e}{-\varphi(s-v)} \int_0^t \mathrm{e}{-\varphi(t-u)} u^{2H} \mathrm{d}u\mathrm{d}v \\
& + \int_0^t \mathrm{e}{-\varphi(t-u)} \int_0^s \mathrm{e}{-\varphi(s-v)} v^{2H} \mathrm{d}v\mathrm{d}u \\
& - \int_0^t \int_0^{\min(s,u)} \mathrm{e}{-\varphi(t-u)-\varphi(s-v)} (u-v)^{2H} \mathrm{d}v\mathrm{d}u \\
& \int_0^t \int_0^v \mathrm{e}{-\varphi(t-u)-\varphi(s-v)} (v-u)^{2H} \mathrm{d}u\mathrm{d}v\bigg].
\end{aligned}
$$

After some tedious calculations (see also the Appendix) these formulas reduce to

$$
\begin{aligned}
& \frac{\varphi\sigma^2}{2}\Big[(\mathrm{e}{-\varphi s} - \mathrm{e}{-\varphi(t-s)})\mathrm{M}(H,\varphi,s) + (\mathrm{e}{-\varphi t} - \mathrm{e}{-\varphi(t+s)})\mathrm{M}(H,\varphi,t) \\
& -\mathrm{e}{-\varphi(t+s)}\frac{1}{2}\left(\mathrm{e}2\varphi t\mathrm{M}(H,-\varphi,t) - \mathrm{M}(H,\varphi,t)\right) \\
& -\mathrm{e}{-\varphi(t+s)}\frac{1}{2}\left(\mathrm{e}2\varphi s\mathrm{M}(H,-\varphi,s) - \mathrm{M}(H,\varphi,s)\right) \\
& +\mathrm{e}{-\varphi(t-s)}\frac{1}{2}\left(\mathrm{e}2\varphi(t-s)\mathrm{M}(H,-\varphi,t-s) - \mathrm{M}(H,\varphi,t-s)\right)\Big].
\end{aligned}
$$

By adding all the terms and simplifying appropriately we obtain formula (6).

Appendix

Some useful formulas

Recall the definition of the confluent hypergeometric function

$$
{}_1\mathrm{F}_1(\alpha,\beta,z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\beta+n)}\frac{z^n}{n!}.
$$

We then define

$$\begin{aligned}
\mathrm{K}(\alpha,\delta,x) &= \frac{1}{\alpha}x^{\alpha}\,{}_1\mathrm{F}_1(\alpha,\alpha+1,\delta x) \\
&= \int_0^x u^{\alpha-1}e^{\delta u}\mathrm{d}u \\
&= x^{\alpha}\sum_{n=0}^{\infty}\frac{(\delta x)^n}{n!(\alpha+n)}.
\end{aligned}$$

Note that

$$\begin{aligned}
\Gamma(\alpha,z) &= \Gamma(\alpha)-\int_0^z u^{\alpha-1}e^{-u}\mathrm{d}u \\
&= \Gamma(\alpha)-\frac{1}{\alpha}z^{\alpha}\,{}_1\mathrm{F}_1(\alpha,\alpha+1,-z) \\
&= \Gamma(\alpha)-\mathrm{K}(\alpha,-1,z),
\end{aligned}$$

where $\Gamma(\alpha,z)=\int_z^\infty u^{\alpha-1}e^{-u}\mathrm{d}u$ (the incomplete gamma function).

Also note that, according to Gradshteyn & Ryzhik (1980) (a consequence of formula 9.212), we have

$$ {}_1\mathrm{F}_1(\alpha,\alpha+1,-z)=e^{-z}\,{}_1\mathrm{F}_1(1,\alpha+1,z). \tag{7}$$

Furthermore define

$$\begin{aligned}
\Lambda(\mu,\alpha,\delta,x) &= B(\mu,\alpha)x^{\mu+\alpha-1}\,{}_1\mathrm{F}_1(\alpha,\mu+\alpha,\delta x) \\
&= \int_0^x u^{\alpha-1}(x-u)^{\mu-1}e^{\delta u}\mathrm{d}u \\
&= x^{\mu+\alpha-1}\sum_{n=0}^{\infty}B(\mu,n+\alpha)\frac{(\delta x)^n}{n!},
\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function.

When $\mu=1$ we have

$$\begin{aligned}
\Lambda(1,\alpha,\delta,x) &= \mathrm{K}(\alpha,\delta,x) \\
&= x^{\alpha}\sum_{n=0}^{\infty}\frac{(\delta x)^n}{n!(\alpha+n)},
\end{aligned}$$

and when $\mu=2$ we have

$$\begin{aligned}
\Lambda(2,\alpha,\delta,x) &= \int_0^x u^{\alpha-1}(x-u)e^{\delta u}\mathrm{d}u \\
&= x^{\alpha+1}\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)(n+\alpha+1)}\frac{(\delta x)^n}{n!}.
\end{aligned}$$

Double summation and the confluent hypergeometric function

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \frac{\delta^j}{j!(\alpha+j)} s^{\alpha+1+j} \sum_{i=0}^{\infty} \frac{(\varepsilon s)^i}{i!(\alpha+1+j+i)} \\
&= \sum_{j=0}^{\infty} \frac{\delta^j}{j!(\alpha+j)} \int_0^s u^{\alpha+j} e^{\varepsilon u} \mathrm{d}u \\
&= \int_0^s u^{\alpha} e^{\varepsilon u} \sum_{j=0}^{\infty} \frac{(\delta u)^j}{j!(\alpha+j)} \mathrm{d}u \\
&= \int_0^s e^{\varepsilon u} \int_0^u x^{\alpha-1} e^{\delta x} \mathrm{d}x \mathrm{d}u \\
&= \int_0^s \int_x^s e^{\varepsilon u} x^{\alpha-1} e^{\delta x} \mathrm{d}u \mathrm{d}x \\
&= \int_0^s x^{\alpha-1} e^{\delta x} \int_x^s e^{\varepsilon u} \mathrm{d}u \mathrm{d}x \\
&= \int_0^s x^{\alpha-1} e^{\delta x} \frac{1}{\varepsilon} (e^{\varepsilon s} - e^{\varepsilon x}) \mathrm{d}x \\
&= \frac{e^{\varepsilon s}}{\varepsilon} \int_0^s x^{\alpha-1} e^{\delta x} \mathrm{d}x - \frac{1}{\varepsilon} \int_0^s x^{\alpha-1} e^{(\delta+\varepsilon)x} \mathrm{d}x \\
&= \frac{s^\alpha}{\alpha\varepsilon} \left\{ e^{\varepsilon s}\, {}_1\mathrm{F}_1(\alpha, \alpha+1, \delta s) - {}_1\mathrm{F}_1(\alpha, \alpha+1, (\delta+\varepsilon)s) \right\}.
\end{aligned}
$$

Specifically, for $\delta = -\varphi$, and $\varepsilon = 2\varphi$, we have the formula

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \frac{(-\varphi)^j}{j!(\alpha+j)} s^{\alpha+1+j} \sum_{i=0}^{\infty} \frac{(2\varphi s)^i}{i!(\alpha+1+j+i)} \\
&= \frac{s^\alpha}{2\alpha\varphi} \left\{ e^{2\varphi s}\, {}_1\mathrm{F}_1(\alpha, \alpha+1, -\varphi s) - {}_1\mathrm{F}_1(\alpha, \alpha+1, \varphi s) \right\}.
\end{aligned}
$$

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Analysing Extremal Behaviour of Financial Time Series

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Large losses in finance as happened for instance during the Black Monday on October 19, 1987, or the LTCM crises during fall 1998 have stimulated discussions about appropriate Risk Management.

According to Richard Felix, chief credit officer at Morgan Stanley, "Risk management is asking what might happen the other 1% of the time". This formulation leads immediately to so-called downside risk measures as the Value-at-Risk (based on a very low quantile of the returns), the expected shortfall or the semivariance, both based on a low quantile (see [1] and [6])).

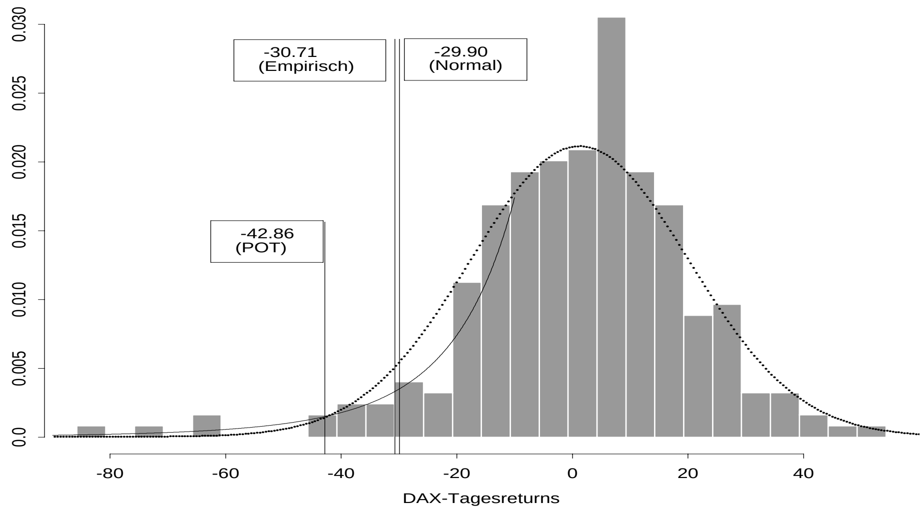

Figure 1: Estimated Value-at-Risk as the 5%-quantile of a DAX portfolio, where the returns of the DAX prices, taken from 29.8.95–26.8.96, are assumed to be iid.

Such a risk measure can certainly not be estimated by a normal model and moment fitting. Quite contrary, the problem suggests the application of stochastic extreme value theory, usually formulated in terms of large order statistics, i.e. in terms of large losses. This means that the most extreme data, which are responsible for far out tail behaviour or high quantiles, are modelled by an appropriate extreme value model. The method is standard for iid data (e.g. [5]).

It is well-known, however, that financial data often show a change in their fluctuations in time and a very special dependence structure, usually captured by volatility modelling, which also may affect tail- and quantile estimation. Common volatility models include diffusion processes (as solutions to SDE's) or ARCH and GARCH processes.

In [3] we investigate the extremal behaviour of a diffusion given by the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t\,, \quad t > 0\,,$$

where $X_0 = x$, B is standard Brownian motion, μ is a drift term and σ is the diffusion coefficient or volatility.

The extremes of such a process can be compared to the extremes of iid random variables with some specific distribution, which in general differs from the stationary distribution of the process, but can explicitly be given for any specific example.

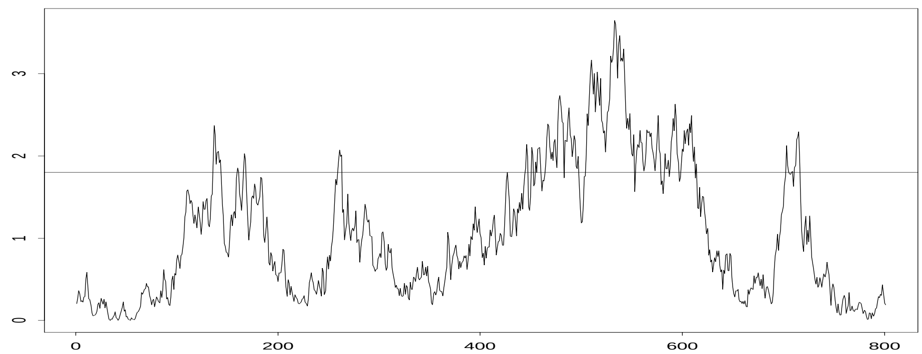

Figure 2: Simulated sample path of the Cox-Ingersoll-Ross model given by the SDE $dX_t = (c - dX_t)\,dt + \sigma\sqrt{X_t}\,dB_t\,, t > 0\,, (X_0 = x)$ and (B_t) standard Brownian motion. (with parameters $c = d = \sigma = 1$).

ARCH and GARCH models describe the volatility as a function of the sample path and the past volatility. These models capture certain stylised facts of financial data like heavy-tailedness, uncorrelatedness and a special dependence structure often found in financial data (see [5], Section 8.4, [7] and references therein). The estimation of risk measures has here to deal with clustering in the extremes, see [2, 4, 7].

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N. N. Leonenko (Kiev):

Homogenization and renormalization of the fractional diffusion equations with random data[‡]*.*

Extended Abstract:

Abstract

Gaussian and non-Gaussian limiting distributions of the rescaled solutions of the fractional in time or in space diffusion equation for Gaussian and non-Gaussian initial data with long-range dependence are described in terms of multiple Wiener-Itô integrals.

1. Introduction

The fractional diffusion equation is obtained from the classical diffusion (or wave) equation by replacing the first or second-order derivatives by fractional derivatives.

The fractional diffusion equation has been proposed by Nigmathulin (1986) to describe diffusion in porous media with fractal geometry. Mainardi (1995) pointed out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media with exhibit a power-law creep, and consequently provided a physical interpretation of this equation in the framework of dynamic viscoelasticity. In the non-stochastic situation the fractional diffusion equation has been studied by Schneider and Wyss (1989), Kochubei (1989, 1990), Schneider (1990, 1992), Fujitu (1990), Mainardi (1996), Saichev and Zaslawsky (1997), Mainardi, Paradisi and Corenlo (1999) and others. More general fractional Burgers equation has been considered by Biler, Funaki and Woyczynski (1998) (see, also, Woyczynski (1998)).

We are interested in fractional in time or in space diffusion equation with singular random initial condition as models of random fields which describe the singular properties of real data arising in ecology, turbulence and finance, for example.

Such data is known to possess long-range dependence (LRD) and/or intermittency. Fractional operators are natural mathematical objects for description of this phenomena.

In particular, Gay and Heyde (1990) introduced a class of random fields that allow LRD via the stochastic operational Laplace equation with fractional Laplace operator. Angulo et al. (1999) introduced the stochastic heat equation in which the Laplacian Δ is replaced by fractional Laplacian of the form $(I-\Delta)^{\alpha/2}(-\Delta)^{\gamma/2}$, $\alpha \geq 0$, $\gamma > 0$. They proved that the stationary solution of such equation displays spatial LRD and intermittency. On the other hand, random fields with singular spectra can be obtained as rescaled solution of the linear diffusion equation with singular initial conditions (see, Albeverio et al. (1994), Leonenko and Woyczynski (1998), Leonenko (1999), Anh and Leonenko(1999a) and the references therein).

[‡]Joint work with V.Anh, Center in Statistical Science and Industrial Mathematics, Queensland.

In this paper we present Gaussian and non-Gaussian scenarios for the rescalled solutions of fractional diffusion-wave equation with singular initial data (see, Anh and Leonenko (1999b)). In a sense, our results are non-central limit theorems (see, Taqqu (1979) or Dobrushin and Major (1979)) for random fields arising as solutions of fractional diffusion equations with singular initial data.

2. Fractional in time diffusion-wave equitation

We consider the fractional (in time) diffusion-wave equitation

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = \mu \frac{\partial^2 u}{\partial x^2}, \quad \mu > 0, \quad 0 < \beta \leq 1, \tag{2.1}$$

where $u = u(t, x, \beta)$, $t > 0$, $x \in R^1$, is the field variable, and the time derivative of order $a = 2\beta$ is defined via Riemann-Liouville calculus (see, for example, Miller and Ross (1993) or Samko et al. (1993)).

Schneider and Wyss (1989) and Mainardi (1996) (see, also Fujitu (1990), Schneider (1990, 1992), Mainardi et al. (1999)) extended the classical analysis to fractional diffusion-wave equation (2.1). In particular, let $g(x)$, $x \in R^1$, be a given sufficiently well-defined function. Consider the Cauchy problem

$$u(0, x, \beta) = g(x), \quad x \in R^1, \quad u(t, \pm\infty, \beta) = 0, \quad t > 0. \tag{2.2}$$

The solution of Cauchy problem (2.1)-(2.2) can be represented as

$$u(t, x, \beta) = \int_{R^1} G(t, y, \beta) g(y) dy, \tag{2.3}$$

where the Green function

$$G(t, x, \beta) = \frac{1}{2t^\beta \sqrt{\mu}} M\left(\frac{|x|}{t^\beta \sqrt{\mu}}; \beta\right), \quad x \in R^1, \quad t > 0,$$

and the function $M(z; \beta)$, $z \geq 0$, $\beta \in (0, 1)$ has the following representation:

$$M(z; \beta) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n! \Gamma(-\beta n + 1 - \beta)}, \quad z \geq 0, \quad \beta \in (0, 1).$$

3. Scaling laws

We consider the to fractional diffusion-wave equation (2.1) subject to the random initial condition

$$u(0, x, \beta) = \nu(\omega, x), \quad \omega \in \Omega, \quad x \in R^1, \tag{3.1}$$

where the random process $\nu(x) = \nu(\omega, x)$, $\omega \in \Omega$, $x \in R^1$, is defined on a suitable complete probability space (Ω, F, P).

The process $\nu(x)$, $x \in R^1$, is assumed to be a separable measurable mean-square continuous, almost sure continuously differentiable, stationary process with expansion $E\nu(x) = 0$ and true covariance function

$$R(x) = cov(\nu(0), \nu(x)) = \int_{R^1} e^{i\lambda x} F(d\lambda), \tag{3.2}$$

where $F(\cdot)$ is the spectral measure defined on a measurable space $(R^1, \mathcal{B}(R^1))$.

In view of Karhunen's Theorem, there exists a complex-valued orthogonally scattered random measure $Z(\Delta) = Z(\omega, \Delta)$, $\omega \in \Omega$, $\Delta \in \mathcal{B}(R^1)$, such that for every $x \in R^1$ the process

$$\nu(x) = \int_{R^1} e^{i\lambda x} Z(d\lambda), \tag{3.3}$$

where $E|Z(\Delta)|^2 = F(\Delta)$, $\Delta \in \mathcal{B}(R^1)$, and the stochastic integral in (3.3) is viewed as an $L_2(\Omega)$-integral with control measure F, $L_2(\Omega)$ being a Hilbert space of complex random variables with finite second order moments.

From (2.3) and (3.3) we obtain that the solution of the initial-value problem (2.1) and (3.1) can be written as

$$u(t, x, \beta) = \int_{R^1} e^{i\lambda y} A(t, \lambda, \beta) Z(d\lambda), \tag{3.4}$$

where

$$A(t, \lambda, \beta) = \int_{R^1} e^{-i\lambda y} G(t, y, \beta) dy =$$

$$= \frac{1}{2} \left[E_\beta \left(i\lambda \sqrt{\mu} t^\beta \right) + E_\beta \left(-i\lambda \sqrt{\mu} t^\beta \right) \right], \tag{3.5}$$

$$t > 0, \quad \lambda \in R^1, \quad \beta \in (0, 1)$$

and

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \beta k)}, \qquad 0 < \beta \leq 1,$$

is Mittag-Leffler function of a complex variable $z \in C$.

Note that for $\beta = \frac{1}{2}$

$$M\left(z; \frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}}, \quad z \in R^1, \quad A(t, \lambda, \beta) = e^{-\lambda^2 \mu t}, \quad \lambda \in R^1,$$

and for $\beta = \frac{1}{3}$

$$M\left(z; \frac{1}{3}\right) = 3^{2/3} A_i \left(z 3^{1/3}\right), \quad z \geq 0, \quad M\left(-z; \frac{1}{3}\right) = M\left(z; \frac{1}{3}\right)$$

where the Airy function $A_i(u)$, $u \geq 0$, is defined via the formula

$$A_i(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}} K_{\frac{1}{3}}\left(\frac{2}{3} z^{\frac{3}{2}}\right), \quad z \geq 0,$$

where

$$K_\nu(z) = \frac{1}{2}\int_0^\infty s^{\nu-1} \exp\left\{-\frac{1}{2}\left(s + \frac{1}{s}\right) z\right\} dz, \quad z \geq 0,$$

is the modified Bessel function of the third kind of order ν (see, for example, Watson (1944)).

If $\nu(x)$, $x \in R^1$, is a stationary Gaussian process with spectral density $f(\lambda)$, $\lambda \in R^1$, then

$$u = u(t, x, \beta), \quad t > 0, \quad x \in R^1, \quad 0 < \beta < 1,$$

is a stationary in x Gaussian field with covariance structure

$$cov\,(u(t,x,\beta), u(t',x',\beta)) = \int e^{i\lambda(x-x')} f(\lambda) A(t,\lambda,\beta)\overline{A(t',\lambda,\beta)} d\lambda,$$

where $A(t,\lambda,\beta)$ is defined in (3.5).

In this paper we consider the limiting distributions of the rescalled solutions of the initial-value problem (2.1)-(3.1) in the case where the stochastic process $\nu(x)$, $x \in R^1$, is a pointwise transformation of a stationary Gaussian process $\xi(x)$, $x \in R^n$ with $n = 1$, i.e.,

$$\nu(x) = h(\xi(x)), \quad x \in R^n, \tag{3.6}$$

where the non-random function $h : R^1 \to R^1$ is such that $Eh^2(\xi(0)) < \infty$.

The underlying stationary field $\xi(x)$, $x \in R^n$, and non-random function $h(\cdot)$ are assumed to satisfy the following conditions

A. The field $\xi(x)$, $x \in R^n$, is a real, measurable, separable, mean-square continuous, a.s. continuous differentiable stationary Gaussian with $E\xi^2(x) = 1$ and covariance function

$$R(x) = cov(\xi(0), \xi(x)) = (1 + |x|^2)^{-\alpha/2}, \quad 0 < \alpha < n, \quad x \in R^n.$$

B. The function $h : R^1 \to R^1$ is such that $Eh^2(\xi(0)) < \infty$.

The (non-linear) function $h(\cdot)$, already assumed to satisfy condition **B**, may be expanded in the series

$$h(u) = \sum_{k=0}^\infty (C_k/k!) H_k(u),$$

$$C_k = \int_{R^1} h(u)\varphi(u) H_k(u)\, du,$$

of orthogonal Chebyshev-Hermite polynomials $\{H_k(u)\}_{k=0}^{\infty}$ which complete an orthogonal system in the Hilbert space $L_2(R^1, \varphi(u)du)$, where

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \quad u \in R^1.$$

C. There exists an integer $m \geq 1$ such that

$$C_1 = \ldots = C_{m-1} = 0, \quad C_m \neq 0.$$

Under the condition **A**, with $n = 1$, the spectral density

$$f_\alpha(\lambda) = f_\alpha(|\lambda|) = \left(2^{\frac{1-\alpha}{2}} / \Gamma(\frac{\alpha}{2})\sqrt{\pi}\right) K_{\frac{1-\alpha}{2}}(|\lambda|)|\lambda|^{\frac{1-\alpha}{2}} =$$

$$= c(\alpha)|)|\lambda|^{1-\alpha}(1 - \theta(\lambda)), \quad 0 < \alpha < 1, \quad \lambda \in R^1, \tag{3.7}$$

where $\theta(\lambda) \to 0$, as $|\lambda| \to 0$, and

$$c(\alpha) = \frac{1}{\Gamma(\alpha) \cos \frac{\alpha\pi}{2}}.$$

From (3.7) we observe that $f_\alpha(|\lambda|) \to \infty$, as $|\lambda| \to 0$, thus we have a random process with LRD.

Our main result is the following:

Theorem 1. *Let $u(t, x; \beta)$, $t > 0$, $x \in R^1$, $0 < \beta < 1$, be a solution of the initial- value problem (2.1) – (3.1) with random initial condition (3.6) with $n = 1$ satisfying conditions* **A** *with $n = 1$,* **B** *and* **C** *with $0 < \alpha < 1/m$. Then the finite-dimensional distributions of the random fields*

$$U_\varepsilon(t, x; \beta) = \frac{1}{\varepsilon^{\frac{m\alpha\beta}{2}}} \left[u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^\beta}; \beta\right) - C_0 \right], \quad t > 0, \quad x \in R^1, \quad 0 < \beta < 1,$$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$U_m(t, x; \beta) = \frac{C_m}{m!} [c(\alpha)]^{m/2} \int_{R^m}' e^{ix(\lambda_1 + \ldots + \lambda_m)} A(t, \lambda_1 + \ldots + \lambda_m; \beta) \frac{W(d\lambda_1) \ldots W(\lambda_m)}{|\lambda_1 \cdots \lambda_m|^{\frac{1-\alpha}{2}}},$$

$t > 0$, $x \in R^1$, $0 < \beta < 1$,where the function $A(t, \lambda; \beta)$ is defined in (3.5), $\int' \ldots$ is the multiple Wiener-Itô integral with respect to complex Gaussian white noise random measure $W(\cdot)$ associated with the Gaussian process $\xi(x)$, $x \in R^1$.

The proof of the Theorem 1 will be given in Anh and Leonenko (1999b). The case $\beta = \frac{1}{2}$, $G(u) = e^{-u}$, $u \in R^1$, was considered by Albeverio et al. (1994) (for the case $m = 1$).

For $\beta = \frac{1}{2}$ and $m \geq 1$ corresponding results are obtained in Leonenko and Woyczynski (1998a) and Anh and Leonenko (1999a).

4. Fractional in space heat equation

We consider the fractional in space diffusion heat equation of the following form

$$\frac{\partial u}{\partial t} = \mu(I-\Delta)^{\kappa/2}(-\Delta)^{\gamma/2}, \ \ \mu > 0, \tag{4.1}$$

where $u = u(t,x)$, $t > 0$, $x \in R^n$, $\mu > 0$, Δ is the Laplasian, I is identity operator and $\kappa \geq 0$, $\gamma \in (0,n)$.

The fundamental solution to equation (4.1) can be represented in the following form (see, Angulo et al. (1999)):

$$G(t,x) = p(t,x,\kappa,\gamma,\mu) = \frac{1}{(2\pi)^n}\int\limits_{R^n} e^{i\langle\lambda,x\rangle-\mu t|\lambda|^{\gamma}(1+|\lambda|^2)^{\kappa/2}}d\lambda. \tag{4.2}$$

Note that

$$p(2t,x;0,2,1) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}, \ \ t > 0, \ \ x \in R^n$$

be fundamental solution of the classical heat equation, and

$$p(2t,x;0,1,1) = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}, \ \ t > 0, \ \ x \in R^n$$

be a Cauchy density. More general

$$p(t,x;0,\gamma,1) = \frac{1}{(2\pi)^n}\int\limits_{R^n} e^{i\langle x,\lambda\rangle-\frac{t}{2}|\lambda|^{\gamma}}d\lambda, \ \ t > 0, \ \ x \in R^n$$

be a density function of a symmetric stable distribution when $0 < \gamma \leq 2$.

Theorem 2. *Let $u(t,x)$, $t > 0$, $x \in R^n$ be a solution of the initial- value problem (4.1)–(3.1), (3.1) with random initial condition (3.6) satisfying conditions* **A**, **B** *and* **C** *with $0 < \alpha < \frac{n}{m}$. Then the finite-dimensional distributions of the random fields*

$$U'_{\varepsilon}(t,x) = \frac{1}{\varepsilon^{\frac{m\alpha}{2\gamma}}}\left[u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{1/\gamma}}\right) - C_0\right],$$

$$t > 0, \ \ x \in R^n, \ \ 0 < \gamma < n, \ \ 0 < \alpha m < n,$$

converge weakly to the finite-dimensional distributions of the random fields

$$U'_m(t,x) = \frac{C_m}{m!}\left[\Gamma\left(\frac{n-\alpha}{2}\right)\Big/\, 2^{\alpha}\pi^{n/2}\Gamma\left(\alpha/2\right)\right]^{m/2} \times$$

$$\times \int'_{R^{mn}} e^{i\langle \lambda_1+\cdots+\lambda_m, x\rangle - \mu t|\lambda_1+\cdots+\lambda_m|^\gamma} \frac{W(d\lambda_1)\dots W(d\lambda_m)}{(|\lambda_1|\cdots|\lambda_m|)^{\frac{n-\alpha}{2}}}, \ \ t>0, \ \ x \in R^n,$$

where $W(\cdot)$ *is the complex Gaussian white noise random measure on* $(R^n, \mathcal{B}(R^n)$ *such that*

$$\xi(x) = \int_{R^n} e^{i\langle \lambda, x\rangle} \sqrt{f_\alpha(\lambda)} W(d\lambda), \ \ x \in R^n$$

and

$$f_\alpha(\lambda) = \left[\pi^{\frac{n}{2}} 2^{\frac{n}{2}+\frac{\alpha-2}{2}} \Gamma(\alpha/2)\right]^{-1} K_{\frac{n-\alpha}{2}}(|\lambda|) \cdot |\lambda|^{\frac{\alpha-n}{2}}, \ \ \lambda \in R^n.$$

The proof of the Theorem 2 will be given elsewhere.

For $m = 1$ the random fields $U_m(t, x; \beta)$ and $U'_m(t, x)$ are Gaussian. For $m \geq 2$ these fields have non-Gaussian structure.

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Francesco Mainardi (Bologna):

Non local transport effects in skewed turbulence via fractional diffusion and Lévy statistics.

EXTENDED ABSTRACT:

Foreword

I would like to recall after Hunt [1] the alleged saying of Einstein that if he could, after solving all the other problems of physics, he would "solve the problem of turbulence". Nowadays, even more so, this saying can be referred also to the "problem of finance".

Einstein's oft-repeated saying has continued to have a powerful effect, because it implies a "solution" in the same way that Einstein solved other problems. However, there are such widely differing views on the objectives of turbulence or finance research, so that, in recent times, scientists prefer to focus their attention on more restricted range of problems rather than on a general theory or general models.

Here we consider the application of fractional calculus and Lévy statistics in a special problem of geophysical turbulence, *i.e.* in modelling the vertical profiles of the effective eddy diffusivity for horizontally homogeneous diffusion of a passive, conservative scalar in the convective boundary layer (CBL) of the atmosphere. In fact, large-eddy simulations have shown that passive, conservative scalars emitted into the CBL have "unusual" (*i.e.* not local) diffusion properties, see *e.g.* [2-4], which can be explained in terms of a fractional diffusion with skewness.

Fractional Fick's law and fractional diffusion

Based on a pioneering work by Feller, Gorenflo and Mainardi, see *e.g.* [5-8], have recently investigated a fractional diffusion equation, which generates all the Lévy stable densities, and have provided original discretization schemes (in time and in space), which can be properly used to simulate the related Lévy flights. This equation reads

$$\frac{\partial}{\partial t}\, u(x,t) = D_\theta^\alpha\, u(x,t)\,, \quad x \in \mathbb{R}, \quad t > 0\,, \tag{1}$$

where $u(x,t)$ is the field variable (concentration) and D_θ^α is the Feller pseudo-differential operator acting with respect to the space variable x, with symbol

$$\widehat{D_\theta^\alpha} = -|\kappa|^\alpha\, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,. \tag{2}$$

The two relevant parameters, α, called the *index of stability*, and θ (related to the asymmetry), referred to as the *skewness*, are real numbers subject to the conditions

$$0 < \alpha \le 2\,; \qquad |\theta| \le \begin{cases} \alpha\,, & \text{if} \quad 0 < \alpha \le 1\,, \\ 2-\alpha\,, & \text{if} \quad 1 < \alpha \le 2\,. \end{cases} \tag{3}$$

So doing Gorenflo and Mainardi have pointed out (in a natural way) a sort of analogy with the classical case of the standard diffusion ($\alpha = 2\,,\ \theta = 0$), where the Gaussian density and Brownian motion are known to play a key role.

Here we would like to extend this argument discussing how Fick's empirical law for the flux $f(u)$ must be suitably generalized by the tools of the fractional calculus in order to obtain the fractional diffusion equation (1) from the continuity equation

$$\frac{\partial}{\partial t}\, u(x,t) + \frac{\partial}{\partial x}\, f[u(x,t)] = 0\,. \tag{4}$$

As a consequence we obtain a model for non local transport effects with skewness not related to pure drift, which is based on Lévy statistics. In our opinion our model, which improves the recent one proposed by Chaves [9], is a good candidate to interpret some basic features of dispersion in geophysical turbulence described in [2-4]. We like also to recall that in the past other models based on Lévy statistics have been applied to turbulence [10].

Acknowledgements

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Rimas Norvaiša (Vilnius):

p-variation, integration and stock price modelling.

Extended Abstract: Continuous–time stochastic processes have become central to many areas of applied mathematics including stock price modelling. The main concepts of stochastic calculus such as filtrations and martingales have a natural interpretation in financial models. It is not surprising that the semimartingale property has become the "underlying" assumption for several models of stock price changes, and processes which are not semimartingales often become unrealistic for stock price modelling. The well known example of a non-semimartingale is a fractional Brownian motion with the Hurst exponent $H > 1/2$. Stochastic processes of this class are sometimes considered as models for returns of a stock price (they also play an essential role in a series of problems in the statistical theory of turbulence since the earlier works of A.N. Kolmogorov on turbulence, see e.g. Obukhov and Yaglom, 1956). Theoretical arguments supporting the semimartingale assumptions often overlook empirical evidence. On the other hand, the econometric evaluation of continuous–time stock price models based on the log returns is not adequate for models other than exponential. A fractional Brownian motion with the Hurst exponent $H > 1/2$ is a pleasant exception for using log returns since the exponential model and the model defined by a linear Riemann–Stieltjes integral equation driven by this process, both coincide. However for recent empirical results see Willinger, Taqqu and Teverovsky (1999). We plan to show that the p-variation and pathwise integration may shed a new light to these types of problems.

The pathwise approach to stock price modelling was initiated by Bick and Willinger (1994). Their approach was motivated by Föllmer (1981) who derived a non–probabilistic version of Itô's formula. An important ingredient in the proof of this Itô formula is the notion of quadratic variation of a function. In this talk we make use of the classical notion of p-variation; in addition to Föllmer's quadratic variation. We consider the class of all functions f having a non-zero Föllmer's quadratic variation such that $f = g + h$, where g has finite p-variation for each $p > 2$ (but may have infinite 2-variation) and h has finite p-variation for some $p < 2$. The function g may be considered as a trajectory of a continuous martingale, while the function h may be considered either as a trajectory of a Lévy process, or as a trajectory of a process which need not be a semimartingale (for example, it may be a trajectory of a fractional Brownian motion with the Hurst exponent $H > 1/2$). We use the integration approach of L.C. Young to deal with functions of bounded p-variation for some $p < 2$. In fact Bick and Willinger (1994) proved that Black and Scholes hedging strategies are self–financing with the portfolio value $(f(T) - K)^+$ at maturity time T whenever a stock price follows a path of a continuous function f which has a suitable non-zero Föllmer's quadratic variation, and $f(T) \neq K$. In particular, this is true for functions $f = g+h$ as above. One can show that this is not so if $g = 0$. That is, Black and Scholes hedging strategies are not self–financing if a path of stock price follows a continuous function with bounded p-variation for some $p < 2$. A probabilistic variant of Bick and Willinger's result was discovered by Schoenmakers and Kloeden (1997).

The primary objective for most financial theories is analysis of asset returns. Therefore it is important to extend the notion of return associated with a stock price to a pathwise

setting. The idea behind this construction is based on known results about a one–to–one correspondence between an evolution and its generator. Let P be a function representing a stock price in the time period $[0, T]$. Then define return R as the generator of the evolution of stock price changes given by $U(t, s) := P(t)/P(s)$ for $0 \leq s \leq t \leq T$. The resulting notion of return appears to be a continuous time analogue of the simple net return used in discrete–time models which differs from the log return in general. The stock price P in this case has a representation of an indefinite product integral which may be considered as an analogue of the stochastic exponent in stochastic calculus. The existence and properties of a price P and its return R, depend on their p-variation. Therefore we consider the problem of estimating the index of p-variation from empirical data of a stock price, where the index of p-variation of a function f is defined to be the greatest lower bound of p such that f has bounded p-variation. This problem is quite different from usual estimation problems because it focus on the analytical properties of sample functions of stochastic processes rather than their distributional properties.

Finally we notice that the p-variation and Riemann–Stieltjes integration with respect to processes with unbounded variation, can be incorporated into stochastic calculus by extending the notion of semimartingale as follows. Given $1 \leq p < 2$, we say that an adapted stochastic process X is the p-semimartingale if there exist stochastic processes M and Z such that $X(t) = X(0) + M(t) + Z(t)$, $t \geq 0$, where M is a local martingale and almost all sample functions of Z have bounded p-variation. So that a 1-semimartingale is the same as classical semimartingale. The Itô formula extends to p-semimartingales by using the Stieltjes integrability theorem of L.C. Young (1936). Since arbitrage is possible when a stock price follows a path of continuous function with bounded p-variation for some $p < 2$ (see Salopek, 1998), it will be of interest to extend the fundamental theorems of financial mathematics to the above context.

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Rudolf Friedrich (Stuttgart) and Joachim Peinke (Oldenburg):

Disentangling determinism and fluctuations
Part I: A new stochastic concept.
Part II: Turbulence and Finance.

Some references for the work presented in these talks are:

R. Friedrich, J. Peinke: *Description of a Turbulent Cascade by a Fokker-Planck equation*, Phys. Rev. L., **78**, (1997), 863–866.

R. Friedrich, J. Peinke: *Statistical properties of a turbulent cascade*, Physica D, **102**, (1997), 147–155.

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A. Naert, B. Castaing, B. Chabaud, B. Hébral, J. Peinke: *Conditional statistics of velocity fluctuations in turbulence*, Physica D, **113**, (1998), 73–78.

A. Naert, R. Friedrich, J. Peinke: *Fokker-Planck equation for the energy cascade in turbulence*, Phys. Rev. E, **56**, (1997), 6719–6722.

Neil Shephard (Nuffield College):

Non-Gaussian OU based models and some of their uses in financial economics[§]*.*

Abstract: Non-Gaussian processes of Ornstein-Uhlenbeck type, or *OU processes* as we shall call them, have considerable potential as building blocks for stochastic models of observational series from a wide range of fields. They offer the possibility of capturing important distributional deviations from Gaussianity and for flexible modelling of dependence structures. This paper develops this potential, drawing on and extending powerful results from probability theory for applications in statistical analysis. We illustrate their power by a sustained application of OU processes within the context of finance and econometrics. Based on well-known (empirical) stylized facts, we construct continuous time stochastic volatility models for financial assets where the volatility processes are superpositions of positive OU processes, and we study these models in relation to financial data and theory.

[§]Based on joint work with Ole E. Barndorff-Nielsen

Albert Shiryaev (Steklov Mathematical Institute, Moscow):

Kolmogorov and the Turbulence.

The manuscript for this talk has appeared in the separate note *Kolmogorov and the Turbulence*, Miscellanea No. 12, May 1999, Centre for Mathematical Physics and Stochastics, University of Aarhus.

Nils Svanstedt (Chalmers and Göteborg University):

Two-scale limits and mean fields in turbulence and finance.

EXTENDED ABSTRACT:

Let us consider the Navier-Stokes equation

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^k \Delta u_\epsilon + \nabla p_\epsilon = f, \\ \\ \operatorname{div} u_\epsilon = 0 \end{cases} \tag{1}$$

and the Euler equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = f, \\ \\ \operatorname{div} u = 0, \end{cases} \tag{2}$$

for incompressible fluids, where $x \in \Omega$ and $t \in \mathbf{R}^+$ and where ϵ^k, $\epsilon > 0$ and k constant, is the magnitude of the viscosity. For small values of ϵ^k (high Reynolds number) it is well known that the fluid velocity u_ϵ in (1) has a tendency to develop turbulent behaviour. Therefore the study of the behaviour of the solution (u_ϵ, p_ϵ) to (1) as $\epsilon \to 0$ is of fundamental importance. A first guess would be that the solution converges to the solution (u, p) to (2). In [8] DiPerna and Majda study this problem for $k = 1$ and no scaling in space and time. They use concentrated compactness (defect measure) methods to show that (u_ϵ, p_ϵ) convergens to (u, p) as a vanishing viscosity limit. In his monograph [9], dedicated to the memory of DiPerna, Evans discusses the same problem and present a somewhat simpler proof. In [2] Bethuel and Ghidaglia consider a sequence of regularized Euler equations and prove compactness, this time by geometric measure theoretic methods (the coarea formula by Federer and Fleming). Note that the regularization in [2] is essential since there is no smoothing heat operator to rely on. The lack of strong convergence for the non-linear inertial term presumably hides many of the mysteries in turbulence, see further comments below. In [3] we employ the technique of Nguetsengs two-scale convergence, see [12], and prove that by choosing $k = 3/2$ and by scaling $u \mapsto \epsilon^{-1/2} u_\epsilon$, $x \mapsto \epsilon^{-1} x = y$ and $t \mapsto \epsilon^{-1/2} t = \tau$, a passage to the limit, sending $\epsilon \to 0$, in (1) yields a viscid two-scales limit with two pressures:

$$\begin{cases} \dfrac{\partial u_0}{\partial \tau} + (u_0 \cdot \nabla_y) u_0 - \Delta_y u_0 + \nabla_y p_1 = f - \nabla p_0, \\ \\ \operatorname{div_y} u_0 = 0, \;\; \operatorname{div}(\displaystyle\int_{T^n} u_0 dy) = 0, \end{cases} \tag{3}$$

where $x \in \Omega$, $\;y \in T^n$ and $\tau \in \mathbf{R}^+$. Here y is the local spatial variable and τ is the scaled fast time variable. T^n, the unit torus in y, is what is referred to as the unit cell in the terminology of homogenization. Moreover subindex y denotes differentiation w.r.t. y. In (3) we have: $u_0 = u_0(x, y, \tau)$, $p_0 = p_0(x, \tau)$ and $p_1 = p_1(x, y, \tau)$ where u_0 and p_1 are T^n-periodic in y if the test functions are chosen to be periodic, but this can be relaxed, see [3] or [5]. The existence theory for (3) is found in [4]. Our compactness proof combines two-scale compactness with the classical compensated compactness argument. One can also apply two-scale compactness with a time-dependent coarea argument, this is work in progress, see [5]. If the periodicity is relaxed, an averaging of (3) in y over the unit torus (denoted bar) results in the mean field

$$\begin{cases} \dfrac{\partial \overline{u}_0}{\partial \tau} + (\overline{u}_0 \cdot \nabla) \overline{u}_0 + \nabla p_0 = f, \\ \\ \operatorname{div} \overline{u}_0 = 0, \end{cases} \tag{4}$$

i.e. the Euler equation. Here we have also used the fast decay of the heat kernel. The irregular behaviour of the mean field is also observed in experiments with Rayleigh-Benard convection in thin layers, see [1]. This has served as a strong motivation for our study. The method we use in [3] works for oscillatory fluids.

The significant convection rolls that develop in Rayleigh-Benard convection enjoy this behaviour, which allows us to prove a Poincare type inequality with constant of order ϵ, c.f. [16]. This inequality plays a crucial role in the homogenization process. The scaling above, which allows the two-scale convergence to work, is nothing but the Kolmogorov scaling and therefore as a benefit our convergence result justifies this scaling in this case. Details can be found in [5]. We consider again the Navier-Stokes equation with a small viscosity.

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^{3/2} \Delta u_\epsilon + \nabla p_\epsilon = f, \\ \\ \operatorname{div} u_\epsilon = 0. \end{cases} \qquad x \in \Omega, \; t \in \mathbf{R}^+. \tag{5}$$

We now assume highly oscillatory data. We also assume a large number of (well separated) spatial scales. A formal expansion

$$u_\epsilon(x, t) = \epsilon^{1/2} \sum_{i=0}^{\infty} \epsilon^i u_i(x, \frac{x}{\epsilon}, \frac{x}{\epsilon^2}, \ldots, \frac{x}{\epsilon^n}, t, \frac{t}{\epsilon^{1/2}}),$$

$$p_\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i p_i(x, \frac{x}{\epsilon}, \frac{x}{\epsilon^2}, \dots, \frac{x}{\epsilon^n}, t, \frac{t}{\epsilon^{1/2}}),$$

together with the chain rule yield a very interesting leading order system

$$\begin{cases} \dfrac{\partial u_0}{\partial \tau} + \displaystyle\sum_{j=1}^{n}[(u_{j-1} \cdot \sum_{k=j}^{n} \nabla_{y_k})u_{k-j}] - \Delta_{y_1} u_0 + \sum_{k=1}^{n} \nabla_{y_k} p_k = f - \nabla_x p_0, \\ \mathrm{div_{y_n}}\, u_0 = 0, \\ \mathrm{div_{y_k}}(\displaystyle\int_{T^3_{k+1}} \cdots \int_{T^3_n} u_0 dy_{k+1} \cdots du_n) = 0, \\ \mathrm{div_x}(\displaystyle\int_{T^3_1} \cdots \int_{T^3_n} u_0 dy_1 \cdots du_n) = 0. \end{cases}$$

where $x \in \Omega$, $y \in T^n$ and $\tau \in \mathbf{R}^+$. Here $y_k = x/\epsilon^k$ are the local spatial scales and $\tau = t/\sqrt{\epsilon}$ is the scaled fast time variable. Moreover, T^3_k denotes the unit torus in y_k. By using methods developed by Allaire and Briane we can prove the existence of the cascade of pressure gradients but the existence theory for the complicated inertial part is widely open. The chain of separated scales corresponds to the cascade of eddies at various scales in turbulent regimes. Similar approaches are now being developed by numerical analysts, see for instance the recent paper [10] on subgrid modeling by my colleagues Johnson and Hoffman at Chalmers Finite Element Center. As we see from the formal expansion, also the leading order inertial term is very complicated and seems to involve all higher order fluid velocities.

In 1973 Black and Scholes [6] came up with an explicit formula for the pricing of European options, the celebrated Black and Scholes model. This model introduces the concept of arbitrage. An accurate pricing mechanism must be arbitrage free in the loosely spoken sense "no profits without risk". In mathematical terms: The correct pricing of an asset say X can not be the expectation $E\{X\}$, since this opens an arbitrage opportunity. Instead diffusion enters the scene and the asset price X is defined as the solution to a stochastic diffusion equation

$$dX_t = \mu X_t dt + \sigma(t, X_t) dB_t, \tag{6}$$

where the first term is called the drift term, the second is called the noise and where B_t represents a Brownian motion. The function σ is called the volatility and it has to be estimated from market data. If the volatility is a deterministic function of the asset price X_t, then the function $C(t,x)$, which gives the no-arbitrage price of a European derivative at time t when $X_t = x$, satisfies the Black-Scholes (BS) partial differential equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(t,x)x^2 \frac{\partial^2 C}{\partial x^2} + r(x\frac{\partial C}{\partial x} - C) = 0, \tag{7}$$

where r is the constant risk free interest rate. One of the key difficulties in the business is the estimation and understanding of the volatility σ. Estimation, from market data, leads to a constant volatility while the intuitive sense of wild swings of the prices indicate that the volatility is random. Models with X_t modelled as a stochastic diffusion driven by a random volatility Ito process were introduced in 1987 by e.g. Hull and White [11]. Stochastic volatility also arises as the continuous limit of discrete models such as ARCH. In [13] Papanicolaou and Sircar take an important step and introduce two time scales in this model, one for the volatility and one for the option price itself, based on the assumptions that they fluctuate on different time scales. They develop an asymptotic analysis for high-frequency oscillating volatility with $\log \sigma$ being a mean-reverting Ornstein-Uhlenbeck process. Under ergodicity assumptions on the processes they derive homogenized limits, which are again BS, where the fluctuations are averaged out. A short description of their approach goes as follows: Suppose the diffusion $Y_t = \log \sigma_t$ satisfies

$$dY_t = \alpha(m - Y_t)dt + \beta dZ_t,$$

for constants α, β and m and a Brownian motion B_t. Then, for "small" $\epsilon > 0$, $\log Y_t^\epsilon$ is described by

$$dY_t^\epsilon = \frac{\alpha}{\epsilon}(m - Y_t^\epsilon)dt + \frac{\beta}{\sqrt{\epsilon}} dZ_t. \tag{8}$$

Here the typical time scale of the lifetime of the contract (typically one year) is of order 1. If (8) is inserted in (6) we get the stochastic differential equation, with random oscillating volatility, for the asset price

$$dX_t^\epsilon = \mu X_t^\epsilon dt + e^{Y_t^\epsilon} X_t^\epsilon dB_t. \tag{9}$$

In order to take the skew effects into account we write

$$Z_t = \rho B_t + \sqrt{1-\rho^2} W_t,$$

where the Brownian motions B_t and W_t are independent. Standard no-arbitrage arguments now yield that the European derivative price $C^\epsilon(t,x,y)$ satisfies the two scales partial differential equation

$$\frac{\partial C^\epsilon}{\partial t} + \frac{e^{2y}x^2}{2}\frac{\partial^2 C^\epsilon}{\partial x^2} + \frac{\rho\beta e^y x}{\sqrt{\epsilon}}\frac{\partial^2 C^\epsilon}{\partial x \partial y} + \frac{\beta^2}{2\epsilon}\frac{\partial^2 C^\epsilon}{\partial y^2} +$$

$$r(x\frac{\partial C^\epsilon}{\partial x} - C^\epsilon) + \left(\frac{\alpha}{\epsilon}(m-y) - \frac{\lambda\beta}{\sqrt{\epsilon}}\right)\frac{\partial C^\epsilon}{\partial y} = 0 \tag{10}$$

$$C^\epsilon(T,x,y) = (x-K)^+.$$

Here λ is called the market price of volatility risk. By the Ito formula one can also write $C^\epsilon(t, X_t^\epsilon, Y_t^\epsilon)$ as a diffusion driven by the Brownian motions above. The averaging procedure is now standard multiple scales homogenization, see e.g. Persson et. al. [14], where the starting point is the ansatz:

$$C^\epsilon(t,x,y) = C_0(t,x,y) + \sqrt{\epsilon} C_1(t,x,y) + \epsilon C_2(t,x,y) + ...$$

and a separation of the differential operators in (9). Formal computations yield the leading order BS equation

$$\frac{\partial C_0}{\partial t} + \frac{1}{2}\overline{\sigma}^2(t,x)x^2\frac{\partial^2 C_0}{\partial x^2} + r(x\frac{\partial C_0}{\partial x} - C_0) = 0, \tag{11}$$

where $C_0 = C_0(t,x)$ and $\log \overline{\sigma}^2$ is the expectation with respect to the invariant measure of the Ornstein-Uhlenbeck process. The second term in the expansion gives a correction to the BS model. This is carried out in some special cases in [13]. The limit analysis in (9), as $\epsilon \to 0$, and its implications is crucial here. In [15] we develop a general stochastic two-scale convergence theory, c.f. [7], for fast mean-reverting stochastic volatility models where we also hope for a better theoretical understanding of e.g. smile curves and volatility clustering for high-frequency data.

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