On Distributions Associated with the Generalized Lévy's Stochastic Area Formula

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Abstract

A closed-form expression is obtained for the conditional probability distribution of $\int_0^t R_s^2\, ds$ given R_t, where $(R_s, s \geq 0)$ is a Bessel process of dimension $\delta > 0$ started from 0, in terms of parabolic cylinder functions. This is done by inverting the following Laplace transform also known as the generalized Lévy's stochastic area formula:

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t R_s^2\, ds\Big) \,|\, R_t = a\Big] = \Big(\frac{\lambda\, t}{\sinh(\lambda\, t)}\Big)^{\delta/2} \exp\Big(-\frac{a^2}{2t}(\lambda\, t\, \coth(\lambda\, t) - 1)\Big)$$

We also examine the joint distribution of $(R_t^2, \int_0^t R_s^2\, ds)$.

Key words and phrases: Bessel process, density/distribution functions, parabolic cylinder functions, Laplace inversion.

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1. Introduction

1.1. If $(R_u, u \geq 0)$ is a Bessel process of dimension $\delta > 0$ started at 0, then the following formula is known to be valid (see e.g. [14]):

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t R_s^2\, ds\Big) \,|\, R_t = a\Big] = \Big(\frac{\lambda\, t}{\sinh(\lambda\, t)}\Big)^{\delta/2} \exp\Big(-\frac{a^2}{2t}(\lambda\, t\, \coth(\lambda\, t) - 1)\Big). \tag{1.1}$$

If $\delta = 1$ and $a = 0$, then (1.1) leads to the distribution of the Brownian bridge $(b_s,\ s \geq 0)$ in the L^2 norm which is identical to Smirnov's distribution for his ω^2-test. We recall below the relation between the integral of the square of the Brownian bridge and the supremum of the absolute value (see e.g. [3]):

$$\int_0^1 b_s^2\, ds + \int_0^1 \tilde{b}_s^2\, ds \overset{\text{law}}{=} \frac{4}{\pi^2} \sup_{0\leq s\leq 1} |b_s|^2 \tag{1.2}$$

where $(\tilde{b}_s,\ 0 \leq s \leq 1)$ is an independent copy of $(b_s,\ 0 \leq s \leq 1)$.

If $\delta = 2$, then (1.1) is the Lévy's stochastic area formula. Indeed, Lévy [10] showed that if $(X(t), Y(t))$ is an $\mathbb{R}^2$-valued Brownian motion, starting from $(0,0)$, then for any $\xi \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$,

$$\mathbb{E}\Big[\exp\Big(i\,\xi \int_0^t (X(u)dY(u) - Y(u)dX(u))\Big) \,|\, X(t) = x, Y(t) = y\Big] \tag{1.3}$$

$$= \mathbb{E}\Big[\exp\Big(-\frac{\xi^2}{2}\int_0^t R^2(u)\, du\Big) \,|\, R(t) = r\Big] \tag{1.4}$$

$$= \Big(\frac{\xi\, t}{\sinh(\xi\, t)}\Big) \exp\Big(-\frac{r^2}{2t}(\xi\, t\, \coth(\xi\, t) - 1)\Big) \tag{1.5}$$

where $R^2 = X^2 + Y^2$ and $r^2 = x^2 + y^2$.

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Lévy's area formula arises naturally in some problems in analysis (explicit formula for the heat kernel corresponding to the Kohn-Laplacian of the Heisenberg group, see [7]), geometry (a probabilistic proof of the well-known index theorems of Atiyah and Singer due to J. M. Bismut, see [4, 5]) and statistical inference (parameter estimation and testing of statistical hypotheses for diffusion-type processes, see chapter 17 in [11]). We also note the close connection between the distributions of subordinated perpetuities and generalized Lévy's formula for the stochastic area of planar Brownian motion (see [16] for details). For a historical account of Lévy's area formula, we refer the interested reader to [9] and [13].

1.2. Equivalently, we can write the generalized Lévy's stochastic area formula (1.1) as follows:

$$\mathbb{E}\Big[\exp\Big(-u\,R_t^2 - v\int_0^t R_s^2\,ds\Big)\Big] = \Big[\cosh(\sqrt{2\,v}\,t) + \frac{2\,u}{\sqrt{2\,v}}\sinh(\sqrt{2\,v}\,t)\Big]^{-\delta/2}. \tag{1.6}$$

In the Brownian case (i.e. $\delta = 1$), the Laplace inversion of (1.6) has been undertaken by Abadir [1, 2] in 1995 who derived the joint density and distribution functions of the following two Brownian functionals:

$$\frac{1}{2}(B_1^2 - 1) = \int_0^1 B_s\,dB_s \qquad \textit{and} \qquad \int_0^1 B_s^2\,ds \tag{1.7}$$

where $B_s, 0 \le s \le 1$ is a standard one-dimensional Brownian motion started at 0. These two functionals play an important role in unit root statistics (see [13]).

1.3. The paper is organized as follows. In Section 2 we derive explicitly the density of $\int_0^t R_s^2\,ds$ given R_t in terms of parabolic cylinder functions. In Section 3 we derive the joint density of $(R_t^2, \int_0^t R_s^2\,ds)$.

2. The density associated with the generalized Lévy's area formula

The following theorem offers a method to invert (1.1); the result may be expressed in terms of parabolic cylinder functions.

Theorem 2.1 *The density $f_{a,\,t}$ of $\int_0^t R_s^2\,ds$ given $R_t = a$ is given by*

$$f_{a,\,t}(x) = \frac{2^{\frac{\delta}{2}}\,t^{\delta/2}}{\sqrt{2\pi}} e^{\frac{a^2}{2t}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\,a^{2j} \sum_{k=0}^{\infty} \frac{(j+\delta/2)_k}{k!}\,x^{-\beta-1}\,e^{-\frac{\alpha^2}{4x}}\,D_{2\beta+1}(\frac{\alpha}{\sqrt{x}}) \tag{2.1}$$

where $\alpha = 2\,k\,t + \frac{a^2}{2} + 2\,j\,t + \frac{\delta}{2}\,t$, $\beta = \frac{j}{2} + \frac{\delta}{4}$, $D_\nu(\xi)$ is a parabolic cylinder function and $(\nu)_k \equiv \nu(\nu+1)\ldots(\nu+k-1) = \Gamma(\nu+k)/\Gamma(\nu)$ is the Pochhammer's symbol.

Proof: First, according to [10; p. 259], we have

$$p^\nu e^{-a\sqrt{p}} \doteq 2^{-\nu-\frac{1}{2}}\pi^{-\frac{1}{2}}t^{-\nu-1}\exp(-\frac{a^2}{8\,t})\,D_{2\nu+1}(\frac{a}{\sqrt{2t}}) \tag{2.2}$$

Using the relation $\coth(x) = 1 + 2(\exp(2x) - 1)^{-1}$, then expanding the exponential:

$$\Big(\frac{\sqrt{2\lambda}\,t}{\sinh(\sqrt{2\lambda}\,t)}\Big)^{\delta/2}\exp\Big(-\frac{a^2}{2t}(\sqrt{2\lambda}\,t\,\coth(\sqrt{2\lambda}\,t) - 1)\Big) \tag{2.3}$$

$$= 2^{3\delta/4}\,t^{\delta/2}\,e^{\frac{a^2}{2t}} \sum_{j=0}^{\infty} \frac{(-\sqrt{2}a^2)^j}{j!}\,\lambda^{\frac{1}{2}\{j+\delta/2\}}\frac{e^{-\sqrt{\lambda}\{\sqrt{2}\frac{a^2}{2}+2\sqrt{2}jt+\frac{\sqrt{2}}{2}\delta t\}}}{(1-e^{-2\sqrt{2}\sqrt{\lambda}t})^{j+\frac{\delta}{2}}} \tag{2.4}$$

$$= 2^{3\delta/4}t^{\delta/2}e^{a^2/2t}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}a^{2j}2^{j/2}\sum_{k=0}^{\infty}\frac{(j+\delta/2)_k}{k!}\lambda^{\frac{1}{2}\{j+\delta/2\}}e^{-\alpha\sqrt{2\,\lambda}} \tag{2.5}$$

the termwise inversion of the series in (2.5) is readily justifiable by elementary estimates. □

Corollary 2.1 *The density $f_{0,\,t}$ of $\int_0^t R_s^2\,ds$ given $R_t = 0$ is given by*

$$f_{0,\,t}(x) = \frac{2^{\frac{\delta}{2}}\,t^{\delta/2}}{\sqrt{2\pi}}\,x^{-\frac{\delta}{4}-1}\sum_{k=0}^{\infty}\frac{(\frac{\delta}{2})_k}{k!}\,e^{-\dfrac{(k+\,\delta/4)^2t^2}{x}}\,D_{\frac{\delta}{2}+1}\Big(\frac{2kt+t\,\delta/2}{\sqrt{x}}\Big) \tag{2.6}$$

where $D_\nu(\xi)$ is a parabolic cylinder function and $(\nu)_k \equiv \nu(\nu{+}1)\ldots(\nu{+}k{-}1) = \Gamma(\nu{+}k)/\Gamma(\nu)$ is the Pochhammer's symbol.

Remark:
L. Tolmatz [15] determined the density (2.6) in the particular case for $\delta = 1$.

3. The joint density of $(R_t^2, \int_0^t R_s^2\,ds)$

Theorem 3.1 *The joint distribution g_t of $(R_t^2, \int_0^t R_s^2\,ds)$ is given by*

$$\begin{aligned} g_t(x,y) = {}& \frac{1}{\sqrt{2\,\pi}\,\Gamma(\frac{\delta}{2})}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\,x^{j+\frac{\delta}{2}-1}\,y^{-\frac{j}{2}-\frac{\delta}{4}-1}\sum_{k=0}^{\infty}\frac{(j+\frac{\delta}{2})_k}{k!} \\ & e^{-\frac{1}{4y}\{2\,(k+j+\frac{\delta}{4})\,t+\frac{x}{2}\}^2}\,D_{\frac{\delta}{2}+j+1}\Big(\frac{2(k+j+\frac{\delta}{4})t\,+\,\frac{x}{2}}{\sqrt{y}}\Big) \end{aligned} \tag{3.1}$$

where $D_\nu(\xi)$ is a parabolic cylinder function and $(\nu)_k \equiv \nu(\nu{+}1)\ldots(\nu{+}k{-}1) = \Gamma(\nu{+}k)/\Gamma(\nu)$ is the Pochhammer's symbol.

Proof: Two methods lead to the same result (3.1). The first method follows from Theorem 2.1 by integrating the conditional density $f_{a^2,t}$ with respect to the law of R_t^2

$$P(R_t^2 \in dx) = (2t)^{-\delta/2}\frac{1}{\Gamma(\frac{\delta}{2})}x^{\delta/2-1}e^{-x/2t}\,dx\,.$$

This leads immediately to (3.1) and the details will be omitted. The second method is based on inverting the Laplace transform (1.6) and this can be done as follows.

Set $X = R_t^2$ and $Y = \int_0^t R_s^2\,ds$. Using formula (1.6), the joint density of X and Y is found to be given by

$$g_t(x,y) = -\frac{1}{4\pi^2}\int_{\beta-i\infty}^{\beta+i\infty}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{xu+yv}\Big[\cosh(\sqrt{2\,v}\,t) + \frac{2\,u}{\sqrt{2\,v}}\sinh(\sqrt{2\,v}\,t)\Big]^{-\delta/2}\,dudv\,.$$

We note that

$$\begin{aligned} &\Big[\cosh(\sqrt{2\,v}\,t) + \frac{2\,u}{\sqrt{2\,v}}\sinh(\sqrt{2\,v}\,t)\Big]^{-\delta/2} \\ &= \sum_{k=0}^{\infty}\frac{(\frac{\delta}{2})_k}{k!}2^{\delta/2}\Big(\sqrt{2v}\Big)^{\delta/2}e^{-\sqrt{2v}(2kt+\frac{\delta}{2}t)}\,\frac{(2u-\sqrt{2v})^k}{(2u+\sqrt{2v})^{k+\frac{\delta}{2}}}\,. \end{aligned} \tag{3.2}$$

Then, according to [10; p. 239], we have

$$(p-a)^{\nu}(p+a)^{-\mu} \doteq \frac{1}{\Gamma(\mu-\nu)}t^{\mu-\nu-1}e^{-at}\,{}_1F_1(-\nu;\mu-\nu;2at) \quad \text{for} \quad \mathcal{R}(\mu-\nu) > 0 \tag{3.3}$$

so it follows that

$$\begin{aligned} &\Big[\cosh(\sqrt{2\,v}\,t) + \frac{2\,u}{\sqrt{2\,v}}\sinh(\sqrt{2\,v}\,t)\Big]^{-\delta/2} \\ &= \int_0^{\infty} dx\,e^{-xu}\sum_{k=0}^{\infty}\frac{(\frac{\delta}{2})_k}{k!}\,\frac{1}{\Gamma(\frac{\delta}{2})}\Big(\sqrt{2v}\Big)^{\delta/2}e^{-\sqrt{2v}(2kt+\frac{\delta}{2}t+\frac{x}{2})}x^{\delta/2-1}\,{}_1F_1\Big(-k;\frac{\delta}{2};x\sqrt{2v}\Big). \end{aligned} \tag{3.4}$$

By expanding Kummer's function:

$${}_1F_1\Big(-k;\frac{\delta}{2};x\sqrt{2v}\Big) = \sum_{j=0}^{\infty}\frac{(-k)_j}{(\frac{\delta}{2})_j}\,\frac{x^j}{j!}\Big(\sqrt{2\,v}\Big)^j \tag{3.5}$$

we conclude as in the proof of Theorem 2.1:

$$(3.6)\quad \begin{aligned} g_t(x,y) = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{\delta}{2})} \sum_{j=0}^{\infty} \frac{1}{(\frac{\delta}{2})_j} \frac{1}{j!}\, x^{j+\frac{\delta}{2}-1}\, y^{-\frac{j}{2}-\frac{\delta}{4}-1} \sum_{i=0}^{\infty} \frac{(\frac{\delta}{2})_i}{i!}\,(-i)_j \\ e^{-\frac{1}{4y}\{2\,(i+\frac{\delta}{4})\,t+\frac{x}{2}\}^2}\, D_{\frac{\delta}{2}+j+1}\Big(\frac{2\,(i+\frac{\delta}{4})\,t+\frac{x}{2}}{\sqrt{y}}\Big)\,. \end{aligned}$$

To show the equivalence between (3.6) and (3.1), let us compare the coefficients of these expressions. Since $(-i)_j = 0 \quad \text{for} \quad i < j$ we see that the second summation in (3.6) takes place only over $i \geq j$, so that by setting $k = i - j$ the coefficients in (3.1) and (3.6) respectively become:

$$\begin{aligned} C(j,k) &= \frac{(-1)^j}{j!} \frac{(j+\frac{\delta}{2})_k}{k!} \\ D(j,k) &= \frac{1}{(\frac{\delta}{2})_j} \frac{1}{j!} \frac{(\frac{\delta}{2})_{j+k}}{(j+k)!}\,\big(-(j+k)_j\big)\,. \end{aligned}$$

It is easily verified that:

$$C(j,k) = \frac{(\frac{\delta}{2}+j+k)}{\Gamma(\frac{\delta}{2}+j)} \frac{(-1)^j}{k!} = D(j,k)\,.$$

□

Remarks:

1. A. Borodin kindly informed us that a similar expression for g_t appears in the new edition of [6] (see 1.9.8 p. 378).

2. Abadir [1] has derived the joint density of $(\sqrt{2}\int_0^1 B_s\,dB_s, 2\int_0^1 B_s^2\,ds) = (\frac{\sqrt{2}}{2}(B_1^2 - 1), 2\int_0^1 B_s^2\,ds)$ which correspond to the case $\delta = 1$.

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