THE LÉVY-ITO DECOMPOSITION OF AN INDEPENDENTLY SCATTERED RANDOM MEASURE

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ABSTRACT. The relations between additive processes and independently scattered random measures are studied. In particular we derive a Lévy-Ito decomposition of independently scattered random measures on $\mathbb{R}^k$.

1. INTRODUCTION

Independently scattered random measures were introduced by Urbanik and Woyczynski (1969) and Rajput and Rosinski (1989). Sato (2002) proved that if $\{\Lambda(A)\colon A \in \mathcal{B}_b(\mathbb{R}_+)\}$ is an atomless independently scattered random measure on $\mathbb{R}_+$, then the process defined by $X_t = \Lambda([0,t])$ is a so-called natural additive process in law, and conversely that any natural additive process in law induces uniquely a continuous independently scattered random measure. Here $\mathcal{B}_b(\mathbb{R}_+)$ denotes the bounded Borel sets in $\mathbb{R}_+$. Sato did not apply the Lévy-Ito decomposition of $\{X_t\colon t \geqslant 0\}$, but a proof of his result can be based on this decomposition in the following way. Let

$$X_t = \int_{[0,t]}\int_{\mathbb{R}^d} y1_{\{|y|\leqslant 1\}}(y)\,\mathrm{d}(j-\nu)(s,y) + \int_{[0,t]}\int_{\mathbb{R}^d} y1_{\{|y|>1\}}(y)\,\mathrm{d}j(s,y) + \int_{[0,t]} \mathrm{d}X_t^{\mathrm{g}} + p_t,$$

where j is the jump measure induced by X, ν the corresponding intensity measure, X^g the Gaussian component of X and $p_t \in \mathbb{R}^d$. Then it is readily seen that the independently scattered random measure induced by $\{X_t\colon t \in \mathbb{R}_+\}$ is

(1.1)

$$\Lambda(A) = \int_A\int_{\mathbb{R}^d} y1_{\{|y|\leqslant 1\}}(y)\,\mathrm{d}(j-\nu)(s,y) + \int_A\int_{\mathbb{R}^d} y1_{\{|y|>1\}}(y)\,\mathrm{d}j(s,y) + \int_A \mathrm{d}X_t^{\mathrm{g}} + \int_A \mathrm{d}p_t.$$

We shall refer to (1.1) as the Lévy-Ito decomposition of $\{\Lambda(A)\colon A \in \mathcal{B}_b(\mathbb{R}_+)\}$.

In this note we consider a generalization to the case where $\mathbb{R}_+$ is replaced by $\mathbb{R}_+^k$ or $\mathbb{R}^k$. We recall in the next section a few properties of independently scattered

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random measures, while Section 3 discusses additive processes on $\mathbb{R}^k_+$. Section 4 contains the main results. We show that a natural additive process on $\mathbb{R}^k_+$ generates an independently scattered random measure and vice versa. Further we we show that any independently scattered random measure on $\mathbb{R}^k$ satisfying a continuity assumption has a Lévy-Ito decomposition similar to (1.1).

Notation. Let d and k denote positive integers. For $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^d)$ in $\mathbb{R}^d$ let $\langle x, y\rangle$ denote their inner product and $|x|$ be the corresponding norm. Let $D = \{x \in \mathbb{R}^d\colon |x| \leqslant 1\}$. Let $\mathcal{B}_0(\mathbb{R}^d)$ be the class of Borel sets B in $\mathbb{R}^d$ with $\inf_{x\in B} |x| > 0$. Let $\mathcal{L}(X)$ denote the law of a random vector X. For a set M and two families $\{X_t\colon t \in M\}$ and $\{Y_t\colon t \in M\}$ of random vectors with X_t and Y_t in $\mathbb{R}^d$ write $\{X_t\colon t \in M\} \overset{\mathrm{d}}{=} \{Y_t\colon t \in M\}$ if the finite-dimensional marginals are the same. We say that $\{X_t\colon t \in M\}$ is a modification of $\{Y_t\colon t \in M\}$ if $(X_{t_1}, \ldots, X_{t_n}) = (Y_{t_1}, \ldots, Y_{t_n})$ *a.s.* for all $n \geqslant 1$ and $t_1, \ldots, t_n \in M$. For probability measures μ_n and μ on $\mathcal{B}(\mathbb{R}^d)$ write $\mu_n \to \mu$ if μ_n converges weakly to μ. Let $\widehat{\mu}$ denote the characteristic function of μ, $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{\mathrm{i}\langle z,x\rangle}\mu(\mathrm{d}x)$ for $z \in \mathbb{R}^d$. Let δ_x denote the point measure at $x \in \mathbb{R}^d$. Let $ID(\mathbb{R}^d)$ denote the class of infinitely divisible distributions. That is, a distribution μ on $\mathcal{B}(\mathbb{R}^d)$ is in $ID(\mathbb{R}^d)$ if and only if $\widehat{\mu}$ is given by $\widehat{\mu}(z) = \exp\left[-\frac{1}{2}z\Sigma z^\top + \mathrm{i}\langle\gamma, z\rangle + \int_{\mathbb{R}^d} g(z,x)\,\nu(\mathrm{d}x)\right]$, $z \in \mathbb{R}^d$, where $\top$ denotes the transpose, $g(z,x) = e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_D(x)$ and (Σ, ν, γ) is the characteristic triplet of μ, that is, Σ is a $d \times d$ nonnegative definite matrix, ν is a Lévy measure on $\mathcal{B}(\mathbb{R}^d)$ and $\gamma \in \mathbb{R}^d$. Denote the entries of Σ by Σ^{ij} and the coordinates of γ by γ^j for $i, j = 1, \ldots, d$. The following is an application of Sato (1999, Theorem 8.7).

Lemma 1.1. *For $n = 1, 2, \ldots$ let $\mu_n \in ID(\mathbb{R}^d)$ have characteristic triplet $(\Sigma_n, \nu_n, \gamma_n)$. The following statements* (i) *and* (ii) *are equivalent.*

(i) $\mu_n \to \delta_0$*;*

(ii) $\gamma^j_n \to 0$ *and* $\Sigma^{ij}_n \to 0$ *for all* $i, j = 1, \ldots d$*, and* $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\,\nu_n(\mathrm{d}x) \to 0$.

If there exists a Lévy measure $\widetilde{\nu}$ on $\mathcal{B}(\mathbb{R}^d)$ such that $\nu_n(B) \leqslant \widetilde{\nu}(B)$ for all $B \in \mathcal{B}(\mathbb{R}^d)$ and $n \in \mathbb{N}$ then (i) *is equivalent to the following condition:*

(iii) $\gamma^j_n \to 0$ *and* $\Sigma^{ij}_n \to 0$ *for all* $i, j = 1, \ldots d$*, and* $\nu_n(B) \to 0$ *for all* $B \in \mathcal{B}_0(\mathbb{R}^d)$.

2. Independently scattered random measures

For $S \in \mathcal{B}(\mathbb{R}^k)$ let $\mathcal{B}_b(S)$ denote the set of bounded Borel sets in S. As in Rajput and Rosinski (1989) and Urbanik and Woyczynski (1969) (in the case $d = 1$) and Sato (2002) (in the case $S = \mathbb{R}_+$) we need the following.

Definition 2.1. Let $S \in \mathcal{B}(\mathbb{R}^k)$ and $\{\Lambda(A)\colon A \in \mathcal{B}_b(S)\}$ denote a family of random vectors in $\mathbb{R}^d$. We call this family *an* $\mathbb{R}^d$*-valued independently scattered random measure on* S if the following three conditions are satisfied: (i) $\mathcal{L}(\Lambda(A)) \in ID(\mathbb{R}^d)$ for all $A \in \mathcal{B}_b(S)$; (ii) $\Lambda(A_1), \dots, \Lambda(A_n)$ are independent whenever $A_1, \dots, A_n \in \mathcal{B}_b(S)$ are disjoint; (iii) $\Lambda(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Lambda(A_n)$ *a.s.* whenever $A_1, A_2, \dots \in \mathcal{B}_b(S)$ are disjoint with $\cup_{n=1}^\infty A_n \in \mathcal{B}_b(S)$. Here the series converges almost surely.

This definition will be used with $S = \mathbb{R}_+^k$ or $S = \mathbb{R}^k$. Let $\{\Lambda(A)\colon A \in \mathcal{B}_b(S)\}$ denote an $\mathbb{R}^d$-valued independently scattered random measure (isrm for short) on S. Let $\mu(A) = \mathcal{L}(\Lambda(A))$ and $(\Sigma_A, \nu_A, \gamma_A)$ denote the characteristic triplet of $\mu(A)$ for $A \in \mathcal{B}_b(S)$. As in Rajput and Rosinski (1989) one shows that $A \to \gamma_A^j$ is a signed measure for $j = 1, \dots, d$, and $A \to \Sigma_A^{ij}$ is a measure for $i = j$ and a signed measure for $i \neq j$. (Notice that $\mathcal{B}_b(S)$ is not a σ-algebra. Hence when writing that $A \to \Sigma_A^{ij}$ is a signed measure we mean that it is a signed measure on $\mathcal{B}(S \cap B(0,r))$ for all $r > 0$, where $B(0,r)$ denotes the ball with center $0 \in \mathbb{R}^k$ and radius r in $\mathbb{R}^k$.) Similarly $A \to \nu_A(B)$ is a measure for $B \in \mathcal{B}(\mathbb{R}^d)$. Hence there exists a unique measure λ on $\mathcal{B}(S)$ satisfying

$$\lambda(A) = \mathrm{trace}(\Sigma_A) + \mathrm{var}\gamma_A + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\, \nu_A(\mathrm{d}x), \quad A \in \mathcal{B}_b(S),$$

where $\mathrm{var}\gamma_A = \sum_{j=1}^d \mathrm{var}\gamma_A^j$ and $\mathrm{var}\gamma^j$ denotes the total variation of the signed measure $A \to \gamma_A^j$. Call λ *the control measure* of $\{\mu(A)\colon A \in \mathcal{B}_b(S)\}$ and of $\{\Lambda(A)\colon A \in \mathcal{B}_b(S)\}$.

Remark 2.2. By a standard extension result there is a unique σ-finite measure ν on $\mathcal{B}(S \times \mathbb{R}^d)$ satisfying $\nu(A \times B) = \nu_A(B)$ for $A \in \mathcal{B}_b(S)$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

3. Additive processes on $\mathbb{R}_+^k$

For $t = (t^1, \dots, t^k) \in \mathbb{R}_+^k$, $a = (a^1, \dots, a^k) \in \mathbb{R}_+^k$ and $b = (b^1, \dots, b^k) \in \mathbb{R}_+^k$ write $a \leqslant b$ if $a^j \leqslant b^j$ for all j and $a < b$ if $a^j < b^j$ for all j, and define the half-open interval $]a,b]$ as $]a,b] = \{t \in \mathbb{R}_+^k\colon a < t \leqslant b\}$. Let $[a,b] = \{t \in \mathbb{R}_+^k\colon a \leqslant t \leqslant b\}$.

For $F = \{F_t\colon t \in \mathbb{R}_+^k\}$ with $F_t \in \mathbb{R}^d$ and $a \leqslant b$ define the increment of F over $]a,b]$, $\Delta_a^b F$, as

$$\Delta_a^b F = \sum_{\epsilon_1=0}^{1} \cdots \sum_{\epsilon_k=0}^{1} (-1)^{\epsilon_1 + \cdots + \epsilon_k} F_{(c^1(\epsilon_1), \dots, c^k(\epsilon_k))},$$

where $c^j(0) = b^j$ and $c^j(1) = a^j$. For example, if $k = 1$ we have $\Delta_a^b F = F_b - F_a$ and when $k = 2$ then $\Delta_a^b F = F_{(b^1,b^2)} + F_{(a^1,a^2)} - F_{(a^1,b^2)} - F_{(b^1,a^2)}$. Notice that if $a \leqslant b$ and $a \not< b$ then $\Delta_a^b F = 0$.

Let $\mathcal{A} = \{t \in \mathbb{R}_+^k : t^j = 0 \text{ for some } j\}$. For $\mathcal{R} = (R_1, \ldots, R_k)$ where R_j is either $\leqslant$ or $>$ write $a\mathcal{R}b$ if $a^j R_j b^j$ for all j.

We say that $F = \{F_t : t \in \mathbb{R}_+^k\}$ is *lamp* if the following three conditions are satisfied: (i) for $t \in \mathbb{R}_+^k$ the limit $F(t, \mathcal{R}) := \lim_{u \to t, t\mathcal{R}u} F_u$ exists for each of the 2^k relations $\mathcal{R} = (R_1, \ldots, R_k)$ where R_j is either $\leqslant$ or $>$; (let $F(t, \mathcal{R}) := F_t$ if there is no u with $t\mathcal{R}u$); (ii) $F_t = F(t, \mathcal{R})$ for $\mathcal{R} = (\leqslant, \ldots, \leqslant)$; (iii) $F_t = 0$ for $t \in \mathcal{A}$.

Here lamp stands for *limits along monotone paths*. See Adler et al. (1984) for references to the literature on lamp trajectories. When F is lamp and $t \in \mathbb{R}_+^k \setminus \mathcal{A}$ define $\Delta_t F := \lim_{n\to\infty} \Delta_{t_n}^t F$ where t_n is a sequence with $t_n \to t$ and $t_n < t$. If F is continuous at the point t then $\Delta_t F = 0$ but the converse is not true, that is, we can have $\Delta_t F = 0$ without F being continuous at t.

Definition 3.1. Let $\{X_t : t \in \mathbb{R}_+^k\}$ be a family of random vectors in $\mathbb{R}^d$. We say that $\{X_t : t \in \mathbb{R}_+^k\}$ is an $\mathbb{R}^d$*-valued additive process in law on* $\mathbb{R}_+^k$ if the following three conditions are satisfied: (i) $X_t = 0$ *a.s.* for $t \in \mathcal{A}$; (ii) $\Delta_{a_1}^{b_1} X, \ldots, \Delta_{a_n}^{b_n} X$ are independent whenever $n \geqslant 2$ and $]a_1, b_1], \ldots,]a_n, b_n]$ are disjoint; (iii) $\{X_t : t \in \mathbb{R}_+^k\}$ is continuous in probability. If, in addition, almost all sample paths of $\{X_t : t \in \mathbb{R}_+^k\}$ are lamp, then $\{X_t : t \in \mathbb{R}_+^k\}$ is called an $\mathbb{R}^d$*-valued additive process on* $\mathbb{R}_+^k$.

This is heavily inspired by the definition given in Adler et al. (1984), p. 5. (Although an additive process would be called a Lévy process by these authors.) For example, the Brownian sheet is an additive process and in the case $k = 1$ the definition above yields usual additive processes, see Sato (1999), Definition 1.6. To describe the characteristic triplets of additive processes we introduce the concept of admissibility.

Definition 3.2. For $t \in \mathbb{R}_+^k$ let (G_t, H_t, p_t) denote the characteristic triplet of a distribution on $\mathcal{B}(\mathbb{R}^d)$, that is, G_t is a $d \times d$ nonnegative definite matrix, H_t is a Lévy measure on $\mathcal{B}(\mathbb{R}^d)$ and $p_t \in \mathbb{R}^d$. We say that $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$ is *admissible* if (i) $(G_t, H_t, p_t) = (0, 0, 0)$ for all $t \in \mathcal{A}$; (ii) $t \to G_t$ and $t \to p_t$ are continuous and $t \to H_t(B)$ is continuous for all $B \in \mathcal{B}_0(\mathbb{R}^d)$; (iii) $(\Delta_a^b G, \Delta_a^b H, \Delta_a^b p)$ is a characteristic triplet for all $a \leqslant b$. We say that $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$ is *natural* if it is admissible and there exist (uniquely) d signed measures $\gamma^1, \ldots, \gamma^d$ on $\mathcal{B}_b(\mathbb{R}_+^k)$ such that $\gamma^j_{[0,t]} = p_t^j$ for all t and $j = 1, \ldots, d$.

Naturalness was introduced by Sato (2002) in the case $k = 1$.

Remark 3.3. Let $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$ be admissible.

(i) There exists uniquely a σ-finite measure ν on $\mathcal{B}(\mathbb{R}_+^k \times \mathbb{R}^d)$ satisfying $\nu([0,t] \times B) = H_t(B)$ for all $t \in \mathbb{R}_+^k$ and $B \in \mathcal{B}(\mathbb{R}^d)$. This ν satisfies in addition $\nu(]a,b] \times B) = \Delta_a^b H(B)$ for all $a \leqslant b$ and $B \in \mathcal{B}(\mathbb{R}^d)$ and $\nu(\mathcal{A} \times \mathbb{R}^d) = \nu(\mathbb{R}_+^k \times \{0\}) = 0$. The existence of ν follows from the fact that $\Delta_a^b H(B) \geqslant 0$ for all $a \leqslant b$ and the continuity of H_t. For $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ define the measure ν_A on $\mathcal{B}(\mathbb{R}^d)$ by $\nu_A(B) = \nu(A \times B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Choosing t such that $A \subseteq [0,t]$ it follows that $\nu_A(B) \leqslant \nu_{[0,t]}(B) = H_t(B)$. Thus ν_A is a Lévy measure. Similarly, there exists uniquely a family $\{\Sigma_A \colon A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ of nonnegative definite $d \times d$ matrices satisfying $\Sigma_{[0,t]} = G_t$ for all $t \in \mathbb{R}_+^k$ and that $A \to \Sigma_A^{ij}$ is a signed measure on $\mathcal{B}_b(\mathbb{R}_+^k)$ for $i, j = 1, \ldots, d$.

(ii) $\{(G_t, H_t, p_t) \colon t \in \mathbb{R}_+^k\}$ can be admissible without being natural. For example, when $k = 1$ we have naturalness if and only if $t \to p_t^j$ is of bounded variation for $j = 1, \ldots, d$.

Remark 3.4. (i) Let $\{X_t \colon t \in \mathbb{R}_+^k\}$ be an additive process in law. Then, obviously, $X_t = \Delta_0^t X$ *a.s* for $t \in \mathbb{R}_+^k$. Moreover, by Adler et al. (1984), Theorem 3.1, $\mathcal{L}(\Delta_a^b X) \in ID(\mathbb{R}^d)$ for all $a \leqslant b$. Let (G_t, H_t, p_t) denote the characteristic triplet of $\mathcal{L}(X_t) = \mathcal{L}(\Delta_0^t X)$. It is then easily seen that for $a \leqslant b$, $(\Delta_a^b G, \Delta_a^b H, \Delta_a^b p)$ is the characteristic triplet of $\mathcal{L}(\Delta_a^b X)$ and that $\mathcal{L}(X_b - X_a)$ has characteristic triplet $(G_b - G_a, H_b - H_a, p_b - p_a)$. Further, $\{(G_t, H_t, p_t) \colon t \in \mathbb{R}_+^k\}$ is admissible. Indeed, we just have to verify Definition 3.2 (ii). Let $t_n \to t$ and define $t_n \wedge t = (t_n^1 \wedge t^1, \ldots, t_n^k \wedge t^k)$. Then $\mathcal{L}(X_{t_n} - X_{t_n \wedge t})$ has characteristic triplet $(G_{t_n} - G_{t_n \wedge t}, H_{t_n} - H_{t_n \wedge t}, p_{t_n} - p_{t_n \wedge t})$. Since $\mathcal{L}(X_{t_n} - X_{t_n \wedge t}) \to \delta_0$ by continuity in probability, it follows from Lemma 1.1 that $G_{t_n} - G_{t_n \wedge t} \to 0$, $p_{t_n} - p_{t_n \wedge t} \to 0$ and $(H_{t_n} - H_{t_n \wedge t})(B) \to 0$ for all $B \in \mathcal{B}_0(\mathbb{R}^d)$. One shows similarly $G_t - G_{t_n \wedge t} \to 0$, $p_t - p_{t_n \wedge t} \to 0$ and $(H_t - H_{t_n \wedge t})(B) \to 0$. Hence $G_t - G_{t_n} \to 0$, $p_t - p_{t_n} \to 0$ and $(H_t - H_{t_n})(B) \to 0$ for all $B \in \mathcal{B}_0(\mathbb{R}^d)$.

(ii) Let $\{(G_t, H_t, p_t) \colon t \in \mathbb{R}_+^k\}$ be admissible. We say that an additive process in law $\{X_t \colon t \in \mathbb{R}_+^k\}$ is *associated with* $\{(G_t, H_t, p_t) \colon t \in \mathbb{R}_+^k\}$ if $\mathcal{L}(X_t)$ has characteristic triplet (G_t, H_t, p_t) for all t. If $\{(G_t, H_t, p_t) \colon t \in \mathbb{R}_+^k\}$ is natural then we say that an additive process in law associated with it is natural.

(iii) Let $\{(G_t, H_t, p_t) \colon t \in \mathbb{R}_+^k\}$ be admissible. Let $\{X_t \colon t \in \mathbb{R}_+^k\}$ be a family of random vectors which satisfies (ii) of Definition 3.1 and that $\mathcal{L}(X_t)$ has characteristic triplet (G_t, H_t, p_t) for all $t \in \mathbb{R}_+^k$. Then $\{X_t \colon t \in \mathbb{R}_+^k\}$ is an additive process in law associated with $\{(G_t, H_t, p_t) \colon t \in \mathbb{R}_+^k\}$. Indeed, we just have to prove that $\{X_t \colon t \in \mathbb{R}_+^k\}$ is continuous in probability. This follows by reverting the argument in the last part of (i).

Remark 3.5. Assume that $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$ is of the form $(G_t, H_t, p_t) = (\widetilde{G}, \widetilde{H}, \widetilde{p})\mathrm{Leb}([0,t])$, where Leb is the Lebesgue measure on $\mathbb{R}^k$ and $(\widetilde{G}, \widetilde{H}, \widetilde{p})$ is the characteristic triplet of a distribution on $\mathcal{B}(\mathbb{R}^d)$. We then have that $(\Delta_a^b G, \Delta_a^b H, \Delta_a^b p) = (\widetilde{G}, \widetilde{H}, \widetilde{p})\mathrm{Leb}(]a,b])$ and $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$ is admissible. Let $\{X_t : t \in \mathbb{R}_+^k\}$ be an additive process (resp. additive process in law) associated with $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$. We say that $\{X_t : t \in \mathbb{R}_+^k\}$ is an *$\mathbb{R}^d$-valued Lévy process on $\mathbb{R}_+^k$* (resp. *Lévy process in law on $\mathbb{R}_+^k$*). Obviously a Lévy process in law is natural.

Whenever $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$ is admissible there exists an additive process associated with it. Indeed, the existence of an additive process in law associated with it follows from Kolmogorov's consistency theorem and the existence of an appropriate modification follows from Adler et al. (1984), Proposition 4.1. In fact, when $H_t = 0$ for all t, then an additive process associated with $\{(G_t, 0, p_t) : t \in \mathbb{R}_+^k\}$ has continuous trajectories almost surely, see Adler et al. (1984), Theorem 3.2.

Theorem 3.6. *Let $\{X_t : t \in \mathbb{R}_+^k\}$ be an additive process associated with $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$. Let*

$$j(C) = \#\{(t, \Delta_t X) : t \in \mathbb{R}_+^k \setminus \mathcal{A},\ (t, \Delta_t X) \in C \text{ and } \Delta_t X \neq 0\} \quad \text{for } C \in \mathcal{B}(\mathbb{R}_+^k \times \mathbb{R}^d).$$

(i) *$\{j(C) : C \in \mathcal{B}(\mathbb{R}_+^k \times \mathbb{R}^d)\}$ is a Poisson random measure with intensity measure ν, where ν is constructed in* Remark 3.3 (i).

(ii) *Let $H_t^1(B) = H_t(B \cap D)$ and $H_t^2(B) = H_t(B \cap D^c)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Define*

$$(3.1)\quad X_t^1 = \int_{[0,t]} \int_{\mathbb{R}^d} y 1_D(y)\, \mathrm{d}(j - \nu)(s, y), \quad X_t^2 = \int_{[0,t]} \int_{\mathbb{R}^d} y 1_{D^c}(y)\, \mathrm{d}j(s, y).$$

We then have that $X_t = X_t^1 + X_t^2 + X_t^{\mathrm{g}} + p_t$, where $\{X_t^{\mathrm{g}} : t \in \mathbb{R}_+^k\}$, $\{X_t^1 : t \in \mathbb{R}_+^k\}$ and $\{X_t^2 : t \in \mathbb{R}_+^k\}$ are independent, $\{X_t^{\mathrm{g}} : t \in \mathbb{R}_+^k\}$ is an additive process associated with $\{(G_t, 0, 0) : t \in \mathbb{R}_+^k\}$ and $\{X_t^i : t \in \mathbb{R}_+^k\}$ is an additive process associated with $\{(0, H_t^i, 0) : t \in \mathbb{R}_+^k\}$ for $i = 1, 2$.

This result is related to Adler et al. (1984), Theorem 4.6. The only difference is that j above is a Poisson random measure on $\mathbb{R}_+^k \times \mathbb{R}^d$ while Theorem 4.6 is formulated in terms of Poisson random measures on $\mathbb{R}^d$. The proofs are essentially the same.

We shall call the process $\{X_t^{\mathrm{g}} : t \in \mathbb{R}_+^k\}$ above *the Gaussian part of* $\{X_t : t \in \mathbb{R}_+^k\}$ and the measure $\{j(C) : C \in \mathcal{B}(\mathbb{R}_+^k \times \mathbb{R}^d)\}$ the *jump measure of* $\{X_t : t \in \mathbb{R}_+^k\}$.

Remark 3.7. Let $\{(G_t, H_t, p_t) : t \in \mathbb{R}_+^k\}$ be natural in the sense of Definition 3.2 and $\gamma = (\gamma^1, \ldots, \gamma^d)$ be the signed measures satisfying $\gamma^j_{[0,t]} = p_t^j$ for all j.

Let $\{X_t\colon t\in\mathbb{R}_+^k\}$ be an additive process associated with $\{(G_t,H_t,p_t)\colon t\in\mathbb{R}_+^k\}$. Let $\mathcal{H}^{\mathrm{g}}$ be the class of measurable functions $f\colon\mathbb{R}_+^k\to\mathbb{R}$ satisfying $\int(f(t))^2\,\mathrm{d}G_t^{jj}<\infty$ for all j, where $\mathrm{d}G_t^{ij}$ denotes integration with respect to the signed measure Σ^{ij} induced by G^{ij} in Remark 3.3 (i). For $f\in\mathcal{H}^{\mathrm{g}}$ we can define the integral $\int f(t)\mathrm{d}X_t^{\mathrm{g}}$ (a random vector in $\mathbb{R}^d$) using a well known route: first one defines the integral when f is simple, that is $f(t)=\sum_{i=1}^n u_i 1_{]a_i,b_i]}(t)$ where $u_i\in\mathbb{R}$ and the half-open intervals are disjoint. For such f we have $\int f(t)\mathrm{d}X_t^{\mathrm{g}}=\sum_{i=1}^n u_i\Delta_{a_i}^{b_i}X^{\mathrm{g}}$. Then by approximating with simple functions one defines the integral for $f\in\mathcal{H}^{\mathrm{g}}$. We have $\mathcal{L}(\int f(t)\,\mathrm{d}X_t^{\mathrm{g}})=N_d(0,\Sigma(f))$, where $\Sigma^{ij}(f)=\int(f(t))^2\,\mathrm{d}G_t^{ij}$. Moreover, there is a version of the Dominated Convergence Theorem: if $\{f_n\}_{n\geqslant 1}$ is a sequence in $\mathcal{H}^{\mathrm{g}}$ with $f_n(t)\to f(t)$ for all t and there exists a function $h\in\mathcal{H}^{\mathrm{g}}$ such that $|f_n(t)|\leqslant|h(t)|$ then $\int f_n(t)\,\mathrm{d}X_t^{\mathrm{g}}\to\int f(t)\,\mathrm{d}X_t^{\mathrm{g}}$ in probability. For $A\in\mathcal{B}_b(\mathbb{R}_+^k)$ write $\int_A \mathrm{d}X_t^{\mathrm{g}}$ for $\int 1_A(t)\,\mathrm{d}X_t^{\mathrm{g}}$. Also define $\int_A \mathrm{d}p_t\colon=(\gamma_A^1,\dots,\gamma_A^d)$.

4. Relations between additive processes and independently scattered random measures

The first part of the following theorem was given by Sato (2002) in the case $k=1$.

Theorem 4.1. *Let $\{X_t\colon t\in\mathbb{R}_+^k\}$ be a natural additive process associated with $\{(G_t,H_t,p_t)\colon t\in\mathbb{R}_+^k\}$.*

(i) *There exists one and up to modification only one isrm $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}_+^k)\}$ satisfying $\Lambda([0,t])=X_t$ a.s. for $t\in\mathbb{R}_+^k$.*

(ii) *Let $\{j(C)\colon C\in\mathcal{B}(\mathbb{R}_+^k\times\mathbb{R}^d)\}$ and $\{X_t^{\mathrm{g}}\colon t\in\mathbb{R}_+^k\}$ be, respectively, the jump measure and the Gaussian part of $\{X_t\colon t\in\mathbb{R}_+^k\}$, and ν denote the measure constructed in* Remark 3.3 (i). *Then $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}_+^k)\}$ is given by*

$$\Lambda(A)=\int_A\int_{\mathbb{R}^d} y1_D(y)\,\mathrm{d}(j-\nu)(t,y)+\int_A\int_{\mathbb{R}^d} y1_{D^c}(y)\,\mathrm{d}j(t,y)+\int_A\mathrm{d}X_t^{\mathrm{g}}+\int_A\mathrm{d}p_t\quad a.s. \tag{4.1}$$

(iii) *Let $\mu(A)=\mathcal{L}(\Lambda(A))$. Then $\mu(A)$ has characteristic triplet $(\Sigma_A,\nu_A,\gamma_A)$, where Σ_A and ν_A are defined in* Remark 3.3 *and $\gamma_A=(\gamma_A^1,\dots,\gamma_A^d)$ in* Definition 3.2.

Proof. Uniqueness follows from Dynkin's lemma. Let $\Lambda(A)$ be defined by (4.1) for all $A\in\mathcal{B}_b(\mathbb{R}_+^k)$. By Theorem 3.6 it is immediate that $\Lambda([0,t])=X_t$ for $t\in\mathbb{R}_+^k$. It is easily verified that the law of $\int_A\int_{\mathbb{R}^d} y1_D(y)\,\mathrm{d}(j-\nu)(t,y)+\int_A\int_{\mathbb{R}^d} y1_{D^c}(y)\,\mathrm{d}j(t,y)$ has characteristic triplet $(0,\nu_A,0)$ and from Remark 3.7 follows that the law of $\int_A\mathrm{d}X_t^{\mathrm{g}}$ has characteristic triplet $(\Sigma_A,0,0)$ for $A\in\mathcal{B}_b(\mathbb{R}_+^k)$. Hence $\mathcal{L}(\Lambda(A))$ has characteristic triplet $(\Sigma_A, \nu_A, \gamma_A)$. Let us verify that $\{\Lambda(A)\colon A \in \mathcal{B}_b(\mathbb{R}^k_+)\}$ is an isrm. Condition (i) in Definition 3.1 is satisfied and condition (ii) follows from the fact that integrals over disjoint sets are independent. Let $A_1, A_2, \ldots \in \mathcal{B}_b(\mathbb{R}^k_+)$ be disjoint such that $\cup_{n=1}^\infty A_n \in \mathcal{B}_b(\mathbb{R}^k_+)$. By the Dominated Convergence Theorem in Remark 3.7 and similar results for integration with respect to (compensated) Poisson random measures we have $\Lambda(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Lambda(A_n)$ *a.s.*, where the sum on the right-hand side exists in probability and hence by independence of the terms also almost surely. This gives condition (iii) in Definition 3.1. $\square$

We call $\{\Lambda(A)\colon A \in \mathcal{B}_b(\mathbb{R}^k_+)\}$ *the isrm induced by* $\{X_t\colon t \in \mathbb{R}^k_+\}$.

For $i = 1, \ldots, k$ and $c \in \mathbb{R}$ let $L(i,c) = \{t = (t^1, \ldots, t^k) \in \mathbb{R}^k\colon t^i = c\}$. Notice that $\mathcal{A} = \cup_{i=1}^k L(i,0) \cap \mathbb{R}^k_+$. We show that an isrm which does not have mass on the sets $L(i,c)$ induces an additive process in law.

Proposition 4.2. *Let $\{\Lambda(A)\colon A \in \mathcal{B}_b(\mathbb{R}^k_+)\}$ be an independently scattered random measure with control measure λ. Let $\widetilde{X}_t = \Lambda([0,t])$ and (G_t, H_t, p_t) be the characteristic triplet of $\mathcal{L}(\widetilde{X}_t)$ for $t \in \mathbb{R}^k_+$.*

(i) *For $a \leqslant b$, $\Delta_a^b \widetilde{X} = \Lambda(]a,b])$ a.s. and the characteristic triplet of $\mathcal{L}(\Delta_a^b \widetilde{X})$ is $(\Delta_a^b G, \Delta_a^b H, \Delta_a^b p)$.*

(ii) *The following statements* (a)–(d) *are equivalent.*

(a) *$\{(G_t, H_t, p_t)\colon t \in \mathbb{R}^k_+\}$ is admissible;*

(b) *$\{\widetilde{X}_t\colon t \in \mathbb{R}^k_+\}$ is an additive process in law;*

(c) *$\lambda(L(i,c) \cap \mathbb{R}^k_+) = 0$ for all $i = 1, \ldots, k$ and $c \geqslant 0$.*

(d) *$\Lambda(A) = 0$ a.s. for all $A \in \mathcal{B}_b(L(i,c) \cap \mathbb{R}^k_+)$, $i = 1, \ldots, k$ and $c \geqslant 0$.*

(iii) *Assume* (a)–(d) *are satisfied. Then $\{\widetilde{X}_t\colon t \in \mathbb{R}^k_+\}$ is natural. Let $\{X_t\colon t \in \mathbb{R}^k_+\}$ be a natural additive process which is a modification of $\{\widetilde{X}_t\colon t \in \mathbb{R}^k_+\}$. Then for $A \in \mathcal{B}_b(\mathbb{R}^k_+)$ $\Lambda(A)$ is given by* (4.1) *where j and X^{g} denote the jump measure and the Gaussian part of $\{X_t\colon t \in \mathbb{R}^k_+\}$, respectively.*

Proof. (i) It is readily seen that $\Delta_a^b \widetilde{X} = \Lambda(]a,b])$ *a.s.* and that $\mathcal{L}(\Delta_a^b \widetilde{X})$ has characteristic triplet $(\Delta_a^b G, \Delta_a^b H, \Delta_a^b p)$. Using Definition 2.1 (ii) it follows that $\{\widetilde{X}_t\colon t \in \mathbb{R}^k_+\}$ satisfies Definition 3.1 (ii).

(ii) Let $(\Sigma_A, \nu_A, \gamma_A)$ denote the characteristic triplet of $\mathcal{L}(\Lambda(A))$. If (c) is satisfied then $(\Sigma_A, \nu_A, \gamma_A) = (0,0,0)$ for all $A \in \mathcal{B}_b(L(i,c) \cap \mathbb{R}^k_+)$. This implies (d). Conversely, if (d) is satisfied then $(\Sigma_A, \nu_A, \gamma_A) = (0,0,0)$ for all $A \in \mathcal{B}_b(L(i,c) \cap \mathbb{R}^k_+)$. Hence

$\lambda(L(i,c)\cap\mathbb{R}_+^k)=0$ by definition of λ. It follows from Remark 3.4 that (a) and (b) are equivalent.

(c) implies (b). It suffices to verify Definition 3.1 (i) and (iii). Since $[0,t]\subseteq\cup_{i=1}^k L(i,0)$ for $t\in\mathcal{A}$ it follows from (d) that $\widetilde{X}_t=\Lambda([0,t])=0$ *a.s.* for all $t\in\mathcal{A}$, which is Definition 3.1 (i). For $t_n,t\in\mathbb{R}_+^k$ with $t_n\to t$ notice that $\limsup\left([0,t]\setminus[0,t_n\wedge t]\right)\subseteq\cup_{i=1}^k L(i,t^i)$. Hence $\limsup\lambda\left([0,t]\setminus[0,t_n\wedge t]\right)\leqslant\lambda\left(\cup_{i=1}^k L(i,t^i)\right)=0$. Let $(\Sigma_n,\nu_n,\gamma_n)$ denote the characteristic triplet of $\mathcal{L}(\Lambda([0,t]\setminus[0,t_n\wedge t])$. Then by definition of λ, condition (ii) in Lemma 1.1 is satisfied. Thus $\widetilde{X}_t-\widetilde{X}_{t_n\wedge t}=\Lambda\left([0,t]\setminus[0,t_n\wedge t]\right)\to 0$ in probability. Similarly $\widetilde{X}_{t_n}-\widetilde{X}_{t\wedge t_n}\to 0$ in probability and hence $\widetilde{X}_{t_n}\to\widetilde{X}_t$ in probability, which is Definition 3.1 (iii).

(b) implies (c). It suffices to prove that for $i=1,\dots,k$ and $c\geqslant 0$ we have $(\Sigma_A,\nu_A,\gamma_A)=(0,0,0)$ for all A given by $A=\{t\in\mathbb{R}_+^k\colon t^i=c \text{ and } t^j\leqslant u^j \text{ for } i\neq j\}$ where $u^j\geqslant 0$. When $c=0$ this follows from the fact that $(G_t,H_t,p_t)=(0,0,0)$ for $t\in\mathcal{A}$. So assume $c>0$. Define $s_n,s\in\mathbb{R}_+^k$ by

$$s^j=\begin{cases} c & \text{if } i=j,\\ u^j & \text{if } i\neq j,\end{cases}\qquad s_n^j=\begin{cases} c-1/n & \text{if } i=j\\ u^j & \text{if } i\neq j.\end{cases}$$

Let $A_n=[0,s]\setminus[0,s_n]$ and notice that $A_n=\{t\in\mathbb{R}_+^k\colon t^i\in]c-1/n,c] \text{ and } t^j\leqslant u^j \text{ for } i\neq j\}$. By continuity in probability of $\{\widetilde{X}_t\colon t\in\mathbb{R}_+^k\}$ follows that $\widetilde{X}_s-\widetilde{X}_{s_n}=\Lambda(A_n)\to 0$ in probability. Since $A_n\downarrow A$ it follows that $\Lambda(A)=0$ *a.s.*, which yields the result.

(iii) follows from Theorem 4.1. □

Remark 4.3. Let $k=1$. Then $L(1,c)$ is simply the one-point set $\{c\}$. Hence (d) above is the condition that $\Lambda(\{t\})=0$ *a.s.* for all $t\in\mathbb{R}_+$. In this case an analogue to the preceding proposition was given in Sato (2002), Theorem 3.2.

Remark 4.4. Let $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}_+^k)\}$ be the isrm induced by a Lévy process $\{X_t\colon t\in\mathbb{R}_+^k\}$ associated with $\{(G_t,H_t,p_t)\colon t\in\mathbb{R}_+^k\}$ where $(G_t,H_t,p_t)=(\widetilde{G},\widetilde{H},\widetilde{p})\mathrm{Leb}([0,t])$ and $(\widetilde{G},\widetilde{H},\widetilde{p})$ is the characteristic triplet of a distribution on $\mathbb{R}^d$. The characteristic triplet of $\mathcal{L}(\Lambda(A))$ is $(\Sigma_A,\nu_A,\gamma_A)=(\widetilde{G},\widetilde{H},\widetilde{p})\mathrm{Leb}(A)$. In particular we see that $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}_+^k\}$ is homogeneous in the sense that $\mathcal{L}(\Lambda(A))=\mathcal{L}(\Lambda(a+A))$ for all $a\in\mathbb{R}_+^k$ and all $A\in\mathcal{B}_b(\mathbb{R}_+^k)$. The control measure of $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}_+^k)\}$ is proportional to the Lebesgue measure.

In the next result we give a Lévy-Ito decomposition of an isrm.

Theorem 4.5. *Let $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}^k)\}$ be an isrm on $\mathbb{R}^k$ with control measure λ. Assume that λ is continuous, i.e. $\lambda(\{t\})=0$ for all $t\in\mathbb{R}^k$. Let $\mu(A)=\mathcal{L}(\Lambda(A))$ and $(\Sigma_A,\nu_A,\gamma_A)$ be the characteristic triplet of $\mu(A)$ for $A\in\mathcal{B}_b(\mathbb{R}^k)$. Then up to modification $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}^k)\}$ is uniquely decomposed as $\Lambda(A)=\Lambda^{\mathrm{g}}(A)+\Lambda^{\mathrm{ng}}(A)+\gamma_A$ a.s. where $\{\Lambda^{\mathrm{g}}(A)\colon A\in\mathcal{B}_b(\mathbb{R}^k)\}$ and $\{\Lambda^{\mathrm{ng}}(A)\colon A\in\mathcal{B}_b(\mathbb{R}^k)\}$ are independent isrms such that $\mathcal{L}(\Lambda^{\mathrm{g}}(A))$ has characteristic triplet $(\Sigma_A,0,0)$ and $\mathcal{L}(\Lambda^{\mathrm{ng}}(A))$ has characteristic triplet $(0,\nu_A,0)$. Moreover, let ν be the measure constructed in* Remark 2.2. *Then there exists a Poisson random measure $\{j(C)\colon C\in\mathcal{B}(\mathbb{R}^k\times\mathbb{R}^d)\}$ with intensity measure ν such that*

$$\Lambda^{\mathrm{ng}}(A)=\int_A\int_{\mathbb{R}^d} y1_D(y)\,\mathrm{d}(j-\nu)(t,y)+\int_A\int_{\mathbb{R}^d} y1_{D^c}(y)\,\mathrm{d}j(t,y).$$

Proof. Step 1. First we prove this result assuming in addition

$$\lambda(L(i,c))=0\quad\text{for } i=1,\dots,k \text{ and } c\in\mathbb{R}. \tag{4.2}$$

To avoid too much notation let us further give the proof in the case $k=2$ only. The idea is to use the preceding results on each of the four quadrants of $\mathbb{R}^2$. Hence define the four quadrants $Q^{\alpha\beta}\subseteq\mathbb{R}^2$ for $\alpha,\beta=+,-$ in the following way: $Q^{++}=[0,\infty[\times[0,\infty[$, $Q^{+-}=[0,\infty[\times]-\infty,0]$, $Q^{-+}=]-\infty,0]\times[0,\infty[$, $Q^{--}=]-\infty,0]\times]-\infty,0]$. Define an isrm $\{\Lambda^{\alpha\beta}(A)\colon A\in\mathcal{B}_b(\mathbb{R}^2)\}$ by $\Lambda^{\alpha\beta}(A)=\Lambda(A\cap Q^{\alpha\beta})$ for $\alpha,\beta=+,-$. Notice that $\Lambda^{\alpha\beta}(A)=0$ *a.s.* for $A\in\mathcal{B}_b(\mathbb{R}^2\setminus Q^{\alpha\beta})$ and if $(\alpha,\beta)\neq(\alpha',\beta')$ then $\Lambda^{\alpha\beta}(A)=\Lambda^{\alpha'\beta'}(A)=0$ *a.s.* for all $A\in\mathcal{B}_b(Q^{\alpha\beta}\cap Q^{\alpha'\beta'})$ by (4.2). In particular the four processes $\{\Lambda^{\alpha\beta}(A)\colon A\in\mathcal{B}_b(\mathbb{R}^2)\}$, $\alpha,\beta=+,-$, are independent with $\Lambda(A)=\sum_{\alpha,\beta=+,-}\Lambda^{\alpha\beta}(A)$ *a.s.* and it suffices to prove the theorem for $\{\Lambda^{\alpha\beta}(A)\colon A\in\mathcal{B}_b(\mathbb{R}^2)\}$ in place of $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}^2)\}$. But, since we can identify $\{\Lambda^{\alpha\beta}(A)\colon A\in\mathcal{B}_b(\mathbb{R}^2)\}$ with an isrm on $Q^{\alpha\beta}$ satisfying an analogue to condition (c) in Proposition 4.2, the theorem follows from this result.

Step 2. Let us prove the result in the general case by induction in k. For $k=1$ we have $\lambda(L(1,c))=0$ for all $c\in\mathbb{R}$ as noticed in Remark 4.3 and we can thus apply Step 1. So, assume that the result is true for any isrm on $\mathbb{R}^{k-1}$ with a continuous control measure and let $\{\Lambda(A)\colon A\in\mathcal{B}_b(\mathbb{R}^k)\}$ be an isrm on $\mathbb{R}^k$ having a continuous control measure λ. There is a countable family $\{(i_l,c_l)\colon l=1,2,\dots\}$ with $i_l\in\{1,\dots,k\}$ and $c_l\in\mathbb{R}$ such that $\lambda(L(i,c))=0$ for $(i,c)\notin\{(i_l,c_l)\colon l=1,2,\dots\}$. Decompose $\Lambda(A)$ as $\Lambda(A)=\Lambda^1(A)+\Lambda^2(A)$ where $\Lambda^1(A)=\Lambda(A\cap(\cup_{l=1}^\infty L(i_l,c_l))^c)$ and $\Lambda^2(A)=\Lambda(A\cap(\cup_{l=1}^\infty L(i_l,c_l)))$. Notice that $\{\Lambda^1(A)\colon A\in\mathcal{B}_b(\mathbb{R}^k)\}$ and $\{\Lambda^2(A)\colon A\in$

$\mathcal{B}_b(\mathbb{R}^k)\}$ are independent and each of them is an isrm. It suffices to establish the theorem for each of these processes separately.

The control measure λ^1 of $\{\Lambda^1(A)\colon A \in \mathcal{B}_b(\mathbb{R}^k)\}$ satisfies $\lambda^1(L(i,c)) = 0$ for all (i,c). The result for $\{\Lambda^1(A)\colon A \in \mathcal{B}_b(\mathbb{R}^k)\}$ thus follows from Step 1.

To prove the theorem for $\{\Lambda^2(A)\colon A \in \mathcal{B}(\mathbb{R}^k)\}$ it suffices to prove it for $\{\Lambda(A \cap L(i_l,c_l))\colon A \in \mathcal{B}_b(\mathbb{R}^k)\}$ for fixed $l = 1,2,\ldots$. Since $L(i_l,c_l)$ is a $(k-1)$-dimensional affine subspace we can identify $\{\Lambda(A \cap L(i_l,c_l))\colon A \in \mathcal{B}_b(\mathbb{R}^k)\}$ with an isrm on $\mathbb{R}^{k-1}$ with a continuous control measure. Then apply the induction hypothesis. $\square$

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