An Extension of Seshadri’s Identities for Brownian Motion

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Abstract

In this note we extend and clarify some identities in law for Brownian motion proved by V. Seshadri [8] using a new identity in law obtained by H. Matsumoto and M. Yor [6].

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1 Introduction

Let $B = (B_t)_{t \geq 0}$ be a one-dimensional standard Brownian motion with $B_0 = 0$. For a real constant $\nu$ and $t \geq 0$, set

$$A_t^{(\nu)} = \int_0^t \exp(2(B_s + \nu s))ds \quad \text{and} \quad A_t = A_t^{(0)}.$$ 

Let $e$ be a standard exponential random variable independent from $B$ and let $L_t$ denote the local time of $B$ at $0$.

Recently, Matsumoto and Yor [6] proved the following result concerning the joint law of $(A_t, B_t)$.

Theorem 1.1 (Matsumoto-Yor) For fixed $t > 0$, the following identity in law holds:

$$(e^e - B_t A_t, B_t) \overset{\text{law}}{=} \cosh(|B_t| + L_t) - \cosh(B_t) B_t.$$  \hspace{1cm} (1.1)

Our aim in this note is to show that this result helps us to find a nontrivial extension of some identities in law (see Theorem 2.1 below) first proved by V. Seshadri. Motivated by the aim to study the joint law of $(A_t, B_t)$ Matsumoto and Yor [6] focused on the left-hand side of the identity (1.1). Here on the contrary we shall be mainly concerned with the right-hand side of this identity.

We also note that Donati-Martin et al. ([1], [2]) used the identity (1.1) in their computations to rederive the expression for the moments of $A_t^{(\nu)}$ earlier obtained by Dufresne [3].

2 Main Result

We begin by recalling Seshadri’s identities in law [8], following closely the presentation given by M. Yor [10].

Theorem 2.1 (Seshadri) For $t \geq 0$ given and fixed, the following identities in law hold:

$$(|B_t| L_t - |B_t|) \overset{\text{law}}{=} (t e/2, B_t)$$  \hspace{1cm} (2.1)

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\[
\left( \frac{2|B_t| + L_t}{2} \right) \overset{\text{law}}{=} \left( \frac{2L_t + |B_t|}{2} \right) \overset{\text{law}}{=} \left( \frac{|B_t|}{2}, L_t \right) \overset{\text{law}}{=} \left( t e/2, |B_t| \right). \tag{2.2}
\]

Remarks:
1) Seshadri’s result (2.1) asserts that for a fixed \( t > 0 \), the two variables \(|B_t|/L_t\) and \( L_t - |B_t|\) are mutually independent, and \(|B_t|/L_t\) is exponentially distributed with parameter \( \lambda = 2/t \). A similar explanation goes for (2.2).
2) Note that \(|B_t|\) and \( L_t\) play a symmetric role in (2.2).

To understand better the title of this note, we reformulate Matsumoto-Yor’s result as follows
\[
\left( \frac{1}{c} \sinh \left( \sqrt{c} \cdot \frac{2|B_t| + L_t}{2} \right) \cdot \sinh \left( \sqrt{c} \cdot \frac{L_t}{2} \right), |B_t| \right) \overset{\text{law}}{=} \left( e/2 e^{-\sqrt{c} B_t}, \int_0^t e^{2 \sqrt{c} B_s} ds, |B_t| \right)
\]
by means of the scaling property of \( B \) and simple hyperbolic identities, where \( c > 0 \). Letting now \( c \) tend to zero, the result 2.2 of Seshadri follows.

Similarly we have:

**Theorem 2.2** For \( t \geq 0 \) given and fixed, the following identity in law holds for all \( c > 0 \):
\[
\left( \frac{1}{c} \sinh \left( \sqrt{c} |B_t| \right) \sinh \left( \sqrt{c} L_t \right), L_t - |B_t| \right) \overset{\text{law}}{=} \left( e/2 e^{-\sqrt{c} B_t}, \int_0^t e^{2 \sqrt{c} B_s} ds, B_t \right) \tag{2.3}
\]

**Proof:** A scaling argument shows that only the case \( c = 1 \) need to be considered. Recalling that \(|B_t| = \beta_t + L_t\) where \( \beta_t = \int_0^t \text{sgn}(B_s) dB_s \) (see e.g. [9]) we can rewrite (2.1) in the following manner:
\[
(|B_t| (|B_t| - \beta_t), -\beta_t) \overset{\text{law}}{=} (|L_t, L_t - |B_t|) \overset{\text{law}}{=} (t e/2, L_t - |B_t|).
\]

Recalling the well-known result concerning the joint law of \((B_t, \beta_t)\) (see e.g. [9])
\[
P(B_t \in dx, \beta_t \in dy) = \frac{2|x| - y}{\sqrt{2\pi t^3}} \exp \left\{ - \frac{(2|x| - y)^2}{2t} \right\} 1_{\{|y| \leq 1\}} \ dx \ dy
\]
and consequently
\[
(B_t \in dx; L_t \in du) = \frac{|x| + u}{\sqrt{2\pi t^2}} \exp \left\{ - \frac{(|x| + u)^2}{2t} \right\} 1_{\{u \geq 0\}} \ dx \ du
\]
it follows that
\[
P(B_t \in dx | \beta_t = y) = \frac{2|x| - y}{t} \exp \left\{ - \frac{(2|x| - y)^2}{2t} + \frac{y^2}{2t} \right\} 1_{\{|y| \geq 1\}} \ dx
\]
for all \( y \in \mathbb{R} \). Thus for every bounded Borel function \( f \) we have for all \( y \) by substituting \( u = 2|x| - y \) and using another well-known hyperbolic identity that
\[
E[f(\sinh(|B_t|) \cdot \sinh(L_t)) \mid L_t = y] = E[f(\sinh(|B_t|) \cdot \sinh(|B_t| - y)) \mid \beta_t = y]
\]
\[
= \int_{|B_t|}^{\infty} \frac{u}{t} \ \exp \left\{ - \frac{u^2}{2t} \right\} \ f(\cosh(u) - \cosh(y))/2) \ du
\]
\[
\int_0^\infty \frac{v + |y|}{t} \ \exp \left\{ - \frac{(v + |y|)^2}{2t} \right\} \ f(\cosh(v + |y|) - \cosh(|y|))/2) \ dv
\]
\[
= E[f(\cosh(|B_t| + L_t) - \cosh(|B_t|)/2) \mid B_t = y]
\]
which by (1.1) equals
\[
E[f(e/2 e^{-B_t} A_t) \mid B_t = y] = E[f \left( e/2 e^{-B_t} \int_0^t e^{2B_s} ds \right) \mid B_t = y].
\]
Alltogether we have proved that for every bounded Borel function \( f \) the following identity
\[
E\left[ f \left( \sinh(|B_t|) \cdot \sinh(L_t) \right) \mid |B_t| - L_t = y \right] = E\left[ f \left( e^{B_t} \cdot e^{-B_t} \int_0^t e^{2B_s} \, ds \right) \mid B_t = y \right]
\]
is true for all \( y \in \mathbb{R} \) from which the result follows observing that \( B_t \) and \(|B_t| - L_t\) are identically distributed.

3 Moments of \( A_t^{(\nu)} \)

In this section we compute moments of certain exponential Brownian functionals connected to the evaluation of Asian options. The techniques used are very simple compared to former proofs (see e.g. [4], [11]) of the same results and furthermore they can be applied in more general situations.

We shall compute all moments of the random variable \( \int_0^t \exp((B_s + \nu s)) \, ds \) i.e. we shall determine the numbers
\[
E\left[ \left( \int_0^t \exp((B_s + \nu s)) \, ds \right)^n \right]
\]
for all \( n \geq 1 \) with \( \nu \in \mathbb{R} \) and \( t > 0 \) given and fixed.

The computation will be based on the following well-known simple fact:

**Lemma 3.1.** If \((M_t)\) is a non-negative right-continuous martingale and \((G_t)\) a continuous increasing process such that \( G_0 = 0 \), then
\[
E\left[ \int_0^t M_s \, dG_s \right] = E[M_t \, G_t]
\]
for all \( t \geq 0 \).

Since the arguments apply not only to the Brownian motion we will assume that we are given a probability space \((\Omega, \mathcal{F}, P)\) and a right-continuous process \( X = (X_s)_{0 \leq s \leq T} \) defined on \((\Omega, \mathcal{F}, P)\) that starts at 0 and has stationary independent increments (shortly called a Lévy process).

Here we assume that the Lévy exponent \( \psi \) of \( X \) defined by
\[
E[\exp(aX_t)] = \exp(t\psi(a))
\]
for \( t \in [0,T] \) and \( a \in \mathbb{R} \) is finite. In the case when \( X \) is a standard Brownian motion we have \( \psi(a) = a^2 / 2 \).

Straightforward calculations show that
\[
(M_t) := (\exp(X_s - s\psi(1)))_{0 \leq s \leq T}
\]
is a non-negative right-continuous martingale starting at 1 and that \((X_s)_{0 \leq s \leq T}\) is a Lévy process on \([0,T] \) under \( \bar{P} \), where \( \bar{P} \) denotes the probability measure on \((\Omega, \mathcal{F})\) defined by
\[
d\bar{P} := M_T \, dP.
\]
The corresponding Lévy exponent \( \bar{\psi} \) is easily seen to be given by
\[
\bar{\psi}(a) = \psi(a + 1) - \psi(1) \quad \text{(for } a \in \mathbb{R} \).
\]

**Theorem 3.1** Let \((X_s)_{0 \leq s \leq T}\) be a Lévy process on \([0,T]\) with exponent \( \psi \). Define for \( n \geq 1, \ t \in [0,T] \) and \( \nu \in \mathbb{R} \)
\[
C_n(t, \nu, \psi) = E\left[ \left( \int_0^t \exp(X_s + \nu s) \, ds \right)^n \right]
\]
Then for $n \geq 2$ we have the following recursive relation:

$$C_n(t, v, \psi) = n \int_0^t C_{n-1}(s, v, \bar{\psi}) \exp(\psi(1)s + vs) \, ds$$  \hspace{1cm} (3.1)$$

for all $t \in [0, T]$ and all $v \in \mathbb{R}$.

**Proof:** Using Lemma 3.1 and the integration by parts formula we obtain for $n \geq 2$:

$$C_n(t, v, \psi) = E\left(\left(\int_0^t \exp(X_u + vu) \, du\right)^n\right)$$

$$= n \cdot E\left[\int_0^t \left(\int_0^s \exp(X_u + vu) \, du\right)^{n-1} \exp(X_s + vs) \, ds\right]$$

$$= n \cdot E\left[\int_0^t M_s \left(\int_0^s \exp(X_u + vu) \, du\right)^{n-1} \exp(\psi(1)s + vs) \, ds\right]$$

$$= n \cdot E\left[M_t \int_0^t \left(\int_0^s \exp(X_u + vu) \, du\right)^{n-1} \exp(\psi(1)s + vs) \, ds\right]$$

$$= n \int_0^t E\left[\left(\int_0^s \exp(X_u + vu) \, du\right)^{n-1}\right] \exp(\psi(1)s + vs) \, ds$$

i.e.

$$C_n(t, v, \psi) = n \int_0^t C_{n-1}(t, v, \bar{\psi}) \exp(\psi(1)s + vs) \, ds.$$

\[ \square \]

Using induction in (3.1) the recursive formula for $(C_n(t, v, \psi))_{n \geq 1}$ can be found, and in the Brownian case we obtain the following closed expression.

**Corollary 3.1** For all $n \geq 1$ and $t \geq 0$ we have:

$$C_n(t, v, a^2/2) = E\left[\left(\int_0^t \exp(B_s + vs) \, ds\right)^n\right] = n! \sum_{j=0}^n \frac{1}{\prod_{i=0, i \neq j}^n (\alpha_i^v - \alpha^v_j)} \exp(t \alpha_j^v)$$

where for each $0 \leq i \leq n$

$$\alpha_i^v = \psi(i) + iv = \frac{v^2}{2} + iv.$$

**Remark:**

A negative answer to the long time unsolved question of whether or not the law of $A^v_j$ is determined by its moments has recently been given by A. Nikeghbali [7].

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**References**


