A Survey and Some Generalizations of Bessel Processes

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2002

Abstract

Bessel processes play an important role in financial mathematics because of their strong relation to financial models like geometric Brownian motion or CIR processes. We are interested in the first time Bessel processes and more generally, radial Ornstein–Uhlenbeck processes hit a given barrier. We give explicit expressions of the Laplace transforms of first hitting times by (squared) radial Ornstein–Uhlenbeck processes, i. e., CIR processes. As a natural extension we study squared Bessel processes and squared Ornstein–Uhlenbeck processes with negative dimensions or negative starting points and derive their properties.

Keywords: First hitting times; CIR processes; Bessel processes; radial Ornstein–Uhlenbeck processes; Bessel processes with negative dimensions

1 Introduction

Bessel processes have come to play a distinguished role in financial mathematics for at least two reasons, which have a lot to do with the models being usually considered. One of these models is the Cox–Ingersoll–Ross (CIR) family of diffusions, also known as square-root diffusions, which solve

$$dX_t = (a + bX_t)\, dt + c\sqrt{|X_t|}\, dB_t\,, \tag{1}$$

with $X_0 = x_0 \geq 0$, $a \geq 0$, $b \in \mathbb{R}$, $c > 0$ and (B_t) standard Brownian motion. For every given value $x_0 \geq 0$, equation (1) admits a unique solution; this solution

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[‡] We would like to thank an anonymous referee whose suggestions were decisive to treat the case $\lambda < 0$ in Section 2.

is strong, i.e. adapted with respect to the natural filtration of (B_t), and takes values in $[0,\infty)$. If $a = 0$ and $x_0 = 0$, the solution of (1) is $X_t \equiv 0$, and from the comparison theorem for one-dimensional diffusion processes (Revuz and Yor [59], Theorem IX.(3.7)), we deduce $X_t \geq 0$ for $a \geq 0$, $x_0 \geq 0$. Hence, in this case, the absolute value in (1) may be omitted a posteriori. Cox, Ingersoll and Ross [10] proposed this family of diffusions for modelling short term interest rates. In recent financial literature they are often studied from different points of view, or serve as important reference processes, see for instance Chen and Scott [8], Deelstra and Parker [12], Delbaen [13], Duffie and Singleton [17], Frydman [23] and Leblanc [40, 39]; they are used for modeling stochastic volatility, see e. g. Ball [3], Genotte and Marsh [25] and Heston [29]. The other even more fundamental model is geometric Brownian motion, standardly used as a model for stock prices

$$S_t = S_0 \exp(\nu\, t + \sigma B_t)\,, \tag{2}$$

with ν, $\sigma \in \mathbb{R}$, (B_t) standard Brownian motion. In both cases, X and S can be represented in terms of (squares of) Bessel processes. We recall the definition of (squares of) Bessel processes (Revuz and Yor [59], Chapter XI).

Definition 1 *For every $\delta \geq 0$ and $x_0 \geq 0$ the unique strong solution to the equation*

$$X_t = x_0 + \delta\, t + 2 \int_0^t \sqrt{|X_s|}\, dB_s \tag{3}$$

is called the square of a δ-dimensional Bessel process *started at x_0 and is denoted by* $\mathrm{BESQ}^{\delta}_{x_0}$.

Clearly, equation (3) is a particular case of equation (1), with $a = \delta$, $b = 0$, $c = 2$. We call the number δ the *dimension* of BESQ^{δ}. This terminology arises from the fact that, in the case $\delta \in \mathbb{N}$, a BESQ^{δ} process X_t can be represented by the square of the Euclidean norm of δ-dimensional Brownian motion B_t: $X_t = |B_t|^2$. The number $\nu \equiv \delta/2 - 1$ is called the *index* of the process BESQ^{δ}. For $\delta \geq 2$, BESQ^{δ} processes will never reach 0 for $t > 0$, and for $0 \leq \delta < 2$ they reach 0 almost surely.

Definition 2 *The square root of* $\mathrm{BESQ}^{\delta}_{y^2}$, $\delta \geq 0$, $y \geq 0$ *is called the* Bessel process of dimension δ *started at y and is denoted by* $\mathrm{BES}^{\delta}_{y}$.

For a study of Bessel processes we refer to Revuz and Yor [59] and Pitman and Yor [53, 54], see also Appendix A. We now show how a general CIR process (1) may be represented in terms of a BESQ process. The relation

$$X_t = e^{bt}\, Y\left(\frac{c^2}{4b}\,(1 - e^{-bt})\right)\,, \tag{4}$$

where Y denotes a squared Bessel process with dimension $\delta = \frac{4a}{c^2}$, clearly establishes a correspondance between the two families of processes. We remark that this relation is used e. g. in Delbaen and Shirakawa [14] and Szatzschneider [67]. For geometric Brownian motion (2), there is the Lamperti relation

$$S_t = \rho^{\left(\frac{\nu}{\sigma^2}\right)}\left(\sigma^2 \int_0^t S_s^2\, ds\right)\,,\ t \geq 0\,, \tag{5}$$

where $\left(\rho^{\left(\frac{\nu}{\sigma^2}\right)}(u), u \geq 0\right)$ denotes a Bessel process with index $\frac{\nu}{\sigma^2}$, with $\rho^{\left(\frac{\nu}{\sigma^2}\right)}(0) = S_0$, see Lamperti [37] and also Williams [72]. The Lamperti representation (5) has been very useful e. g. in connection with Asian option pricing, see Geman and Yor [24] and Yor [75], Chapter 6. For a multivariate extension of the Lamperti relation we refer to Jacobsen [32]. It may be helpful to indicate that $A_t = \sigma^2 \int_0^t S_s^2\, ds$ admits as its inverse process

$$u \to \int_0^u \frac{dh}{\sigma^2 \left(\rho_h^{\left(\frac{\nu}{\sigma^2}\right)}\right)^2}$$

We now make some remarks about the range of the values of the parameters a, b, c, σ and ν which appear in (1), (2), (3), (4) and (5). The signs of c and σ are irrelevant since $B \overset{(d)}{=} -B$. We assume $c, \sigma > 0$. In fact, by scaling we may and will restrict ourselves to $c = 2$ and $\sigma = 1$. For $c = 2$, CIR processes defined by (1) coincide with squared radial Ornstein–Uhlenbeck processes which are studied in detail in Section 2, see (7), and in the following we will use either one or the other terminology. The sign of b influences the behaviour of a CIR process X, since in the case $a > 0$ there exists a unique stationary density of X only if $b < 0$; we note that stationary CIR processes also enjoy the ergodic property. In the case $a \geq \frac{c^2}{2} = 2$, a CIR process starting in $x_0 \geq 0$ stays strictly positive for $t > 0$; for $a < 2$, a CIR process X_t starting in $x_0 \geq 0$ hits 0 with probability $p \in]0, 1[$ if $b > 0$ and almost surely if $b \leq 0$. Note that in the case $a > 0$ the boundary 0 is instantaneously reflecting, whereas in the case $a = 0$, as soon as a CIR process X_t hits 0 it is extinct, i.e. it remains at 0.

Until now, studies of square-root diffusions defined by (1) have always assumed $a \geq 0$ and $x_0 \geq 0$. Under these assumptions, we have seen how to deduce $X_t \geq 0$ for all $t \geq 0$ from the comparison theorem for one-dimensional diffusion processes and the absolute value in the square root term can be omitted. But it seems to be natural also to consider $a < 0$ or to start the process in $x_0 < 0$. In such cases one should be already careful about the formulation of the square root term; for the introduction we consider the case $a < 0$ and $x_0 > 0$ with the formulation (1). But see also Section 3 for a more general substitution, i. e., we replace $\sigma(x) \equiv c\sqrt{|x|}$ by $\tilde{\sigma}(x) \equiv c\sqrt{\alpha x^+ + \beta x^-}$, α, $\beta \geq 0$. The process (X_t), $t \geq 0$, defined by (1) with $a < 0$, $b \in \mathbb{R}$, $c > 0$ and starting in $x_0 > 0$ remains in $\mathbb{R}_+$ until $T_0 = \inf\{t > 0 | X_t = 0\}$. Then, since $Y_t \equiv -X_{T_0+t}$ satisfies

$$dY_t = (-a - bY_t)\, dt + c\sqrt{|Y_t|}\, d\tilde{B}_t\,,$$

with $(\tilde{B}_t)$ standard Brownian motion, we know that $Y_t \geq 0$ for all $t \geq 0$, thus $(X_t,\, t \geq T_0)$ ranges in $\mathbb{R}_-$ and (Y_t) is a CIR process (1) with parameters $-a > 0$, $-b \in \mathbb{R}$ and $c > 0$.

Consider formula (5) with $\nu < 0$. We know from Dufresne [18] (see also Pollack and Siegmund [58] and Yor [76], Théorème 1)

$$\int_0^\infty \exp 2(B_s + \nu\, s) ds \overset{(d)}{=} \frac{1}{2Z_{(-\nu)}}\,,$$

where Z_μ, $\mu > 0$, is a Gamma variable with index μ, i. e.,

$$P(Z_\mu \in dt) = \frac{t^{\mu-1}e^{-t}}{\Gamma(\mu)}\, dt\,.$$

Since $S_t \to 0$, as $t \to \infty$, then $\rho_u^{(\nu)} \to 0$ as u converges to $\int_0^\infty \exp 2(B_s + \nu s)ds$, and $\tilde{T}_0 \equiv \inf\{t | \rho_t^{(\nu)} = 0\} = \int_0^\infty \exp 2(B_s + \nu s)ds$. It seems natural to consider $(\rho_{\tilde{T}_0 + u}, u \geq 0)$. The case $\nu < -1$ corresponds to a Bessel process $\rho^{(\nu)}$ with dimension $\delta = 2(\nu + 1) < 0$. In Section 3, we will derive and discuss properties of negative-dimensional squared Bessel processes, and also squared radial Ornstein–Uhlenbeck processes with $\delta < 0$, with starting points in $\mathbb{R}$.

The above discussion shows how first hitting times of squared radial Ornstein–Uhlenbeck processes, i.e. CIR processes, may arise. In Section 2, we present a survey of the explicit computations of the Laplace transform of first hitting times of (squared) radial Ornstein–Uhlenbeck processes, by exploiting the relation between radial Ornstein–Uhlenbeck processes and Bessel processes. Note that because of the previous discussions about the sign and behaviour of X in the negative dimensional case we will have no difficulties in computing the Laplace transforms of first hitting times T_y of a negative dimensional process X starting in x_0 in all possible cases, say $x_0 > 0 > y$.

From a financial point of view we are interested in the quantity

$$E_a[1_{(T_x < t)}(R_t - k)^+]\,,$$

with a radial Ornstein–Uhlenbeck process (R_t) starting in $a \geq 0$, see Section 2, and $k \in \mathbb{R}^+$. We remark that the quantity $E[1_{(T_x<t)}(S_t - k)^+]$, expressing values of barrier options with underlying stock price process (S_t) as in (2), is investigated in Chesney *et al.* [9] by considering Laplace transforms with respect to time. The Laplace transform of $E_a[1_{(T_x<t)}(R_t - k)^+]$ with respect to time is

$$\begin{aligned}\int_0^\infty e^{-\alpha t} E_a[1_{(T_x<t)}(R_t - k)^+] dt &= E_a[\int_{T_x}^\infty e^{-\alpha t}(R_t - k)^+ dt] \\ &= E_a[e^{-\alpha T_x}\int_0^\infty e^{-\alpha u}(R_{T_x+u} - k)^+ du]\,.\end{aligned}$$

Using the strong Markov property this equals

$$E_a[e^{-\alpha T_x}]\,E_x[\int_0^\infty e^{-\alpha u}(R_u-k)^+du]\,, \tag{6}$$

which gives a clear motivation for our next computations of the Laplace transforms of first hitting times of radial Ornstein–Uhlenbeck processes, as well as to that of their resolvents, here applied to the function $r \to (r-k)^+$; as for the resolvent we refer to Remark 6 in Section 2.
To close up this introduction let us mention that we have preferred, in this paper, the use of stochastic arguments, i.e. Itô's formula, Doob's h-transform, time reversal, etc. to that of differential equations arguments which nonetheless play an important role throughout.

2 First hitting times of radial Ornstein–Uhlenbeck processes

As motivated in the introduction, we are interested in the law of first hitting times of (squared) radial Ornstein–Uhlenbeck processes. For general discussions of first hitting times of diffusions we refer to Arbib [2], Breiman [7], Horowitz [30], Kent [35], Nobile *et al.* [48], Novikov [49, 50, 51], Pitman and Yor [56, 57], Ricciardi and Sato [60], Rogers [61], Salminen [62], Shepp [64], Siegert [66], Truman and Williams [69] and Yor [73]. More general discussions of inverse local times and occupation times $\int_0^T 1_{(X_s\le y)}ds$, when X is a diffusion and T a particular stopping time, are dealt with in Hawkes and Truman [28], Truman [68] and Truman *et al.* [70], with particular emphasis on the Ornstein–Uhlenbeck case.
First we recall the definition of (squared) radial Ornstein–Uhlenbeck processes. We will use another notation than in (1) which is related to the notation in Definition 1. Let (W_t) be a one-dimensional Brownian motion, $\lambda \in \mathbb{R}$, $\delta \ge 0$ and $z \ge 0$. The solution to the equation

$$Z_t = z + \int_0^t (\delta - 2\lambda Z_s)\,ds + 2\int_0^t \sqrt{|Z_s|}\,dW_s \tag{7}$$

is unique and strong (Revuz and Yor [59], Chapter IX §3); as in the discussion of equation (1) we deduce $Z_t \ge 0$. It is called a *squared δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$* and its law on $C(\mathbb{R}_+,\mathbb{R})$ is denoted by ${}^{-\lambda}Q_z^\delta$. It is a Markov process; hence, the square root of this process is also a Markov process and is called a *δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$*. Denote its law on $C(\mathbb{R}_+,\mathbb{R})$ by ${}^{-\lambda}P_x^\delta$ where $x=\sqrt{z}$. The following application of Girsanov's theorem relates ${}^{-\lambda}P_x^\delta$ to P_x^δ, the law of a $\mathrm{BES}^\delta(x)$ process, hence obviously ${}^{-\lambda}Q_z^\delta$ to Q_z^δ, the law of a squared $\mathrm{BES}^\delta(z)$ process.

Proposition 1 *For every $\lambda \in \mathbb{R}$ and $x \geq 0$, the following relationship holds:*

$$ {}^{-\lambda}P_x^\delta\big|_{\mathcal{F}_t} = \exp\left\{-\frac{\lambda}{2}[R_t^2 - x^2 - \delta\, t] - \frac{\lambda^2}{2}\int_0^t R_s^2 ds\right\} \cdot P_x^\delta\big|_{\mathcal{F}_t} \,. \tag{8} $$

Proof: The only care needed to justify (8) is that the Girsanov local martingale which appears in (8) is in fact a martingale. But this follows from the fact that both diffusions with laws ${}^{-\lambda}P_x^\delta$ and P_x^δ are non-explosive (see e.g. McKean's presentation of Girsanov's theorem in the case of explosion [44], Section 3.7). This closes the proof.

Remark 1 More generally, we can replace in (8) the restrictions to the σ-fields $\mathcal{F}_t$ by restrictions to $\mathcal{F}_T \cap (T < \infty)$, for any $(\mathcal{F}_t)$-stopping time T.

Corollary 1 *For $\delta \geq 2$ (resp. $\delta < 2$), and $x > 0$, $(R_t, t \geq 0)$ does not visit 0 a.s. (resp. visits 0 a.s.) under ${}^{-\lambda}P_x^\delta$.*

Proof: The result for $\lambda = 0$ is well-known, and it extends to any $\lambda \in \mathbb{R}$, with the help of (8). This closes the proof.
The following consequence of Proposition 1 will be helpful later to transfer results valid for ${}^{-\lambda}P_x^\delta$, with $\lambda > 0$, to ${}^{\lambda}P_x^\delta$.

Corollary 2 *For every $\lambda \in \mathbb{R}$ and every stopping time T with respect to $(\mathcal{F}_t)$, there is the absolute continuity relationship:*

$$ {}^{\lambda}P_x^\delta\big|_{\mathcal{F}_T\cap(T<\infty)} = \exp(\lambda(R_T^2 - x - \delta\, T)) \cdot {}^{-\lambda}P_x^\delta\big|_{\mathcal{F}_T\cap(T<\infty)} \tag{9} $$

Because $R_t = \sqrt{Z_t}$ reaches 0 a.s. for $\delta < 2$, we need some care to use Itô's formula and express (R_t) as the solution of some stochastic equation: For $\delta > 1$, it is the solution to the equation

$$ dR_t = \left(\frac{\delta - 1}{2R_t} - \lambda R_t\right) dt + dW_t \,, \quad R_0 = x = \sqrt{z}\,. $$

For $\delta = 1$ we have with Itô-Tanaka's formula

$$ |R_t| = |x| - \lambda \int_0^t |R_s|\, ds + \tilde{W}_t + L_t, $$

where (R_t) is a ${}^{-\lambda}P_x^1$ process, $\tilde{W}_t \equiv \int_0^t \operatorname{sgn}(R_s) dW_s$ is standard Brownian motion and L_t is the local time of Brownian motion. For $\delta < 1$ we obtain from (48)

$$ R_t = x + \frac{\delta - 1}{2}\,\text{p.v.}\int_0^t \frac{ds}{R_s} - \lambda \int_0^t R_s ds + \hat{W}_t \,, $$

where $(\hat{W}_t)$ is standard Brownian motion under ${}^{-\lambda}P^\delta$ and p.v. denotes the principal value.

Our aim is to find the law of

$$T_{x\to y} \equiv \inf\{t \,|\, R_t = y\}\,, \tag{10}$$

the first time a radial Ornstein–Uhlenbeck process (R_t) with parameter $-\lambda$ starting in $x \geq 0$ hits the level y. We distinguish between the cases $0 \leq y < x$ and $0 \leq x < y$. For $\delta < 2$, we have ${}^{-\lambda}P_x^\delta(T_{x\to 0} < \infty) > 0$, that is, the process (R_t) may reach 0; if $\delta < 2$ and $\lambda > 0$, then ${}^{-\lambda}P_x^\delta(T_{x\to 0} < \infty) = 1$, that is, (R_t) reaches 0 almost surely and hence every y almost surely, $0 \leq y < x$. For $\delta \geq 2$, we have ${}^{-\lambda}P_x^\delta(T_{x\to y} < \infty) = 1$ almost surely for every $y \geq x$.
Call ${}^{-\lambda}P_x^\delta$ the law of a δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$. The density ${}^{-\lambda}p_x^\delta(t)$ of the first hitting time of 0 of a radial Ornstein–Uhlenbeck process is calculated in Elworthy *et al.* [21], Corollary 3.10, by using a time reversal argument from $T_{x\to 0}$, that is, for $y = 0$ the problem is already solved:

$${}^{-\lambda}p_x^\delta(t) = \frac{x^{2-\delta}}{2^\nu \Gamma(\nu)} \exp\left[\tfrac{\lambda}{2}(\delta t + x^2(1-\coth(\lambda t)))\right] \left[\frac{\lambda}{\sinh(\lambda t)}\right]^{\frac{4-\delta}{2}}\,, \tag{11}$$

where $\delta < 2$, $\lambda > 0$, $x > 0$ and $\nu = \frac{4-\delta}{2} - 1$.
We have for $0 \leq y \leq x$

$$T_{x\to 0} \overset{\text{(law)}}{=} T_{x\to y} + T_{y\to 0}\,,$$

where $T_{x\to y}$ and $T_{y\to 0}$ may be assumed to be independent because of the strong Markov property. Hence, with (11) we obtain for the Laplace transform (LT) of $T_{x\to y}$

$${}^{-\lambda}E_x^\delta\left[\exp\left(-\mu T_{x\to y}\right)\right] = \frac{\phi_x(\mu)}{\phi_y(\mu)}\,, \tag{12}$$

where, still in the case $\lambda > 0$, we find

$$\phi_x(\mu) = {}^{-\lambda}E_x^\delta\left[\exp\left(-\mu T_{x\to 0}\right)\right] = \int_0^\infty \exp(-\mu t)\ {}^{-\lambda}p_x^\delta(t)\, dt.$$

Results for the case $\lambda < 0$ will be derived from those for $\lambda > 0$ with the help of Corollary 2. Our main results are Theorem 1, 2, 3, 4 and Corollary 3 and 4, where we derive explicit expressions of the LTs of first hitting times by radial Ornstein–Uhlenbeck processes with arbitrary $x, y \geq 0$. But let us first concentrate on the case $\lambda = 0$, i.e. Bessel processes.

2.1 First hitting times of Bessel processes.

In order to find an expression of the law and/or the LT of certain first hitting times of Bessel processes, it is convenient to consider time reversed Bessel processes. More generally, for time reversed diffusion processes, see Appendix B.

Let (X_t) be a Bessel process with dimension $\delta < 2$ starting in $x > 0$. The time reversed process $(X_{(T_{x\to 0})-u}\,,\ u \le T_{x\to 0})$ enjoys the following relationship with $(\hat{X}_u,\ u \ge 0)$, a $\hat{\delta}$-dimensional Bessel process starting in 0, with $\hat{\delta} \equiv (4-\delta)$, see Appendix B:

$$\left(X_{(T_{x\to 0})-u},\ u \le T_{x\to 0}\right) \overset{\text{(law)}}{=} \left(\hat{X}_u,\ u \le \hat{L}_{0\to x}\right)\,,$$

where

$$\hat{L}_{0\to x} = \sup\{t \,|\, \hat{X}_t = x\},$$

which is finite since $\{\hat{X}_u, u \ge 0\}$ is transient (which follows from $\hat{\delta} > 2$). In particular, as remarked in Getoor and Sharpe [27], Sharpe [63], $\hat{L}_{0\to x}$ under $P_0^{\hat{\delta}}$ has the same law as $T_{x\to 0}$ under P_x^{δ}. We know (Getoor [26], Pitman and Yor [53]):

$$\hat{L}_{0\to x} \overset{\text{(law)}}{=} \frac{x^2}{2Z_{\hat{\nu}}}\,, \tag{13}$$

where $Z_{\hat{\nu}}$ is a Gamma variable with parameter $\hat{\nu} \equiv \frac{\hat{\delta}}{2} - 1$, i.e.:

$$P(Z_{\hat{\nu}} \in dt) = \frac{t^{\hat{\nu}-1}e^{-t}}{\Gamma(\hat{\nu})}\,dt\,, \tag{14}$$

and hence

$$P_x^{\delta}(T_{x\to 0} \in dt) = P_0^{\hat{\delta}}(\hat{L}_{0\to x} \in dt) = \frac{1}{t\Gamma(\hat{\nu})}\left(\frac{x^2}{2t}\right)^{\hat{\nu}} e^{-\frac{x^2}{2t}}dt\,. \tag{15}$$

We note that (11) was obtained in Elworthy *et al.* [21] in an analogous way, additionally using Girsanov's transformation.

Since, for $y \neq 0$, $y < x$, the first hitting time $T_{x\to y}$ is not distributed as $\hat{L}_{y\to x}$, the last exit time of x by $(\hat{X}_u)$ starting in y, we cannot use the same time reversal argument as above in order to find an expression for its LT. $T_{x\to y}$ has the same law as $\hat{L}_{y\nearrow x}$ where $\hat{L}_{y\nearrow x}$ denotes the last exit time of x by $(\hat{X}_u)$ starting in y at the last exit time of y, i.e., we consider the process starting in y conditioned on never visiting y again. We have

$$\hat{L}_{0\to x} \overset{\text{(law)}}{=} \hat{L}_{0\to y} + \hat{L}_{y\nearrow x}\,, \tag{16}$$

where $\hat{L}_{0\to y}$ and $\hat{L}_{y\nearrow x}$ are independent because of the strong Markov property (or path decomposition) at last exit times (for a survey see Millar [45]). Figure 1 illustrates the situation.

Our aim is to find an expression of the LT of $\hat{L}_{y\nearrow x}$:

$$E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{y\nearrow x}\right)\right] = \frac{E_0^{\hat{\delta}}\left[\exp\left(-\frac{\alpha^2}{2}\hat{L}_{0\to x}\right)\right]}{E_0^{\hat{\delta}}\left[\exp\left(-\frac{\alpha^2}{2}\hat{L}_{0\to y}\right)\right]}\,.$$

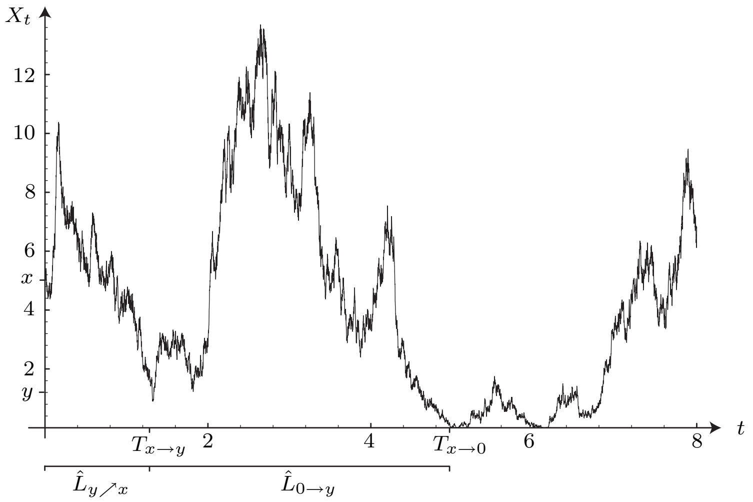

Figure 1: A simulated path of a BES_x^δ process, $\delta < 2$, starting in $x = 5$.

With (15), and with the Sommerfeld integral representation of the Macdonald function $K_{\hat{\nu}} = K_\nu$, also known as the modified Bessel function of the third kind of order $\hat{\nu} = -\nu$, (see e.g. Lebedev [38] p. 119, (5.10.25))

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}} t^{-\nu-1} dt\,, \tag{17}$$

we obtain the LT of $\hat{L}_{0\to y}$:

$$E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{0\to y}\right)\right] = \frac{(\alpha y)^{\hat{\nu}} K_{\hat{\nu}}(\alpha\, y)}{2^{\hat{\nu}-1}\Gamma(\hat{\nu})}.$$

Thus (see also Getoor [26], Getoor and Sharpe [27], Kent [34] (3.7) and Pitman and Yor [53]) for $y < x$, $y \neq 0$:

$$\begin{aligned} E_x^\delta\left[\exp\left(-\tfrac{\alpha^2}{2}T_{x\to y}\right)\right] &= E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{y\nearrow x}\right)\right] \\ &= \left(\frac{x}{y}\right)^{\hat{\nu}} \frac{K_{\hat{\nu}}(\alpha\, x)}{K_{\hat{\nu}}(\alpha\, y)} = \left(\frac{y}{x}\right)^{\nu} \frac{K_\nu(\alpha\, x)}{K_\nu(\alpha\, y)}. \end{aligned}$$

This formula might also have been obtained by looking for a function ϕ such that $\phi(X_t)\exp(-\frac{\alpha^2}{2}t)$ is a local martingale.
Likewise, in the case $0 < x < y$ we have, see Kent [34] (3.8) and Pitman and

Yor [53]):

$$E_x^\delta\left[\exp\left(-\tfrac{\alpha^2}{2}T_{x\to y}\right)\right] = \left(\frac{y}{x}\right)^\nu \frac{I_\nu(\alpha\, x)}{I_\nu(\alpha\, y)}\,,$$

with I_ν the modified Bessel function of the first kind of order ν. Since

$$\lim_{\varepsilon\downarrow 0}\frac{I_\nu(\mu\varepsilon)}{\varepsilon^\nu} = \left(\frac{\mu}{2}\right)^\nu \Gamma^{-1}(\nu+1) \tag{18}$$

we obtain for $x = 0$, $y > 0$

$$E_0^\delta\left[\exp\left(-\tfrac{\alpha^2}{2}T_{0\to y}\right)\right] = \left(\frac{\alpha}{2}\right)^\nu \Gamma^{-1}(\nu+1)\frac{y^\nu}{I_\nu(\alpha y)}\,.$$

We remark that for Brownian motion, that is, in the case $\delta = 1$, we have

$$T_{x\to 0} \overset{(\text{law})}{=} T_{y\to 0} + (\tilde{T}_{(x-y)\to 0}),$$

where $x > y \geq 0$ and $(\tilde{T}_{(x-y)\to 0})$ denotes an independent copy of $T_{x\to y}$. We know the density of $T_{x\to 0}$ for Brownian motion

$$\sigma_x(t) = \frac{x}{\sqrt{2\pi t^3}}\, e^{-\frac{x^2}{2t}}\,,$$

i. e., we know the density of $T_{(x-y)\to 0}$ and hence the density of $T_{x\to y}$. Note that

$$E_x^1\left[\exp\left(-\tfrac{\alpha^2}{2}T_{x\to 0}\right)\right] = \int_0^\infty e^{-\frac{\alpha^2 t}{2}}\sigma_x(t)\,dt = e^{-\alpha x}.$$

2.2 First hitting times of radial Ornstein–Uhlenbeck processes.

Our aim is to find an explicit expression of the LT (12) of $T_{x\to y}$ of a radial Ornstein–Uhlenbeck process R. We pursue an idea we got by studying Breiman [7] and exploit the relation between CIR and BES processes. Breiman considers an Ornstein–Uhlenbeck process $Y(u) = e^{-u}B(\frac{1}{2}e^{2u})$ (our $\frac{1}{2}$ corrects a misprint in [7]) with B standard Brownian motion, $B(1) = 0$, and gives the LT of the first hitting time T_c for Y from the boundaries $\pm c$

$$E(e^{-\alpha T_c}) = \tilde{D}_\alpha^{-1}(c)\,,$$

where

$$\tilde{D}_\alpha(c) = \frac{2^{1-\alpha/2}}{\Gamma(\frac{\alpha}{2})}\int_0^\infty e^{-\frac{t^2}{2}}t^{\alpha-1}\cosh(ct)\,dt\,,$$

see also Borodin and Salminen [6], II.7.2.0.1, p. 429. First hitting times of Ornstein–Uhlenbeck processes are also studied by Ricciardi and Sato [60] and Horowitz [30].

A δ-dimensional radial Ornstein–Uhlenbeck process R_t with parameter $-\lambda$, $R_0 = 0$, can be written as

$$R_t = e^{-\lambda t} X\left(\frac{e^{2\lambda t}-1}{2\lambda}\right), \tag{19}$$

where X is a BES$^\delta$ process, $X_0 = 0$. We are interested in the first time T_x, R_t hits the level $x > 0$. Although some of our next arguments may work for $\lambda < 0$, we restrict ourselves for now to the case $\lambda > 0$. Later, to deal with the case $\lambda < 0$, we shall use Corollary 4. Analogously to Breiman [7] we assume that D_α is a function such that $\{D_\alpha(R_t)e^{-\alpha t},\, t \geq 0\}$ is a martingale with respect to the filtration of $X\left(\frac{e^{2\lambda t}-1}{2\lambda}\right)$. Equivalently, $\{D_\alpha(\frac{1}{\sqrt{2\lambda u+1}}X_u)(2\lambda u+1)^{-\frac{\alpha}{2\lambda}},\, u \geq 0\}$ is a martingale with respect to the filtration of X_u. This means via Itô's formula that $\{D_\alpha(\frac{r}{\sqrt{2\lambda u+1}})(2\lambda u+1)^{-\frac{\alpha}{2\lambda}},\, r \geq 0,\, u \geq 0\}$ solves $\frac{1}{2}\frac{d^2}{dr^2} + \frac{\delta-1}{2r}\frac{d}{dr} + \frac{d}{du} = 0$, i. e., it is a space-time harmonic function with respect to $\frac{1}{2}\frac{d^2}{dr^2} + \frac{\delta-1}{2r}\frac{d}{dr} + \frac{d}{du}$. Further we assume that $(D_\alpha(r), r \leq \rho)$ is bounded for every ρ. Then $\{D_\alpha(R_t)e^{-\alpha t}, t < T_x\}$ is bounded. Applying the optional stopping theorem (see e. g. Revuz and Yor [59], §II.3) we have

$${}^{-\lambda}E_0^\delta[D_\alpha(R_{T_x})e^{-\alpha T_x}] = D_\alpha(0)\,,$$

thus

$${}^{-\lambda}E_0^\delta[e^{-\alpha T_x}] = \frac{D_\alpha(0)}{D_\alpha(x)}.$$

Now we are motivated to find the function D_α explicitly. First, we derive the space-time harmonic functions for Bessel processes.

We know by Widder [71], Theorem 14.1.1, that the most general $\mathbb{R}^+$-valued space-time harmonic function $h(x,t)$ with $x \in \mathbb{R}^n$, $t \geq 0$, with respect to $\frac{1}{2}\Delta_x + \frac{d}{dt}$ is of the form

$$h(x,t) = \int_{\mathbb{R}^n} \exp\left(\xi \cdot x - \frac{|\xi|^2 t}{2}\right) m(d\xi), \tag{20}$$

where m is a positive measure. We assume h is not identically zero. From (20) we obtain the general positive space-time harmonic function $(\tilde{h}(r,t),\, r,\, t \in \mathbb{R}_+)$ with respect to $\frac{1}{2}\frac{d^2}{dr^2} + \frac{n-1}{2r}\frac{d}{dr} + \frac{d}{dt}$ via

$$\tilde{h}(r,t) = \int_{S^{n-1}} h(\theta r, t)\,\sigma(d\theta) = \int_{\mathbb{R}^n}\int_{S^{n-1}} \exp(\xi\cdot\theta r - \frac{|\xi|^2 t}{2})\,\sigma(d\theta)\,m(d\xi), \tag{21}$$

where σ is the uniform probability measure on the unit sphere S^{n-1}. We know the following integral representation for the Bessel function I_ν

$$I_\nu(|\eta|) = \frac{|\frac{\eta}{2}|^\nu}{\Gamma(\nu+1)}\int_{S^{n-1}} \exp(\eta\cdot\theta)\,\sigma(d\theta)\,,$$

$\eta \in \mathbb{R}^n$, $\nu = \frac{n}{2} - 1$, see Lebedev [38], §5.10, and (5.c2) in Pitman and Yor [53], and inserting in (21) we have

$$\tilde{h}(r,t) = \Gamma(\nu+1)\, 2^{\nu} \int_{\mathbb{R}^n} I_\nu(|\xi| r) \exp\left(-\frac{|\xi|^2 t}{2}\right) (|\xi| r)^{-\nu}\, m(d\xi)\,.$$

Substituting $|\xi|$ by $u \in \mathbb{R}_+$ we obtain a space time harmonic function $\tilde{h}$ with respect to a Bessel process with index ν

$$\tilde{h}(r,t) = \Gamma(\nu+1)\, 2^{\nu} \int_0^\infty I_\nu(ur) e^{-\frac{u^2 t}{2}} (ur)^{-\nu}\, \tilde{m}(du)\,.$$

We believe this formula is true for all dimensions, although a simple proof still eludes us. In fact, we only need to find a function D_α such that

$$\begin{aligned} D_\alpha\left(\frac{r}{\sqrt{2\lambda t+1}}\right)(2\lambda t+1)^{-\frac{\alpha}{2\lambda}} &= \tilde{h}(r,t) \\ &= \Gamma(\nu+1)\, 2^{\nu} \int_0^\infty I_\nu(ur) e^{-\frac{u^2 t}{2}} r^{-\nu} f(u)\, du \end{aligned}$$

with an L^1-function $f : \mathbb{R}_+ \to \mathbb{R}_+$; then $\{D_\alpha(R_t)e^{-\alpha t}, t \le T_x\}$ is a bounded martingale and the optional stopping theorem applies.
Let us first look for D_α such that

$$D_\alpha\left(\frac{r}{\sqrt{2\lambda t+1}}\right)(2\lambda t+1)^{-\frac{\alpha}{2\lambda}} = \Gamma(\nu+1)\left(\frac{2}{r}\right)^{\nu} \int_0^\infty I_\nu(ru)\, e^{-\frac{u^2 t}{2}} f(u)\, du\,,$$

equivalently,

$$\begin{aligned} D_\alpha(y) &= (2\lambda t+1)^{\frac{\alpha}{2\lambda}-\frac{\nu}{2}}\, \Gamma(\nu+1)\left(\frac{2}{y}\right)^{\nu} \int_0^\infty I_\nu(\sqrt{2\lambda t+1}\; yu)\, e^{-\frac{u^2 t}{2}} f(u)\, du \\ &= (2\lambda t+1)^{\frac{\alpha}{2\lambda}-\frac{\nu+1}{2}}\, \Gamma(\nu+1)\left(\frac{2}{y}\right)^{\nu} \int_0^\infty I_\nu(\eta y)\, e^{-\frac{\eta^2}{4\lambda}}\, g\left(\frac{\eta}{\sqrt{2\lambda t+1}}\right) d\eta, \qquad (22) \end{aligned}$$

with $g(v) = e^{-\frac{v^2}{4\lambda}} f(v)$. The righthandside of (22) should not depend on t, hence we may choose $g(v) \equiv c_0\, v^{\frac{\alpha}{\lambda}-\nu-1}$, with a constant $c_0 > 0$, or

$$f(v) \equiv c_0\, v^{\frac{\alpha}{\lambda}-\nu-1} e^{-\frac{v^2}{4\lambda}}, \qquad (23)$$

($\in \mathcal{L}^1$). We see that $D_\alpha(\frac{r}{\sqrt{2\lambda t}})(2\lambda t)^{-\frac{\alpha}{2\lambda}} = \tilde{h}(r,t)$, and certainly also $\tilde{h}(r, t+c)$ with a constant c, is a space-time harmonic function with respect to $\frac{1}{2}\frac{d^2}{dr^2} + \frac{\delta-1}{2r}\frac{d}{dr} + \frac{d}{dt}$. Thus

$$\begin{aligned} & D_\alpha\left(\frac{r}{\sqrt{2\lambda t+1}}\right)(2\lambda t+1)^{-\frac{\alpha}{2\lambda}} \\ &\qquad = \Gamma(\nu+1)\left(\frac{2}{r}\right)^{\nu} \int_0^\infty I_\nu(ru)\, e^{-\frac{u^2}{2}\left(t+\frac{1}{2\lambda}\right)} f(u)\, du \end{aligned}$$

with f in (23) and we find

$$D_\alpha(y) = \Gamma(\nu+1)\left(\frac{2}{y}\right)^{\nu} \int_0^\infty I_\nu(\eta y)\, e^{-\frac{\eta^2}{4\lambda}}\, \eta^{\frac{\alpha}{\lambda}-\nu-1}\, d\eta. \tag{24}$$

With (18) we see

$$D_\alpha(0) = \int_0^\infty e^{-\frac{\eta^2}{4\lambda}}\, \eta^{\frac{\alpha}{\lambda}-1} d\eta = 2^{\frac{\alpha}{\lambda}-1}\, \lambda^{\frac{\alpha}{2\lambda}}\, \Gamma(\tfrac{\alpha}{2\lambda})\,.$$

Recalling the definition of the confluent hypergeometric function ϕ

$$\phi(a,b;z) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j}\frac{z^j}{j!},$$

where $b \neq 0, -1, -2, \ldots$ and

$$(r)_0 = 1,\ (r)_j = \frac{\Gamma(r+j)}{\Gamma(r)} = r(r+1)\ldots(r+j-1),$$

$j = 1, 2, \ldots$, see e.g. Lebedev [38] (9.9.1), and of the Whittaker's functions

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}}\, e^{-\frac{1}{2}z}\, \phi(\mu - k + \tfrac{1}{2}, 2\mu+1; z),$$

see e.g. Lebedev [38] (9.13.16), finally, we obtain together with the Bateman Manuscript Project [4], 4.16.(20), the following theorem.

Theorem 1 *The LT of the first time $T_x = \inf\{t \,|\, R_t = x\}$ a δ-dimensional radial Ornstein–Uhlenbeck process R_t starting in 0 with parameter $-\lambda$ hits the level x is*

$$\begin{aligned}
{}^{-\lambda}E_0^{\delta}(e^{-\alpha T_x}) &= \frac{2^{\frac{\alpha}{\lambda}-\nu-1}\, x^{\nu}\, \Gamma(\frac{\alpha}{2\lambda})\, \lambda^{\frac{\alpha}{2\lambda}}}{\Gamma(\nu+1)\displaystyle\int_0^\infty I_\nu(\eta x)\, e^{-\frac{\eta^2}{4\lambda}}\, \eta^{\frac{\alpha}{\lambda}-\nu-1} d\eta} \\
&= \frac{(\sqrt{\lambda}x)^{\,\nu+1}\, e^{-\frac{\lambda x^2}{2}}}{M_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1),\frac{\nu}{2}}\,(\lambda x^2)} \\
&= \frac{1}{\phi(\frac{\alpha}{2\lambda}, \nu+1; \lambda x^2)}\,.
\end{aligned}$$

For $0 < y < x$ we have

$$T_{0\to x} = T_{0\to y} + T_{y\to x}$$

where $T_{0\to y}$ and $T_{y\to x}$ are independent because of the strong Markov property. We deduce

Corollary 3 *The LT of the first time $T_x = \inf\{t \,|\, R_t = x\}$ a δ-dimensional radial Ornstein–Uhlenbeck process R_t starting in y, $0 < y < x$, with parameter $-\lambda$ hits the level x is*

$$\begin{aligned}
{}_{-\lambda}E_y^{\delta}(e^{-\alpha T_x}) &= \left(\tfrac{x}{y}\right)^{\nu+1} e^{\frac{\lambda}{2}(y^2-x^2)}\, \frac{M_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1),\frac{\nu}{2}}\,(\lambda y^2)}{M_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1),\frac{\nu}{2}}\,(\lambda x^2)} \\
&= \frac{\phi(\frac{\alpha}{2\lambda},\nu+1;\lambda y^2)}{\phi(\frac{\alpha}{2\lambda},\nu+1;\lambda x^2)}\,.
\end{aligned}$$

Analogously, we obtain the LT of T_x in the case $0 < x < y$. Note that here we have to use the modified Bessel functions of the third kind K_ν instead of the modified Bessel functions of the first kind I_ν such that the required uniform integrability assumption is satisfied. With the confluent hypergeometric function of the second kind ψ

$$\psi(a,b;z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}\,\phi(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)}\,z^{1-b}\,\phi(1+a-b,2-b;z),$$

see e.g. Lebedev [38], (9.10.3), and with Whittaker's functions

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}}\,e^{-\frac{1}{2}z}\,\psi(\mu-k+\tfrac{1}{2};2\mu+1;z),$$

see e.g. Lebedev [38], (9.13.16), we obtain with Bateman Manuscript Project [4], 4.16.(37), the following theorem.

Theorem 2 *The LT of the first time $T_x = \inf\{t \,|\, R_t = x\}$ a δ-dimensional radial Ornstein–Uhlenbeck process R_t starting in y, $0 < x < y$, with parameter $-\lambda$ hits the level x is*

$$\begin{aligned}
{}_{-\lambda}E_y^{\delta}(e^{-\alpha T_x}) &= \left(\tfrac{x}{y}\right)^{\nu} \frac{\int_0^\infty K_\nu(\eta y)e^{-\frac{\eta^2}{4\lambda}}\,\eta^{\frac{\alpha}{\lambda}-\nu-1}d\eta}{\int_0^\infty K_\nu(\eta x)e^{-\frac{\eta^2}{4\lambda}}\,\eta^{\frac{\alpha}{\lambda}-\nu-1}d\eta} \\
&= \left(\tfrac{x}{y}\right)^{\nu+1} e^{\frac{\lambda}{2}(y^2-x^2)}\, \frac{W_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1),\frac{\nu}{2}}\,(\lambda y^2)}{W_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1),\frac{\nu}{2}}\,(\lambda x^2)} \\
&= \frac{\psi(\frac{\alpha}{2\lambda},\nu+1;\lambda y^2)}{\psi(\frac{\alpha}{2\lambda},\nu+1;\lambda x^2)}\,.
\end{aligned}$$

With

$$K_\nu(x) \sim \frac{\Gamma(-\nu)}{2^{\nu+1}}x^\nu$$

for $x \to 0$ and $\nu < 0$, we obtain immediately

Corollary 4 *The LT of the first time* $T_0 = \inf\{t \,|\, R_t = 0\}$ *a* δ*-dimensional radial Ornstein–Uhlenbeck process* R_t *starting in* y *with* $\delta < 2$ *with parameter* $-\lambda$ *hits* 0 *is*

$$\begin{aligned} {}^{-\lambda}E_y^\delta(e^{-\alpha T_0}) &= \frac{2^{\nu-\frac{\alpha}{\lambda}+2}\, y^{-\nu}\, \lambda^{-\frac{\alpha}{2\lambda}}}{\Gamma\left(\frac{\alpha}{2\lambda}\right)\Gamma(-\nu)} \int_0^\infty K_\nu(\eta y)\, e^{-\frac{\eta^2}{4\lambda}}\, \eta^{\frac{\alpha}{\lambda}-\nu-1} d\eta \\ &= 4\,(\sqrt{\lambda} y)^{-\nu-1}\, e^{\frac{\lambda y^2}{2}}\, \frac{\Gamma(\frac{\alpha}{2\lambda}-\nu)}{\Gamma(-\nu)}\, W_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1),\frac{\nu}{2}}(\lambda y^2) \\ &= 4\,\frac{\Gamma(\frac{\alpha}{2\lambda}-\nu)}{\Gamma(-\nu)}\,\psi(\frac{\alpha}{2\lambda}, \nu+1; \lambda y^2) \end{aligned}$$

For $\delta \geq 2$ *we have* $T_0 = \infty$ *almost surely, i.e.,* ${}^{-\lambda}E_y^\delta(e^{-\alpha T_0}) = 0$.

Remark 2 In the case treated in Corollary 4 we know the density of T_0 by (11).

Remark 3 The Laplace transforms of first hitting times of *squared* δ-dimensional Bessel or radial Ornstein–Uhlenbeck processes are obtained immediately. Moreover, from the investigation of the behaviour of squared Bessel or radial Ornstein–Uhlenbeck processes with negative dimensions or negative starting points, see the introduction and Section 3, we have no difficulties in extending the results above to, e. g., the case $x > 0 > y$.

Remark 4 In disguised versions the formulae above also appear in Pitman and Yor [53], Chapter 12, and Eisenbaum [19].

We now show how to deduce the laws of first hitting times in the case $\lambda < 0$ for our previous formulae.

Theorem 3 *(*$\lambda < 0$*): Let* $\mu = -\lambda > 0$*. The LT of the first time* $T_x = \inf\{t \,|\, R_t = x\}$ *a* δ*-dimensional radial Ornstein–Uhlenbeck process* R_t *starting in* 0 *with parameter* μ *hits the level* x *is*

$$ {}^{\mu}E_0^\delta(e^{-\alpha T_x}) = \frac{e^{\mu x^2}}{\phi(\frac{\alpha+\delta\mu}{2\mu}, \frac{\delta}{2}; \mu x^2)} \,. $$

Theorem 4 *(*$\lambda < 0$*): Let* $\mu = -\lambda > 0$*. The LT of the first time* $T_x = \inf\{t \,|\, R_t = x\}$ *a* δ*-dimensional radial Ornstein–Uhlenbeck process* R_t *starting in* y*,* $0 < x < y$*, with parameter* μ *hits the level* x *is*

$$ {}^{\mu}E_y^\delta(e^{-\alpha T_x}) = e^{\mu(x^2-y^2)}\,\frac{\psi(\frac{\alpha+\delta\mu}{2\mu}, \frac{\delta}{2}; \mu y^2)}{\psi(\frac{\alpha+\delta\mu}{2\mu}, \frac{\delta}{2}; \mu x^2)} \,. $$

The statements of the $\lambda < 0$ version of Corollary 3 and 4 are left to the reader. Proof of Theorems 3 and 4: From Corollary 2 we know that for any $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $f(R_t, t)$ is a ${}^{-\lambda}P_x^\delta$-martingale if and only if $f(R_t, t)\exp(-\lambda(R_t^2 - \delta t))$ is a ${}^{-\mu}P_x^\delta$-martingale. In other terms, a function ${}^{-\lambda}H : \mathbb{R}_+ \to \mathbb{R}_+$ is such that:

$$ {}^{-\lambda}H(r)e^{-\alpha t} \text{ is space time harmonic wrt. the } \left({}^{-\lambda}P_x^\delta\right)_{x\geq 0}\text{-family,} \tag{25} $$

if and only if, ${}^{-\lambda}H(r)e^{\mu r^2}e^{-(\alpha+\delta\mu)t}$ is space time harmonic with respect to the $\left({}^{-\mu}P_x^\delta\right)_{x\geq 0}$-family.
Consequently, if $F_\beta(r)e^{-\beta t}$ is space time harmonic with respect to $\left({}^{-\mu}P_x^\delta\right)_{x\geq 0}$, then:

$$ {}^{-\lambda}H_\alpha(r) = F_{\frac{\alpha+\delta\mu}{2\mu}}(r)e^{-\mu r^2} $$

satisfies (25). Now, Theorems 3 and 4 follow from Theorems 1 and 2 applied with the parameter $\mu(= -\lambda) > 0$. This closes the proof.

Remark 5 We mention another approach to obtain results concerning first hitting times of radial Ornstein–Uhlenbeck processes. Consider a δ-dimensional radial Ornstein–Uhlenbeck process (R_t), starting in $x > 0$, $\delta < 2$. We are interested in $T_y = \inf\{t | R_t = y\}$, $x > y \geq 0$. In analogy to the case of BES processes in Section 2.1 we consider the process (R_t) time reversed, starting from $T_0 = \inf\{t | R_t = 0\}$, see Appendix B. Denote the time reversed process by $(\hat{R}_t)$. With the same notation as in (16) we are interested in $\hat{L}_{x\nearrow y}$, i. e., in the process $(\hat{R}_t)$ after $\hat{L}_{0\to y}$. By using the technique of enlargement of filtration, we can write $(\hat{R}_t)$ after $\hat{L}_{0\to y}$ as a diffusion. For a treatment see e.g. Jeulin [33] and Yor [78], §12. Heuristically spoken, we enlarge the original filtration progressively, so that the last exit time $\hat{L}_{0\to y}$ becomes a stopping time. Applying Theorem 12.4 in Yor [78] we obtain

Proposition 2 *For the diffusion process* $\tilde{R}_u \equiv \hat{R}_{(\hat{L}_{0\to y}+u)}$ *we have*

$$ \tilde{R}_u = y + \int_0^u b(\tilde{R}_v)dv - \int_0^u \frac{s'(\tilde{R}_v)}{s(y) - s(\tilde{R}_v)}\, 1_{(\tilde{R}_v > y)}dv + \tilde{W}_u, \tag{26} $$

where $u \geq 0$*,* b *is the drift and* s *is the scale function of the transient diffusion* $\hat{R}$*.*

As an illustration consider the process $\hat{R}$ to be a transient BES process $\hat{X}$, i.e. a BES process with index $\nu > 0$, started in 0. Its scale function may be chosen as $s(x) = -x^{-2\nu}$ and we obtain from (26)

$$ \tilde{X}_u \equiv \hat{X}_{(\hat{L}_{0\to y}+u)} = y + \int_0^u \frac{1}{\tilde{X}_v}\,\frac{(\nu + \frac{1}{2})\tilde{X}_v^{2\nu} + (\nu - \frac{1}{2})y^{2\nu}}{\tilde{X}_v^{2\nu} - y^{2\nu}}\,dv + \tilde{W}_u. \tag{27} $$

For a $\mathrm{BES}^3(0)$ process $\hat{X}$ (27) reduces to

$$\tilde{X}_u \equiv \hat{X}_{(\hat{L}_{0\to y}+u)} = y + \int_0^u \frac{dv}{\tilde{X}_v - y} + \tilde{W}_u .$$

Hence for a $\mathrm{BES}^3(0)$ process X we have

$$(X_{L_y+u} - y,\, u \geq 0) \overset{(\text{law})}{=} (X_u,\, u \geq 0)\, .$$

In general, the transition density $\tilde{p}$ of the diffusion $\tilde{R}$ in (26) is unknown. Note that if it were known, we would obtain the density of the last hitting time of x by the process $\tilde{R}$ immediately from the formula

$$P_y(\tilde{L}_x \in dt) = -\frac{1}{2s(x)}\, \tilde{p}_t(y,x)\, dt\, ,$$

see Borodin and Salminen [6] IV.43, Revuz and Yor [59] VII.(4.16), where $\tilde{R}_0 = y$ and s is the scale function of $\tilde{R}$ with $\lim_{a\downarrow 0} s(a) = -\infty$ and $s(\infty) = 0$.

Remark 6 As motivated in the end of Section 1 we are interested in the resolvent of radial Ornstein–Uhlenbeck processes. We obtain for the resolvent (see (6)) with $f(r) \equiv (r-k)^+$

$$E_x[\int_0^\infty e^{-\alpha t} f(R_t) dt] = \int_0^\infty f(r) \int_0^\infty e^{-\alpha t} p_t(x,r)\, dt\, dr$$

with p_t the transition density of R_t, leaving us with the computation of $\int_0^\infty e^{-\alpha t} p_t(x,y)\, dt$. First, we consider BES^δ_x processes and obtain with Appendix A.2

$$\int_0^\infty e^{-\alpha t}\, \frac{1}{t}\, y \left(\frac{y}{x}\right)^\nu \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right) dt = 2\left(\frac{y}{x}\right)^\nu y\, I_\nu(z_1)\, K_\nu(z_2),$$

where

$$z_1 = \sqrt{2\alpha}\, \min(x,y)$$

and

$$z_2 = \sqrt{2\alpha}\, \max(x,y),$$

see Oberhettinger and Badii [52] (I.1.15.55). For a treatment of the resolvent for Bessel processes we refer to Itô and McKean [31], and also to De Schepper *et al.* [11] and Alili and Gruet [1]. Using relation (19) we obtain for a δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t}\, 2y\, \lambda\, \frac{e^{2\lambda t}}{e^{2\lambda t}-1} \left(\frac{e^{\lambda t} y}{x}\right)^\nu \exp\left(-\lambda \frac{x^2 + e^{2\lambda t} y^2}{e^{2\lambda t}-1}\right) I_\nu\left(\frac{2\lambda x y e^{\lambda t}}{e^{2\lambda t}-1}\right) dt \\
&= y \left(\frac{y}{x}\right)^\nu \int_0^\infty \frac{(u+1)^{\frac{1}{2}(\nu - \frac{\alpha}{\lambda})}}{u} \exp\left(-\lambda \frac{x^2 + (u+1) y^2}{u}\right) I_\nu\left(\frac{2\lambda x y \sqrt{u+1}}{u}\right) du.
\end{aligned}$$

3 BESQ processes with negative dimensions and extensions

Bessel processes with nonnegative dimension $\delta \geq 0$ and starting point $x \geq 0$ are well-studied, e.g. Revuz and Yor [59], Chapter XI. As already pointed out in the introduction it seems to be quite natural also to consider Bessel processes with negative dimensions or negative starting points. Therefore we are motivated to extend the definition of BESQ_x^δ processes and to introduce the class of BESQ_x^δ processes with arbitrary δ, $x \in \mathbb{R}$.

Definition 3 *The solution to the stochastic differential equation*

$$dX_t = \delta\, dt + 2\sqrt{|X_t|}\, dW_t\,, \quad X_0 = x\,, \tag{28}$$

where $\{W_t\}$ is a one-dimensional Brownian motion, $\delta \in \mathbb{R}$ and $x \in \mathbb{R}$, is called the square of a δ-dimensional Bessel process*, starting in x, and is denoted by* BESQ_x^δ.

Moreover, we generalize the definition of a CIR process, i.e., a squared δ-dimensional radial Ornstein-Uhlenbeck process with parameter $-\lambda$ in (7), by allowing the starting point and δ to be in $\mathbb{R}$.

Definition 4 *The solution to the stochastic differential equation*

$$dX_t = (\delta + 2\lambda\, X_t)\, dt + 2\sqrt{|X_t|}\, dW_t, \quad X_0 = x\,, \tag{29}$$

where δ, λ, $x \in \mathbb{R}$ and $\{W_t\}$ is a one-dimensional Brownian motion, is called a squared δ-dimensional radial Ornstein–Uhlenbeck process with parameter λ.

First, we will derive and discuss properties of BESQ_x^δ processes (28) with $\delta, x \in \mathbb{R}$, and as an extension we study CIR processes (29) in Section 3.1.
As mentioned in the introduction equation (28) is not the only possible way of defining BESQ^δ processes with $\delta \in \mathbb{R}$. We will discuss this point after investigating the behaviour of BESQ^δ processes with $\delta \in \mathbb{R}$ defined by (28). Equation (28) has a unique strong solution (Revuz and Yor [59], Chapter IX §3). Denote its law by Q_x^δ. First, we want to investigate the behaviour of a BESQ_x^δ process, starting in $x > 0$ with dimension $\delta \leq 0$. In the case $\delta = 0$ the process reaches 0 in finite time and stays there. As for the case $\delta < 0$, we deduce from the comparison theorem that 0 is also reached in finite time. Let us consider the behaviour of a BESQ_x^δ process $\{X_t\}$ with $\delta < 0$ and $x > 0$ after it reached 0; we find:

$$\tilde{X}_u \equiv X_{T_0+u} = \delta\, u + 2\int_{T_0}^{T_0+u} \sqrt{|X_s|}\, dW_s\,, \quad u \geq 0\,, \tag{30}$$

where $T_0 = \inf\{t \,|\, X_t = 0\}$ denotes the first time the process $\{X_t\}$ hits 0. With the notation $\gamma \equiv -\delta$ we obtain from (30)

$$-\tilde{X}_u = \gamma\, u + 2\int_0^u \sqrt{|\tilde{X}_s|}\, d\tilde{W}_s\,, \quad u \geq 0\,,$$

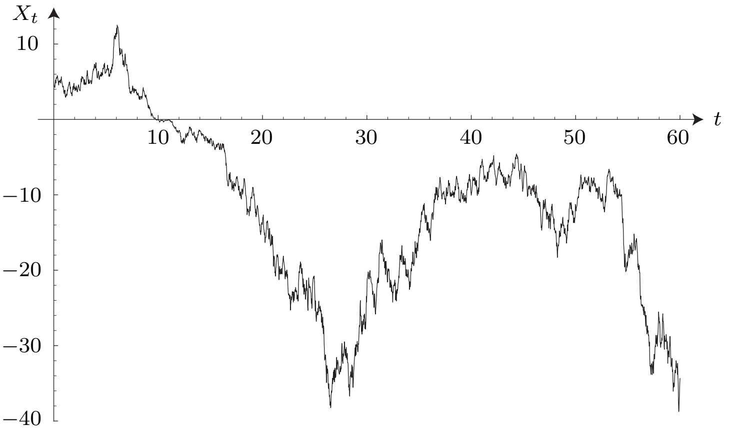

Figure 2: A simulated path of a $\mathrm{BES}^{-\gamma}$ process with $\gamma \geq 2$, $x_0 = 5$.

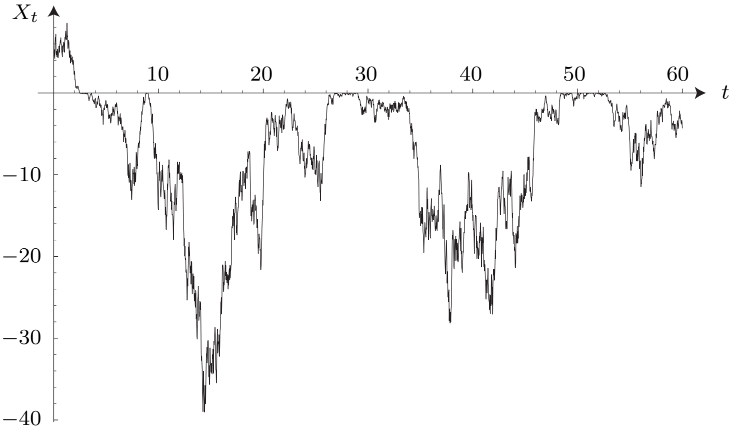

Figure 3: A simulated path of a $\mathrm{BES}^{-\gamma}$ process with $\gamma < 2$, $x_0 = 5$.

where $\tilde{W}_s \equiv -(W_{s+T_0} - W_{T_0})$, that is, after the $\mathrm{BESQ}_x^{-\gamma}$ process $\{X_t\}$ hits 0, it behaves as a $-\mathrm{BESQ}_0^{\gamma}$ process. Two simulated paths of a $\mathrm{BES}_x^{-\gamma}$ process with $\gamma \geq 2$, respectively $\gamma < 2$, are shown in Figures 2 and 3.

From the above discussion we deduce that a BESQ_x^{δ} process with $\delta < 0$ and $x \leq 0$ behaves as $[-\mathrm{BESQ}_{-x}^{-\delta}]$, especially it never becomes positive. For a BESQ_x^{δ} process with dimension $\delta \geq 0$ and starting point $x \leq 0$, we obtain with the same argument, that it behaves as a $-\mathrm{BESQ}_{-x}^{-\delta}$ process; this means, until it hits 0 for the first time it behaves as a $-\mathrm{BESQ}_{-x}^{-\delta}$ process, and after that it behaves as a BESQ_0^{δ} process.

Let us now pursue another way of extending Definition 1 to BESQ^{δ} processes with $\delta \in \mathbb{R}$. Instead of a BESQ^{δ} process in (28) with diffusion coefficient $\sigma(x) = 2\sqrt{|x|}$ we consider the process

$$X_t = x_0 - \delta t + 2 \int_0^t \sqrt{\alpha X_s^+ + \beta X_s^-}\, dB_s \,,$$

where α, $\beta \geq 0$, $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, and as before $x_0 > 0$, $\delta > 0$. For $t \leq T_0$, we have

$$X_t = x_0 - \delta t + 2 \int_0^t \sqrt{\alpha X_s}\, dB_s \,. \tag{31}$$

After T_0 the process evolves as

$$X_{T_0+t} = -\delta t + 2 \int_{T_0}^{T_0+t} \sqrt{\alpha X_s^+ + \beta X_s^-}\, dB_s \,,$$

and with $Y_t \equiv -X_{T_0+t}$ we have

$$Y_t = \delta t + 2 \int_0^t \sqrt{\alpha Y_s^- + \beta Y_s^+}\, d\tilde{B}_s \,.$$

This admits only one solution which is the positive process

$$Y_t = \delta t + 2 \int_0^t \sqrt{\beta Y_s}\, d\tilde{B}_s \,. \tag{32}$$

Note that the parameter β in (32), as well as the parameter α in (31), can be thought of as coming from a time transformation of a BESQ^{γ} process Z

$$Z_t = \gamma t + 2 \int_0^t \sqrt{Z_s}\, dB_s \,,$$

since

$$Z_{c^2 t} = \gamma c^2 t + 2 \int_0^t \sqrt{c^2 Z_{c^2 s}}\, d\bar{B}_s \,.$$

After this remark let us now continue to investigate properties of processes defined by (28). An important and well-known property of squared Bessel processes with nonnegative dimensions is their additivity property, see Appendix A.4. The additivity property is no longer true for BESQ^{δ} processes with $\delta \in \mathbb{R}$ arbitrary. Consider the BESQ_x^{β} process Z^{β} and the $\mathrm{BESQ}_y^{\tilde{\beta}}$ process $Z^{\tilde{\beta}}$, where

$\beta > 0$, $\tilde{\beta} \equiv -\gamma < 0$ with $\beta \geq \gamma$, $x \geq 0$ and $y \leq 0$. Assuming that the additivity property holds, would yield:

$$Z^{\beta} + Z^{\tilde{\beta}} \overset{\text{(law)}}{=} Z^{\beta} - Z^{\gamma} \overset{\text{(law)}}{=} Z^{\beta-\gamma} \geq 0,$$

since $\beta \geq \gamma$, which cannot be true because of the independence of Z^{β} and Z^{γ}. Our aim is to find the semigroup of a $\mathrm{BESQ}_x^{-\gamma}$ process $\{X_t\}$ with $\gamma \geq 0$, $x \geq 0$. Our result is the following

Proposition 3 *The semigroup of a BESQ*$_x^{-\gamma}$ *process,* $\gamma > 0$, $x \geq 0$, *is given by:*

$$Q_x^{-\gamma}(X_t \in dy) = q_t^{-\gamma}(x,y)dy\,,$$

with

$$q_t^{-\gamma}(x,y) = q_t^{4+\gamma}(y,x)1_{(y>0)} + \int_0^t (\gamma+2)\, q_s^{4+\gamma}(0,x)\, q_{t-s}^{\gamma}(0,-y)1_{(y<0)}\, ds\,.$$

Proof: a) We decompose the process (X_t) before and after T_0, its first hitting time of 0; we obtain

$$E_x^{-\gamma}[f(X_t)] = A + B\,,$$

where

$$\begin{aligned} A &= E_x^{-\gamma}[f(X_t)\,1_{(t<T_0)}]\,, \\ B &= E_x^{-\gamma}[f(X_t)\,1_{(T_0<t)}] = E_x^{-\gamma}[f(X_{T_0+(t-T_0)})\,1_{(T_0<t)}] \\ &= \int_0^t Q_x^{-\gamma}(T_0 \in ds) \int_{-\infty}^0 f(y)\, q_{t-s}^{\gamma}(0,-y)dy\,, \end{aligned}$$

since $(-X_{(T_0+u)}, u \geq 0)$ is distributed as a BESQ_0^{γ} process.
b) We shall compute A with either of the following arguments:

(b.1) We use the absolute continuity relationship:

$$Q_x^{-\gamma}\big|_{\mathcal{F}_t \cap (t<T_0)} = \left(\frac{X_t}{x}\right)^{-\nu} \cdot Q_x^{4+\gamma}\big|_{\mathcal{F}_t}\,,$$

where $4+\gamma = 2(1+\nu)$, and we note, from formula (49), that:

$$\left(\frac{y}{x}\right)^{-\nu} q_t^{4+\gamma}(x,y) = q_t^{4+\gamma}(y,x)\,.$$

Thus, we obtain:

$$A = \int_0^{\infty} f(y)\, q_t^{4+\gamma}(y,x)\, dy\,. \tag{33}$$

(b.2) The following time reversal argument shall confirm formula (33). First, we note

$$\begin{aligned} E_x^{-\gamma}[f(X_t)\, 1_{(t<T_0)}] &= E_0^{4+\gamma}[f(X_{L_x - t})\, 1_{(t<L_x)}] \\ &= \int_t^\infty q_x(s)\, E_0^{4+\gamma}[f(X_{s-t}) |\, X_s = x]\, ds\,, \quad (34) \end{aligned}$$

where $q_x(s)\, ds \equiv Q_0^{4+\gamma}(L_x \in ds)$. Then, standard Markovian computations show that:

$$E_0^{4+\gamma}[f(X_{s-t}) |\, X_s = x] = \int f(y) \frac{q_{s-t}^{4+\gamma}(0,y)\, q_t^{4+\gamma}(y,x)}{q_s^{4+\gamma}(0,x)}\, dy\,. \tag{35}$$

Moreover, we find, by comparison of (15) and (50), say, that:

$$q_x(s) = (2+\gamma)\, q_s^{4+\gamma}(0,x)\,. \tag{36}$$

Putting together (34), (35) and (36), we obtain:

$$E_x^{-\gamma}[f(X_t)\, 1_{(t<T_0)}] = \int_t^\infty (2+\gamma) \int f(y)\, q_{s-t}^{4+\gamma}(0,y)\, q_t^{4+\gamma}(y,x)\, dy\, ds\,.$$

Using Fubini's theorem, integrating in (ds), and using (36) with y instead of x, we obtain finally:

$$A = \int_0^\infty f(y) q_t^{4+\gamma}(y,x)\, dy\,,$$

that is, formula (33).

c) The computation of B is done with the formula $Q_x^{-\gamma}(T_0 \in ds) = q_s(s)ds$, followed by the use of formula (36).
This closes the proof.
In order to make the semigroup formula $q_t^{-\gamma}(x,y)$ more explicit, we need to compute

$$\int_0^t q_s^{4+\gamma}(0,x)\, q_{t-s}^{\gamma}(0,\bar{y})\, ds\,,$$

where $\bar{y} = -y$, for $y < 0$. Elementary computations yield:

$$q_t^{(-\gamma)}(x,y) = k(x,y,\gamma,t) e^{-\alpha-\beta} \int_0^\infty \frac{(w+1)^{2m}}{w^m} \exp\left(-\beta w - \frac{\alpha}{w}\right) dw\,, \tag{37}$$

with

$$k(x,y,\gamma,t) \equiv \Gamma^{-2}\left(\frac{\gamma}{2}\right) \frac{2^{-\gamma}}{\gamma}\, x^{\frac{\gamma}{2}+1}\, |y|^{\frac{\gamma}{2}-1}\, t^{-\gamma-1},$$

and

$$m \equiv \frac{\gamma}{2}\,,\ \alpha \equiv \frac{|y|}{2t}\,,\ \beta \equiv \frac{x}{2t}\,.$$

We expand formula (37) for $\gamma \in \mathbb{N}$ as follows:

$$\begin{aligned} q_t^{(-\gamma)}(x,y) &= k(x,y,\gamma,t)\, e^{-\alpha-\beta} \sum_{k=0}^{2m} \binom{2m}{k} \int_0^\infty w^{m-k} \exp\left(-\beta w - \frac{\alpha}{w}\right) dw \\ &= k(x,y,\gamma,t)\, e^{-\alpha-\beta}\, 2 \sum_{k=0}^{2m} \binom{2m}{k} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}(k-m-1)} K_{k-m-1}(2\sqrt{\alpha\beta})\,, \end{aligned}$$

where $K_\nu(z)$ denotes the Macdonald function with index ν, see formula (17), taken from Lebedev [38], (5.10.25).
As a test on our above description of the law $Q_x^{-\gamma}$ of a $\mathrm{BESQ}_x^{-\gamma}$ process $(Z_t)_{t\geq 0}$ with $\gamma, x > 0$, we now make some computations involving linear functionals of this process; see also Revuz and Yor [59] (Exercise (XI.1.34)), and for an improvement of this exercise see Föllmer *et al.* [22]. Since $(-Z_{T_0+t}, t \geq 0)$ is a BESQ_0^γ process, independent of the past of (Z_t) up to T_0, we have

$$\begin{aligned} Q_x^{-\gamma}\left[\exp(-\int_0^\infty Z_u\, f(u)\, du)\right] = \int_0^\infty Q_x^{-\gamma}\left[\exp\,(-\int_0^t Z_u\, f(u)\, du)\middle|\, T_0 = t\right] \\ \times Q_0^\gamma\left[\exp(\int_0^\infty X_v\, f(t+v)\, du)\right]\, Q_x^{-\gamma}[T_0 \in dt], \end{aligned}$$

with a BESQ_0^γ process $(X_v)_{v\geq 0}$ and a positive Borel function f. In the following, negative dimensional BESQ processes are denoted by (Z_t) and positive dimensional ones by (X_t). We obtain

$$Q_x^{-\gamma}\left[\exp\,(-\int_0^t Z_u\, f(u)\, du)\,\middle|\, T_0 = t\right] = Q_0^{4+\gamma}\left[\exp\,(-\int_0^t X_u\, f(t-u)\, du)\,\middle|\, L_x = t\right]$$

and with the equalities

$$\begin{aligned} & Q_0^{4+\gamma}\left[\exp\,(-\int_0^t X_u\, f(t-u)\, du)\,\middle|\, L_x = t\right] \\ &= Q_0^{4+\gamma}\left[\exp\,(-\int_0^t X_u\, f(t-u)\, du)\,\middle|\, X_t = x\right] \\ &= Q_x^{4+\gamma}\left[\exp\,(-\int_0^t X_{t-u}\, f(t-u)\, du)\,\middle|\, X_t = 0\right] \\ &= Q_x^{4+\gamma}\left[\exp\,(-\int_0^t X_u\, f(u)\, du)\,\middle|\, X_t = 0\right]\,, \end{aligned}$$

finally, we have

$$Q_x^{-\gamma}\left[\exp(-\int_0^\infty Z_u\, f(u)\, du)\right] = \int_0^\infty Q_x^{4+\gamma}\left[\exp\,(-\int_0^t X_u\, f(u)\, du)\,\middle|\, X_t = 0\right] \\ \times Q_0^\gamma\left[\exp(\int_0^\infty X_v\, f(t+v)\, dv)\right]\, q_x(t)\, dt\,. \quad (38)$$

Our aim is to determine

$$Q_x^{4+\gamma}\left[\exp\,(-\int_0^t X_u\, f(u)\, du)\,\middle|\, X_t = 0\right]$$

and

$$Q_0^\gamma\left[\exp(\int_0^\infty X_v\, f(t+v)\, dv)\right]$$

more explicitly. Using the well-known fact that for a function $h : \mathbb{R}^+ \to \mathbb{R}$ in L^1 with $h(x) \leq c$ for all $x \in \mathbb{R}^+$ and continuous in a neighbourhood of 0:

$$\lim_{\lambda\to\infty}\int_0^\infty \lambda e^{-\lambda y} h(y)dy = h(0)\,,$$

we deduce that the ratio

$$\frac{Q_x^\delta(\exp(-\lambda X_t - \int_0^t X_u\, f(u)\, du))}{Q_x^\delta(\exp(-\lambda X_t))} \\ = \frac{\int_0^\infty e^{-\lambda y} q_t^\delta(x,y) Q_x^\delta(\exp(-\int_0^t X_u\, f(u)\, du)|X_t = y)\, dy}{\int_0^\infty e^{-\lambda y} q_t^\delta(x,y)\, dy}, \quad (39)$$

as λ tends to infinity, converges to

$$Q_x^\delta\left(\exp(-\int_0^t X_u\, f(u)\, du)|X_t = 0\right)\,.$$

Let us consider the numerator and denominator of ratio (39) separately. Via Pitman and Yor [55] (formula (*1.h*), and Revuz and Yor [59] (Theorem (XI.1.7)) we know

Lemma 1 *Consider a positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $\int_0^n f(s)ds < \infty$ for all n. Then*

$$Q_x^\delta\left[\exp(-\int_0^t X_u f(u)du - \frac{\lambda}{2}X_t)\right] \\ = (\psi'_f(t) + \lambda\psi_f(t))^{-\delta/2}\exp\frac{x}{2}\left[\phi'_f(0) - \frac{(\phi'_f + \lambda\phi_f)(t)}{(\psi'_f + \lambda\psi_f)(t)}\right], (40)$$

where ϕ_f *is the unique solution of the Sturm-Liouville equation*

$$\frac{1}{2}\phi_f''(s) = f(s)\phi_f(s),$$

with $s \in (0,\infty)$, $\phi_f(0) = 1$, *which is positive and non increasing, and* $\phi_f'(0)$ *is the right derivative in 0, and*

$$\psi_f(t) \equiv \phi_f(t)\int_0^t \frac{ds}{\phi_f^2(s)}\,.$$

Furthermore

$$Q_x^\delta\left(\exp(-\int_0^\infty X_u f(u)du)\right) = \phi_f(\infty)^{\delta/2}\exp\left(\frac{x}{2}\phi_f'(0)\right), \qquad (41)$$

where $\phi_f(\infty) \in [0,1]$ *is the limit at infinity of* $\phi_f(s)$.

From the formula following Corollary (XI.1.3) in Revuz and Yor [59] we know

$$Q_x^\delta(\exp(-\frac{\lambda}{2}X_t)) = (1+\lambda t)^{-\frac{\delta}{2}}\exp\left(\frac{-\frac{\lambda}{2}x}{1+\lambda t}\right),$$

and together with (40) we obtain

$$\begin{aligned} &Q_x^\delta(\exp(-\int_0^t X_u\, f(u)\, du)|X_t = 0) \\ &\qquad = \left(\frac{t}{\psi_f(t)}\right)^{\delta/2}\exp\left(\frac{x}{2}(\phi_f'(0) - \frac{\phi_f(t)}{\psi_f(t)} + \frac{1}{t})\right). \qquad (42) \end{aligned}$$

Now we will determine $Q_x^\delta(\exp(\int_0^\infty X_s f(t+s)ds))$ more explicitly. Note that the function

$$\phi_t(s) \equiv \frac{\phi_f(t+s)}{\phi_f(t)}$$

solves

$$\frac{1}{2}\phi_t''(s) = f(t+s)\phi_t(s),$$

and that formula (41) leads to

$$\begin{aligned} Q_x^\delta\left(\exp - \int_0^\infty X_s f(t+s)ds)\right) &= \phi_t(\infty)^{\delta/2}\exp\left(\frac{x}{2}\phi_t'(0)\right) \\ &= \left(\frac{\phi_f(\infty)}{\phi_f(t)}\right)^{\delta/2}\exp\left(\frac{x}{2}\frac{\phi_f'(t)}{\phi_f(t)}\right). \end{aligned}$$

Lemma 2 *For a positive decreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that*

$$Q_x^\delta \left(\exp \int_0^\infty X_s f(s) ds \right) < \infty, \tag{43}$$

there exists a function $\tilde{\phi}$ which is on $(0, \infty)$ the unique solution of

$$\frac{1}{2} \tilde{\phi}''(s) = -f(s) \tilde{\phi}(s),$$

with $\tilde{\phi}(0) = 1$ such that

$$Q_x^\delta \left(\exp \int_0^\infty X_u f(t+u) du \right) = \left(\frac{\tilde{\phi}(\infty)}{\tilde{\phi}(t)} \right)^{\delta/2} \exp \left(\frac{x}{2} \frac{\tilde{\phi}'(t)}{\tilde{\phi}(t)} \right).$$

Proof: Concerning verifications of condition (43) we refer to McGill [43]. We want to determine functions $\alpha(t)$ and $\beta(t)$ with

$$Q_x^\delta \left(\exp \int_0^\infty X_u f(t+u) du \right) = (\beta(t))^{\delta/2} \exp(x \alpha(t)). \tag{44}$$

Using the Markov property we have

$$\begin{aligned} & Q_x^\delta \left(\exp \int_0^\infty X_u f(u) du | \mathcal{F}_t \right) \\ & \qquad = \exp \left(\int_0^t X_u f(u) du \right) Q_{X_t}^\delta \left(\exp \int_0^\infty X_u f(t+u) du \right). \end{aligned}$$

Denote

$$h(x, t; \omega) \equiv (\beta(t))^{\delta/2} \exp(x \alpha(t)) \exp \left(\int_0^t X_u f(u) du \right).$$

Since $Q_x^\delta \left(\exp \int_0^\infty X_u \, f(u) \, du | \mathcal{F}_t \right)$ is a martingale, with Itô's formula we see that the functions α and β have to fulfill the following condition in terms of h

$$\frac{\partial}{\partial t} h(x,t) + 2x \frac{\partial^2}{\partial x^2} h(x,t) + \delta \frac{\partial}{\partial x} h(x,t) = 0.$$

This implies

$$\alpha'(t) + f(t) + 2\alpha^2(t) = 0, \; \alpha(t) + \frac{\beta'(t)}{2\beta(t)} = 0.$$

Hence

$$\alpha(t) = -\frac{\beta'(t)}{2\beta(t)}, \; f(t) = \frac{\beta''(t)}{2\beta(t)} - \left(\frac{\beta'(t)}{\beta(t)} \right)^2.$$

Consider the function $\tilde{\phi} : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\tilde{\phi}(s) \equiv \frac{\beta(0)}{\beta(s)}.$$

Then $\tilde{\phi}$ is on $(0,\infty)$ the unique solution of

$$\frac{1}{2}\tilde{\phi}''(t) = -f(t)\tilde{\phi}(t),$$

with $\tilde{\phi}(0) = 1$. We obtain

$$\alpha(t) = \frac{\tilde{\phi}'(t)}{2\tilde{\phi}(t)}.$$

From (44) we deduce by dominated convergence that $\lim_{t\to\infty}\beta(t) = 1$, and hence $\tilde{\phi}(\infty) \equiv \lim_{t\to\infty}\tilde{\phi}(t) = \beta(0)$.
Altogether we obtain for (38) with (42) and Lemma 2

Theorem 5

$$\begin{aligned} Q_x^{-\gamma}\left(\exp(-\int_0^\infty Z_u f(u)du)\right) &= \frac{1}{\Gamma(\frac{\gamma}{2}+1)}\left(\frac{x}{2}\right)^{\frac{\gamma}{2}+1} \\ \times \int_0^\infty \left(\frac{1}{\psi_f(t)}\right)^{\frac{\gamma}{2}+2} &\left(\frac{\tilde{\phi}(\infty)}{\tilde{\phi}(t)}\right)^{\gamma/2} \exp\left(\frac{x}{2}\left(\phi_f'(0) - \frac{\phi_f(t)}{\psi_f(t)}\right)\right)\, dt\,, \end{aligned}$$

with ϕ_f and ψ_f from Lemma 1, and $\tilde{\phi}$ from Lemma 2.

Example: Consider

$$f(u) \equiv \frac{\lambda^2}{2}\, 1_{[0,a]}(u)\,, \quad a,\, \lambda > 0\,.$$

We know (Pitman and Yor [54], p. 432 (2.m))

$$\begin{aligned} &Q_x^{4+\gamma}\left(\exp(-\frac{\lambda^2}{2}\int_0^t X_u du)\middle|\, X_t = 0\right) \\ &\qquad = \left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{4+\gamma}{2}} \exp\left(-\frac{x}{2t}(\lambda t\coth(\lambda t) - 1)\right)\,, \end{aligned}$$

and hence we have

$$\begin{aligned} Q_x^{-\gamma}\left(\exp(-\frac{\lambda^2}{2}\int_0^a Z_t\, dt)\right) &= \int_0^a \left(\tfrac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{4+\gamma}{2}} \exp\left(-\frac{x}{2t}(\lambda t\coth(\lambda t) - 1)\right) \\ &\quad \times Q_0^{\gamma}\left(\exp(\frac{\lambda^2}{2}\int_0^{a-t} X_v dv)\right) q_x(t)\, dt \\ &\quad + \int_a^\infty Q_x^{4+\gamma}\left(\exp(-\frac{\lambda^2}{2}\int_0^a X_u du)\middle|\, X_t = 0\right) q_x(t)\, dt. \end{aligned}$$

We investigate more deeply

$$Q^{4+\gamma}_{\substack{x\to 0\\ (t)}}\left(\exp(-\frac{\lambda^2}{2}\int_0^a X_u du)\right) \equiv Q_x^{4+\gamma}\left(\exp(-\frac{\lambda^2}{2}\int_0^a X_u du)\middle|\, X_t = 0\right)\,,$$

where a continuous process with law $Q^{4+\gamma}_{\underset{(t)}{x\to 0}}$ is called the *Squared Bessel Bridge* from x to 0 over $[0,t]$, see e.g. Revuz and Yor [59], XI.3. For $\mathcal{A}\in\mathcal{F}_a$ we have

$$\begin{aligned}
Q^{4+\gamma}_{\underset{(t)}{x\to 0}}(\mathcal{A}) &= \lim_{y\downarrow 0}\frac{E^{4+\gamma}_x\left(1_{\mathcal{A}}1_{[0,y]}(X_t)\right)}{Q^{4+\gamma}_x(X_t\in[0,y])}\\
&= \lim_{y\downarrow 0}\frac{E^{4+\gamma}_x\left(1_{\mathcal{A}}E^{4+\gamma}_x(1_{[0,y]}(X_t)|\mathcal{F}_a)\right)}{Q^{4+\gamma}_x(X_t\in[0,y])}\\
&= \lim_{y\downarrow 0}E^{4+\gamma}_x\left(1_{\mathcal{A}}\frac{Q^{4+\gamma}_{X_a}(X_{t-a}\in[0,y])}{Q^{4+\gamma}_x(X_t\in[0,y])}\right)\,,
\end{aligned}$$

and

$$\lim_{y\downarrow 0}\frac{Q^{4+\gamma}_{X_a}(X_{t-a}\in[0,y])}{Q^{4+\gamma}_x(X_t\in[0,y])}=\frac{q^{4+\gamma}_{t-a}(X_a,0)}{q^{4+\gamma}_t(x,0)}\equiv\psi(a,X_a)\,,$$

where $q^\delta_t(x,y)$ is the transition density of BESQ$^\delta$, $\delta>0$. From Yor [77] (proof of the theorem in 0.5) we obtain

$$\lim_{y\downarrow 0}\frac{q^{4+\gamma}_{t-a}(X_a,y)}{q^{4+\gamma}_t(x,y)}=\left(\frac{t}{t-a}\right)^{\frac{4+\gamma}{2}}\exp\left(-\frac{X_a}{2(t-a)}\right)\exp\left(\frac{x}{2t}\right)\,.$$

Formula (2.k) in Pitman and Yor [54] gives us

$$\begin{aligned}
&Q^{4+\gamma}_x\left(\exp\left(-\frac{X_a}{2(t-a)}-\frac{\lambda^2}{2}\int_0^a X_u\,du\right)\right)\\
&=\left(\cosh(\lambda a)+\frac{1}{\lambda(t-a)}\sinh(\lambda a)\right)^{-\frac{4+\gamma}{2}}\exp\left(-\frac{\lambda x}{2}\cdot\frac{1+\frac{\coth(\lambda a)}{\lambda(t-a)}}{\coth(\lambda a)+\frac{1}{\lambda(t-a)}}\right)\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
Q^{4+\gamma}_{\underset{(t)}{x\to 0}}\left(\exp(-\frac{\lambda^2}{2}\int_0^a X_u du)\right) &= \left(\left(\frac{t-a}{t}\right)\cosh(\lambda a)+\frac{1}{\lambda t}\sinh(\lambda a)\right)^{-\frac{4+\gamma}{2}}\\
&\quad\cdot\exp\left(\frac{x}{2t}-\frac{\lambda x}{2}\frac{1+\frac{\coth(\lambda a)}{\lambda(t-a)}}{\coth(\lambda a)+\frac{1}{\lambda(t-a)}}\right)\,.
\end{aligned}$$

If in addition $\lambda(a-t)<\frac{\pi}{2}$, we obtain

$$Q^{\gamma}_0\left(\exp(\frac{\lambda^2}{2}\int_0^{a-t}X_v dv)\right)=\cos(\lambda(a-t))^{-\frac{\gamma}{2}}\,,$$

and finally,

$$Q^{-\gamma}_x\left(\exp(-\frac{\lambda^2}{2}\int_0^a Z_t dt)\right)$$

$$= \frac{1}{\Gamma(\frac{\gamma}{2}+1)} \left(\frac{x}{2}\right)^{\frac{\gamma}{2}+1} \left[\lambda^{\frac{\gamma}{2}+2} \int_0^a \exp\left(-\frac{x}{2}\lambda\coth(\lambda t)\right) \frac{\cos(\lambda(a-t))^{-\frac{\gamma}{2}}}{\sinh(\lambda t)^{\frac{\gamma}{2}+2}}\, dt \right.$$
$$\left. + \int_0^\infty [\cosh(\lambda a)\, u + \frac{1}{\lambda}\sinh(\lambda a)]^{-\frac{\gamma}{2}-2} \exp\left(-\frac{\lambda x}{2}\,\frac{\lambda\, u + \coth(\lambda a)}{\lambda\coth(\lambda a)\, u + 1}\right) du \right].$$

3.1 Extension to squared radial Ornstein–Uhlenbeck processes.

As an extension to BESQ_x^δ processes in (28) with δ, $x \in \mathbb{R}$ we now investigate squared δ-dimensional radial Ornstein–Uhlenbeck processes, i.e., CIR processes, defined by (29) with δ, $x \in \mathbb{R}$. We consider the case $\delta < 0$ and $x > 0$; for $\lambda = 0$ this corresponds to a BESQ_x^δ process with negative dimension δ. We call ${}^\lambda Q_x^\delta$ the law on $C(\mathbb{R}_+, \mathbb{R})$ and, as before, we denote $Q_x^\delta \equiv {}^0 Q_x^\delta$. Via Girsanov transformation we obtain the relationship

$${}^\lambda Q_x^\delta|_{\mathcal{F}_t} = \exp\left(\lambda \int_0^t \sqrt{|X_s|}\, \mathrm{sgn}\,(X_s)\, dW_s - \frac{\lambda^2}{2}\int_0^t |X_s|\, ds\right)\, Q_x^\delta|_{\mathcal{F}_t}\ . \quad (45)$$

Note that because no explosion occurs on both sides of this formula, the density is a true martingale. We may also write the stochastic integral $\int_0^t \mathrm{sgn}\,(X_s)\sqrt{|X_s|}\, dW_s$ in a simpler form, since we have from Itô's formula:

$$|X_t| = |x| + \int_0^t \mathrm{sgn}\,(X_s)\,(\delta\, ds + 2\sqrt{|X_s|}\, dW_s) + L_t^0(X)\,,$$

where $L_t^0(X)$ is the semimartingale local time of X in 0. For $L_t^0(X)$ we obtain with Revuz and Yor [59], Corollary VI.1.9,

$$L_t^0(X) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t 1_{[0,\varepsilon]}(X_s)\, d\langle X, X\rangle_s = \lim_{\varepsilon\downarrow 0}\frac{4}{\varepsilon}\int_0^t |X_s|\, 1_{[0,\varepsilon]}(X_s)\, ds$$
$$\le \lim_{\varepsilon\downarrow 0}\left(4\int_0^t 1_{[0,\varepsilon]}(X_s)\, ds\right) = 0\,.$$

Hence we have

$$\int_0^t \mathrm{sgn}\,(X_s)\sqrt{|X_s|}\, dW_s = \frac{1}{2}\left(|X_t| - |x| - \delta\int_0^t \mathrm{sgn}\,(X_s)\, ds\right)\ .$$

Thus, (45) takes the form

Lemma 3

$${}^\lambda Q_x^\delta|_{\mathcal{F}_t} \equiv \exp\left(\frac{\lambda}{2}\left(|X_t| - |x| - \delta\int_0^t \mathrm{sgn}\,(X_s)\, ds\right) - \frac{\lambda^2}{2}\int_0^t |X_s|\, ds\right)\, Q_x^\delta|_{\mathcal{F}_t}\ .$$

Applying Lemma 3 we obtain the conditional expectation formula

$$
\begin{aligned}
{}^{\lambda}q_t^{\delta}(x,y) &= q_t^{\delta}(x,y)\, \exp\left(\frac{\lambda}{2}\left(|y|-|x|\right)\right) \\
&\quad\times Q_x^{\delta}\left(\exp\left(-\frac{\delta\lambda}{2}\int_0^t \operatorname{sgn}(X_s)\,ds - \frac{\lambda^2}{2}\int_0^t |X_s|\,ds\right)\middle|\; X_t = y\right) \quad (46)
\end{aligned}
$$

where q_t^{δ} denotes the semigroup density in y of a BESQ^{δ} process, $\delta < 0$, given by (37).

Using the time-space transformation from a BESQ^{δ} process (X_t^{δ}) to a squared radial Ornstein–Uhlenbeck process $({}^{\lambda}X_t^{\delta})$

$$
{}^{\lambda}X_t^{\delta} = e^{2\lambda t}\, X^{\delta}_{\left(\frac{1-e^{-2\lambda t}}{2\lambda}\right)},
$$

we also have together with the relationship (46)

$$
{}^{\lambda}q_t^{\delta}(x,y) = e^{-2\lambda t}\, q^{\delta}_{\left(\frac{1-e^{-2\lambda t}}{2\lambda}\right)}(x,\, e^{-2\lambda t}y)\,,
$$

from which ${}^{\lambda}q_t^{\delta}(x,y)$ is obtained since $q_t^{\delta}(x,y)$ is known, see (37). Hence we obtain from (46)

Theorem 6 *We have*

$$
\begin{aligned}
&Q_x^{\delta}\left(\exp\left(-\frac{\delta\lambda}{2}\int_0^t \operatorname{sgn}(X_s)\,ds - \frac{\lambda^2}{2}\int_0^t |X_s|\,ds\right)\middle|\; X_t = y\right) \\
&= q^{\delta}_{\left(\frac{1-e^{-2\lambda t}}{2\lambda}\right)}(x,\, e^{-2\lambda t}y)\exp\left(-2\lambda t - \frac{\lambda}{2}\left(|y|-|x|\right)\right)/q_t^{\delta}(x,y),
\end{aligned}
$$

with $q_t^{\delta}(x,y)$ given by (37).

Appendices

A Some properties of Bessel processes

A.1

In addition to Definitions 1 and 2 we give explicit expressions for Bessel processes. For $\delta > 1$ a $\mathrm{BES}^{\delta}_{x_0}$ process X_t satisfies $E[\int_0^t (ds/X_s)] < \infty$ and is the solution to the equation

$$X_t = x_0 + \frac{\delta - 1}{2} \int_0^t \frac{ds}{X_s} + W_t \,. \tag{47}$$

For $\delta \leq 1$ the situation is less simple. For $\delta = 1$ the role of (47) is played by

$$X_t = |W_t| = \tilde{W}_t + L_t \,,$$

where $\tilde{W}_t \equiv \int_0^t \mathrm{sgn}(W_s) dW_s$ is a standard Brownian motion, and L_t is the local time of Brownian motion. For a treatment of local times see e.g. Revuz and Yor [59], Chapter VI. For $\delta < 1$ we have

$$X_t = x_0 + \frac{\delta - 1}{2} \, \mathrm{p.v.} \int_0^t \frac{ds}{X_s} + W_t \,, \tag{48}$$

where the principal value is defined as

$$\mathrm{p.v.} \int_0^t \frac{ds}{X_s} \equiv \int_0^\infty x^{\delta - 2} (L_t^x - L_t^0) dx$$

and the family of local times $(L_t^x, x \geq 0)$ is defined as

$$\int_0^t \varphi(X_s) ds = \int_0^\infty \varphi(x) L_t^x x^{\delta - 1} dx$$

for all Borel functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, see Bertoin [5]. The decomposition (48) was obtained using the fact that a power of a Bessel process is another Bessel process time-changed:

$$q X_\nu^{1/q}(t) = X_{\nu q} \left(\int_0^t \frac{ds}{X_\nu^{2/p}(s)} \right) ,$$

where $\frac{1}{p} + \frac{1}{q} = 1, \nu > -\frac{1}{q}$, see e.g. Revuz and Yor [59], Proposition (XI.1.11).

A.2 Transition densities.

(Squared) Bessel processes are Markov processes and their transition densities are known explicitly. For $\delta > 0$, the transition density for BESQ^{δ} is equal to

$$q_t^\delta(x, y) = \frac{1}{2t} \left(\frac{y}{x} \right)^{\frac{\nu}{2}} \exp \left\{ -\frac{x + y}{2t} \right\} I_\nu \left(\frac{\sqrt{xy}}{t} \right) , \tag{49}$$

where $t > 0$, $x > 0$, $\nu \equiv \frac{\delta}{2} - 1$ and I_ν is the modified Bessel function of the first kind of index ν. For $x = 0$ we have

$$q_t^\delta(0, y) = (2t)^{-\frac{\delta}{2}} \Gamma\left(\delta/2\right)^{-1} y^{\frac{\delta}{2}-1} \exp\left\{-\frac{y}{2t}\right\} . \tag{50}$$

For the case $\delta = 0$, the semi-group of BESQ^0 is equal to

$$Q_t^0(x, \cdot) = \exp\left\{-\frac{x}{2t}\right\} \varepsilon_0 + \tilde{Q}_t(x, \cdot), \tag{51}$$

where ε_0 is the Dirac measure in 0 and $\tilde{Q}_t(x, \cdot)$ has the density

$$q_t^0(x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{-\frac{1}{2}} \exp\left\{-\frac{x+y}{2t}\right\} I_1\left(\frac{\sqrt{xy}}{t}\right) .$$

The transition density for BES^δ is obtained from (49), (50) resp. (51) and is equal to

$$p_t^\delta(x, y) = \frac{1}{t} \left(\frac{y}{x}\right)^\nu y \exp\left\{-\frac{x^2+y^2}{2t}\right\} I_\nu\left(\frac{xy}{t}\right) ,$$

with $t > 0$, $x > 0$, and

$$p_t^\delta(0, y) = 2^{-\nu} t^{-(\nu+1)} \Gamma(\nu+1)^{-1} y^{2\nu+1} \exp\left\{-\frac{y^2}{2t}\right\} ,$$

for $\delta > 0$, and the semi-group for BES^0 is equal to

$$P_t^0(x, \cdot) = \exp\left\{-\frac{x^2}{2t}\right\} \varepsilon_0 + \tilde{P}_t(x, \cdot) ,$$

where ε_0 is the Dirac measure in 0 and $\tilde{P}_t(x, \cdot)$ has the density

$$p_t^0(x, y) = \frac{x}{t} \exp\left\{-\frac{x^2+y^2}{2t}\right\} I_1\left(\frac{xy}{t}\right) .$$

A.3 Scaling property.

BES^δ processes have the Brownian scaling property, i.e. if X is a BES_x^δ, then the process $c^{-1} X_{c^2 t}$ is a $\mathrm{BES}_{x/c}^\delta$ for any $c > 0$. BESQ^δ processes have the following scaling property: if X is a BESQ_x^δ, then the process $c^{-1} X_{ct}$ is a $\mathrm{BES}_{x/c}^\delta$.

A.4 Additivity property of squared Bessel processes.

An important and well-known property of BESQ^δ processes with $\delta \geq 0$ is the following additivity property.

Theorem 7 (Shiga and Watanabe [65]) *For every* δ, $\delta' \geq 0$ *and* x, $x' \geq 0$*:*

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'} \,,$$

where $Q_x^\delta * Q_{x'}^{\delta'}$ *denotes the convolution of* Q_x^δ *and* $Q_{x'}^{\delta'}$*.*

For a proof see Shiga and Watanabe [65] or Revuz and Yor[59], Theorem (XI.1.2).

B Time reversal

Consider a transient diffusion X, living on $\mathbb{R}_+$, with $X_0 = x_0 \geq 0$. Denote its last exit time of $a \geq 0$ by $L_a = \sup\{u \mid X_u = a\}$, where $\sup \emptyset = 0$. For a fixed, L_a is finite almost surely, and for $x_0 < a$, $L_a > 0$ almost surely. We consider the time reversed process $\tilde{X}$, where

$$\tilde{X}_t(\omega) \equiv \begin{cases} X_{L_a(\omega)-t}(\omega)\,, & \text{if } 0 < t < L_a(\omega)\,, \\ \partial & \text{if } L_a(\omega) \leq t \text{ or } L_a(\omega) = \infty\,, \end{cases} \tag{52}$$

where ∂ denotes the 'cemetery', and $\tilde{X}_0(\omega) = X_{L_a(\omega)}(\omega)$, if $0 < L_a(\omega) < \infty$, else $\tilde{X}_0(\omega) = \partial$. As a consequence we have the equality

$$\{X_u,\, u \leq L_a\} = \{\tilde{X}_{T_0-u},\, u \leq T_0\}\,, \tag{53}$$

where we assume X starting in 0 and $T_0 \equiv \inf\{u | \tilde{X}_u = 0\}$.
We remark that a diffusion may be reversed at cooptional times, a more general class than last exit times, see Nagasawa [46, 47], Revuz and Yor [59], Chapter VII.4. However, for our purposes here it is reasonable to restrict ourselves to last exit times. In the following, we state a general time reversal result (see Nagasawa [46, 47], Sharpe [63], Getoor and Sharpe [27], Revuz and Yor [59]). Denote the semi-group of X by (P_t), the potential kernel of X by U, and let $\tilde{\mathcal{F}}_t = \sigma(\tilde{X}_s,\, s \leq t)$ be the natural filtration of $\tilde{X}$.

Theorem 8 *We assume that there is a probability measure* μ *such that the potential* $\nu = \mu U$ *is a Radon measure. Further we assume that there is a second semi-group on* $\mathbb{R}^+$*, denoted by* $(\hat{P}_t)$*, such that* $\hat{P}_t f$ *is right-continuous in* t *for every continuous function* f *with compact support on* $\mathbb{R}^+$ *and such that the resolvents* (U_p) *and* $(\hat{U}_p)$ *are in duality with respect to* ν*, i.e.,*

$$\int U_p f \cdot g \, d\nu = \int f \cdot \hat{U}_p g \, d\nu \tag{54}$$

for every $p > 0$ *and every positive Borel functions* f *and* g*. Equality (54) can also be written as*

$$\langle U_p f, g \rangle_\nu = \langle f, \hat{U}_p g \rangle_\nu \,.$$

Then under P_μ, the process $\tilde{X}$ is a Markov process with respect to $(\tilde{\mathcal{F}}_t)$ with transition semi-group $(\hat{P}_t)$ and we have the duality

$$\langle P_\phi f, g\rangle_\nu = \langle f, \hat{P}_\phi g\rangle_\nu \,,$$

for any positive Borel function ϕ on $\mathbb{R}^+$ where $P_\phi f(x) = \int_0^\infty \phi(t) P_t f(x) dt$.

We will obtain explicit formulae for time-reversed diffusions via Doob's h-transform.

B.1 Doob's h transform.

Consider a one-dimensional diffusion X, with sample space $(I^{\partial,\infty}, \mathcal{F}^c_\infty)$, where $I \subseteq [-\infty, \infty]$ and $I^{\partial,\infty} := \{\omega : [0,\infty) \mapsto I \cup \{\partial\}\}$, $\mathcal{F}^c_\infty := \sigma\{\omega(t) | \, t \geq 0\}$.

Definition 5 *A non-negative measurable function $h : I \mapsto \mathbb{R} \cup \{\infty\}$ is called α-excessive for X, $\alpha \geq 0$, if*
a) $e^{-\alpha t} E_x(h(X_t)) \leq h(x)$, for all $x \in I$, $t \geq 0$,

b) $e^{-\alpha t} E_x(h(X_t)) \to h(x)$, for all $x \in I$ as $t \downarrow 0$.

A 0-excessive function is simply called excessive.

Let h be an α-excessive function for a diffusion X. The life time of a path $\omega \in I^{\partial,\infty}$ is defined by $\zeta(\omega) := \inf\{t \,|\, \omega_t = \partial\}$. We construct a new probability measure P^h by

$$P^h_x|_{\mathcal{F}_t} = e^{-\alpha t} \, \frac{h(\omega(t))}{h(x)} \, P_x|_{\mathcal{F}_t} \,, \tag{55}$$

for $t < \zeta$ and $x \in I$. The process under the new measure P^h is a regular diffusion and is called *Doob's h-transform* of X. As for Doob's h-transform we refer to Doob [16] and Dellacherie and Meyer [15]; in presenting Doob's h-transform we followed Borodin and Salminen [6].
As an application let us consider a transient diffusion X under probability measure P with scale function s. Doob's h-transform of X with the excessive function $h \equiv s$, that is, the process under the new measure P^h, is a process which reaches 0 almost surely. We have

$$P_x|_{\mathcal{F}_t} = \frac{\left(\frac{1}{s}\right)(X_{t\wedge T_0})}{\left(\frac{1}{s}\right)(x)} \; P^h_x|_{\mathcal{F}_t} \,. \tag{56}$$

Note that we obtain

$$\frac{s\,(X_t)}{s(x)} \; P_x|_{\mathcal{F}_t} \equiv 1_{(t<T_0)} \; P^h_x|_{\mathcal{F}_t} \,,$$

and hence,

$$Q^h_t f(x) \equiv \frac{1}{s(x)} \, E_x\left[(fs)(X_t)\right] \equiv \frac{1}{s(x)} P_t(fs)(x) \,, \tag{57}$$

is the semigroup of the process under P^h killed when it reaches 0. In other words, we have the following time reversal result: Doob's h-transform of the transient process under P with $h = s$ is the process under P^h killed at T_0; this is the process $\tilde{X}$ in our former notation.
From formula (56) we can obtain explicit formulae of the diffusion processes via Girsanov's theorem. Assume, a process (X_t) under P_x has the form

$$X_t = x + \int_0^t \beta(X_u)\, du + \int_0^t \alpha(X_u)\, dB_u\,,$$

then via Girsanov's theorem we obtain with (56) that the process under P_x^h has the form

$$X_t = x + \int_0^t \left(\beta + \left(\alpha \frac{s'}{s}\right)\right)(X_u)\, du + \int_0^t \alpha(X_u)\, d\hat{B}_u, \quad t \le T_0\,. \tag{58}$$

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