Realised power variation and stochastic volatility models

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Abstract

Limit distribution results on realised power variation, that is sums of absolute powers of increments of a process, are derived for certain types of semimartingale with continuous local martingale component, in particular for a class of flexible stochastic volatility models. The theory covers, for example, the cases of realised volatility and realised absolute variation. Such results should be helpful in, for example, the analysis of volatility models using high frequency information.

Some key words: Absolute returns; Mixed asymptotic normality; Realised volatility; p-variation; Quadratic variation; Semimartingale.

1 Introduction

Stochastic volatility processes play an important role in financial economics, generalising Brownian motion to allow the scale of the increments (or returns in economics) to change through time in a stochastic manner. We show such intermittency can be coherently measured using sums of absolute powers of increments, which we name realised power variation. This paper derives limit theorems for these measures, over a fixed interval of time, as the number of high frequency increments goes off to infinity.

This paper has six other sections. In Section 2 we establish our idea of realised power variation as well as define the regularity assumptions we need to derive our limit theorems. Section 3 contains our main results, while the proofs of them are given in Section 4. Section 5 gives some examples of the processes covered by our theory, while a Monte Carlo experiment to assess the accuracy of our limit theory approximations is conducted in Section 6. Finally, Section 7 gives some concluding remarks including a discussion of the use of these ideas in other areas of study, for instance turbulence and image analysis.

2 Models, notation and regularity conditions

We first introduce some notation for realised power variation quantities of an arbitrary semimartingale x. Let δ be positive real and, for any $t \geq 0$, define

$$x_\delta(t) = x(\lfloor t/\delta \rfloor \, \delta),$$

where $\lfloor a \rfloor$ for any real number a denotes the largest integer less than or equal to a. The process $x_\delta(t)$ is a discrete approximation to $x(t)$. Further, for r positive real we define the *realised power variation of order* r[1] or *realised* r*-tic variation* of $x_\delta(t)$ as

$$\begin{aligned} [x_\delta]^{[r]}(t) &= \sum_{j=1}^{M} |x_\delta\,(j\delta) - x_\delta((j-1)\delta)|^r \\ &= \sum_{j=1}^{M} |x\,(j\delta) - x((j-1)\delta)|^r \end{aligned} \tag{1}$$

where $M = M(t) = \lfloor t/\delta \rfloor$. Then, in particular, for $M \to \infty$, *realised quadratic variation*

$$[x_\delta]^{[2]}(t) \xrightarrow{p} [x](t),$$

where $[x]$ is the *quadratic variation* process of the semimartingale x. Note also that,

$$[x_\delta]^{[2]} = [x_\delta].$$

Henceforth, for simplicity of exposition, we fix t and take δ so that $M = \lfloor t/\delta \rfloor$ is an integer (and then $\delta M = t$).

Our detailed results will be established for the stochastic volatility (SV) model where basic Brownian motion is generalised to allow the volatility term to vary over time and there to be a rather general drift. Then the y^* follows

$$y^*(t) = \alpha(t) + \int_0^t \sigma(s)\mathrm{d}w(s), \qquad t \geq 0, \tag{2}$$

where $\sigma > 0$ and α are assumed to be stochastically independent of the standard Brownian motion w. Throughout this paper we will assume that the processes $\tau = \sigma^2$ and α are of locally bounded variation. This implies that τ and α are locally bounded Riemann integrable functions and that y^* is a semimartingale with a continuous local martingale component. We

[1] The similarly named p-variation, $0 < p < \infty$, of a real-valued function f on $[a, b]$ is defined as

$$\sup_\kappa \sum |f(x_i) - f(x_{i-1})|^p \, ,$$

where the supremum is taken over all subdivisions κ of $[a, b]$. If this function is finite then f is said to have bounded p-variation on $[a, b]$. The case of $p = 1$ gives the usual definition of bounded variation.

This condition has been studied recently in the probability literature. See the work of, for example, Lyons (1994) and Mikosch and Norvaisa (2000).

call σ the *spot volatility* process and α the *mean* or *risk premium* process. (For some general information on processes y^* of this type, see for example Ghysels, Harvey, and Renault (1996) and Barndorff-Nielsen and Shephard (2001a)). By allowing the spot volatility to be random and serially dependent, this model will imply its increments will exhibit volatility clustering and have unconditional distributions which are fat tailed. This allows it to be used in finance and econometrics as a model for log-prices. In turn, this provides the basis for option pricing models which overcome some of the major failings in the Black-Scholes option pricing approach. Leading references in this regard include Hull and White (1987), Heston (1993) and Renault (1997). See also the recent work of Nicolato and Venardos (2001).

For the price process (2) the *realised power variation of order* r of y^* is, at time t and discretisation δ, $[y_\delta^*]^{[r]}(t)$. Letting

$$y_j(t) = y^*(j\delta) - y^*((j-1)\delta)$$

we have that

$$[y_\delta^*]^{[r]}(t) = \sum_{j=1}^{M} |y_j(t)|^r .$$

We use the notation $\tau(t) = \sigma^2(t)$ and

$$\tau^*(t) = \int_0^t \tau(s)\mathrm{d}s$$

and, more generally, we consider the *integrated power volatility of order* r

$$\tau^{r*}(t) = \int_0^t \tau^r(s)\mathrm{d}s.$$

That τ^r is Riemann integrable for every $r > 0$ follows from the assumed locally bounded variation of τ and the fact, due to Lebesgue, that a bounded function f on a finite interval I is Riemann integrable on I if and only if the Lebesgue measure of the set of discontinuity points of f is equal to 0 (see Hobson (1927, pp. 465–466), Munroe (1953, p. 174, Theorem 24.4) or Lebesgue (1902)). In our case the latter property follows immediately from the bounded variation of τ (any function of bounded variation is the difference between an increasing and a decreasing function and any monotone function has at most countably many discontinuities).

Throughout the following, r denotes a positive number. Moreover we shall refer to the following conditions on the volatility and mean processes:

(V) The volatility process $\tau = \sigma^2$ is (pathwise) locally bounded away from 0 and has, moreover, the property

$$p\text{-}\lim_{\delta \downarrow 0} \delta^{1/2} \sum_{j=1}^{M} |\tau^r(\eta_j) - \tau^r(\xi_j)| = 0 \qquad (3)$$

for some $r > 0$ (equivalently for all $r > 0$)[2] and for any ξ_j and η_j such that

$$0 \leq \xi_1 \leq \eta_1 \leq \delta \leq \xi_2 \leq \eta_2 \leq 2\delta \leq \cdots \leq \xi_j \leq \eta_j \leq M\delta = t.$$

(M) The mean process α satisfies (pathwise)[3]

$$\overline{\lim_{\delta \downarrow 0}} \max_{1 \leq j \leq M} \delta^{-1} |\alpha(j\delta) - \alpha((j-1)\delta)| < \infty. \tag{4}$$

These regularity conditions are quite mild.[4] Of some special interest are cases where α is of the form

$$\alpha(t) = \int_0^t g(\sigma(s))\mathrm{d}s,$$

for g a smooth function. Then regularity of τ will imply regularity of α.

Note that the assumptions allow the spot volatility to have, for example, deterministic diurnal effects, jumps, long memory, no unconditional mean or to be non-stationary.

3 Results

Our main theoretical result is

Theorem 1 For $\delta \downarrow 0$ and $r \geq 1/2$, under conditions **(V)** and **(M)**,

$$\mu_r^{-1} \delta^{1-r/2} [y_\delta^*]^{[r]}(t) \xrightarrow{p} \tau^{r/2*}(t) \tag{5}$$

and

$$\frac{\mu_r^{-1} \delta^{1-r/2} [y_\delta^*]^{[r]}(t) - \tau^{r/2*}(t)}{\mu_r^{-1} \delta^{1-r/2} \sqrt{\mu_{2r}^{-1} v_r [y_\delta^*]^{[2r]}(t)}} \xrightarrow{\mathcal{L}} N(0,1), \tag{6}$$

where $\mu_r = \mathrm{E}\{|u|^r\}$ and $v_r = \mathrm{Var}\{|u|^r\}$, with $u \sim N(0,1)$. $\square$

In the proof of this theorem, to be given in the next section, the only place where the assumption $r \geq 1/2$ is needed is where Lemma 3 is invoked.

[2]The equivalence follows on noting that for each j there exists an ω_j with

$$\inf_{(j-1)\delta \leq s \leq j\delta} \tau(s) \leq \omega_j \leq \sup_{(j-1)\delta \leq s \leq j\delta}$$

such that

$$|\tau^r(\eta_j) - \tau^r(\xi_j)| = r\omega_j^{r-1} |\tau(\eta_j) - \tau(\xi_j)|$$

and then using that τ is pointwise bounded away from 0 and ∞.

[3]This condition is implied by Lipschitz continuity and itself implies continuity of α.

[4]Condition **(V)** is satisfied in particular if τ is of OU type, cf. Example 1 below, and condition **(M)** is valid if, for instance, α is the intOU process plus drift, cf. Example 2.

This theorem tells us that, for $\delta \downarrow 0$, scaled realised power variation converges in probability to integrated power volatility and follows asymptotically a normal variance mixture distribution with variance distributed as

$$\delta\mu_r^{-2} v_r \tau^{r*}(t),$$

which is consistently estimated by the square of the denominator in (6). Hence the limit theory is statistically feasible and does not depend upon knowledge of α or σ^2.

Leading cases are *realised quadratic variation*, which is usually called *realised volatility* in the finance and econometrics literature,

$$[y_\delta^*]^{[2]}(t) = \sum_{j=1}^{M} y_j^2(t),$$

in which case

$$\frac{\sum_{j=1}^{M} y_j^2(t) - \tau^*(t)}{\sqrt{\frac{2}{3}\sum_{j=1}^{M} y_j^4(t)}} \xrightarrow{\mathcal{L}} N(0,1), \tag{7}$$

and *realised absolute variation*

$$[y_\delta^*]^{[1]}(t) = \sum_{j=1}^{M} |y_j(t)|,$$

when

$$\frac{\sqrt{\pi/2}\sqrt{\delta}\sum_{j=1}^{M} |y_j(t)| - \sigma^*(t)}{\sqrt{(\pi/2-1)\,\delta\sum_{j=1}^{M} y_j^2(t)}} \xrightarrow{\mathcal{L}} N(0,1). \tag{8}$$

In the case of $r = 2$ the result considerably strengthens the well known quadratic variation result that realised quadratic variation converges in probability to integrated volatility $\int_0^t \sigma^2(s)\mathrm{d}s$ — which was highlighted in concurrent and independent work by Andersen and Bollerslev (1998a) and Barndorff-Nielsen and Shephard (2001a). The asymptotic distribution of realised quadratic variation was discussed by Barndorff-Nielsen and Shephard (2002) in the special case where $\alpha(t) = \mu t + \beta\int_0^t \sigma^2(s)\mathrm{d}s$. To our knowledge the probability limit of (normalised) realised absolute variation has not been previously derived, let alone its asymptotic distribution.

Taking sums of squares of increments of log-prices has a very long tradition in financial economics — see, for example, Poterba and Summers (1986), Schwert (1989), Taylor and Xu (1997), Christensen and Prabhala (1998), Dacorogna, Muller, Olsen, and Pictet (1998), Andersen, Bollerslev, Diebold, and Labys (2001) and Andersen, Bollerslev, Diebold, and Ebens (2001). However, for a long time no theory was known for the behaviour of such sums outside the Brownian motion case. Since the link to quadratic variation has been made there has been a remarkably fast development in this field. Contributions include Corsi, Zumbach, Muller, and Dacorogna (2001), Andersen, Bollerslev, Diebold, and Labys (2001), Andersen, Bollerslev, Diebold, and

Ebens (2001), Barndorff-Nielsen and Shephard (2002), Andreou and Ghysels (2001), Bai, Russell, and Tiao (2000), Maheu and McCurdy (2001), Areal and Taylor (2001), Galbraith and Zinde-Walsh (2000), Bollerslev and Zhou (2001) and Bollerslev and Forsberg (2001).

Andersen and Bollerslev (1998b) and Andersen and Bollerslev (1997) empirically studied the properties of $\sum_{j=1}^{M} |y_j(t)|$ computed using sums of absolute values of intra-day returns on speculative assets (many authors in finance have based their empirical analysis on absolute values of returns — see, for example, Taylor (1986, Ch. 2), Cao and Tsay (1992), Ding, Granger, and Engle (1993), West and Cho (1995), Granger and Ding (1995), Jorion (1995), Shiryaev (1999, Ch. IV) and Granger and Sin (1999)). This was empirically attractive, for using absolute values is less sensitive to possible large movements in high frequency data. There is evidence that if returns do not possess fourth moments then using absolute values rather than squares would be more reliable (see, for example, the work on the distributional behaviour of the correlogram of squared returns by Davis and Mikosch (1998) and Mikosch and Starica (2000)). However, the approach was abandoned in their subsequent work reported in Andersen and Bollerslev (1998a), Andersen, Bollerslev, Diebold, and Ebens (2001) and Andersen, Bollerslev, Diebold, and Labys (2001) due to the lack of appropriate theory for the sum of absolute returns as $\delta \downarrow 0$, although recently Andreou and Ghysels (2001) have performed some interesting Monte Carlo studies in this context, while Shiryaev (1999, pp. 349–350) mentions an interest in the limit of sums of absolute returns. Our work provides a theory for the use of sums of absolute returns.

4 Proofs

Since the mean and volatility processes α and τ are jointly independent of the Brownian motion w we need only argue conditionally on (α, τ). Define α_j, τ_j and σ_j by

$$\alpha_j = \alpha(j\delta) - \alpha((j-1)\delta),$$

$$\tau_j = \tau^*(j\delta) - \tau^*((j-1)\delta)$$

and

$$\sigma_j = \sqrt{\tau_j}.$$

As a preliminary step we show

Lemma 1 For $\delta \to 0$,

$$\delta^{1-r}[\tau_\delta^*]^{[r]}(t) \to \tau^{r*}(t), \tag{9}$$

pathwise. □

Proof By the boundedness of $\tau(t)$ we have that for every $j = 1, ..., M$ there exists a constant θ_j such that

$$\inf_{(j-1)\delta \le s \le j\delta} \tau(s) \le \theta_j \le \sup_{(j-1)\delta \le s \le j\delta} \tau(s)$$

and

$$\tau_j = \theta_j \delta, \tag{10}$$

and using this and the Riemann integrability of $\tau^r(t)$ we obtain

$$\begin{aligned} \delta^{1-r}[\tau_\delta^*]^{[r]} &= \delta^{1-r} \sum_{j=1}^{M} \tau_j^r \\ &= \sum_{j=1}^{M} (\tau_j/\delta)^r \delta \\ &= \sum_{j=1}^{M} \theta_j^r \delta \\ &\to \int_0^t \tau^r(s) \mathrm{d}s \\ &= \tau^{r*}(t). \end{aligned}$$

□

It is now convenient to write $y^*(t)$ as

$$y^*(t) = \alpha(t) + y_0^*(t),$$

where

$$y_0^*(t) = \int_0^t \sigma(s) \mathrm{d}w(s)$$

and to introduce

$$y_{0j} = y_0^*(j\delta) - y_0^*((j-1)\delta).$$

The joint law of $y_{01}, ..., y_{0M}$ is equal to that of $v_1, ..., v_M$ where

$$v_j = \sigma_j u_j$$

and $u_1, ..., u_M$ are i.i.d. standard normal and independent of the process σ. It follows that

$$[y_{0\delta}^*]^{[r]}(t) \stackrel{\mathcal{L}}{=} \sum_{j=1}^{M} \tau_j^{r/2} |u_j|^r . \tag{11}$$

The conditional mean and variance of $[y_{0\delta}^*]^{[r]}(t)$ are then

$$\mathrm{E}\{[y_{0\delta}^*]^{[r]}(t)|\tau\} = \mu_r \sum_{j=1}^{M} \tau_j^{r/2} = \mu_r [\tau_\delta^*]^{[r/2]}(t) \tag{12}$$

and

$$\mathrm{Var}\{[y_{0\delta}^*]^{[r]}(t)|\tau\} = v_r \sum_{j=1}^{M} \tau_j^r = v_r [\tau_\delta^*]^{[r]}(t). \tag{13}$$

Hence, letting

$$D_0 = \mu_r^{-1}[y_{0\delta}^*]^{[r]}(t) - [\tau_\delta^*]^{[r/2]}(t)$$

we have

$$\mathrm{E}\{D_0|\tau\} = 0$$

and

$$\mathrm{Var}\{D_0|\tau\} = \mu_r^{-2} v_r [\tau_\delta^*]^{[r]}(t).$$

By Lemma 1 as $\delta \to 0$,

$$\delta^{1-r}\mathrm{Var}\{D_0|\tau\} \to \mu_r^{-2} v_r \tau^{r*}(t)$$

indicating the validity of

Proposition 1 For $\delta \to 0$,

$$\delta^{(1-r)/2} \frac{\mu_r^{-1}[y_{0\delta}^*]^{[r]}(t) - [\tau_\delta^*]^{[r/2]}(t)}{\sqrt{\mu_r^{-2} v_r \tau^{r*}(t)}} \xrightarrow{\mathcal{L}} N(0,1). \tag{14}$$

□

Proof To establish this proposition we recall Taylor's formula with remainder term:

$$f(x) = f(0) + f'(0)x + x^2 \int_0^1 (1-s) f''(sx)\mathrm{d}s. \tag{15}$$

By (11) and (12) we find, using (15), that the conditional cumulant transform of D_0 is of the form

$$\begin{aligned} \log \mathrm{E}\{\exp(i\zeta D_0)|\tau\} &= \zeta^2 \sum_{j=1}^{M} \tau_j^r \int_0^1 (1-s)\kappa_r''(\tau_j^{r/2}\zeta s)\mathrm{d}s \\ &= \zeta^2 \delta^r \sum_{j=1}^{M} \theta_j^r \int_0^1 (1-s)\kappa_r''(\delta^{r/2}\theta_j^{r/2}\zeta s)\mathrm{d}s, \end{aligned}$$

where θ_j is given by (10) and κ_r denotes the cumulant transform of $\mu_r^{-1}|u|^r$ for u a standard normal random variable. Consequently,

$$\log \mathrm{E}\{\exp(i\zeta\delta^{(1-r)/2} D_0)|\tau\} = \frac{1}{2}\zeta^2 \delta R$$

where

$$R = 2\sum_{j=1}^{M} \theta_j^r \int_0^1 (1-s)\kappa_r''(\delta^{1/2}\theta_j^{r/2}\zeta s)\mathrm{d}s.$$

The boundedness of τ on $[0,t]$ implies

$$\lim_{\delta\downarrow 0} \delta^{1/2} \max_j \theta_j^{r/2} = 0$$

and hence, for $\delta \downarrow 0$,

$$\begin{aligned} \delta R &\to 2\int_0^1 (1-s)\mathrm{d}s\kappa_r''(0) \lim_{\delta\downarrow 0} \sum_{j=1}^M \theta_j^r \delta \\ &= \kappa_r''(0)\tau^{r*}(t) = -\mu_r^{-2} v_r \tau^{r*}(t). \end{aligned}$$

Therefore

$$\log \mathrm{E}\{\exp(i\zeta\delta^{(1-r)/2} D_0)|\tau\} = -\frac{1}{2}\zeta^2\mu_r^{-2} v_r \tau^{r*}(t) + o(1) \tag{16}$$

and Proposition 1 follows. $\square$

Lemma 1 uses only the local boundedness and Riemann integrability of τ. Invoking condition **(V)** we may strengthen the result (9) as follows.

Lemma 2 Under condition **(V)** we have

$$\delta^{1-r}[\tau_\delta^*]^{[r]}(t) - \tau^{r*}(t) = o_p(\delta^{1/2}).$$

$\square$

PROOF For each j there exists a number ψ_j such that

$$\inf_{(j-1)\delta\le s\le j\delta} \tau(s) \le \psi_j \le \sup_{(j-1)\delta\le s\le j\delta} \tau(s)$$

and

$$\int_{(j-1\delta)}^{j\delta} \tau^r(s)\mathrm{d}s = \psi_j^r \delta. \tag{17}$$

Using this and (10) we find

$$\begin{aligned} \delta^{1-r}[\tau_\delta^*]^{[r]}(t) - \tau^{r*}(t) &= \delta^{1-r}\sum_{j=1}^M \tau_j^r - \int_0^t \tau^r(s)\mathrm{d}s \\ &= \delta\sum_{j=1}^M (\theta_j^r - \psi_j^r), \end{aligned} \tag{18}$$

and the conclusion now follows from assumption **(V)**. $\square$

Proposition 2 Under condition **(V)**, for $\delta \to 0$,

$$\frac{\delta^{1-r/2}\mu_r^{-1}[y_{0\delta}^*]^{[r]}(t) - \tau^{r/2*}(t)}{\delta^{1/2}\sqrt{\mu_r^{-2} v_r \tau^{r*}(t)}} \xrightarrow{\mathcal{L}} N(0,1). \tag{19}$$

□

Proof We may rewrite the left hand side of (14) as

$$\frac{\delta^{1-r/2}\mu_r^{-1}[y_{0\delta}^*]^{[r]}(t) - \delta^{1-r/2}[\tau_\delta^*]^{[r/2]}(t)}{\delta^{1/2}\sqrt{\mu_r^{-2}v_r\tau^{r*}(t)}} = \frac{\delta^{1-r/2}\mu_r^{-1}[y_{0\delta}^*]^{[r]}(t) - \tau^{r/2*}(t)}{\delta^{1/2}\sqrt{\mu_r^{-2}v_r\tau^{r*}(t)}} + \frac{\delta^{1-r/2}[\tau_\delta^*]^{[r/2]}(t) - \tau^{r/2*}(t)}{\delta^{1/2}\sqrt{\mu_r^{-2}v_r\tau^{r*}(t)}}$$

and Lemma 2 then implies the result. □

As an immediate consequence of Proposition 2 we have

Corollary 1 Under condition **(V)**, for $\delta \to 0$,

$$\delta^{1-r}\mu_{2r}^{-1}[y_{0\delta}^*]^{[2r]}(t) \xrightarrow{p} \tau^{r*}(t).$$

□

In other words, when normalised, $[y_{0\delta}^*]^{[2r]}(t)$ provides a consistent estimate of $\tau^{r*}(t)$. Combining this with (19) yields the conclusion of Theorem 1 for the special case where the mean process α is identically 0.

The remaining task is to show that, to the order concerned, α does not affect the asymptotic limit behaviour, provided conditions **(V)** and **(M)** hold. For this it suffices to show that, under **(V)** and **(M)**,

$$\delta^{(1-r)/2}\left\{[y_\delta^*]^{[r]}(t) - [y_{0\delta}^*]^{[r]}(t)\right\} = o_p(1),$$

cf. Proposition 1.

We shall in fact prove the following stronger result.

Proposition 3 Under conditions **(V)** and **(M),**

$$\delta^{-r/2}\left\{[y_\delta^*]^{[r]}(t) - [y_{0\delta}^*]^{[r]}(t)\right\} = O_p(1).$$

□

Proof Let

$$\underline{\tau} = \inf_{0\le s\le t}\tau(s) \quad \text{and} \quad \bar{\tau} = \sup_{0\le s\le t}\tau(s)$$

and

$$\gamma_j = \delta^{-1}\alpha_j$$

and note that (pathwise for (α, τ)), by assumption,

$$0 < \underline{\tau} \leq \bar{\tau} < \infty,$$

implying

$$0 < \min_j \theta_j \leq \max_j \theta_j < \infty,$$

while, due to assumption **(M)**, there exists a constant c for which

$$\max_j |\gamma_j| \leq c,$$

whatever the value of M.

We have

$$\begin{aligned}
[y_\delta^*]^{[r]}(t) - [y_{0\delta}^*]^{[r]}(t) &= \sum_{j=1}^{M} (|\alpha_j + y_{0j}|^r - |y_{0j}|^r) \\
&= \sum_{j=1}^{M} \left(|\delta\gamma_j + \delta^{1/2}\theta_j^{1/2}u_j|^r - |\delta^{1/2}\theta_j^{1/2}u_j|^r \right) \\
&= \delta^{r/2} \sum_{j=1}^{M} \theta_j^{r/2} \left\{ \left| \left(\gamma_j/\theta_j^{1/2}\right) \delta^{1/2} + u_j \right|^r - |u_j|^r \right\}
\end{aligned}$$

and hence

$$\delta^{-r/2} \left\{ [y_\delta^*]^{[r]}(t) - [y_{0\delta}^*]^{[r]}(t) \right\} \stackrel{\mathcal{L}}{=} \sum_{j=1}^{M} \theta_j^{r/2} h_r \left(u_{0j}; \gamma_j/\theta_j^{1/2} \right)$$

where

$$h_r(u;\rho) = \left| \rho\delta^{1/2} + u \right|^r - |u|^r.$$

The conclusion of Proposition 3 now follows from Lemma 3 below. $\square$

Lemma 3 For $r \geq 1/2$, u a standard normal random variable and ρ constant,

$$\mathrm{E}\{h_r(u;\rho)\} = O(\delta)$$

and

$$\mathrm{Var}\{h_r(u;\rho)\} = O(\delta).$$

$\square$

PROOF With φ denoting the standard normal density we obtain

$$\begin{aligned}
\mathrm{E}\{|\rho\delta^{1/2}+u|^r\} &= \int_{-\infty}^{\infty}\left|\rho\delta^{1/2}+x\right|^r\varphi(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty}|x|^r\varphi(x)e^{\rho\delta^{1/2}x}\mathrm{d}xe^{-\rho^2\delta/2} \\
&= \int_{-\infty}^{\infty}|x|^r\varphi(x)e^{\rho\delta^{1/2}x}\mathrm{d}x+O(\delta) \\
&= \mathrm{E}\{|u|^r\}+\int_{-\infty}^{\infty}|x|^r\varphi(x)\left(e^{\rho\delta^{1/2}x}-1\right)\mathrm{d}x+O(\delta),
\end{aligned}$$

i.e.

$$\begin{aligned}
\mathrm{E}\{h_r(u;\rho)\} &= \delta^{1/2}\int_{-\infty}^{\infty}x|x|^r\varphi(x)\frac{e^{\rho\delta^{1/2}x}-1}{\delta^{1/2}x}\mathrm{d}x+O(\delta) \\
&= \delta^{1/2}\int_{-\infty}^{\infty}x|x|^r\varphi(x)\frac{e^{\rho\delta^{1/2}x}-1-\rho\delta^{1/2}x}{\delta^{1/2}x}\mathrm{d}x+\rho\delta^{1/2}\int_{-\infty}^{\infty}x\,|x|^r\,\varphi(x)\mathrm{d}x+O(\delta) \\
&= O(\delta).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{E}\{h_r(u;\rho)^2\} &= \mathrm{E}\{h_{2r}(u;\rho)\}+2\mathrm{E}\{|u|^r\left(|u|^r-|\rho\delta^{1/2}+u|^r\right)\} \\
&= O(\delta)+2\mathrm{E}\{|u|^r\left(|u|^r-|\rho\delta^{1/2}+u|^r\right)\} \qquad (20)
\end{aligned}$$

by the previous result. Here

$$\begin{aligned}
\mathrm{E}\{|u|^r|\rho\delta^{1/2}+u|^r\} &= \int_{-\infty}^{\infty}|x|^r|\rho\delta^{1/2}+x|^r\varphi(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty}|x^2-\frac{1}{4}\rho^2\delta|^r\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}xe^{-\frac{1}{8}\rho^2\delta} \\
&= \int_{-\infty}^{\infty}|x^2-\frac{1}{4}\rho^2\delta|^r\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}x+O(\delta) \\
&= \int_{-\infty}^{\infty}|x|^{2r}\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}x+O(\delta) \\
&\quad+\int_{-\infty}^{\infty}\left(\left|x^2-\frac{1}{4}\rho^2\delta\right|^r-|x^2|^r\right)\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}x. \qquad (21)
\end{aligned}$$

Now, for a and b nonnegative numbers we have the inequality

$$|b^r-|b-a|^r|\leq\begin{cases} a^r & \text{for} \quad 0\leq b\leq a \\ rb^{r-1}a & \text{for} \quad b>a. \end{cases} \qquad (22)$$

Using this and $r\geq 1/2$ we find that

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\left(|x^2-\frac{1}{4}\rho^2\delta|^r-|x^2|^r\right)\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}x\right| \\
\leq\ & \left(\frac{r}{4}\rho^2\delta\right)^r\int_{|x|\leq\rho\delta^{1/2}/2}\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}x+\frac{r}{4}\rho^2\delta\int_{-\infty}^{\infty}|x|^{2(r-1)}\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}x \qquad (23) \\
=\ & O(\delta). \qquad (24)
\end{aligned}$$

Thus, combining (20), (21) and (24),we have

$$\begin{aligned}
\mathrm{E}\{h_r(u;\rho)^2\} &= 2\left\{\int_{-\infty}^{\infty}|x|^{2r}\varphi(x)\mathrm{d}x - \int_{-\infty}^{\infty}|x|^{2r}\varphi(x)e^{\frac{1}{2}\rho\delta^{1/2}x}\mathrm{d}x\right\} + O(\delta) \\
&= 2\int_{-\infty}^{\infty}|x|^{2r}\varphi(x)\left(1-e^{\frac{1}{2}\rho\delta^{1/2}x}\right)\mathrm{d}x + O(\delta) \\
&= 2\delta^{1/2}\int_{-\infty}^{\infty}x|x|^{2r}\varphi(x)\frac{1-e^{\frac{1}{2}\rho\delta^{1/2}x}}{\delta^{1/2}x}\mathrm{d}x + O(\delta) \\
&= O(\delta),
\end{aligned}$$

as was to be shown. □

5 Examples

The following two examples show that conditions **(V)** and **(M)** are satisfied for OU models used by Barndorff-Nielsen and Shephard (2001a) in the context of SV models.

Example 1 *Volatility process τ of OU type.* Without loss of generality we suppose that τ satisfies the equation

$$\tau(t) = e^{-\lambda t}\tau(0) + \int_0^t e^{-\lambda(t-s)}\mathrm{d}z(\lambda s),$$

where z is a subordinator, i.e. a positive Lévy process, which is referred to as the BDLP (background driving Lévy process).

The number of jumps of the BDPL z on the interval $[0,t]$ is at most countable. Let $z_1 \geq z_2 \geq \ldots$ denote the jump sizes given in decreasing order and let $u_1, u_2, \ldots$ be the corresponding jump times. Then, for any $s \in [0,t]$,

$$\tau(s) = \tau(0)e^{-\lambda s} + \sum_{n=1}^{\infty} z_n e(s;u_n),$$

where

$$e(s;u) = e^{-\lambda(s-u)}\mathbf{1}_{[u,t]}(s)$$

and we have

$$\sum_{j=0}^{M}|e(j\delta;u) - e((j-1)\delta;u)| \leq 2.$$

Hence

$$\sum_{j=0}^{M}|\tau(j\delta) - \tau((j-1)\delta)| \leq 2\left\{\tau(0) + \sum_{n=1}^{\infty} z_n\right\} = 2\left\{\tau(0) + z(\lambda t)\right\},$$

showing that condition **(V)** is amply satisfied. □

Example 2 *OU volatility and intOU risk premium* In this particular case the volatility process τ is as in the previous example and the mean process is of the form

$$\alpha(t) = \mu\delta + \beta\tau^*(t),$$

where μ and β are arbitrary real parameters.

We then have

$$\alpha_j = \delta\{\mu + \beta\theta_j\},$$

implying

$$\max_{1\leq j\leq M} \delta^{-1}|\alpha(j\delta) - \alpha((j-1)\delta)| \leq |\mu| + |\beta|\bar{\tau} < \infty,$$

so that condition **(M)** is indeed satisfied. $\square$

6 A Monte Carlo experiment

6.1 Multiple realised power variations

We have stated the definition and results for realised power variation for a single fixed t. It is clear that the theory can also be applied repeatedly on non-overlapping increments to the process. Let us write $\Delta > 0$ as a time interval and focus on the n-th such interval. Then define the intra-Δ increments as

$$y_{j,n} = y^*\left((n-1)\Delta + \frac{\Delta j}{M}\right) - y^*\left((n-1)\Delta + \frac{\Delta(j-1)}{M}\right), \qquad \delta = \Delta M^{-1},$$

which allows us to construct the n-th realised power variation

$$[y_\delta^*]_n^{[r]} = \sum_{j=1}^{M} |y_{j,n}|^r .$$

The implication is that

$$\frac{\left[\mu_r^{-1}\left(\Delta M^{-1}\right)^{1-r/2}[y_\delta^*]_n^{[r]}\right] - \int_{(n-1)\Delta}^{n\Delta}\sigma^r(s)\mathrm{d}s}{\mu_r^{-1}\left(\Delta M^{-1}\right)^{1-r/2}\sqrt{\frac{v_r}{\mu_{2r}}[y_\delta^*]_n^{[2r]}}} \xrightarrow{\mathcal{L}} N(0,1).$$

An important observation is that the *realised power variation errors*

$$\left\{\mu_r^{-1}\left(\Delta M^{-1}\right)^{1-r/2}[y_\delta^*]_n^{[r]}\right\} - \sigma_n^{[r]}, \tag{25}$$

where $\sigma_n^{[r]}$, the *actual power volatility*, is defined as

$$\sigma_n^{[r]} = \sigma^{r*}(n\Delta) - \sigma^{r*}((n-1)\Delta), \qquad \text{where} \qquad \sigma^{r*}(t) = \int_0^t \sigma^r(s)\mathrm{d}s.$$

will be asymptotically uncorrelated through n, although they will not be independent.

6.2 Simulated example

6.2.1 Realised power variation and actual power volatility

The above distribution theory says in particular that realised power variation error will converge in probability to zero as $\delta \downarrow 0$. To see the magnitude of this error we have carried out a simulation. This will allow us to see how accurate our asymptotic analysis is in practice. Throughout we have set the mean process $\alpha(t)$ to zero. Our experiments could have been based on the familiar constant elasticity of variance (CEV) process which is the solution to the SDE

$$\mathrm{d}\sigma^2(t) = -\lambda \left\{\sigma^2(t) - \xi\right\} \mathrm{d}t + \omega\sigma(t)^\eta \mathrm{d}b(\lambda t), \qquad \eta \in [1, 2],$$

where b is standard Brownian motion uncorrelated with w. Of course the special cases of $\eta = 1$ delivers the square root process, while when $\eta = 2$ we have Nelson's GARCH diffusion. These models have been heavily favoured by Meddahi and Renault (2000) in the context of SV models. Instead of this we will work with the non-Gaussian Ornstein-Uhlenbeck process, or OU process for short, which is the solution to the stochastic differential equation

$$\mathrm{d}\sigma^2(t) = -\lambda\sigma^2(t)\mathrm{d}t + \mathrm{d}z(\lambda t), \tag{26}$$

where z is a subordinator (that is a Lévy process with non-negative increments). These models have been developed in this context by Barndorff-Nielsen and Shephard (2001a). In Figure 1(a), (c), (e) we have drawn a curve to represent a simulated sample path of $\sigma_n^{[2]}$ from an OU process where $z(t)$ has a $\Gamma(t4, 8)$ marginal distribution, $\lambda = -\log(0.99)$ and $\Delta = 1$, along with the associated realised quadratic variation (depicted using crosses) computed using a variety of values of M. It is helpful to keep in mind that

$$\mathrm{E}\left(\sigma^2(t)\right) = \mathrm{E}\left(z(1)\right) = \frac{1}{2} \quad \text{and} \quad \mathrm{Var}\left(\sigma^2(t)\right) = \frac{1}{2}\mathrm{Var}\left(z(1)\right) = \frac{1}{32}.$$

The corresponding results for $\sigma_n^{[1]}$ and realised absolute variation is given in Figure 1(b), (d), (f). We see that as M increases the precision of realised power variation increases, while Figure 1 shows that the variance of the realised power variation increases with the level of volatility. This is line with the prediction from the asymptotic theory, for the denominator increases with the level of volatility.

6.2.2 QQ plots

To assess the finite sample performance of the asymptotic distributions of the realised quadratic and absolute variation we have constructed some QQ plots based upon the standardised errors (7) and (8). Our main focus will be on the absolute variation case, leaving the quadratic case to

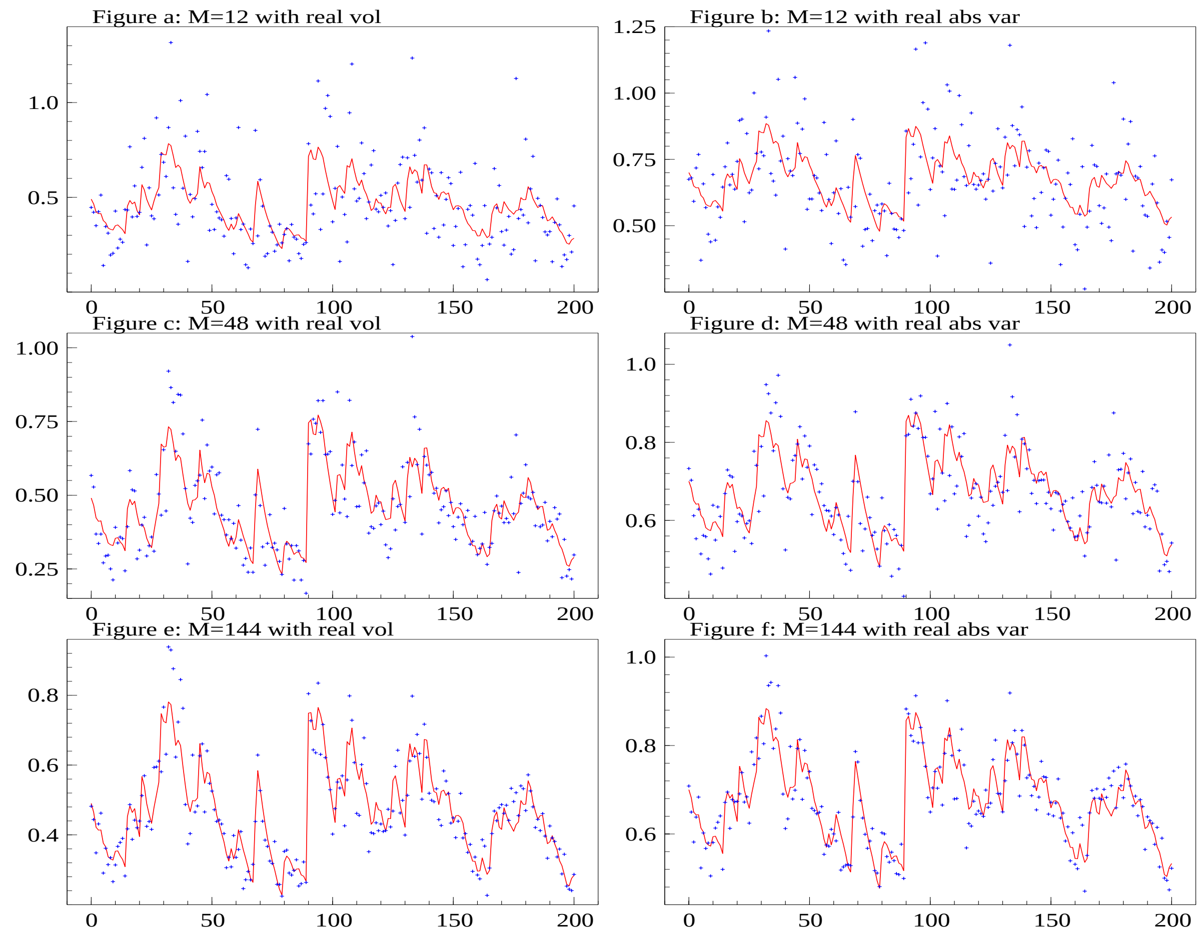

Figure 1: *Simulation from an OU-$\Gamma(4,8)$ process. Drawn is $\sigma_n^{[2]}$ and $\sigma_n^{[1]}$ against time, together with their associated realised power variation estimators. Graph is computed for $M = 12$, 48 and 144.*

be covered in detail in a follow up paper by Barndorff-Nielsen and Shephard (2001b). However, to start off with we give both cases, in order to allow an easy comparison.

Figure 2 gives QQ plots based on the simulation experiment reported in the previous subsection, with a sample size of $10,000$. Again we vary M over 12, 48 and 144. The left hand side graphs give the realised quadratic variation errors $[y_\delta^*]_n^{[2]} - \sigma_n^{[2]}$ as well as plus and minus two times the asymptotic standard errors. The plot is based on the first 600 simulations. This graph shows how much the standard errors change through time. This continues to happen with large values of M and reflects the stochastic denominator in the limit theory.

The middle graphs of Figure 2 give the corresponding realised absolute variation errors

$$\sqrt{\frac{\pi\delta}{2}}\sum_{j=1}^{M}|y_{j,n}| - \sigma_n^{[1]},$$

together with twice standard error bounds. The conditional standard errors are more stable

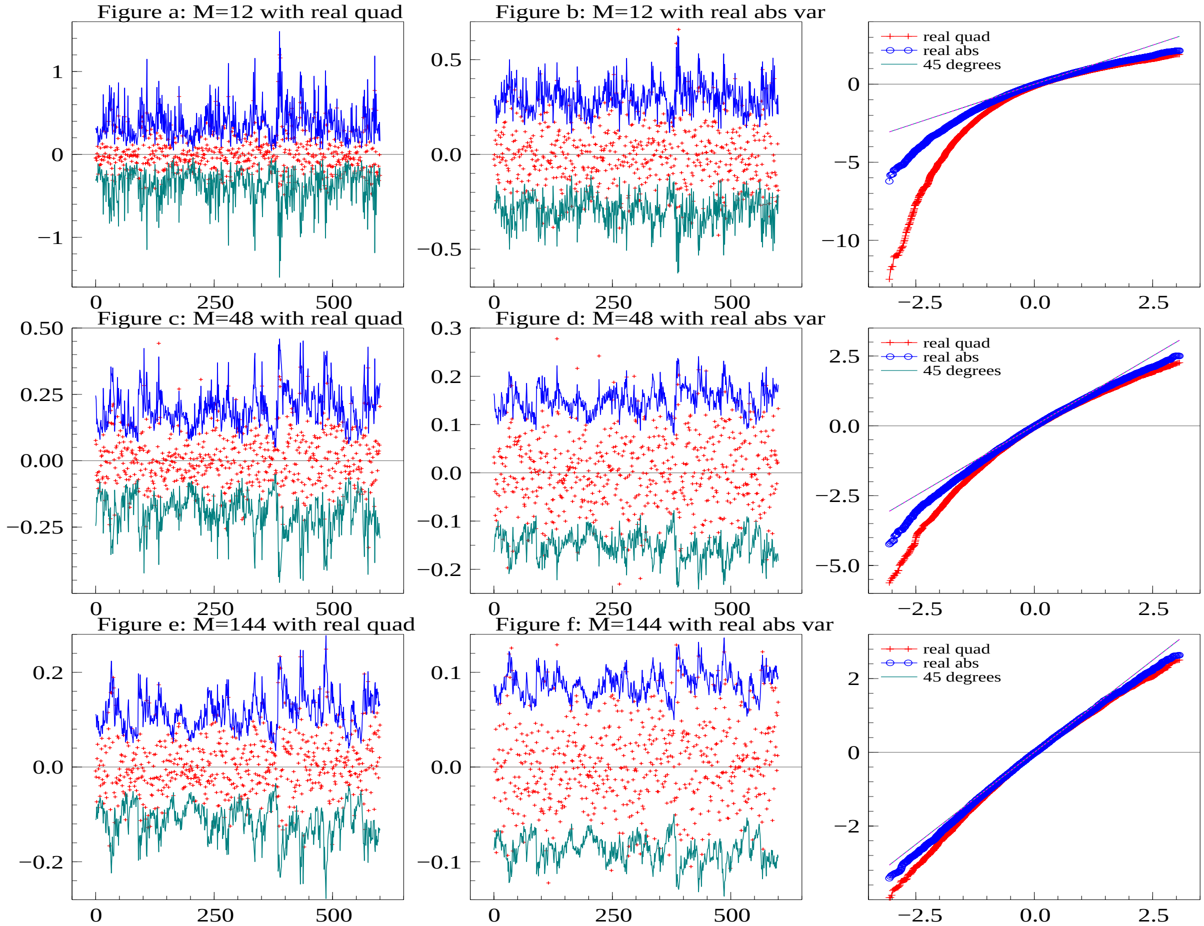

Figure 2: *Plots for the realised quadratic variation error and the realised absolute variation error plus twice their asymptotic standard errors. (a) and (b) have $M = 12$, while (c) and (d) has $M = 48$ and while (e) and (f) has $M = 144$. Corresponding QQ plots are on the right hand side, based on the standardised realised quadratic variations and the realised absolute variations.*

through time, especially when M is small. The unconditional variance of the errors is approximately

$$\delta\left(\mu_1^{-2}-1\right)\Delta \mathrm{E}\left\{\sigma^2(t)\right\}.$$

The right hand side of Figure 2 gives the associated QQ plots for the standardised residuals from the realised quadratic and absolute variation measures. These use all $10,000$ observations. The results are clear, in comparison with the asymptotic limit laws both random variables are too fat tailed in small samples, with this problem reducing as M increases. The realised absolute variation version of the statistic has much better finite sample behaviour, while the realised quadratic variation is quite poorly behaved.

6.2.3 Logarithmic transformation

The realised power variation $[y_\delta^*]^{[r]}(t)$ is the sum of non-negative items and so is non-negative. It would seem sensible to transform this variable to the real line in order to improve its finite sample performance. Hence we use the standard logarithmic transformation (that is for a consistent estimator $\widehat{\theta}$ of θ we approximate $\log(\widehat{\theta})$ by $\log(\theta)+\left(\widehat{\theta}-\theta\right)/\theta$, hence the asymptotic distribution of $\widehat{\theta}-\theta$ can be used to deduce the asymptotic distribution of $\log(\widehat{\theta})-\log(\theta)$). For the general realised power variation this implies

$$\frac{\log\left[\mu_r^{-1}\delta^{1-r/2}[y_\delta^*]^{[r]}(t)\right]-\log\int_0^t\sigma^r(s)\mathrm{d}s}{\mu_r^{-1}\delta^{1-r/2}\sqrt{\frac{v_r}{\mu_{2r}}\frac{[y_\delta^*]^{[2r]}(t)}{\left[\mu_r^{-1}\delta^{1-r/2}[y_\delta^*]^{[r]}(t)\right]^2}}}\overset{\mathcal{L}}{\to}N(0,1). \tag{27}$$

Referring to the denominator we should note that

$$\frac{\mu_{2r}^{-1}\delta^{1-r}[y_\delta^*]^{[2r]}(t)}{\left[\mu_r^{-1}\delta^{1-r/2}[y_\delta^*]^{[r]}(t)\right]^2}\overset{\mathcal{L}}{\to}\frac{\int_0^t\sigma^{2r}(s)\mathrm{d}s}{\left\{\int_0^t\sigma^r(s)\mathrm{d}s\right\}^2}\geq 1,$$

by Jensen's inequality. In the realised absolute variation case (27) simplifies to

$$\frac{\log\left\{\sqrt{\frac{\pi\delta}{2}}\sum_{j=1}^M|y_j(t)|\right\}-\log\int_0^t\sigma(s)\mathrm{d}s}{\sqrt{\delta\left(\frac{\pi}{2}-1\right)\frac{\sum_{j=1}^M y_j^2(t)}{\left\{\sqrt{\frac{\pi\delta}{2}}\sum_{j=1}^M|y_j(t)|\right\}^2}}}\overset{\mathcal{L}}{\to}N(0,1). \tag{28}$$

Using the same simulation setup as that employed in the previous subsection, we plot in Figure 3 the log version of the realised absolute variation error

$$\log\left\{\sqrt{\frac{\pi\delta}{2}}\sum_{j=1}^M|y_{j,n}|\right\}-\log\sigma_n^{[1]},$$

plus and minus twice the corresponding standard errors using (28). The standard errors have now stabilised, almost not moving with n. This is not surprising for t times

$$\frac{t^{-1}\int_0^t\sigma^2(s)\mathrm{d}s}{\left\{t^{-1}\int_0^t\sigma(s)\mathrm{d}s\right\}^2}$$

is empirically very close to one for small values of t while for large t it converges to

$$\frac{\mathrm{E}\left\{\sigma^2(t)\right\}}{\left[\mathrm{E}\left\{\sigma(t)\right\}\right]^2},$$

so long as the volatility process is ergodic and the moments exist.

The corresponding QQ plots in Figure 3 have also improved, with the normality approximation being accurate even for moderate values of M. This result carries over to wider simulations we have conducted.

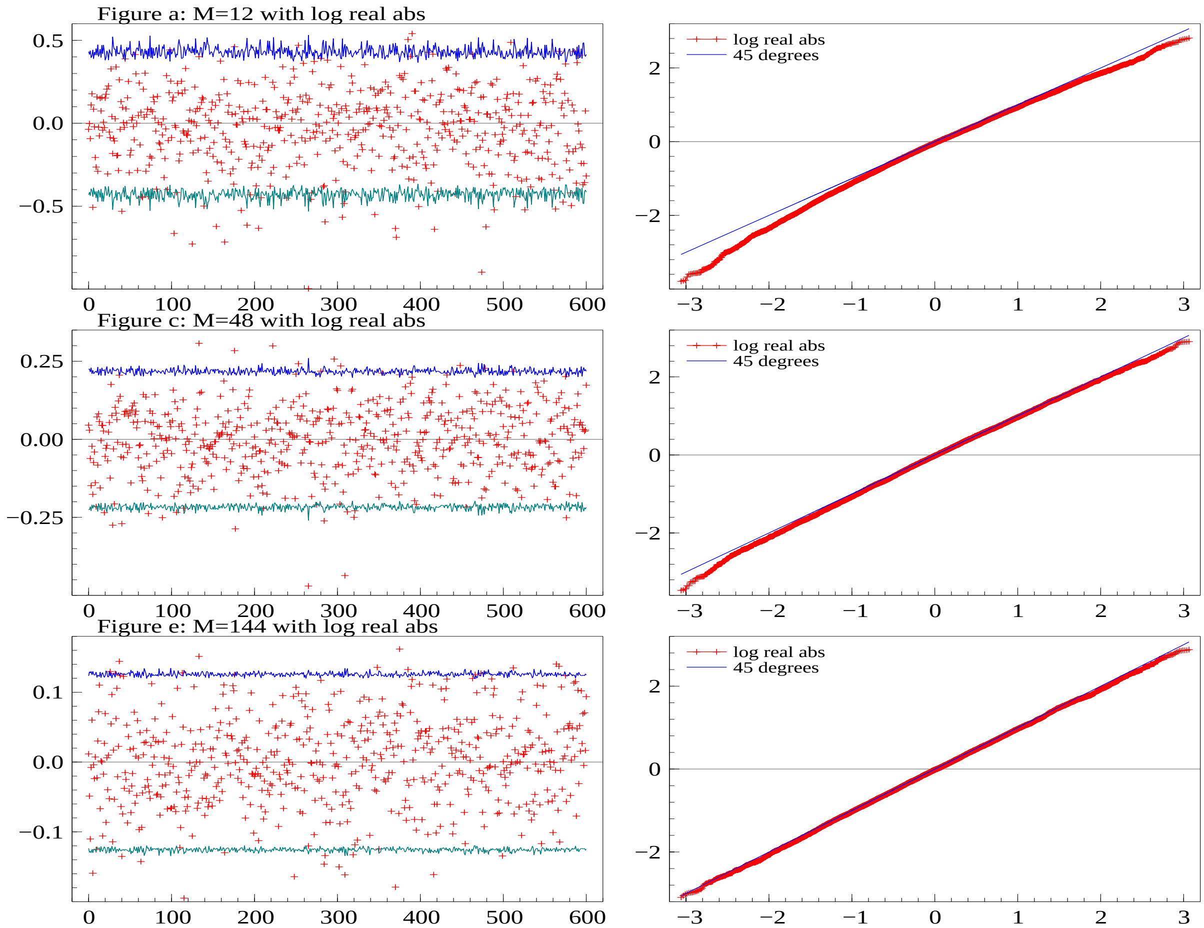

Figure 3: *Plots for the log transform of the realised absolute variation. Left hand plots are the errors plus and minus twice their asymptotic standard errors. Corresponding QQ plots are on the right hand side, based on the standardised log realised absolute variations.*

7 Conclusions

This paper has introduced the idea of realised power variation, which generalises the concept of realised volatility. The asymptotic analysis we provide, for $\delta \downarrow 0$, represents a significant extension of the usual quadratic variation result. Further, we provide a limiting distribution theory which considerably strengthens the consistency result and allows us to understand the variability of the difference between the realised power variation and the actual power volatility.

We have seen that when we take a log transformation of the realised power variation then the finite sample performance of the asymptotic approximation to the distribution of this estimator improves and seems to be accurate even for moderate values of M.

Finally, our motivation for the study reported in this paper came originally from mathematical finance and financial econometrics where volatility is a key object of study. However, stochastic models in the form of a 'signal' α plus a noise term e where e is (conditionally) Gaussian with a variance that varies from 'site' to 'site' are ubiquitous in the natural and technical sciences, and we believe that results similar to those discussed here will be of interest for applications in a variety of other fields, for instance in turbulence and in spatial statistics.

8 Acknowledgments

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