PSEUDO-DIFFUSIONS AND QUADRATIC TERM STRUCTURE MODELS

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Abstract. The non-gaussianity of processes observed in financial markets and relatively good performance of gaussian models can be reconciled by replacing the Brownian motion with Lévy processes whose Lévy densities decay as $\exp(-\lambda|x|)$ or faster, where $\lambda > 0$ is large. This leads to asymptotic pricing models. The leading term, P_0, is the price in the Gaussian model with the same instantaneous drift and variance. The first correction term depends on the instantaneous moments of order up to three, that is, the skewness is taken into account, the next term depends on moments of order four (kurtosis) as well, etc. In empirical studies, the asymptotic formula can be applied without explicit specification of the underlying process: it suffices to assume that the instantaneous moments of order greater than two are small w.r.t. moments of order one and two, and use empirical data on moments of order up to three or four. As an application, the bond pricing problem in the non-Gaussian quadratic term structure model is solved.

Key words: Quadratic term structure models, Lévy processes, asymptotic solution

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1. Introduction

To account for fat tails, skewness and excessive kurtosis of empirical probability distributions of returns in real Financial Markets, it has become increasingly popular to model the dynamics of market factors as a Lévy process. Lévy models are more realistic than Gaussian ones but the latter are much more tractable. Indeed, in the Gaussian framework, explicit pricing formulas are known for a wide range of options and other contingent claims both without and with early exercise features, whereas in the Lévy models, most of the pricing formulas have been obtained for contingent claims of the European type, with the deterministic life-span. There are some explicit analytic results for options with early exercise features: see Boyarchenko and Levendorskiĭ (2000, 2001, 2002a, b), Mordecki (2002) and the bibliography therein for pricing of perpetual American options, and Boyarchenko and Levendorskiĭ (2002b, c) for pricing of barrier options and first touch digitals. However, the pricing formulas are complicated and difficult for numerical implementation except for a rather special case of pricing of perpetual American options under exponential jump-diffusions or spectrally one-sided processes.

Another obstacle for non-Gaussian modelling arises when one considers more general Markov processes. The explicit pricing formulas in affine term structure models and certain Lévy-driven Ornstein-Uhlenbeck models are known in the case of contingent claims with the deterministic life span only – see Duffie et al. (2000, 2002), Chasko and Das (2002), and Barndorff-Nielsen and Shephard (2001b), respectively. In the general case, the dependance on the state variable does not allow one to obtain explicit analytical answers. Barndorff-Nielsen and Levendorskiĭ (2001) notice that typically a good fit to the data can be achieved with Lévy processes whose Lévy densities decay as $\exp(-\lambda|x|)$ or faster, where $\lambda > 0$ (the steepness parameter of the exponential Lévy process) is large. They used this property to derive an asymptotic pricing formula for European options under certain class of Feller processes. The same observation was used in Boyarchenko and Levendorskiĭ (2002a,b,d) and Kudryavzev and Levendorskiĭ (2002) to derive efficient approximate formulas for perpetual American and Bermudan options, and first-touch-digitals, respectively.

It was observed in Boyarchenko and Levendorskiĭ (2002a, b) that the simple approximate formula is of the same form as the corresponding formula in a Gaussian model even when the underlying Lévy process has no Gaussian component. It can be shown that the leading term of the approximate pricing formula in Barndorff-Nielsen and Levendorskiĭ (2001) can also be written as the pricing formula in a Gaussian model. These observations can serve as an analytical explanation of relatively good performance of Gaussian models in apparently non-Gaussian situations. Thus, as far as pricing formulas are concerned, Lévy processes with large steepness parameters behave almost as the Brownian motion, and Feller processes with large steepness parameters considered in Barndorff-Nielsen and Levendorskiĭ (2001) behave almost as Gaussian diffusions. It seems reasonable to use the nomer *pseudo-diffusions* for Lévy processes and more general Lévy-like Feller processes with large steepness parameters .

As was observed above, the modelling with pseudo-diffusions allows one to obtain an efficient approximation to the price; in some situations, the asymptotic expansion of the price can be obtained, of the form

$$P(x,t) \sim P_0(x,t)(1 + \lambda^{-1}P_1(x,t) + \lambda^{-2}P_2(x,t) + \cdots), \tag{1.1}$$

where the leading term, P_0, is the price in the Gaussian model with the same instantaneous drift and variance. The first correction term takes into account the moments of order three as well (skewness), the second correction term accounts for moments of order four, etc. Notice that though the leading term looks as the pricing formula in the Gaussian model, the "drift" and "variance-covariance matrix" used in the formula for the leading term are not the same as the ones of the Gaussian component of the process unless it is purely Gaussian. Indeed, the Lévy process may have no diffusion component at all.

The aim of the paper is to consider quadratic term structure models (QTSM) when the stochastic factor follows a mean-reverting pseudo-diffusion process of the simplest form (it is unlikely that in the QTSM model, an explicit pricing formula can be obtained unless the process process is Gaussian), and derive a pricing formula of the form (1.1). For the discussion about advantages of the Gaussian QTSM model, see Ahn et al (2002) and the bibliography therein.

1.1. **Plan of the paper.** In Section 2, we list families of exponential Lévy processes used in empirical studies of financial markets. In Section 3, we formulate the pricing problem for an interest rate derivative of the European type, and by using the Feynman-Kac theorem, reduce the pricing problem to the boundary problem for an integro-differential equation. We also explain the scheme of the asymptotic pricing. In Section 4, we recall the solution of the bond pricing problem in the one-factor Gaussian case, and indicate the properties of the solution which are crucial for our asymptotic method. In Section 5, we demonstrate our method in the simplest case of the one-factor Lévy model for the bond price, and in Section 6, we present numerical examples. In Section 7, we summarize our results and suggest a procedure of parameter fitting based on the asymptotic expansion. In the appendix, we prove technical results.

2. Lévy processes in financial modelling

As early as in 1963, Mandelbrot suggested to use stable Lévy processes. The modelling with stable Lévy processes is not quite realistic since the tails of Lévy stable distributions are too fat (polynomially decaying), whereas the tails of distributions of returns observed in real financial markets exhibit exponential decay. Moreover, the second moment of a Lévy stable distribution is infinite (unless it is a Gaussian one). This contradicts the observed convergence to the Gaussian distribution over a longer time scale, and even worse, the underlying stock itself should have the infinite price under the stable Lévy process, which makes the model inconsistent for pricing purposes. Starting with the beginning of the 90th, several families of Lévy processes with probability distributions having exponentially decaying tails have been used to describe the behavior of stock prices in real financial markets:

- Variance Gamma Processes (VGP) constructed and used by Madan and co-authors in a series of papers during 90th (see Madan et al. (1998) and the bibliography therein);
- Hyperbolic Processes (HP) were constructed and used by Eberlein and co-authors (see Eberlein et al. (1998), Eberlein and Prause (1999)); hyperbolic distributions were constructed by Barndorff-Nielsen (1977));
- Normal Inverse Gaussian Processes (NIG) were introduced by Barndorff-Nielsen (1998) and used to model German stocks by Barndorff-Nielsen and Jiang (1998);
- Truncated Lévy Processes (TLP) constructed by Koponen (1995) were used for modeling in real financial markets by Bouchaud and Potters (1997), Cont et al (1997) and Matacz (2001); the extended Koponen family was constructed in Boyarchenko and Levendorskiǐ (2000) (the generalization was needed since probability distribution of Koponen's family have tails of the same rate of exponential decay whereas in real financial markets, the left tail is usually much fatter; in Boyarchenko and Levendorskiǐ (2002a,b), the name KoBoL processes is used).
- Normal Tempered Stable Lévy processes were constructed in Barndorff-Nielsen and Levendorskiǐ (2001) and Barndorff-Nielsen and Shephard (2001a); they contain NIG as a subclass.

In Boyarchenko and Levendorskiǐ (2000), a general class of Lévy processes, which contained all the classes listed above modulo certain reservation about VGP was introduced, under the name Generalized Truncated Lévy Processes. Later, in Barndorff-Nielsen and Levendorskiǐ (2001), the name: "Regular Lévy processes of exponential type" (RLPE) was suggested. For a more detailed exposition, see Boyarchenko and Levendorskiǐ (2002a, 2002b). In order to present examples, recall that a Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\langle\xi,X(t)\rangle}] = e^{-t\psi(\xi)}$. The characteristic exponent is given by the Lévy-Khintchine formula

$$\psi(\xi) = -i\langle b,\xi\rangle + \frac{1}{2}||\Sigma\xi||^2 + \int_{\mathbf{R}^n}(1 + i\langle\xi,y\rangle\mathbf{1}_{|\cdot|\le 1}(y) - e^{i\langle\xi,y\rangle})F(dy), \tag{2.1}$$

where $A := \Sigma^T\Sigma$ is the variance-covariance matrix of the Gaussian component, $b \in \mathbf{R}^n$, and $F(dx)$ is the Lévy density (density of jumps), which satisfies

$$\int_{\mathbf{R}^n} \min\{|x|^2, 1\}F(dx) < \infty,$$

Any *generating triplet* $A, b, F(dx)$ with these properties defines a Lévy process (see e.g. Sato (1999)). If $\Sigma = 0$, then we have a pure jump process.

Wide families of jump-diffusion processes are subclasses of the class of RLPE; they are RLPE of order 2.

Example 2.1. Let X be a Lévy process with the Lévy density

$$F(dx) = c_+\lambda_+ e^{\lambda_+ x}\mathbf{1}_{(-\infty,0)}(x)dx + c_-(-\lambda_-)e^{\lambda_- x}\mathbf{1}_{(0,+\infty)}(x)dx,$$

where $\lambda_+ > 0, \lambda_- < -1$ and $c_\pm > 0$. Then

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $\sigma^2 \geq 0$ and $b \in \mathbf{R}$ are the variance and drift of the Gaussian component. The $\psi(\xi)$ is analytic in the strip $\Im\xi \in (\lambda_-, \lambda_+)$.

Example 2.2. The characteristic exponent of a process of KoBoL family in 1D is of the form

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{2.2}$$

where $\nu \in (0, 2), \nu \neq 1, c > 0, \lambda_- < 0 < \lambda_+$, and $\mu \in \mathbf{R}$; it is analytic in a strip $\Im\xi \in (\lambda_-, \lambda_+)$, and (3.8)-(3.9) are satisfied in this strip.

Example 2.3. The characteristic exponent of a Normal Inverse Gaussian process in 1D is of the form

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{2.3}$$

where $\nu \in (0, 2), \delta > 0$, and $\alpha > |\beta|$; it is analytic in the strip $\Im\xi \in (-\alpha + \beta, \alpha + \beta)$, and (3.8)-(3.9) are satisfied in this strip, with $\nu = 1$.

Since the sum of the characteristic exponents of two RLPE's is the characteristic exponent of an RLPE, the list of model examples can easily be expanded. For multi-dimensional examples, see Boyarchenko and Levendorskiĭ (2002b).

Examples 2.1–2.3 are examples of pseudo-diffusions if $\lambda_+, |\lambda_-|$, and $\alpha \pm \beta$ are large. Typically, processes observed in empirical studies of financial markets (hyperbolic processes and variance gaussian processes including) enjoy this property.

The majority of papers on Lévy models deal with asset pricing. Eberlein and Raible (1999) consider the HJM-model driven by a Lévy process (see also Eberlein and Özkan (2001)). For the usage of jump-diffusion processes and more general Lévy processes in affine term structure models of interest rates, see Duffie et al. (2000, 2002), Chasko and Das (2002) and the bibliography therein. Barndorff-Nielsen and Shephard (2001b) suggested to use Lévy-driven Ornstein-Uhlenbeck processes for interest rate modelling purposes.

3. The model

3.1. **Lévy-driven QTSM.** In the Gaussian QTSM, the instantaneous interest rate is represented as a quadratic function of the state variables, and the latter are specified as diffusions. We assume that under an EMM chosen by the market, the SDE of the state variables can be written as

$$dX(t) = (\tilde{\theta}(t) - \kappa X(t))dt + dZ(t), \tag{3.1}$$

where $\{Z(t)\}$ is an n-dimensional Lévy process, $\tilde{\theta} : \mathbf{R}^n \to \mathbf{R}$ is a continuous vector-function, and κ is a constant $n \times n$ matrix, whose eigenvalues λ_j satisfy the condition

$$\Re\lambda_j > 0. \tag{3.2}$$

The interest rate is modelled as

$$r(X(t)) = R_0 + 2\langle R_1, X(t)\rangle + \langle \Gamma X(t), X(t)\rangle, \tag{3.3}$$

where $R_0 \in \mathbf{R}$, $R_1 \in \mathbf{R}^n$ are constant scalar and vector, $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbf{R}^n$, and Γ is positively definite symmetric matrix. The last condition ensures that

$$r(X(t)) = \langle \Gamma(X(t) + \Gamma^{-1}R_1), X(t) + \Gamma^{-1}R_1\rangle + R_0 - ||\Gamma^{-1}R_1||^2$$

is semi-bounded from below. By choosing R_0, R_1 and Γ appropriately, one can ensure any lower bound on $r(X(t))$. Notice that if one wishes to price a derivative of a stock whose dynamics is characterized by X, then one may allow r to depend only on some of the factors $X_j(t)$, say, $r = r(X_1(t), \dots, X_m(t))$, where $m < n$; in this case, in (3.3), $R_1 \in \mathbf{R}^m$, and Γ is an $m \times m$ matrix.

If Z has no jump component then the bond pricing problem reduces to a system of ODE (Riccati equations), which can easily be solved numerically, and in the one-factor case, even analytically. It seems unlikely that a reasonably simple exact solution exists for a general Lévy process but we manage to obtain an *asymptotic solution* if X is a pseudo-diffusion, that is, the Lévy density of Z decays exponentially, and the steepness parameter is large. The leading term of the asymptotics is the price in the Gaussian model with the same instantaneous moments of order one and two, and the correction terms are polynomials in the factors with coefficients depending on the time to expiry. After the leading term is found, they can be calculated recursively, by using only integration procedures. Thus, the suggested method is relatively simple (though in multi-factor models, the number of additional integration procedures may be rather large). In the one-factor case, the first correction term is proportional to skewness, and the second one depends on the skewness and kurtosis; to be more precise, the first correction is proportional to skewness, and the second one is the sum of two terms, one of which is proportional to the square of the skewness, and the other to the kurtosis. In many cases, the contribution of the kurtosis is small relative to the other terms; if we omit the last term, then the pricing formula becomes a sum of the leading term which looks as the price in the Gaussian model, and the correction term, which is a quadratic polynomial w.r.t. to skewness. We obtain a formula, where skewness (or the excess of the historic skewness over the skewness under an EMM) can be used as a parameter characterizing the risk.

Similar formula for the forward rate and numerical examples show that the first correction term has a pronounced upward hump, if the skewness is negative; in the result, the corrected forward rate curve can be hump shaped even when the Gaussian forward rate curve is not, and all parameters of the model are time-independent. By changing the parameters, various shapes of the forward rate curve can be obtained.

Empirical studies show that both skewness and kurtosis can be fairly large, and hence, the corrections to the Gaussian price quite sizable. Consider, for instance, the statistics for the daily change interest rates (dr) from Table 1 in Das (2002). (The table presents descriptive statistics for the Fed Funds rate over the period January 1988 to December 1997, and the unit is 1 percent). Mean: $m = -0.0005$; standard deviation: $\sigma = 0.2899$; skewness: $\lambda_3 = 0.3950$; excess kurtosis: $k_4 = 19.8667$. Recall that for probability distribution $P(dx)$,

$$m := \langle x \rangle := \int_{-\infty}^{+\infty} xP(dx), \quad \sigma^2 := \langle (x-m)^2 \rangle$$

$$\lambda := \langle (x-m)^3 \rangle / \sigma^3, \quad k_4 := \langle (x-m)^4 \rangle / \sigma^4 - 3,$$

and that if $P(dx) = P_{\Delta t}(dx)$ is the probability distribution of a Lévy process with the characteristic exponent ψ, then

$$m(\Delta t)/\Delta t = i\psi'(0); \quad \sigma^2(\Delta t)/\Delta t = \psi''(0);$$

$$\langle (x-m)^3 \rangle (\Delta t)/\Delta t = -i\psi^{(3)}(0); \quad [\langle (x-m)^4 \rangle (\Delta t) - 3\sigma^4(\Delta t)]/\Delta t = -\psi^{(4)}(0).$$

We see that the coefficients in the third and fourth terms in the Taylor series for ψ around zero are smaller than the second one but non-negligible whereas in the Gaussian case all coefficients starting from the third one are zero.

The skewness and kurtosis of the process under an EMM can assume essentially arbitrary values provided they are small w.r.t. variance; in particular, one should expect that the skewness of the process under EMM is negative even when the one under historic measure is positive as in the empirical example above. This means that even the one-factor approximate non-Gaussian model has two free additional parameters (albeit small) which can be used to get a better fit to the data than in the Gaussian model. In multi-factor models, the number of additional free parameters is larger still.

3.2. **Reduction of a pricing problem to a boundary problem.** Consider a contingent claim with the maturity date T and payoff $g(X(T))$. Its price at time $t < T$ is given by

$$f(X(t), t) = E_t[e^{-\int_t^T r(X(s))ds} g(X(T))]. \tag{3.4}$$

In applications, the payoff g is measurable (usually, continuous), and it may grow at infinity. In the latter case, additional conditions on Z may be needed. For instance, if g grows not faster than an exponential:

$$|g(x)| \le Ce^{\omega|x|}, \tag{3.5}$$

where C and $\omega > 0$ are independent of x, then it suffices to assume that there exists $\lambda > \omega$ such that for all μ in the ball $|\mu| < \lambda$, and some $t > 0$,

$$E[e^{\langle \mu, Z(t) \rangle}] < \infty, \tag{3.6}$$

which implies that the tails of probability densities of the process Z decay exponentially: faster than $\exp(-\rho|x|)$, for any $\rho < \lambda$.

It follows from (3.6), that for any $\xi = \eta + i\tau \in \mathbf{C}^n$ from the tube domain $\mathbf{R}^n + iU_\lambda := \{\xi \mid |\Im\xi| = |\tau| < \lambda\}$ (in the one-factor case, a tube domain is a strip), and any $t > 0$,

$$E[e^{i\langle\xi, Z(t)\rangle}] < \infty. \tag{3.7}$$

(Instead of balls U_λ, one can use more general open sets containing the origin.) It is immediate from (3.7), that $\psi(\xi)$ and its derivatives w.r.t. the complex argument ξ are well-defined in the same tube domain $\mathbf{R}^n + iU_\lambda$ (one says that $\psi(\xi)$ is *analytic* in $\mathbf{R}^n + iU_\lambda$), and we may use the latter condition on ψ instead of the former condition (3.7). To justify the use of the Feynman-Kac formula, we assume that Z is a regular Lévy process of exponential type (RLPE). This means that ψ admits a representation

$$\psi(\xi) = -i\langle\mu, \xi\rangle + \phi(\xi), \tag{3.8}$$

where $\mu \in \mathbf{R}^n$, and ϕ satisfies the following condition: there exist $c > 0, \nu \in (0, 2]$ and $\nu_1 < \nu$ such that as $\xi \to \infty$ in the tube domain $\mathbf{R}^n + iU_\lambda$,

$$\phi(\xi) = c|\xi|^\nu + O(|\xi|^{\nu_1}) \tag{3.9}$$

(see Boyarchenko and Levendorskiǐ (2002b)). The ν and U_λ are called the order and type of the process.

To simplify the justification of the use of the Feynman-Kac formula, we add unnecessary condition: for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, there exists a constant C_α such that for all ξ in the tube domain $\mathbf{R}^n + iU_\lambda$,

$$|\partial^\alpha \phi(\xi)| \le C_\alpha (1 + |\xi|)^{\nu - |\alpha|}, \tag{3.10}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Notice that this condition holds for all model classes of RPPE's.

In the appendix, by making use of the Feynman-Kac formula, we will prove the following theorem.

Theorem 3.1. *Let the stochastic factor satisfy (3.1), (3.2), (3.3), (3.6), (3.8), (3.9), and (3.10), let r be given by (3.3), and let g be a continuous function, which admits a bound (3.5).*

Then a) the stochastic expression (3.4) defines a continuous function f, which admits an estimate

$$|f(x, t)| \le C_1 e^{\omega|x|}, \tag{3.11}$$

where C_1 is independent of x and $t \le T$;

b) f is a unique solution to the following problem:

$$(\partial_t + \langle\tilde\theta(t) - \kappa x, \partial_x\rangle + L - r(x)) f(x, t) = 0, \quad t < T, \tag{3.12}$$

$$f(x, T) = g(x), \tag{3.13}$$

where L is the infinitesimal generator of Z.

Recall that the infinitesimal generator of the Lévy process Z, L, can be represented in the form of a pseudo-differential operator (PDO) with the symbol $-\psi$: $L = -\psi(D_x)$. A PDO $A = a(D)$ with the symbol a acts on sufficiently regular functions as follows:

$$(Au)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} a(\xi)\hat{u}(\xi)d\xi,$$

where $\hat{u}$ is the Fourier transform of u:

$$\hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-i\langle x,\xi\rangle} u(x)dx.$$

In particular, the partial derivative ∂_x is the PDO with the symbol $i\xi$.

3.3. **Asymptotic pricing.** The asymptotic pricing formulas will be derived under the following conditions. Assume that the characteristic exponent of the driving Lévy process depends on a small parameter $\epsilon > 0$: $\psi(\xi) = \psi(\epsilon, \xi)$ and satisfies the following three conditions. First, we require that the λ in the definition of the tube domain $\mathbf{R}^n + iU_\lambda$ satisfies $\lambda >> \epsilon^{-1/2}$. The next two conditions are formulated for ξ in the tube domain $\mathbf{R}^n + iU_\lambda$:

1) in the region $|\xi| > \epsilon^{-1/2}$, $\psi(\epsilon, \xi)$ admits an estimate

$$\Re\psi(\epsilon, \xi) \geq c|\xi|^\nu, \tag{3.14}$$

where $\nu \in (0, 2]$ and $c > 0$ are independent of (ϵ, ξ) in the region;

2) in the region $|\xi| \leq \epsilon^{-1/2}$, $\psi(\epsilon, \xi)$ admits an asymptotic expansion: in the one-factor case,

$$\psi(\epsilon, \xi) = -i\mu\xi + \frac{\sigma^2}{2}\xi^2 - \sum_{j=3}^{\infty} \epsilon^{j-2} k_j \cdot (i\xi)^j, \tag{3.15}$$

where the coefficients k_j are uniformly bounded:

$$|k_j| \leq C\sigma^2/2, \tag{3.16}$$

where C is independent of j; in the multi-variate case, (3.15) is replaced with

$$\psi(\epsilon, \xi) = -i\langle \mu, \xi\rangle + \frac{1}{2}||\Sigma\xi||^2 - \sum_{j=3}^{\infty} \epsilon^{j-2} k_j(i\xi), \tag{3.17}$$

where $k_j(\xi)$ is a homogeneous polynomial of order j, which admits a bound

$$|k_j(\Sigma^{-1} i\xi)| \leq C|\xi|^j, \tag{3.18}$$

where C is independent of j.

The asymptotic solution will be found in the following sections. Here we explain the main idea in the one-factor case. We look for the solution in the form

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots. \tag{3.19}$$

From (3.15), we can formally write

$$L = \mu\partial_x + \frac{\sigma^2}{2}\partial_x^2 + \sum_{j=3}^{\infty} \epsilon^{j-2} k_j \partial_x^j, \tag{3.20}$$

and by substituting (3.19) and (3.20) into (3.12), we obtain a formal equality

$$(L_0 + \sum_{j=1}^{\infty} \epsilon^j L_j)(f_0 + \sum_{l=1}^{\infty} \epsilon^l f_l) = 0,\ t < T, \tag{3.21}$$

where

$$L_0 = \partial_t + (\tilde{\theta}(t) + \mu - \kappa x)\partial_x + \frac{\sigma^2}{2}\partial_x^2 - r(x)$$

is of the same form as the operator in the Gaussian model, and

$$L_l = k_{l+2}\partial_x^{l+2}, \quad l = 1, 2, \dots.$$

By multiplying out in (3.21) and gathering terms of the same order in ϵ, we obtain the following series of problems. The leading term of the asymptotics is found from

$$\begin{aligned} L_0 f_0(x,t) &= 0,\ t < T; \\ f_0(x,T) &= g(x), \end{aligned}$$

which is the pricing problem in the Gaussian model; and the following terms are found step by step, by solving problems

$$\begin{aligned} L_0 f_l(x,t) &= -\sum_{j=1}^{l} k_{j+2}\partial_x^{j+2} f_{l-j}(x,t), \quad t < T, \\ f_l(x,T) &= 0, \end{aligned}$$

for $l = 1, 2, \dots$. We believe that for practical purposes, it suffices to use an approximate formula (3.19) with terms up to order 2; this allows one to take into account the skewness and kurtosis. This approximate solution can be written as

$$f \approx f_0 + \epsilon k_3 f_1 + (\epsilon k_3)^2 f_{21} + \epsilon^2 k_4 f_{22}, \tag{3.22}$$

where f_1, f_{21} and f_{22} solve equations

$$\begin{aligned} L_0 f_1(x,t) &= -\partial_x^3 f_0(x,t), \\ L_0 f_{21}(x,t) &= -\partial_x^3 f_1(x,t), \\ L_0 f_{22}(x,t) &= -\partial_x^4 f_0(x,t) \end{aligned}$$

in the half-space $t < T$, subject to zero boundary condition. The explicit formulas for the bond price can be found in Section 4 and Section 5. Formula (3.22) may seem somewhat inconvenient for practical applications since it depends on the small parameter ϵ, which is not explicitly specified. Notice, however, that

$$\epsilon k_3 = -i\psi^{(3)}(0)/3!, \quad \epsilon^2 k_4 = -\psi^{(4)}(0)/4!,$$

and the derivatives of the characteristic exponent at 0 can be inferred from empirical data - see Introduction. Thus, we may write (3.22) without ϵ:

$$f \approx f_0 - i\frac{\psi^{(3)}(0)}{3!}f_1 - \left(\frac{\psi^{(3)}(0)}{3!}\right)^2 f_{21} - \frac{\psi^{(4)}(0)}{4!}f_{22}. \tag{3.23}$$

By using (3.23), the influence of the moments of order 3 and 4 on the price can be explicitly analyzed; and this influence is highly non-linear in (x, t), since the functions in (3.23) are.

If $P := f$ is the bond price, then we can derive from (3.23) similar approximate formulas for the yield and forward rate.

The final remark is: in order to find a current term of the asymptotics, we have to differentiate the previous terms, therefore an asymptotic solution with several terms of may produce serious errors in the neighborhood of a point where the pay-off g is not sufficiently smooth. Indeed, one can hardly expect that a formula which is polynomial in x, can give a high order approximation in this case. Hence, in a neighborhood of such a point, a different asymptotic formula should be written. It will be derived in a subsequent publication on interest rate derivatives.

4. Bond pricing: Gaussian model, one-factor case

In this section, we recall the solution of problem (3.12)-(3.13) in the one-factor Gaussian case, when $\psi(\xi) = -i\mu\xi + \sigma^2\xi^2/2$, and $L = \mu\partial_x + \frac{\sigma^2}{2}\partial_x^2$, and indicate the properties of the solution which are crucial for the asymptotic method to work.

We assume that the interest rate is a quadratic function of the stochastic factor $X(t)$:

$$r(X(t)) = R_0 + 2R_1X(t) + X(t)^2. \tag{4.1}$$

The dynamics of the stochastic factor, X, is governed by (3.1), where $\tilde{\theta}$ is a scalar function, and κ is a positive scalar. The bond price is given by (3.4) with $g(x) \equiv 1$, hence it is a bounded solution to problem (3.12)-(3.13) with $g(x) = 1$ in the RHS of (3.13).

Set $\tau = T - t$, $\theta(\tau) = \tilde{\theta}(T - t) + \mu$, and with some abuse of notation, write $f(x, \tau)$ instead of $f(x, T - \tau)$. We look for the bounded solution to the problem

$$(-\partial_\tau + (\theta(\tau) - \kappa)\partial_x + \frac{\sigma^2}{2}\partial_x^2 - r(x))f(x, \tau) = 0, \quad \tau > 0, \tag{4.2}$$

$$f(x, 0) = g(x) \tag{4.3}$$

in the form

$$f(x, \tau) = \exp \Phi_0(x, \tau), \tag{4.4}$$

where

$$\Phi_0(x, \tau) = A(\tau)x^2 + B(\tau)x + C(\tau). \tag{4.5}$$

By substituting (4.4) into (4.2), we obtain

$$(\exp(-\Phi_0)\mathcal{L}\exp\Phi_0)(x,\tau) - r(x) = 0, \tag{4.6}$$

where

$$\mathcal{L} = -\partial_\tau + (\theta(\tau)-\kappa)\partial_x + \frac{\sigma^2}{2}\partial_x^2, \tag{4.7}$$

subject to $\Phi_0(x,0)=0$. Straightforward calculations yield the following system of ODE with zero initial data:

$$-A'(\tau) - 2\kappa A(\tau) + 2\sigma^2 A(\tau)^2 - 1 = 0, \tag{4.8}$$

$$-B'(\tau) - \kappa B(\tau) + 2\sigma^2 A(\tau)B(\tau) + 2\theta(\tau)A(\tau) - 2R_1 = 0, \tag{4.9}$$

$$-C'(\tau) + \sigma^2 A(\tau) + \frac{\sigma^2}{2}B(\tau)^2 + \theta(\tau)B(\tau) - R_0 = 0. \tag{4.10}$$

Equation (4.8) is solved by separation of variables:

$$A(\tau) = A_1A_2\frac{1-e^{\omega\tau}}{A_2 - A_1e^{\omega\tau}}, \tag{4.11}$$

where $A_1 < 0 < A_2$ are roots of the quadratic equation $2\sigma^2A^2 + 2\kappa A - 1 = 0$, and $\omega = 2\sigma^2(A_1 - A_2) < 0$. $A(\tau)$ having being found, we can calculate $B(\tau)$ from the linear equation (4.9):

$$B(\tau) = \frac{2e^{\omega_1\tau}(A_2I_1(\tau) - A_1I_2(\tau))}{(A_2-A_1)(A_2-A_1e^{\omega\tau})}, \tag{4.12}$$

where $\omega_1 = 2\sigma^2A_1 - \kappa$, and

$$\begin{aligned} I_1(\tau) &= \int_0^\tau (A_1\theta(s) - R_1)e^{-\omega_1 s}ds, \\ I_2(\tau) &= \int_0^\tau (A_2\theta(s) - R_1)e^{(\omega-\omega_1)s}ds. \end{aligned}$$

If θ is independent of τ, then I_j can be calculated explicitly:

$$\begin{aligned} I_1(\tau) &= \frac{A_1\theta - R_1}{\omega_1}(1-e^{-\omega_1\tau}), \\ I_2(\tau) &= \frac{A_2\theta - R_1}{\omega_1 - \omega}(1-e^{(\omega-\omega_1)\tau}), \end{aligned}$$

and therefore

$$\begin{aligned} B(\tau) &= \frac{2}{(A_2-A_1)(A_2-A_1e^{\omega\tau})} \\ &\quad\times\left[\frac{A_2(A_1\theta-R_1)}{\omega_1}(e^{\omega_1\tau}-1) - \frac{A_1(A_2\theta-R_1)}{\omega_1-\omega}(e^{\omega_1\tau}-e^{\omega\tau})\right]. \end{aligned} \tag{4.13}$$

Finally, we find $C(\tau)$ from (4.10) by integration:

$$C(\tau) = \int_0^\tau (\sigma^2 A(s) + \frac{\sigma^2}{2} B(s)^2 + \theta(s)B(s) - R_0)ds. \tag{4.14}$$

If θ is constant, then $C(\tau)$ can be calculated explicitly. In order that $B(\tau)$ and $C(\tau)$ can be calculated explicitly, θ need not to be a constant; for instance, one can use exponential polynomials.

To end this section, we make the crucial remark on the properties of the solution. First, from (4.11),

$$A(\tau) < 0, \quad \forall\, \tau > 0, \tag{4.15}$$

and as $\tau \to 0$,

$$A(\tau) \sim A_1 A_2 \frac{-\omega}{A_2 - A_1} \tau = A_1 A_2 2\sigma^2 \tau = -\tau. \tag{4.16}$$

Hence, for any $\tau \in (0, T]$, $f(x, \tau)$ decays as $\exp(-\tau x^2)$, as $x \to \pm\infty$, and $\hat{f}(\xi, \tau)$, the Fourier transform of $f(x, \tau)$ w.r.t. the first argument, decays as $\tau^{-1/2} \exp(-\xi^2/(4\tau)$, as $\xi \to \pm\infty$. To be more specific,

$$\begin{aligned} \hat{f}(\xi, \tau) &= \int_{-\infty}^{+\infty} e^{-ix\xi + A(\tau)x^2 + B(\tau)x + C(\tau)} dx \\ &= \frac{1}{-\pi A(\tau)} \exp[C(\tau) + (\xi + iB(\tau))^2/(4A(\tau))]. \end{aligned} \tag{4.17}$$

We conclude that for any N, in the region $\tau \in (0, T]$, $|\xi| > \epsilon^{-1/2}$,

$$|\xi^N \hat{f}(\xi, \tau)| \le C_N e^{-\xi^2/(8\tau)}, \tag{4.18}$$

where C_N is independent of τ and ξ. Notice that the RHS of (4.18) is negligible.

5. Bond pricing: Lévy model, one-factor case

5.1. **The leading term of the asymptotics.** We assume that (3.14), (3.15) and (3.16) hold. Take μ and σ^2 from (3.15), and denote by f_0 the solution to the Gaussian bond pricing problem (4.2)-(4.3); it is given by (4.4), (4.11), (4.12) and (4.14), and satisfies (4.15), (4.16), (4.17) and (4.18). Introduce $f^1 := f - f_0$. Since f_0 and f are solutions to problems (4.2)-(4.3) and (3.12)-(3.13), respectively, and (3.15) holds, we conclude that f^1 is the solution to the following problem: in the half-plane $\tau > 0$,

$$(-\partial_\tau + (\theta(\tau) - \kappa x)\partial_x - \psi(D_x) - r(x))f^1(x, \tau) = -\mathcal{D}_1(\epsilon, D_x)f_0(x, \tau), \tag{5.1}$$

where

$$\begin{aligned}\mathcal{D}_1(\epsilon, D_x) &:= - \quad \psi(\epsilon,\xi) + \frac{\sigma^2}{2}\xi^2 - i\mu\xi \\ &= \quad \sum_{j=3}^{\infty} \epsilon^{j-2} k_j (i\xi)^j \\ &= \quad \epsilon \sum_{j=3}^{\infty} \epsilon^{j-3} k_j (i\xi)^j.\end{aligned}$$

We also have the initial condition

$$f^1(x,0) = 0. \tag{5.2}$$

From (3.16) and (4.18), the following estimate for the RHS in (5.1) follows:

$$|\mathcal{D}_1(\epsilon,\xi)\hat{f}_0(\xi,\tau)| \le C_0\epsilon\tau^{-1/2}\exp(-\xi^2/(8\tau)), \tag{5.3}$$

where C_0 is independent of $\epsilon \in (0,1)$ and $\tau \in (0,T]$. By making the inverse Fourier transform, we obtain

$$||\mathcal{D}_1(\epsilon, D_x) f_0(x,\tau)||_{C(\mathbf{R}\times[0,T])} \le C\epsilon, \tag{5.4}$$

where C is independent of $\epsilon \in (0,1)$. By applying the Feynman-Kac theorem to (5.1)-(5.2), the representation of f^1 in the form of the stochastic integral results:

$$f^1(x,\tau) = E_{-\tau}\left[\int_{-\tau}^{0} e^{-\int_{-\tau}^{s} r(X(s'))ds'} \mathcal{D}(\epsilon, D_x) f_0(x,s)ds\right],$$

and from (5.4), we derive an estimate

$$|f^1(x,\tau)| \le C\epsilon, \tag{5.5}$$

where C is independent of $\epsilon \in (0,1)$, $x \in \mathbf{R}$ and $\tau \in (0,T]$.

5.2. **First correction term.** Estimate (5.5) shows that f_0 is indeed the leading term of the asymptotics of f as $\epsilon \to 0$, and in view of (3.15), it is natural to look for the first correction term in the form ϵf_1, where f_1 is the solution to the following problem:

$$(-\partial_\tau + (\theta(\tau) - \kappa x)\partial_x + \frac{\sigma^2}{2}\partial_x^2 - r(x))f_1(x,\tau) = -k_3\partial_x^3 f_0(x,\tau), \tag{5.6}$$

in the half-plane $\tau > 0$, subject to

$$f_1(x,0) = 0. \tag{5.7}$$

We look for f_1 in the form

$$f_1(x,\tau) = k_3 f_0(x,\tau)\tilde{f}_1(x,\tau) = k_3 e^{\Phi_0(x,\tau)}\tilde{f}_1(x,\tau), \tag{5.8}$$

where Φ_0 is given by (4.5). By substituting into (5.6), we obtain that $\tilde{f}_1$ solves the problem

$$(\mathcal{L} + \sigma^2(\partial_x\Phi_0)\partial_x + e^{-\Phi_0}(\mathcal{L}e^{\Phi_0}) - r(x))\tilde{f}_1 = -e^{-\Phi_0}\partial_x^3 e^{\Phi_0}, \tag{5.9}$$

in the half-plane $\tau > 0$, subject to

$$\tilde{f}_1(x, \tau) = 0, \tag{5.10}$$

where $\mathcal{L}$ is defined by (4.7). Equation (4.6) allows us to simplify (5.9):

$$(\mathcal{L} + \sigma^2(2A(\tau)x + B(\tau))\partial_x)\tilde{f}_1 = -e^{-\Phi_0}\partial_x^3 e^{\Phi_0}.$$

We calculate the operator

$$\begin{aligned} e^{-\Phi_0}\partial_x^3 e^{\Phi_0} &= (e^{-\Phi_0}\partial_x e^{\Phi_0})^3 \\ &= (\partial_x + 2A(\tau)x + B(\tau))^3, \end{aligned}$$

and by using $\partial_x 1 = 0$, rewrite (5.9) as

$$\mathcal{L}_1\tilde{f}_1(x, \tau) = \tilde{g}_0(x, \tau), \tag{5.11}$$

where

$$\begin{aligned} \tilde{g}_0(x, \tau) &:= 8A^3x^3 + 12A^2Bx^2 + (12A^2 + 6AB^2)x + 6AB + B^3, \\ \mathcal{L}_1 &:= \partial_\tau + (-\theta_1(\tau) + \kappa_1(\tau)x)\partial_x - \frac{\sigma^2}{2}\partial_x^2, \\ \theta_1(\tau) &:= \theta(\tau) + \sigma^2 B(\tau), \\ \kappa_1(\tau) &:= \kappa - 2\sigma^2 A(\tau), \end{aligned}$$

and $A = A(\tau)$, $B = B(\tau)$. Clearly, we may look for the solution to (5.11) in the form of a polynomial of degree 3, with coefficients vanishing at $\tau = 0$:

$$\tilde{f}_1(x, \tau) = a_{13}(\tau)x^3 + a_{12}(\tau)x^2 + a_{11}(\tau)x + a_{10}(\tau). \tag{5.12}$$

By substituting into (5.11), we obtain a system of linear ODE:

$$a'_{13} + 3\kappa_1 a_{13} = 8A^3, \tag{5.13}$$

$$a'_{12} + 2\kappa_1 a_{12} - 3\theta_1 a_{13} = 12A^2B, \tag{5.14}$$

$$a'_{11} + \kappa_1 a_{11} - 2\theta_1 a_{12} - 3\sigma^2 a_{13} = 12A^2 + 6AB^2, \tag{5.15}$$

$$a'_{10} - \theta_1 a_{11} - \sigma^2 a_{12} = 6AB + B^3, \tag{5.16}$$

which can easily be integrated step by step. Namely, let

$$\kappa_2(s) = \int_0^s \kappa_1(s')ds'; \tag{5.17}$$

then

$$a_{13}(\tau) = 8e^{-3\kappa_2(\tau)}\int_0^\tau e^{3\kappa_2(s)}A(s)^3ds, \tag{5.18}$$

$$a_{12}(\tau) = e^{-2\kappa_2(\tau)}\int_0^\tau e^{2\kappa_2(s)}(12A(s)^2B(s)+3\theta_1(s)a_{13}(s))ds, \tag{5.19}$$

$$\begin{aligned} a_{11}(\tau) = e^{-\kappa_2(\tau)}\int_0^\tau e^{\kappa_2(s)}(12A(s)^2+6A(s)B(s)^2 \\ +2\theta_1(s)a_{12}(s)+3\sigma^2a_{13}(s))ds, \end{aligned} \tag{5.20}$$

$$a_{10}(\tau) = \int_0^\tau (6A(s)B(s)+B(s)^3+\theta_1(s)a_{11}(s)+\sigma^2a_{12}(s))ds. \tag{5.21}$$

Formulas (5.8), (5.12) and (5.18)-(5.21) give the first order approximation

$$f \approx f_0\cdot(1+\epsilon k_3\tilde{f}_1) \tag{5.22}$$

to the bond price. The proof similar to the one of estimate (5.5) albeit more involved shows that the error of approximation (5.22) is

$$|f(x,\tau)-f_0(x,\tau)(1+\epsilon k_1\tilde{f}_1(x,\tau))| \le C\epsilon^2(1+|x|^2)^{3/2}, \tag{5.23}$$

where C is independent of $\epsilon\in(0,1)$, $x\in\mathbf{R}$ and $\tau\in(0,T]$.

Unlike (5.5), we have a polynomially growing factor $(1+|x|^2)^{3/2}$ in the RHS of (5.23). Notice, however, that for practical purposes, one needs to know the bond price for small values of $r(X(t))$, hence for small values of $X(t)$, and therefore the polynomially growing factor $(1+|x|^2)^{3/2}$ does not matter much.

5.3. **First correction term II: the derivation based on the change of variables.** To simplify the calculation of the next terms of the asymptotics, it is advantageous to change the variables in equations similar to (5.11):

$$x = -\theta_2(\tau)+e^{\kappa_2(\tau)}y, \tag{5.24}$$

where κ_2 is given by (5.17), and θ_2 is the solution to the Cauchy problem

$$\begin{aligned} \theta_2'(\tau)-\kappa_1(\tau)\theta_2(\tau) &= \theta_1(\tau), \\ \theta_2(0) &= 0, \end{aligned}$$

that is,

$$\theta_2(\tau) = e^{\kappa_2(\tau)}\int_0^\tau e^{-\kappa_2(s)}\theta_1(s)ds.$$

The same change of the variables simplifies the calculation of $\tilde{f}_1$. Introduce an operator S by $S(f)(y,\tau) = f(x(y),\tau)$. Under the change of variables (5.24), $-\partial_\tau+(\theta_1(\tau)-\kappa_1(\tau))\partial_x \mapsto -\partial_\tau$ and $\partial_x\mapsto e^{-\kappa_2(\tau)}\partial_y$, therefore

$$\mathcal{L}_2 := S^{-1}\mathcal{L}_1S = \partial_\tau - \frac{\sigma^2}{2}e^{-2\kappa_2(\tau)}\partial_y^2,$$

and we can rewrite (5.11) in the form

$$\mathcal{L}_2 F_1(y,\tau) = G_0(y,\tau), \tag{5.25}$$

where $F_1 = S\tilde{f}_1$, $G_0 = S\tilde{g}_0$. Clearly, G^0 is a polynomial in y of the same order as $\tilde{g}_0$:

$$G_0(y,\tau) = G_{0,3}(\tau)y^3 + G_{0,2}(\tau)y^2 + G_{0,1}(\tau)y + G_{0,0}(\tau),$$

and the coefficients $G_{0,j}$ can easily be calculated by using formulas for the coefficients of $\tilde{g}_0$ or, better, independently. Under the change of variables (5.24), $\partial_x + 2A(\tau)x + B(\tau)$ becomes

$$\mathcal{D} := e^{-\kappa_2(\tau)}(\partial_y + A_1(\tau)y + B_1(\tau)),$$

where

$$A_1(\tau) = 2e^{2\kappa_2(\tau)}A(\tau), \quad B_1(\tau) = e^{\kappa_2(\tau)}(B(\tau) - 2A(\tau)\theta_2(\tau)),$$

therefore

$$\begin{aligned} G_0 &= \mathcal{D}^3 \cdot 1 \\ &= e^{-3\kappa_2}(A_1^3 y^3 + 3A_1^2 B_1 y^2 + (3A_1^2 + 3A_1 B_1^2)y + 3A_1 B_1 + B_1^3), \end{aligned} \tag{5.26}$$

where $\kappa_2 = \kappa_2(\tau), A_1 = A_1(\tau)$ and $B_1 = B_1(\tau)$. The solution of (5.25) subject to $F_1(y,0) = 0$ is a polynomial in y of the same order as G_0:

$$F_1(y,\tau) = F_{1,3}(\tau)y^3 + F_{1,2}(\tau)y^2 + F_{1,1}(\tau)y + F_{1,0}(\tau),$$

whose coefficients can easily be found by integration:

$$F_{1,3}(\tau) = \int_0^\tau G_{0,3}(s)ds, \tag{5.27}$$

$$F_{1,2}(\tau) = \int_0^\tau G_{0,2}(s)ds, \tag{5.28}$$

$$F_{1,1}(\tau) = \int_0^\tau (G_{0,1}(s) + 3\sigma^2 e^{2\kappa_2(\tau)}G_{0,3}(s))ds, \tag{5.29}$$

$$F_{1,0}(\tau) = \int_0^\tau (G_{0,0}(s) + \sigma^2 e^{2\kappa_2(\tau)}G_{0,2}(s))ds. \tag{5.30}$$

After that we make the inverse change of variables $y = e^{-\kappa_2(\tau)}(x + \theta_2(\tau))$, and calculate $\tilde{f}_1 = S^{-1}F_1$.

5.4. **Next terms of the asymptotics.** Suppose that the approximation of order $j \ge 1$ has been found:

$$f = f_0 \cdot (1 + \sum_{l=1}^{j} \epsilon^l k_{l+2}\tilde{f}_l) = f_0 \sum_{l=0}^{j} \epsilon^l k_{l+2}\tilde{f}_l,$$

where $k_2 = 1$, $f_0 = e^{\Phi_0}$, $\tilde f_0 \equiv 1$, and $\tilde f_l$, $1 \le l \le j$, are polynomials in x with coefficients depending on on τ:

$$\tilde f_l(x,\tau) = \sum_{s=0}^{m_l} a_{ls}(\tau)x^s. \tag{5.31}$$

We look for the next term of the asymptotics in the form $\epsilon^{j+1}k_{j+3}f_0\tilde f_{j+1}$, where $f_{j+1} := f_0\tilde f_{j+1}$ is the solution to the problem

$$(\mathcal{L} - r(x))f_{j+1}(x,\tau) = -g_j(x,\tau),\ \tau > 0, \tag{5.32}$$

$$f_{j+1}(x,0) = 0, \tag{5.33}$$

where $\epsilon^{j+1}g_j$ is the collection of terms of order ϵ^{j+1} in the expression

$$\sum_{p=3}^{\infty} \epsilon^{p-2}k_p\partial_x^p(\sum_{l=0}^{j}\epsilon^l k_{l+2}f_0\tilde f_l),$$

that is,

$$g_j = \sum_{p=3}^{j+3} k_p k_{j+5-p}\partial_x^p(f_0\tilde f_{j+3-p}).$$

Set

$$\begin{aligned}\tilde g_j &= \frac{1}{f_0}g_j \\ &= \sum_{p=3}^{j+3} k_p k_{j+5-p}(e^{-\Phi_0}\partial_x e^{\Phi_0})^p\tilde f_{j+3-p} \\ &= \sum_{p=3}^{j+3} k_p k_{j+5-p}(\partial_x + 2A(\tau)x + B(\tau))^p\tilde f_{j+3-p},\end{aligned} \tag{5.34}$$

and multiply (5.32) by $e^{-\Phi_0}$. We obtain

$$(\mathcal{L} + \sigma^2(\partial_x\Phi_0)\partial_x + e^{-\Phi_0}(\mathcal{L}e^{\Phi_0}) - r(x))\tilde f_{j+1} = -\tilde g_j. \tag{5.35}$$

Equation (4.6) allows us to simplify (5.35) and obtain a problem in the half-space $\tau > 0$ with the unknown $\tilde f_{j+1}$:

$$(-\partial_\tau + (\theta_1(\tau) - \kappa_1(\tau)x)\partial_x + \frac{\sigma^2}{2}\partial_x^2)\tilde f_{j+1} = -\tilde g_j,\ \tau > 0, \tag{5.36}$$

$$\tilde f_{j+1}(x,0) = 0. \tag{5.37}$$

From (5.34), $\tilde g_j$ is a polynomial in x as well:

$$\tilde g_j(x,\tau) = \sum_{s=0}^{m'_j} b_{j,s}(\tau)x^s, \tag{5.38}$$

where $m'_j = \max_{0\le l\le j}(m_l - l + j + 3)$, and any of the coefficients $b_{j,s}$ may be zero, that is, the order of $\tilde{g}_j$ may be less than m'_j. Denote by m_j the order of $\tilde{g}_j$.

By making the change of variables (5.24), we simplify (5.36):

$$\mathcal{L}_2 F_{j+1} = G_j, \tag{5.39}$$

where $F_j = S\tilde{f}_j, G_j = S\tilde{g}_j$. Since G_j is a polynomial in y of order m_j:

$$G_j(y,\tau) = \sum_{l=0}^{m_j} G_{j,l}(\tau) y^l$$

(the coefficients $G_{j,l}$ will be calculated in Subsection 5.5), F_{j+1} also is:

$$F_{j+1}(y,\tau) = \sum_{l=0}^{m_j} F_{j+1,l}(\tau) y^l,$$

and the coefficients $F_{j+1,l}$ are easily found by integration:

$$F_{j+1,l}(\tau) = \int_0^\tau G_{j,l}(s)ds, \ \ l = m_j, m_j - 1; \tag{5.40}$$

$$F_{j+1,l}(\tau) = \int_0^\tau \left(\frac{\sigma^2}{2} e^{2\kappa_2(s)}(l+2)(l+1)F_{j+1,l+2}(s) + G_{j,l}(s) \right) ds, \tag{5.41}$$

for $l = m_j - 2, m_j - 3, \dots 0$.

5.5. **Calculation of $G_{j,l}$.** We can rewrite (5.34) as

$$G_j = \sum_{p=3}^{j+3} k_p k_{j+5-p} \mathcal{D}^p F_{j+3-p}, \quad j = 0, 1, \dots. \tag{5.42}$$

By using the initial data $F_0 = 1$, we find G_j and F_{j+1} from (5.42) and (5.40)-(5.41) step by step. In particular, G_0 is given by (5.26), and

$$G_1 = k_3^2 G_{11} + k_4 G_{12}, \tag{5.43}$$

where

$$G_{11} = \mathcal{D}^3 F_1, \quad \text{and } G_{12} = \mathcal{D}^4 F_0 = \mathcal{D}^4 \cdot 1 \tag{5.44}$$

are polynomials of degree 6 and 4, respectively. Hence,

$$F_1 = k_3^2 F_{21} + k_4 F_{22}, \tag{5.45}$$

where F_{21} and F_{22} are polynomials of degree 6 and 4, respectively, which solve (5.39) with G_{11} and G_{12} in the RHS (and satisfy the initial condition $F_{2j}(y,0) = 0$). The reader can use (5.40)-(5.41) to obtain explicit formulas for coefficients of F_{2j} in terms of coefficients of G_{1j}. Notice that though the latter can be written explicitly, in practical implementation of the method, it is simpler to write a program which calculates the coefficients of a polynomial $\mathcal{D}P$, given coefficients of a polynomial P, and use this program to calculate G_{1j} (and G_l for $l > 1$ should one wish it, though it is not a reasonable thing to do in applications).

5.6. **Second order approximation for the bond price, yield and forward rate.** We see that the second order approximation can be written in the form

$$f \approx e^{\Phi_0}(1 + \epsilon k_3 \tilde{f}_1 + (\epsilon k_3)^2 \tilde{f}_{21} + \epsilon^2 k_4 \tilde{f}_{22}), \tag{5.46}$$

where Φ_0, $\tilde{f}_1 = S^{-1}F_1$, $\tilde{f}_{21} = S^{-1}F_{21}$ and $\tilde{f}_{22} = S^{-1}F_{22}$ are polynomials in x with coefficients depending on τ - and parameters $\tilde{\theta}$, κ, $\mu := i\psi'(0)$, $\sigma^2 := \psi''(0)$. For practical applications,(5.46) can be rewritten in the form

$$f \approx e^{\Phi_0}\left(1 + K_3 \tilde{f}_1 + K_3^2 \tilde{f}_{21} + K_4 \tilde{f}_{22}\right), \tag{5.47}$$

where

$$\begin{aligned} K_3 : &= \epsilon k_3 = -i\psi^{(3)}(0)/3!, \\ K_4 : &= \epsilon^2 k_4 = -\psi^{(4)}(0)/4! \end{aligned}$$

can be inferred from the data.

Denote by $P := f$ is the price of the bond, and by $P_0 := \exp(\Phi_0)$ the price in the Gaussian model; then (5.47) becomes

$$P \approx P_0 + K_3 P_1 + K_3^2 P_{21} + K_4 P_{22}, \tag{5.48}$$

where

$$P_1 = P_0 \tilde{f}_1,\ P_{21} = P_0 \tilde{f}_{21},\ P_{22} = P_0 \tilde{f}_{22}.$$

By using the formulas for the yield

$$R(x, \tau) = -\frac{\ln P(x, \tau)}{\tau}$$

and forward rate

$$F(x, \tau) = -\frac{\partial}{\partial \tau} \ln P(x, \tau),$$

we obtain approximate formulas

$$R \approx R_0 + K_3 R_1 + K_3^2 R_{21} + K_4 R_{22}, \tag{5.49}$$

where

$$\begin{aligned} R_0(x, \tau) &= -\Phi_0(x, \tau)/\tau, \\ R_1(x, \tau) &= -\tilde{f}_1(x, \tau)/\tau, \\ R_{21}(x, \tau) &= (\tilde{f}_1(x, \tau)^2/2 - \tilde{f}_{21}(x, \tau))/\tau, \\ R_{22}(x, \tau) &= -\tilde{f}_{22}(x, \tau)/\tau, \end{aligned}$$

and

$$F \approx F_0 + K_3 F_1 + K_3^2 F_{21} + K_4 F_{22}, \tag{5.50}$$

where

$$\begin{aligned}
F_0(x,\tau) &= -\frac{\partial}{\partial\tau}\Phi_0(x,\tau),\\
F_1(x,\tau) &= -\frac{\partial}{\partial\tau}\tilde{f}_1(x,\tau),\\
F_{21}(x,\tau) &= \frac{\partial}{\partial\tau}(\tilde{f}_1(x,\tau)^2/2-\tilde{f}_{21}(x,\tau)),\\
F_{22}(x,\tau) &= -\frac{\partial}{\partial\tau}\tilde{f}_{22}(x,\tau).
\end{aligned}$$

6. Numerical examples

(The author thanks Nina Boyarchenko for writing the programs for numerical examples and checking the algebra in the previous two sections.) We take the simplest model for r: $r = x^2$, and constant $\tilde{\theta}(t) = 0.06$, $\kappa = 0.3$, $\mu = 0$ and $\sigma^2 = 0.08$. We also fix $x = 0.25$, and study shapes of correction terms to the bond price, yield and forward rates in (5.48), (5.49) and (5.50) (see Figures 1–3).

From Figure 3, we clearly see that it is the first correction term, F_1, that can account for for the hump of the forward rate curve - provided the skewness is negative and not too small in modulus. In the next three Figures 4–6, we plot the bond price, yield and forward rate; first, the leading term (dots), then the formula with the first correction term taken into account (solid line), and finally, the formula with the two correction terms (dotted line). We take the same parameters as above, and $K_3 = -\sigma^2/8 = -0.005$, $K_4 = K_3/20 = 2.5\cdot 10^{-4}$. We see that fairly large skewness does produce a hump-shaped forward rate curve, when the Gaussian curve has no hump; the asymptotic formulas are applicable since K_3 is small w.r.t. σ^2, and K_4 is small w.r.t. K_3.

In the last series of figures (Figures 7–9), we fix small $K_4 = \sigma^2/200$, and show how the shapes of the curves vary with the skewness. We take $K_3 = \sigma^2/16$ (dashes), $K_3 = \sigma^2/32$ (dotted line), $K_3 = -\sigma^2/32$ (dots), and $K_3 = -\sigma^2/16$ (crosses). The solid line is the Gaussian curve.

7. Conclusion

We constructed a class of QTSM models with a regular Lévy process of exponential type in place of the Gaussian one in standard QTSM. By using the Feynman-Kac formula, we have reduced the pricing problem for an interest rate derivative to a boundary problem for a pseudo-differential operator. In the case of the bond, we found an approximate solution to the boundary problem assuming that the tails of probability densities of a process decay sufficiently fast. The leading term of the approximate solution looks as in the Gaussian model (even when the underlying process has no Gaussian component), and the correction terms depend on skewness and kurtosis. In cases when the instantaneous fourth central moment is small w.r.t. the third one, the skewness of the process under EMM can be used as the parameter, which characterizes the risk.

Numerical examples are produced to show that by changing skewness and kurtosis, various shapes of the forward rate curve can be obtained. In particular, negative skewness can produce a hump-shaped forward rate curve even when the Gaussian curve has no hump: the very non-Gaussianity of the process is (one of) causes of the hump of the forward rate curve. Bond prices and the yield curve also change but the types of the shape of the curves remain essentially the same.

Guided by the asymptotic decomposition of the bond price, we suggest the following procedure of the parameter fitting to an empirical bond price curve $P(x, \tau)$:

1. Infer parameters of the Gaussian model from the data by using any standard procedure.
2. Given parameters $\tilde{\theta}, \kappa, \sigma$ of the Gaussian model, calculate correction terms P_1, P_{21} and P_{22}.
3. Choose an appropriate metrics in the space of price curves, and find a pair (K_3, K_4), which minimizes the distance between the error curve of the Gaussian model, $P - P_0$, and the two-dimensional surface

$$V = \{K_3 P_1 + K_3^2 P_{21} + K_4 P_{22} \mid K_3, K_4 \in \mathbf{R}\}.$$

In order that this procedure be consistent, the resulting K_3 and K_4 must be small relative to $\sigma^2/2$.

Similar procedure can be formulated in the multi-factor model; since the manifold V is now multi-dimensional, the fitting quality is expected to be much better.

Appendix A. Proofs of technical results

A.1. **Proof of Theorem 3.1.** By using the decomposition $g = g_+ - g_-$, where $g_+(x) := \max\{g(x), 0\}$ and $g_- = g_+ - g$ are non-negative, we see that it suffices to prove Theorem 3.1 for continuous non-negative g. Fix $\chi \in C^\infty(\mathbf{R})$ such that $0 \le \chi(x) \le 1$ for all x, $\chi(x) = 1, x \le 1$, $\chi(x) = 0, x \ge 2$, and for any $m > 0$, set $g^m(x) := \chi(|x|/m)g(x)$. Then g^m is a continuous function with the compact support. Define f^m by (3.4) with g^m in the RHS. For any x, $g^m(x) \uparrow g(x)$ as $m \to \infty$, and by the Monotone Convergence Theorem, $f^m(x) \uparrow f(x)$. Notice that $f^m \to f$ in the sense of generalized functions: for any non-negative $u \in C_0^\infty(\mathbf{R}^n \times (0, T))$,

$$\int_{\mathbf{R}^n} f^m(x)u(x)dx \to \int_{\mathbf{R}^n} f(x)u(x)dx, \quad m \to \infty.$$

Below we will show that

(i) let g be continuous and satisfy (3.5); then problem (3.12)-(3.13) has a unique continuous solution $f(g; x, t)$, which satisfies (3.11);
(ii) in the half-space $t < T$, f is of the class $C^{2,1}$ w.r.t. (x, t);
(iii) $f(g^m; \cdot, \cdot) \to f(g; \cdot, \cdot)$ as $m \to \infty$, in the sense of generalized functions;
(iv) if g is a continuous function of the compact support, then $f(x, t) := f(g; x, t)$ is given by (3.4).

By (iv), $f(g^m;\cdot,\cdot) = f^m(\cdot,\cdot)$, by (iii) and (ii), the limit in (iii) is a continuous function, and since we already know that $f^m(x,t) \to f(x,t)$ pointwise, we conclude that $f(\cdot,\cdot) = f(g;\cdot,\cdot)$ solves the problem (3.12)-(3.13), and finish the proof of Theorem 3.1.

It remains to prove (i)-(iv). We start with the proof of (i). Assume that κ is diagonalizable: there exists a matrix C such that $\kappa_C := C^{-1}\kappa C$ is a diagonal matrix with the diagonal entries κ_j (if κ is not diagonalizable, an additional step is to be made - see the end of the proof of (i)). By making the change of variables $x = Cy$, we reduce to the case of the diagonal matrix $\kappa(= \kappa_C)$; the $\psi(\xi)$ becomes $\psi_C(\xi) := \psi((C')^{-1}\xi)$. To simplify the notation below (and without loss of generality), we assume that κ itself is diagonal. In (3.12)-(3.13), change the variables:

$$t = T - \tau, \quad x_j = -\tilde{\theta}_{2j}(\tau) + e^{\kappa_j\tau}y_j, \ j = 1, \dots, n,$$

where

$$\tilde{\theta}_{2j}(\tau) = e^{\tau\kappa_j}\int_0^\tau e^{-s\kappa_j}\tilde{\theta}_j(T-s)ds$$

is the solution to ODE

$$\tilde{\theta}'_{2j}(\tau) - \kappa\tilde{\theta}_{2j}(\tau) = \tilde{\theta}_j(T-\tau),$$

subject to $\tilde{\theta}_{2j}(0) = 0$, and set

$$\begin{aligned} v(y,\tau) &= f(x,t), \\ e^{\kappa\tau} &= \mathrm{diag}(e^{\kappa_j\tau}), \\ r_1(y,\tau) &= r(-\tilde{\theta}_2(\tau) + e^{\kappa\tau}y). \end{aligned}$$

We obtain

$$(\partial_\tau + \psi(e^{-\kappa\tau}D_y) + r_1(y,\tau))v(y,\tau) = 0, \ \tau > 0, \tag{A.1}$$

$$v(y,0) = g(y). \tag{A.2}$$

Notice that $\tilde{\theta}_1 \in C^2([0,T])$ since $\tilde{\theta} \in C([0,T])$, and r_1 satisfies estimate

$$|\partial_y^\alpha \partial_\tau^s r_1(y,\tau)| \le C_{\alpha,s}(1+|y|)^{2-|\alpha|}, \quad |\alpha|, s = 0,1,2, \tag{A.3}$$

$$c_0|y|^2 - C_0 \le r_1(y,\tau), \tag{A.4}$$

where $c_0 > 0$ and $C_0, C_{\alpha,s}$ depend on T but not on $x \in \mathbf{R}^n$ and $\tau \in [0,T]$.

Estimates (3.8), (3.9), (3.10), and (A.3)-(A.4) allows one to apply the standard technique of construction of the inverse to the operator of a boundary problem for PDO to problem (A.1)-(A.2). This technique is based on the construction of an appropriate partition of unity, localization and patching of an approximate inverse from local inverses; for the realization of this general scheme for many classes of PDO see Levendorskiǐ (1993). In op. cit., boundary value problems in L_p-based spaces were considered whereas here we need corresponding results for C^s-based spaces. This modification is straightforward: see e.g. the modification in Barndorff-Nielsen and Levendorskiǐ (2001), for a different class of PDO.

In the result, we obtain that v, the continuous solution to problem (A.1)-(A.2), which admits estimate (3.11), exists and it is unique. Moreover, it is of the class $C^{2,1}$ in the half-plane $\tau > 0$, and satisfies estimate

$$\sup_{\mathbf{R}^n\times[0,T]} |e^{-\omega|y|}v(y,\tau)| \le C \sup_{\mathbf{R}^n} |e^{-\omega|y|}g(y)|, \tag{A.5}$$

where C depends on T, κ, $\tilde{\theta}$ and the constants in estimates for ψ and r. By making the inverse changes of variables and unknowns, we obtain (i) and (ii).

If κ is not diagonalizable, then prior to the change of variables $x = Cy$, an additional change of variables $x_j \mapsto e^{\rho_j}x_j, j = 1,\dots,n$, where $\rho_j > 0$, is needed. The κ will be replaced by $\kappa - \mathrm{diag}(\rho_j)$, which generically has pairwise distinct eigenvalues and hence, diagonalizable; $\tilde{\theta}$ will change as well, and $\psi(D_x)$ becomes $\psi(e^{-\rho_1\tau}D_{x_1},\dots,e^{-\rho_n\tau}D_{x_n})$. After that we make the same changes of variables (using the new κ and $\tilde{\theta}$).

To prove (iii), we take $\omega_1 \in (\omega,\lambda)$, and apply the argument above starting with $g - g^m$ instead of g and ω_1 instead of ω. Since

$$\sup_{\mathbf{R}^n} |e^{-\omega_1|x|}(g(x) - g^m(x))| \to 0 \quad \text{as } m \to \infty,$$

estimate (A.5) implies that

$$\sup_{\mathbf{R}^n\times[0,T]} |e^{-\omega_1|x|}(f(x,t) - f^m(x,t))| \to 0 \quad \text{as } m \to \infty,$$

which proves (iii).

It remains to prove (iv). We have seen that for a continuous g with compact support, f is continuous in the half-plane $t < T$, and of the class $C^{2,1}$ in the open half-plane. Moreover, $f(x,\tau)$ decays faster than $e^{-\omega|x|}$ as $x \to \infty$, for any $\omega > -\lambda$ (notice that a continuous g of the compact support satisfies (3.5) for any ω, and in order that the proof of estimate (3.11) remain valid, we may use any $\omega > -\lambda$, negative ones in particular). Further, r is continuous and semi-bounded from below. These conditions are more than sufficient for the Feynman-Kac theorem to be applicable (for instance, at this stage, we can repeat the proof on p.274 in Rogers and Williams (1994)), which gives (iv).

Theorem 3.1 has been proved.

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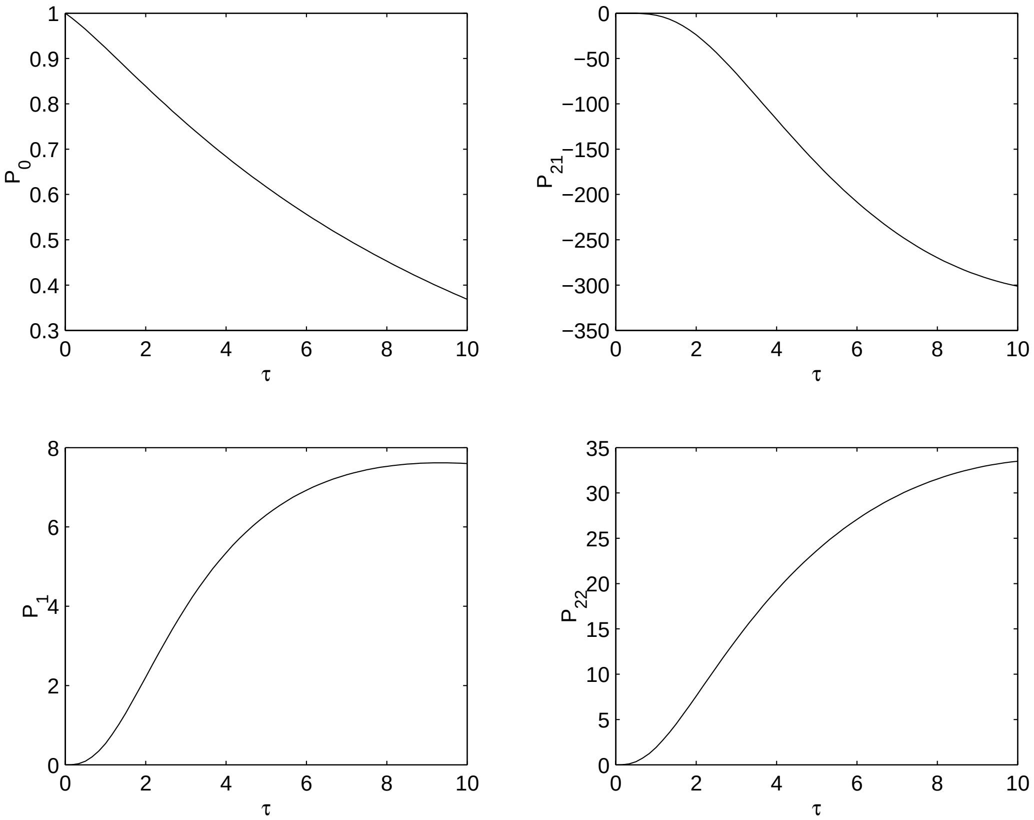

FIGURE 1. Components of the asymptotics of the bond price. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08$.

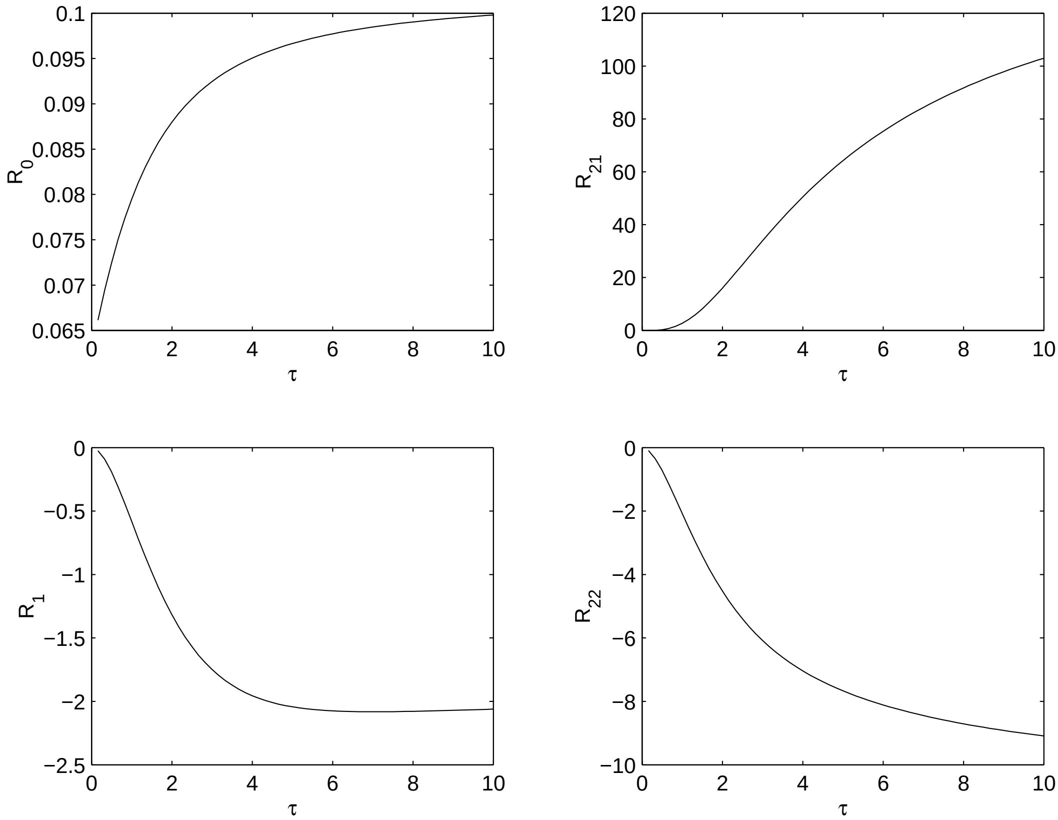

FIGURE 2. Components of the asymptotics of the yield. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08$.

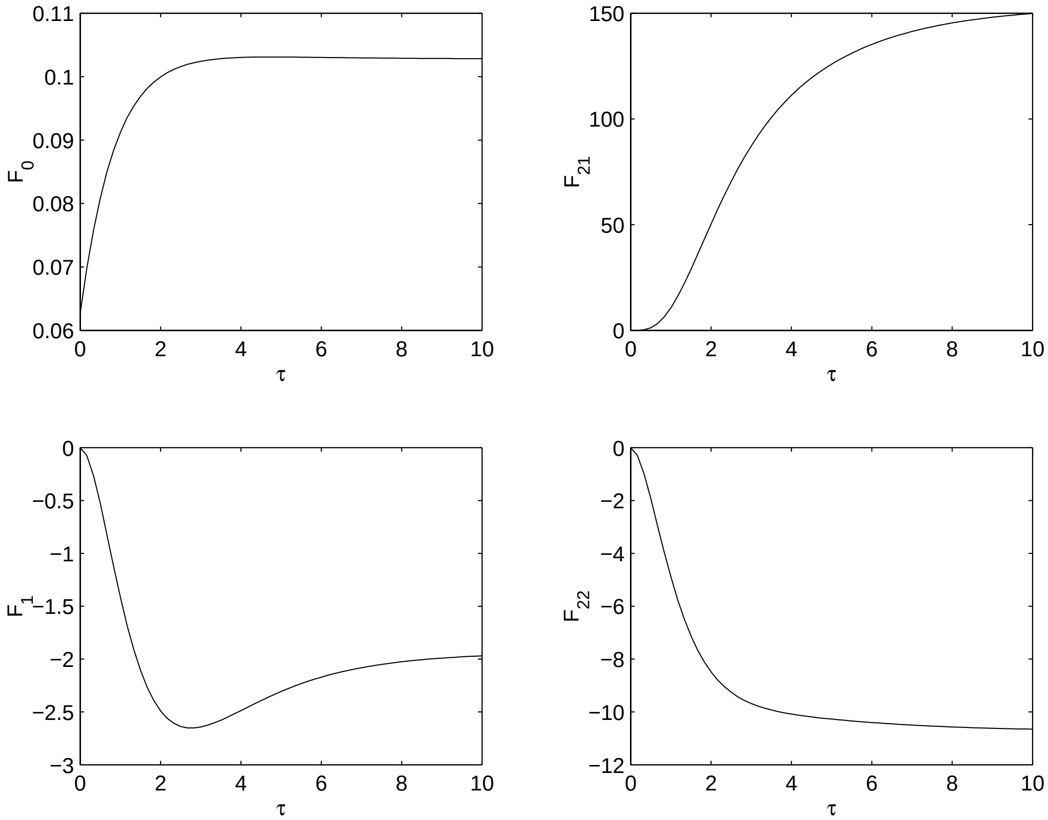

FIGURE 3. Components of the asymptotics of forward rate. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08$.

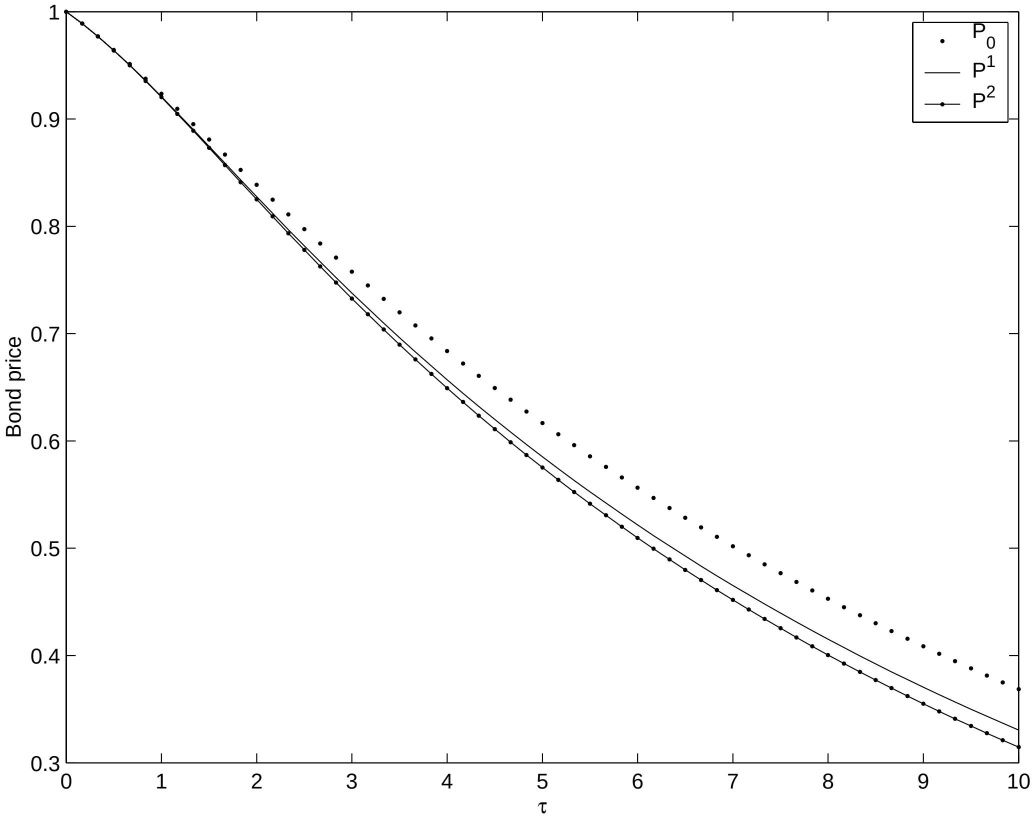

FIGURE 4. Bond price: leading term, first and second approximations. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08, K_3 = -0.005, K_4 = 2.5 \cdot 10^{-4}$.

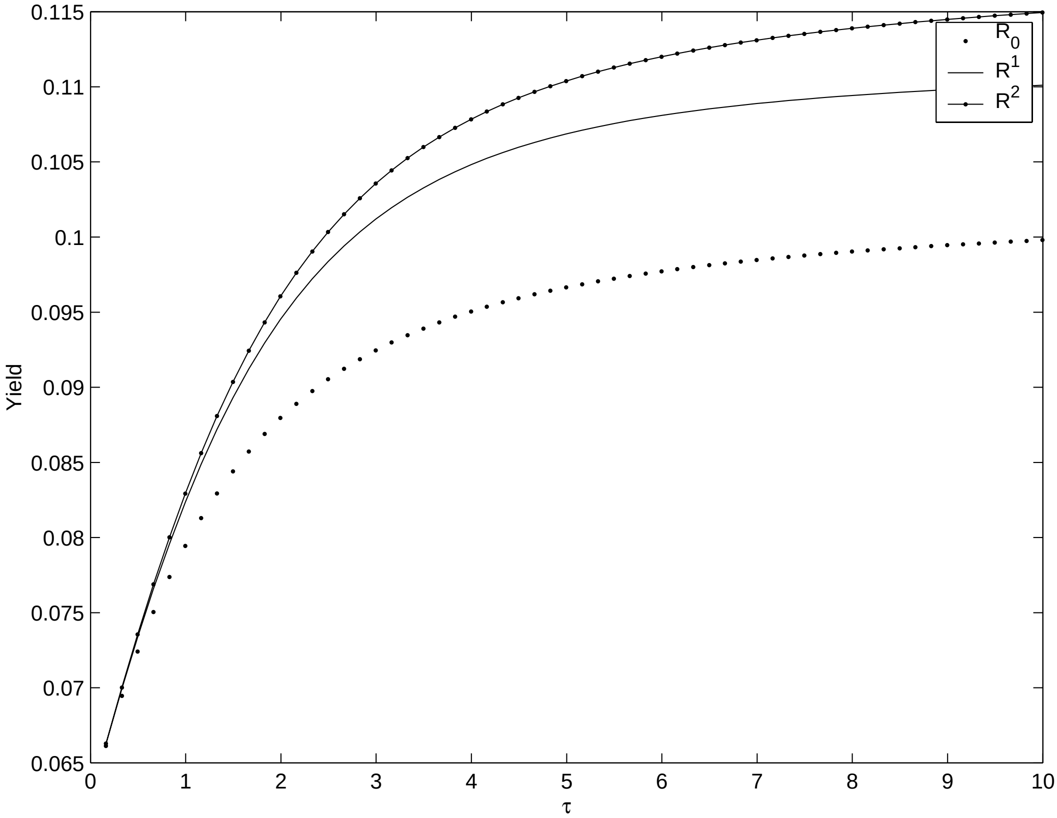

FIGURE 5. Yield: leading term, first and second approximations. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08, K_3 = -0.005, K_4 = 2.5 \cdot 10^{-4}$.

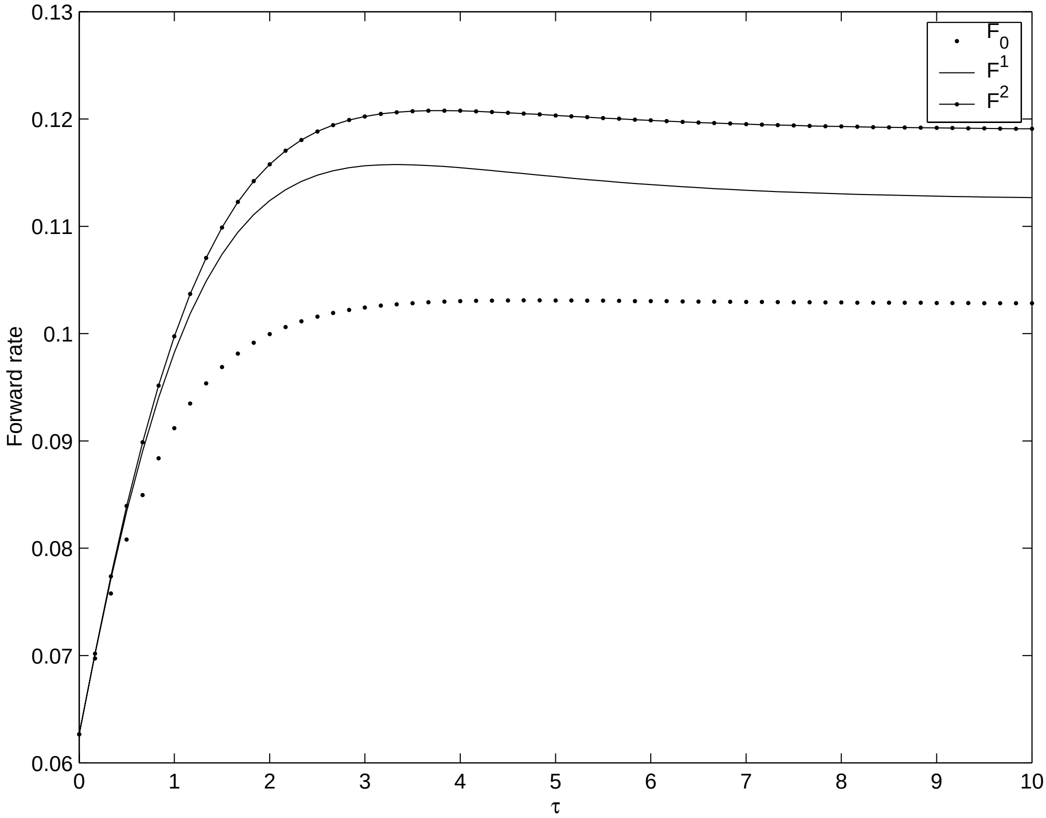

FIGURE 6. Forward rate: leading term, first and second approximations. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08, K_3 = -0.005, K_4 = 2.5 \cdot 10^{-4}$.

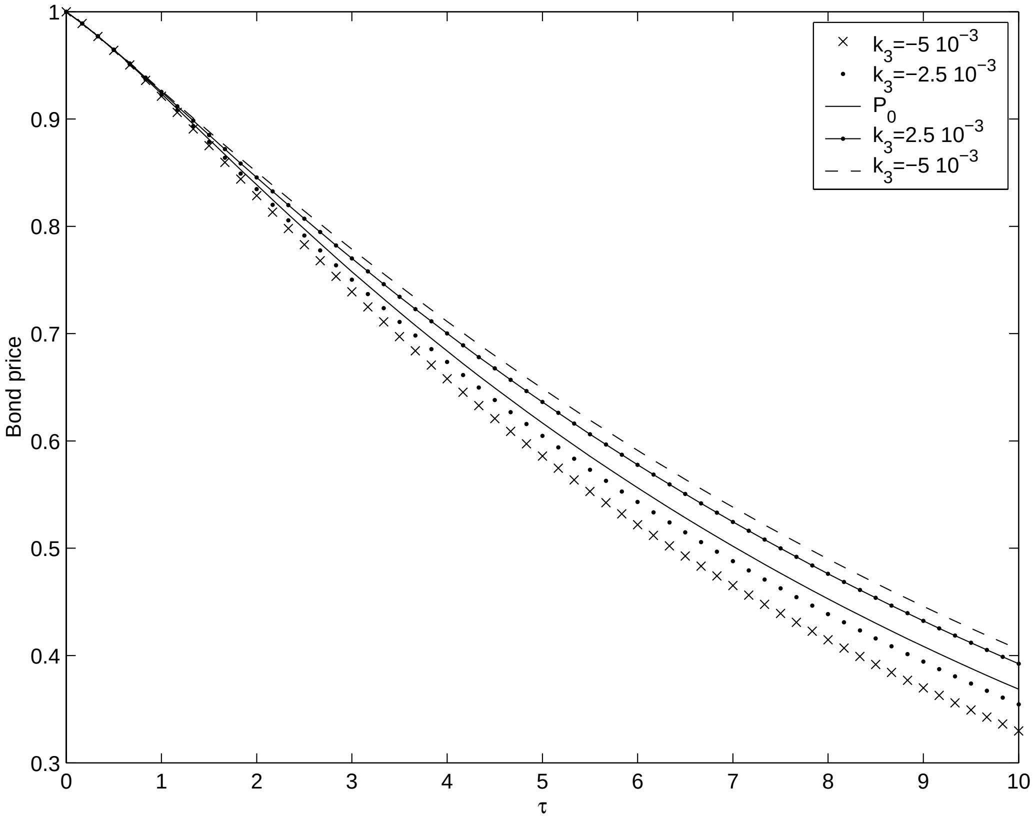

FIGURE 7. Bond price. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08, K_4 = 4 \cdot 10^{-4}$.

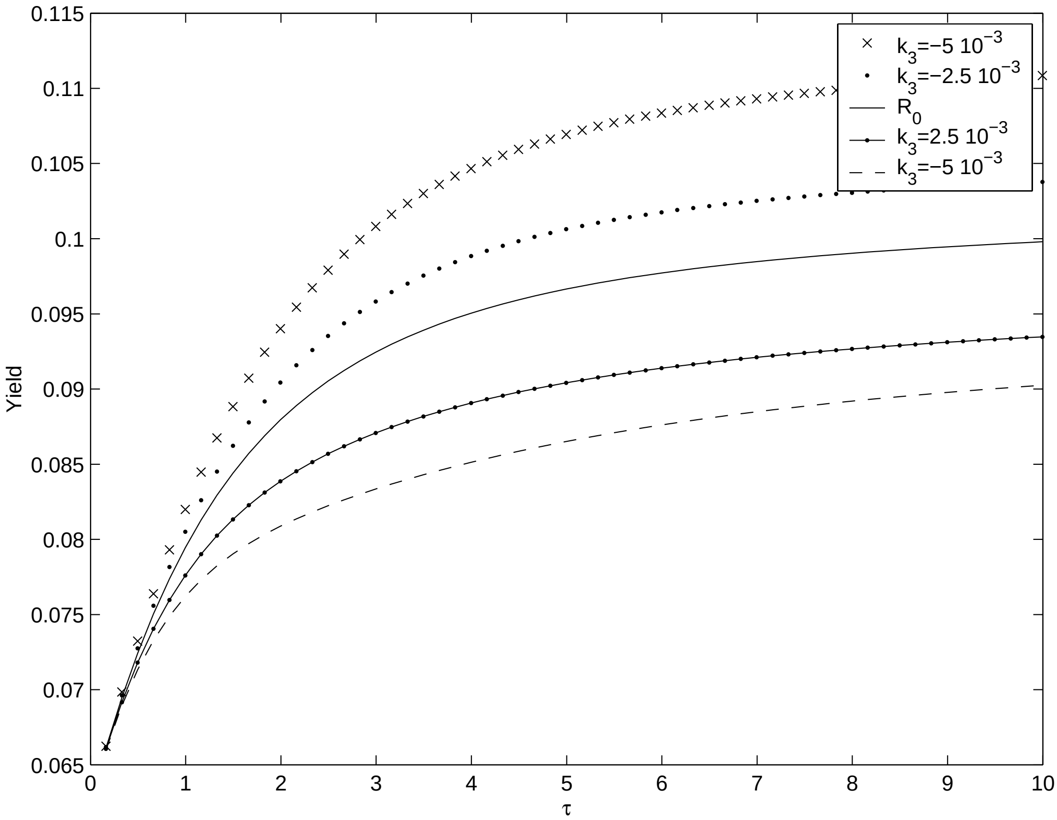

FIGURE 8. Yield. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08, K_4 = 4 \cdot 10^{-4}$.

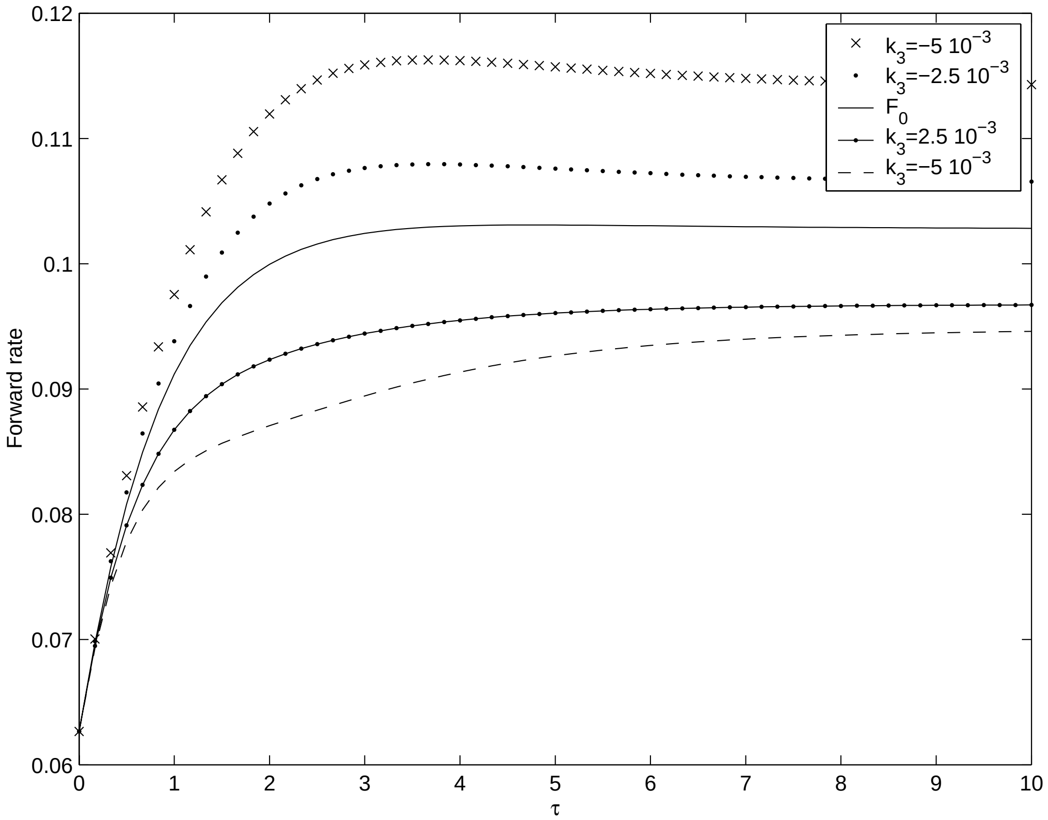

FIGURE 9. Forward rate. Parameters: $r = x^2 = 0.0625, \tilde{\theta} = 0.06, \kappa = 0.3, \mu = 0, \sigma^2 = 0.08, K_4 = 4 \cdot 10^{-4}$.