On Nonlinear Integral Equations Arising in Problems of Optimal Stopping

J. L. PEDERSEN *and* G. PESKIR

Abstract

Let $B = (B_t)_{0 \le t \le 1}$ be a standard Brownian motion started at zero, let $\lambda > 0$ be given and fixed, and let $G : [0,1] \times I\!R \rightarrow I\!R$ be a measurable function. Consider the optimal stopping problem:

$$V = \sup_{0 \le \tau \le 1} E\Big(e^{-\lambda \tau} G(\tau, B_\tau) \Big)$$

where τ is a stopping time of B. Then, under certain regularity conditions on the map G, the optimal stopping time is given by

$$\tau_b = \inf \{ \, 0 \le t \le 1 \mid B_t \ge b(t) \, \}$$

where the optimal stopping boundary $t \mapsto b(t)$ is characterized as a unique solution of the nonlinear integral equation:

$$\int_0^{1-t} e^{-\lambda s} E\bigg(\Big(G_t + \tfrac{1}{2} G_{xx} - \lambda G \Big)(t+s, b(t)+B_s) \cdot I\Big(b(t)+B_s > b(t+s) \Big) \bigg)\, ds$$
$$= e^{-\lambda(1-t)} E\Big(G\big(1, b(t)+B_{1-t}\big) \Big) - G\big(t, b(t)\big)$$

being valid for all $t \in [0,1]$. The key argument in the proof is based upon an extension of the Itô-Tanaka formula yielding local times of B at curved boundaries.

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