Martingales and the Distribution of the Time to Ruin

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Abstract

We determine the ultimate ruin probability and the Laplace transform of the distribution of the time to ruin in the classical risk model, where claims arrive according to a renewal process, with waiting times that are of phase-type, while the claims themselves follow a distribution with a Laplace transform that is a rational function. The main tools are martingales, the optional sampling theorem and results from the theory of piecewise deterministic Markov processes.

Key words and phrases. Risk process, probability of ruin, time to ruin, martingales, optional sampling, piecewise deterministic Markov processes

1 Introduction

In this paper we study the classical risk model

$$X_t = x_0 + \alpha t - \sum_{n=1}^{N_t} U_n$$

where N is a renewal counting process (delayed so that $N_0 \equiv 0$) and the claims $(U_n)_{n\geq 1}$ form a sequence of iid strictly positive random variables,

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independent of N. The purpose is to determine the distribution of $\mathrm{T}_{\mathrm{ruin}}$, the time to ruin,

$$\mathrm{T}_{\mathrm{ruin}} = \inf \{t : X_t \leq 0\}$$

for as large a class of interarrival time distributions (that describe N) and claim size distributions as possible, with $x_0 > 0$ an arbitrary initial reserve. More precisely, we shall give an exact description of the Laplace transform $\mathcal{L}$ for $\mathrm{T}_{\mathrm{ruin}}$ when the interarrival distribution is of phase-type while the claims have a distribution with a Laplace transform that is a rational function (in particular the claims can be of phase-type). In this generality there is no hope of obtaining closed form expressions for $\mathcal{L}$, but in the description we give of $\mathcal{L}(\theta)$ for an arbitrary θ, the only non-explicit part consists in finding all roots with strictly negative real parts for a given polynomium, which makes it easy to evaluate each $\mathcal{L}(\theta)$-value numerically.

As an easy byproduct we also find the probability $p_{\mathrm{ruin}} = \mathbb{P}^{x_0}(\mathrm{T}_{\mathrm{ruin}} < \infty)$ of ultimate ruin.

The technique used for obtaining the results is standard: we determine a suitable class of martingales and use the optional sampling theorem. Here it is absolutely essential that we consider martingales not just with respect to the filtration generated by X, but also use the continuous time Markov chain J that generates the phase-type interarrival times. The idea of using this larger filtration we believe to be new.

The main problem in applying the martingale method is the problem of undershoot, i.e. at the time of ruin the risk process makes a downward jump to a random level ≤ 0 where for a direct and successful use of the martingale approach one would need $X_{\mathrm{T}_{\mathrm{ruin}}}$ to be a fixed, non-random quantity. This problem is resolved by replacing X by a suitable piecewise deterministic Markov process, absorbed at the time of ruin, that also involves the Markov chain J. (The idea of using a martingale for X itself absorbed at $\mathrm{T}_{\mathrm{ruin}}$ goes back at least to Gerber [6] who used it in particular to find p_{ruin} when N is a homogeneous Poisson process and the U_n are exponential).

For existing results about the model and problem studied here, see Asmussen [1], Chapter 5. For the simplest case with N Poisson and the claims exponential, the distribution of the time to ruin is known. Other special cases have been studied by Dickson and Hipp [4] and [5] who obtained expressions for p_{ruin} and also certain quantities related to the claim causing ruin. In all however it is perhaps fair to say that not too many exact results are available concerning the distribution of $\mathrm{T}_{\mathrm{ruin}}$. In the literature, approximations have been studied extensively (see Asmussen [1] and the references there), but what there is of precise results deal typically with the double Laplace

transform
$$\int_0^\infty e^{-\vartheta x_0}\mathbb{E}^{x_0}\left[e^{-\theta \mathrm{T}_{\mathrm{ruin}}};\mathrm{T}_{\mathrm{ruin}}<\infty\right]\,dx_0$$
and then only for special models.

For recent results on the distributions of passage times and overshoot for Lévy processes with two-sided jumps of phase-type, see Asmussen et al [2]. Also see Winkel [9] for explicit results on level passage events (including the passage time itself) for Lévy processes that are subordinators, .

It is feasible (as suggested to the author by S. Asmussen and M. Bladt), that the methods used in this paper may allow one to drop the assumption that claims arrive according to a renewal process and use the much more general setup with a Markov arrival process (MAP) instead – however it is as yet not clear to the author how to do this in detail.

2 The model

Let $(V_n)_{n\geq 1}$ be a sequence of independent random variables, the *interarrival times* between claims, $0<V_n<\infty$, such that the V_n for $n\geq 2$ are iid, and consider the delayed *renewal counting process* $N=(N_t)_{t\geq 0}$,
$$N_t=\sum_{n=1}^{\infty}1_{(T_n\leq t)}$$
where $T_n=V_1+\cdots+V_n$ is the time of the n'th claim.

Further, let $(U_n)_{n\geq 1}$ be a sequence of iid random variables $0<U_n<\infty$, independent of the sequence (V_n), let $\alpha>0$, $x_0>0$ and consider the *risk process* $X=(X_t)_{t\geq 0}$ with initial state x_0,
$$X_t=x_0+\alpha t-\sum_{n=1}^{N_t}U_n.$$
Thus x_0 is the *initial capital* or *reserve* and α is the *premium rate.*

The *time to ruin* is
$$\mathrm{T}_{\mathrm{ruin}}=\inf\{t:X_t\leq 0\}\leq\infty.$$
(using $\inf\emptyset=\infty$) with Laplace transform
$$\mathcal{L}(\theta):=\mathbb{E}^{x_0}\left[e^{-\theta\mathrm{T}_{\mathrm{ruin}}};\mathrm{T}_{\mathrm{ruin}}<\infty\right]\quad(\theta\geq 0). \tag{1}$$
Mostly we shall study $\mathcal{L}$ for $\theta>0$ in which case
$$\mathcal{L}(\theta)=\mathbb{E}^{x_0}\left[e^{-\theta\mathrm{T}_{\mathrm{ruin}}}\right]\quad(\theta>0). \tag{2}$$

From (1) we in particular obtain the *ruin probability*

$$p_{\text{ruin}} := \mathbb{P}^{x_0}\left(\mathrm{T}_{\text{ruin}} < \infty\right) = \mathcal{L}(0) \tag{3}$$

which can also be obtained using (2),

$$p_{\text{ruin}} = \lim_{\theta \to 0, \theta > 0} \mathbb{E}^{x_0}\left[e^{-\theta \mathrm{T}_{\text{ruin}}}\right]. \tag{4}$$

Of course the Laplace transform $\mathcal{L}$ is defective precisely when $p_{\text{ruin}} < 1$.

Throughout the paper it is assumed that the interarrival times V_n for $n \geq 2$ are of *phase-type.* V_1 will always have a distribution closely related to that of the other V_n, in particular of course the distribution may be the same.

As introduced by Neuts [8] the phase-type distribution comes from a Markov chain $J^* = \left(J_t^*\right)_{t \geq 0}$ with finite state space $\{1, \ldots, p\} \cup \{\nabla\} = E \cup \{\nabla\}$ with $p \geq 1$ and ∇ an absorbing state. Considering an initial distribution $a = \left(a_i\right)_{i \in E}$ for J^* that is concentrated on E, the phase-type distribution is then the distribution of T_{abs}, the time to absorption in ∇.

With this structure for J^*, the intensity matrix $\bar{Q}$ for J^* has the form (the first p rows and columns corresponding to the states $1, \ldots, p$ and the last row and column corresponding to ∇),

$$\bar{Q} = \begin{pmatrix} Q & q \\ \mathbf{0}_{1 \times p} & 0 \end{pmatrix} \tag{5}$$

with $Q \in \mathbb{R}^{p \times p}$ satisfying for all $i, j \in E$ that

$$Q_{ii} = -\lambda_i \leq 0, \quad Q_{ij} \geq 0 \text{ for } i \neq j, \quad \sum_{j \in E} Q_{ij} \leq 0, \tag{6}$$

and with $q \in \mathbb{R}^{p \times 1}$ determined so that $\bar{Q}$ becomes a true intensity matrix and hence is given by

$$q = -Q\mathbf{1}, \tag{7}$$

$\mathbf{1}$ denoting the column vector with all entries $= 1$.

It is assumed throughout that *absorption is possible from any* $i \in E$, i.e. all $\lambda_i > 0$ and the transition matrix $\Pi = \left(\pi_{ij}\right)_{i,j \in E}$ that governs the jumps for J^* satisfies that for all $i \in E$ there exists $n \in \mathbb{N}$ such that

$$\sum_{j \in E} \pi_{ij}^{(n)} < 1, \tag{8}$$

$\pi_{ij}^{(n)}$ denoting the (i,j)'th element of the n'th power Π^n of Π. Equivalently, the transition probabilities $p_{ij}(t) = \mathbb{P}\left(J_{s+t}^* = j \,|\, J_s^* = i\right)$ for J^* satisfy that

$$\sum_{j \in E} p_{ij}(t) < 1 \quad (t > 0, i \in E). \tag{9}$$

For later reference, recall that Π is given by

$$\pi_{ii} = 0, \quad \pi_{ij} = \frac{1}{\lambda_i} Q_{ij} \ (i \neq j), \quad \sum_{j \in E} \pi_{ij} = 1 - \frac{q_i}{\lambda_i}.$$

We shall call Q a *sub-intensity matrix* if Q can be used to form the intensity matrix for a Markov chain with an absorbing state as in (5) *and* such that for this chain, absorption into ∇ is possible – and therefore eventually certain – from any $i \in E$.

Before proceeding, here are some of the formulas pertinent to the phase-type distribution: if the initial distribution for J^* is $a = (a_i)_{i \in E}$ (written as a column vector) with $a_i \geq 0$, $\sum a_i = 1$, then the distribution of $\mathrm{T}_{\mathrm{abs}}$ has survivor function

$$\mathbb{P}(\mathrm{T}_{\mathrm{abs}} > t) = a^T e^{tQ} \mathbf{1} \quad (t \geq 0), \tag{10}$$

density

$$f_V(t) = -a^T e^{tQ} Q \mathbf{1} = a^T e^{tQ} q \tag{11}$$

and Laplace transform

$$L_V(\mu) = a^T (Q - \mu I)^{-1} Q \mathbf{1} = 1 + \mu a^T (Q - \mu I)^{-1} \mathbf{1} \quad (\mu \geq 0). \tag{12}$$

Lemma 1 (Facts about sub-intensity matrices). *Let Q be a sub-intensity matrix.*

(i) $\lambda_i = -Q_{ii} > 0$ *for all* $i \in E$.

(ii) Q *is non-singular.*

(iii) $Q - \delta I$ *is a sub-intensity matrix for all* $\delta > 0$.

(iv) $Q^{-1} v < 0$ *for all* $v \in \mathbb{R}^{p \times 1}$ *with* $v > 0$.

(v) *All coordinates of the vector* $\delta (Q - \delta I)^{-1} \mathbf{1}$ *are strictly decreasing functions of* $\delta > 0$, *decreasing from* 0 *to* -1.

(vi) *All coordinates of the vector* $\delta (Q - \delta I)^{-1} \mathbf{1}$ *are strictly convex functions of* $\delta > 0$.

Notation. If $u \in (u_i)_{i \in E}$ is a vector we write $u > 0$, respectively $u \geq 0$, if $u_i > 0$ for all i, respectively $u_i \geq 0$ for all i.

Proof. (i) $\lambda_i = 0$ makes i absorbing in contradiction to the assumption that it is possible to reach $\nabla \neq i$ from i.

(ii) also requires the basic assumption that Q be a sub-intensity matrix: by (9) $e^{tQ}\mathbf{1} < \mathbf{1}$ and with

$$\|v\| = \max\{|v_i| : i \in E\}, \quad \left\|e^{tQ}\right\| = \sup\left\{\left\|e^{tQ}v\right\| : \|v\| \leq 1\right\}$$

it follows that $\left\|e^{tQ}\right\| < 1$. But then, if $Qv_0 = 0$, since $e^{tQ}v_0 = v_0$ we deduce that $\|v_0\| = \left\|e^{tQ}v_0\right\| < \|v_0\|$ if $v_0 \neq 0$, hence $v_0 = 0$.

(iii) is obvious: (6) holds for $Q - \delta I$ and (8) also holds since direct absorption into ∇ from any i is now possible: $\sum_j (Q - \delta I)_{ij} < 0$ for all i.

(iv) Consider v with $v > 0$ and write $Q^{-1}v = u$. Then $v_i = \sum_j Q_{ij}u_j$ and determining i_0 such that $u_{i_0} = \max\{u_i : i \in E\}$, in particular

$$\begin{aligned} 0 < v_{i_0} &= -\lambda_{i_0}u_{i_0} + \sum_{j \neq i_0} Q_{i_0 j}u_j \\ &\leq -\lambda_{i_0}u_{i_0} + \left(\sum_{j \neq i_0} Q_{i_0 j}\right)u_{i_0}. \end{aligned}$$

Since $0 \leq \sum_{j \neq i_0} Q_{i_0 j} \leq \lambda_{i_0}$ the last expression would be ≤ 0 were $u_{i_0} \geq 0$. Thus $u_{i_0} < 0$ as wanted.

(v) and (vi) follow from the right-most expression in (12) with $a = \varepsilon_{i_0}$ for an arbitrary $i_0 \in E$ together with the fact that Laplace transforms for strictly positive random variables decrease strictly from 1 to 0 and are strictly convex. ∎

In order to find the distribution of T_{ruin} when the V_n are phase-type, we shall use a joint process $(X, J) = (X_t, J_t)_{t \geq 0}$ with fixed initial state $(X_0, J_0) \equiv (x_0, i_0)$. Here X is going to be the risk process from above, J is independent of the sequence (U_n) and J_t is defined for all t by using independent versions of the Markov chain J^* as follows: suppose given an *entrance law* $a = (a_i)_{i \in E}$, i.e. all $a_i \geq 0$, $\sum a_i = 1$. Start J in state i_0 and let it follow J^* until the time of absorption, at which point in time V_1 the first claim U_1 is triggered and J is returned to E using the entrance law a. Now let J follow a new copy of J^* (independent of everything else) until its time of absorption V_2, then trigger U_2 as a claim at time $V_1 + V_2$ and return J to E using the distribution a etc. Thus in particular

$$\mathbb{P}^{x_0, i_0}(V_1 > s) = \sum_{j \in E}\left(e^{sQ}\right)_{i_0 j} = \left(e^{sQ}\mathbf{1}\right)_{i_0}$$

while for $n \geq 2$, the V_n are iid with the distribution described by either of (10), (11) or (12).

Note that for $n \geq 2$,

$$\xi := \mathbb{E}^{x_0,i_0} V_n = -a^T Q^{-1} \mathbf{1}$$

and that (essentially) by the law of large numbers

$$p_{\text{ruin}} = 1 \quad \text{iff} \quad \alpha\xi \leq \mathbb{E}U_1. \tag{13}$$

We shall denote by $\mathbb{P}^{x_0}$ the probability

$$\mathbb{P}^{x_0} = \sum_{i_0 \in E} a_{i_0} \mathbb{P}^{x_0,i_0},$$

so that under $\mathbb{P}^{x_0}$ the time until the first claim V_1 is independent of and has the same distribution as the V_n for $n \geq 2$.

The process (X, J) with fixed initial state (x_0, i_0) is a time-homogeneous piecewise deterministic Markov process (PDMP, see Davis [3]) (with transitions that do not depend on the initial state (x_0, i_0)) with state space $\mathbb{R} \times E$ determined as follows, cf. Jacobsen [7]: piecewise deterministic behaviour

$$\phi_t(x, i) = (x + \alpha t, i),$$

total intensity for a jump

$$q(x, i) = \lambda_i$$

and jump probabilities $r((x, i), \cdot)$ given by

$$\begin{aligned} r((x,i), \{(x,j)\}) &= \frac{1}{\lambda_i} Q_{ij} \quad (i \neq j), \\ r((x,i),]-\infty, x'] \times \{j\}) &= \frac{q_i}{\lambda_i} a_j (1 - F_U(x - x')) \quad (x' \leq x,\ i, j \in E) \end{aligned}$$

with F_U the distribution function for the U_n. (The first line describes the jump of J when there is no absorption for J^*, the second what happens when absorption occurs, i.e. a renewal takes place and one of the claims U_n is triggered).

With the fixed initial value (x_0, i_0), (X, J) may be identified with the marked point process (T'_n, Y'_n) where T'_n is the time of the n'th jump for (X, J) (which comes before the n'th jump for X), $Y'_n = \left(X_{T'_n}, J_{T'_n}\right)$ the state reached by that jump and

$$(X_t, J_t) = \phi_{t - T'_{N'_t}} \left(Y'_{N'_t}\right),$$

using $T'_0 \equiv 0$, $Y'_0 \equiv (x_0, i_0)$. Here $N'_t = \sum_{n=1}^{\infty} 1_{(T'_n \leq t)}$ counts the total number of jumps for (X, J).

From the description of (X, J) one finds that the space-time generator has the form, see e.g. Davis [3], Section 26 or Jacobsen [7], Section 6.3.,

$$\begin{aligned} \mathcal{A}g(t;x,i) &= D_t g(t;x,i) + \alpha D_x g(t;x,i) + \sum_{j\in E} Q_{ij} g(t;x,j) \\ &\quad + q_i \sum_{j\in E} a_j \int_{]0,\infty[} F_U(dy)\; g(t;x-y,j). \end{aligned}$$

If in particular g is bounded and $\mathcal{A}g \equiv 0$, then the process $(g(t;X_t,J_t))_{t\geq 0}$ is a true martingale.

We shall exploit the martingale technique, but not on the PDMP (X, J). Define $\left(\tilde{X}, \tilde{J}\right)$ by

$$\tilde{X}_t = \begin{cases} X_t & \text{if } t < \mathrm{T}_{\mathrm{ruin}} \\ 0 & \text{if } t \geq \mathrm{T}_{\mathrm{ruin}}, \end{cases} \qquad \tilde{J}_t = J_{\mathrm{T}_{\mathrm{ruin}} \wedge t}.$$

Since ruin can only occur at a renewal epoch, it is clear that $\left(\tilde{X}, \tilde{J}\right)$ is a time-homogeneous PDMP with state space $\mathbb{R}_0 \times E$, initial state (x_0, i_0) and determined by deterministic behaviour $\tilde{\phi}$, total jump intensity $\tilde{q}$ and jump probabilities $\tilde{r}$ as follows:

$$\tilde{\phi}_t(x,i) = \begin{cases} (x+\alpha t, i) & \text{if } x > 0 \\ (0,i) & \text{if } x = 0, \end{cases}$$

$$\tilde{q}(x,i) = \begin{cases} \lambda_i & \text{if } x > 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and for $x > 0$, provided F_U is continuous

$$\begin{aligned} \tilde{r}((x,i),\{(x,j)\}) &= \frac{1}{\lambda_i} Q_{ij} \quad (i \neq j), \\ \tilde{r}((x,i),[0,x'] \times \{j\}) &= \frac{q_i}{\lambda_i} a_j (1 - F_U(x - x')) \quad (0 \leq x' \leq x,\; i,j \in E). \end{aligned}$$

(In the last line, with a jump corresponding to a new claim occurring from (x, i) the distribution of the new x coordinate is that of $(x - U_1) \vee 0$. In particular $\tilde{r}((x,i),\{(0,j)\}) = \frac{q_i}{\lambda_i} a_j (1 - F_U(x))$. If F_U is not continuous the F_U values should be replaced by limits from the left).

The space-time generator for $\left(\tilde{X}, \tilde{J}\right)$ has the form

$$\begin{aligned} \tilde{\mathcal{A}}g(t;x,i) &= D_t g(t;x,i) + \alpha D_x g(t;x,i) + \sum_{j\in E} Q_{ij} g(t;x,j) \\ &\quad + q_i \sum_{j\in E} a_j \left(\int_{]0,x[} F_U(dy)\; g(t;x-y,j) + (1 - F_U(x))\, g(t;0,j) \right) \end{aligned} \tag{14}$$

for $x>0$ and $\widetilde{\mathcal{A}}g\left(t;0,i\right)=D_{t}g\left(t;0,i\right).$

What is done in the sequel is to hunt for bounded g such that

$$\widetilde{\mathcal{A}}g\left(t;x,i\right)\equiv 0$$

when $x>0$ (but not for $x=0$!) so that by Itô's formula (Jacobsen [7], Theorem 6.3.1), writing T as short for $\mathrm{T}_{\mathrm{ruin}}$,

$$\begin{aligned} g\left(t;\tilde{X}_{t},\tilde{J}_{t}\right) &= g\left(0;x_{0},i_{0}\right)+\int_{0}^{t}ds\,\widetilde{\mathcal{A}}g\left(s;\tilde{X}_{s},\tilde{J}_{s}\right)+M_{t}^{\mathrm{loc}} \\ &= g\left(0;x_{0},i_{0}\right)+\int_{\mathrm{T}\wedge t}^{t}ds\,\widetilde{\mathcal{A}}g\left(s;0,\tilde{J}_{\mathrm{T}\wedge s}\right)+M_{t}^{\mathrm{loc}} \\ &= g\left(0;x_{0},i_{0}\right)+1_{\left(\mathrm{T}\leq t\right)}\left(g\left(t;0,\tilde{J}_{\mathrm{T}}\right)-g\left(\mathrm{T};0,\tilde{J}_{\mathrm{T}}\right)\right)+M_{t}^{\mathrm{loc}}, \end{aligned}$$

where M^{loc} is a local martingale. With g and $\widetilde{\mathcal{A}}$ bounded, M^{loc} becomes uniformly bounded on finite time intervals and is therefore a true martingale M with expectation 0. By optional sampling, for any t, $\mathbb{E}^{x_0,i_0}M_{\mathrm{T}\wedge t}=0$ and since the term involving the indicator $1_{(\mathrm{T}\leq t)}$ vanishes when evaluated with t replaced by $\mathrm{T}\wedge t$, it follows that the process $\left(g\left(\mathrm{T}\wedge t;\tilde{X}_{\mathrm{T}\wedge t},\tilde{J}_{\mathrm{T}\wedge t}\right)\right)_{t\geq 0}$ is a martingale (with respect in fact to the filtration generated by the joint process $\left(\tilde{X},\tilde{J}\right)$) and therefore

$$\mathbb{E}^{x_0,i_0}g\left(\mathrm{T}_{\mathrm{ruin}}\wedge t;\tilde{X}_{\mathrm{T}_{\mathrm{ruin}}\wedge t},\tilde{J}_{\mathrm{T}_{\mathrm{ruin}}\wedge t}\right)=g\left(0;x_{0},i_{0}\right)\quad\left(x_{0}>0,i_{0}\in E,t\geq 0\right), \tag{15}$$

an identity that is essential for what follows.

3 Exponential claims

In this section we discuss the special case where the U_n are assumed to be exponential while the V_n are of phase-type as described in the previous section. This will serve to demonstrate the martingale method in a simple framework and also motivate the approach used for the general result in the next section.

Theorem 2 *Consider the joint process (X,J) with initial state (x_0,i_0) and assume that the claims U_n are exponential at rate $\beta>0$, $P\left(U_n>u\right)=e^{-\beta u}$ for $u\geq 0$. Then for every $\theta>0$*

$$\mathcal{L}\left(\theta\right)=\mathbb{E}^{x_0,i_0}e^{-\theta\mathrm{T}_{\mathrm{ruin}}}=-v_{\gamma,i_0}e^{\gamma x_0} \tag{16}$$

where $\gamma = \gamma(\theta)$ *is the unique solution from the interval* $]-\beta, 0[$ *to the equation*

$$\gamma = \beta\left(\theta - \alpha\gamma\right) a^T \left(Q - \left(\theta - \alpha\gamma\right) I\right)^{-1} \mathbf{1} \tag{17}$$

and $v_\gamma = v_{\gamma(\theta)}$ *is the vector*

$$v_\gamma = \left(Q - \left(\theta - \alpha\gamma\right) I\right)^{-1} q. \tag{18}$$

In particular, if the V_n *are iid (*J *is started according to the entrance law* a*),*

$$\mathbb{E}^{x_0} e^{-\theta \mathrm{T}_{\mathrm{ruin}}} = \frac{\beta + \gamma}{\beta} e^{\gamma x_0}. \tag{19}$$

For fixed initial state, the ruin probability is

$$p_{\mathrm{ruin}} = \mathbb{P}^{x_0, i_0}\left(\mathrm{T}_{\mathrm{ruin}} < \infty\right) = -v_{\gamma(0+), i_0} e^{\gamma(0+) x_0} \tag{20}$$

where $\gamma\left(0+\right) := \lim_{\theta \downarrow 0, \theta > 0} \gamma(\theta)$*, while if the* V_n *are iid,*

$$p_{\mathrm{ruin}} = \mathbb{P}^{x_0}\left(\mathrm{T}_{\mathrm{ruin}} < \infty\right) = \frac{\beta + \gamma(0+)}{\beta} e^{\gamma(0+) x_0}. \tag{21}$$

Here $\gamma\left(0+\right) = 0$ *and* $p_{\mathrm{ruin}} = 1$ *(in both (20) and (21)) if and only if* $-\alpha\beta a^T Q^{-1} \mathbf{1} \leq 1$ *(i.e. iff (13) holds), and* $\gamma\left(0+\right) < 0$ *and* $p_{\mathrm{ruin}} < 1$ *if and only if* $-\alpha\beta a^T Q^{-1}\mathbf{1} > 1$ *in which case* $\gamma\left(0+\right)$ *is the only strictly negative solution to the equation*

$$1 = -\alpha\beta a^T \left(Q + \alpha\gamma I\right)^{-1} \mathbf{1}. \tag{22}$$

Proof. Consider the function

$$g\left(t; x, i\right) = \begin{cases} c_i e^{\gamma x - \theta t} & \text{if } x > 0 \\ K e^{-\theta t} & \text{if } x = 0. \end{cases} \tag{23}$$

For $x > 0$,

$$\begin{aligned} \widetilde{\mathcal{A}} g\left(t; x, i\right) &= \left(-\theta + \alpha\gamma\right) c_i e^{\gamma x - \theta t} + \left(Qc\right)_i e^{\gamma x - \theta t} \\ &\quad + \sum_{j \in E} q_i a_j e^{-\theta t} \left(\int_0^x dy\, \beta e^{-\beta y} c_j e^{\gamma(x-y)} + e^{-\beta x} K\right). \end{aligned} \tag{24}$$

Given $\theta > 0$ the problem is to find scalars γ and K and a (column) vector c so that this expression vanishes for all $x > 0$ and all $i \in E$. The factor $e^{-\theta t}$

cancels out and we are left with exponential terms $e^{\gamma x}$ and $e^{-\beta x}$ – equating the two coefficients to 0 yields the conditions

$$(\alpha\gamma-\theta)\,c+Qc+\frac{\beta}{\beta+\gamma}\left(a^{T}c\right)q \;=\; 0 \tag{25}$$

$$\sum_{j} q_i a_j\left(-\frac{\beta}{\beta+\gamma}c_j+K\right) \;=\; 0. \tag{26}$$

In order for (25) to hold, the first two terms must yield a vector proportional to q. With $\gamma\le 0$ write $\delta=\theta-\alpha\gamma$ which is >0 if $\gamma\le 0$. The matrix $Q_\delta=Q-\delta I$ is then a sub-intensity matrix and in particular invertible (Lemma 1(ii)), and defining

$$v_\gamma=Q_\delta^{-1}q=-Q_\delta^{-1}Q\mathbf{1}=-\mathbf{1}-\delta Q_\delta^{-1}\mathbf{1} \tag{27}$$

it is seen that (25) holds with $c=v_\gamma$ provided

$$\frac{\beta}{\beta+\gamma}a^{T}v_\gamma=-1 \tag{28}$$

and using (27) it is seen directly that this is equivalent to (17). Further (26) holds if

$$K=\frac{\beta}{\beta+\gamma}a^{T}v_\gamma=-1.$$

We now show that (17) has exactly one solution $\gamma\in\,]-\beta,0[$. By Lemma 1(v) and (vi), the right hand side of (17) is strictly increasing and convex as a function of $\gamma<0$. It remains to note that as $\gamma\uparrow 0$ the left hand side (limit 0) is $>$ the right hand side (limit $\beta\theta a^{T}\left(Q-\theta I\right)^{-1}\mathbf{1}<0$ by Lemma 1(iv)), while for $\gamma=-\beta$ the left hand side (value $-\beta$) is $<$ the right hand side (value $\beta\delta_\beta a^{T}\left(Q_{\delta_\beta}\right)^{-1}\mathbf{1}>-\beta$ by Lemma 1(v), writing $\delta_\beta=\theta+\alpha\beta$).

To establish the final claim of the theorem, we saw already that the right hand side of (17) is strictly increasing as a function of γ and (Lemma 1(iv)) strictly decreasing as a function of $\theta>0$. It follows that $\theta\mapsto\gamma\left(\theta\right)$ is strictly decreasing, i.e. $\gamma\left(\theta\right)\uparrow\gamma\left(0+\right)$ as $\theta\downarrow 0$, and letting $\theta\downarrow 0$ in (17) we see that $\gamma\left(0+\right)$ solves the equation

$$\gamma=-\beta\alpha\gamma a^{T}\left(Q+\alpha\gamma\right)^{-1}\mathbf{1}, \tag{29}$$

which always has the solution $\gamma=0$, but where it is clear that we need the strictly negative solution if it exists. The right hand side of (29) is strictly increasing and strictly convex as a function of $\gamma\le 0$ with the value 0 for $\gamma=0$ and therefore $\gamma\left(0+\right)<0$ iff

$$1<D_\gamma\left(-\beta\alpha\gamma a^{T}\left(Q+\alpha\gamma\right)^{-1}\mathbf{1}\right)|_{\gamma=0}\,.$$

Since the derivative equals $-\beta\alpha a^T Q^{-1}\mathbf{1}$, the proof is complete. ■

Example 3 *The simplest application of the theorem is when $p = 1$. Then all V_n (including V_1) are iid exponential at rate $\lambda > 0$ corresponding to $Q = -\lambda$, $q = \lambda$. Also, necessarily $a_1 = 1$.*

The equation (17) for $\gamma = \gamma(\theta)$ becomes

$$\gamma = \frac{\beta(\theta - \alpha\gamma)}{-\lambda - \theta + \alpha\gamma}$$

which has one positive and one negative solution, where we need the negative root,

$$\gamma = \frac{1}{2\alpha}\left(\lambda + \theta - \alpha\beta - \sqrt{(\lambda + \theta - \alpha\beta)^2 + 4\alpha\beta\theta}\right). \tag{30}$$

Also $v_\gamma = -\frac{\beta+\gamma}{\beta}$ so

$$E^{x_0} e^{-\theta \mathrm{T}_{\mathrm{ruin}}} = \frac{\beta + \gamma}{\beta} e^{\gamma x_0}$$

for $x_0 > 0$, $\theta > 0$ with γ given by (30). Letting $\theta \downarrow 0$ yields the well known result

$$p_{\mathrm{ruin}} = \begin{cases} 1 & \text{if } \lambda \geq \alpha\beta \\ \frac{\lambda}{\alpha\beta} e^{\left(\frac{\lambda}{\alpha} - \beta\right)x_0} & \text{if } \lambda < \alpha\beta. \end{cases}$$

For a different derivation of the results from this example, see Asmussen [1], Proposition IV.1.2

4 The general case

This section contains the main result of the paper, Theorem 6. This result generalizes Theorem 2 to allow for non-exponential claims, more precisely it is assumed that the Laplace transform of the distribution for the U_n is a rational function, while the V_n are of general phase-type as described in Section 2. In particular the class of claim size distributions includes all phase-type distributions. (For an extensive discussion of rational Laplace transforms and matrix-exponential distributions, see Asmussen [1], Section VIII.6).

With Theorem 2 and its proof in mind it seems natural to look for martingales of the form $\left(g\left(\mathrm{T}_{\mathrm{ruin}} \wedge t; \tilde{X}_{\mathrm{T}_{\mathrm{ruin}}\wedge t}, \tilde{J}_{\mathrm{T}_{\mathrm{ruin}}\wedge t}\right)\right)_{t\geq 0}$ with, for a given $\theta > 0$,

$$g(t; x, i) = \begin{cases} e^{-\theta t} \sum_{k=1}^m c_{ik} e^{\gamma_k x} & (x > 0), \\ K e^{-\theta t} & (x = 0), \end{cases} \tag{31}$$

where γ_k, c_{ik}, K – but not m – depend on θ, Theorem 2 corresponding to the case $m = 1$. We shall of course need g bounded so the $\gamma_k \in \mathbb{C}$ must satisfy $\operatorname{Re}\gamma_k \leq 0$ – in fact it will turn out that $\operatorname{Re}\gamma_k < 0$. (As will be shown, it is for $m \geq 2$ essential to allow for complex valued γ_k which if present will appear in conjugate pairs).

The main difficulty for $m \geq 2$ is to arrive at the martingale condition

$$\widetilde{\mathcal{A}}g\,(t; x, i) = 0 \quad (t \geq 0, x > 0, i \in E)\,, \tag{32}$$

cf. the discussion in Section 2, more specifically the question is which claim size distributions F_U to consider, and how, corresponding to a given F_U, one should find the 'parameters' γ_k, K and c_{ik} as functions of $\theta > 0$ – apriori it is not at all clear that this is possible in any generality for $m \geq 2$, at least there is no simple algebraic correspondence between the F_U required for $m \geq 2$ and the simple exponentials used for $m = 1$, even though the extension to g of the form (31) from the case $m = 1$, see (23), is simple and linear.

In the discussion that follows it will be useful to think of all the γ_k as distinct. This is indeed what happens for most values of θ, but for special θ-values it may happen that two or more γ_k collapse into the same value, a situation that creates some special technical problems.

Writing out (32) when g is given by (31) yields, cf. (24),

$$\begin{aligned} &- \sum_{k=1}^{m} (\theta - \alpha\gamma_k)\, c_{ik} e^{\gamma_k} x + \sum_{j\in E} \sum_{k=1}^{m} Q_{ij} c_{jk} e^{\gamma_k x} \\ + \quad & q_i \sum_{j\in E} a_j \left(\int_{]0,x[} F_U\,(dy) \sum_{k=1}^{m} c_{jk} e^{\gamma_k (x-y)} + (1 - F_U(x))\, K \right) = 0. \end{aligned} \tag{33}$$

For $\theta > 0$ fixed, put $\delta_k = \theta - \alpha\gamma_k$ and define (cf. (27)) the k'th column of the matrix (c_{ik}) by

$$c_{|k} = r_k Q_{\delta_k}^{-1} q \tag{34}$$

where the r_k are real or complex numbers to be determined later. (Even though δ_k may now be complex, a suitable version of Lemma 1 still holds, in particular Q_{δ_k} is invertible as long as $\operatorname{Re}\delta_k \geq 0$). Inserting (34) into (33) shows the latter to be implied by (a common factor q_i appears and can therefore be divided away),

$$\sum_{k=1}^{m} r_k e^{\gamma_k x} + \sum_{k=1}^{m} s_k \int_{]0,x[} F_U\,(dy)\; e^{\gamma_k (x-y)} + (1 - F_U\,(x))\, K = 0 \quad (x > 0) \tag{35}$$

where

$$s_k = \sum_{j\in E} a_j c_{jk} = r_k a^T Q_{\delta_k}^{-1} q. \tag{36}$$

It is from (35) that we shall shortly deduce the required structure for F_U, i.e. that the corresponding Laplace transform L_U be a rational function. After that, with F_U given, the task is to show how the γ_k, r_k and K should be determined in order for (35) to hold for an arbitrary $\theta > 0$.

First note that K is found easily: let $x \downarrow 0$ in (35) to obtain

$$K = -\sum_{k=1}^{m} r_k. \tag{37}$$

Inserting this into (35), the expression on the left becomes linear in the r_k implying that they can only be determined up to proportionality.

Looking at (35), it is clear that it becomes much more tractable by taking Laplace transforms with $\int_0^\infty (1 - F_U(x))\, dx < \infty$ (i.e. $\mathbb{E}U_n < \infty$) the only condition needed in order to obtain an equivalent formulation of (35). This integrability condition we therefore assume for now, and note that it is automatic for all F_U considered in Theorem 6. So let $\nu \geq 0$, multiply by $e^{-\nu x}$ and integrate x from 0 to ∞. Since

$$\begin{aligned} \int_0^\infty dx\, e^{-\nu x} \int_0^x F_U(dy)\, e^{\gamma_k (x-y)} &= \frac{L_U(\nu)}{\nu - \gamma_k}, \\ \int_0^\infty dx\, e^{-\nu x} (1 - F_U(x)) &= \frac{1}{\nu} (1 - L_U(\nu)), \end{aligned}$$

(35) becomes

$$\sum_{k=1}^{m} \frac{r_k}{\nu - \gamma_k} + L_U(\nu) \sum_{k=1}^{m} \frac{s_k}{\nu - \gamma_k} + \frac{K}{\nu} (1 - L_U(\nu)) = 0 \quad (\nu \geq 0)$$

and solving this for L_U and using (37) gives

$$L_U(\nu) = \frac{-\sum_{k=1}^{m} r_k \frac{\gamma_k}{\nu - \gamma_k}}{\sum_{k=1}^{m} \left(s_k \frac{\nu}{\nu - \gamma_k} + r_k \right)} \tag{38}$$

which is certainly a rational function of ν. More precisely, for $\underline{\gamma} = (\gamma_k)$ such that $\mathrm{Re}\, \gamma_k < 0$ for all k, introduce the polynomials

$$\pi^{\underline{\gamma}}(\nu) = \prod_{k=1}^{m} (\nu - \gamma_k), \quad \pi^{\underline{\gamma}}_{\setminus k}(\nu) = \prod_{1 \leq \ell \leq m, \ell \neq k} (\nu - \gamma_\ell)$$

and multply by $\pi^{\underline{\gamma}}(\nu)$ in the numerator and denominator of (38) to obtain

$$L_U(\nu) = \frac{-\sum_{k=1}^{m} r_k \gamma_k \pi^{\underline{\gamma}}_{\setminus k}(\nu)}{\sum_{k=1}^{m} ((r_k + s_k)\nu - r_k \gamma_k)\, \pi^{\underline{\gamma}}_{\setminus k}(\nu)}, \tag{39}$$

a rational function with the denominator a polynomium in ν of degree (at most) m and the numerator a polynomium of degree $\leq m-1$.

Having thus argued that L_U must be rational, we of course now assume that this is indeed the case,

$$L_U(\nu) = \int_0^\infty e^{-\nu x}\, F_U(dx) = \frac{P_U(\nu)}{R_U(\nu)} \quad (\nu \geq 0) \tag{40}$$

where R_U is a polynomium of degree exactly $m \geq 1$ and P_U is a polynomium of degree $\leq m-1$. As polynomiums $P_U(z)$ and $R_U(z)$ are defined for all $z \in \mathbb{C}$ and to normalize we now further assume that they have no common roots and that the leading term of R_U is z^m ($R_U(z)-z^m$ is of degree $\leq m-1$). Then

(i) $P_U(0) = R_U(0) \neq 0$ (because $L_U(0) = 1$ and P_U, R_U do not have a common root at 0);

(ii) all the m roots z (counted with multiplicity) of R_U satisfy that $\operatorname{Re} z < 0$ (by analytic extension

$$L_U(z) = \int_0^\infty e^{-zx}\, F_U(dx) = \frac{P_U(z)}{R_U(z)}$$

is well defined with $|L_U(z)| \leq 1$ provided $\operatorname{Re} z \geq 0$. In particular, for such z, $R_U(z) = 0$ is impossible since $R_U(z) = 0$ implies $P_U(z) \neq 0$ by assumption);

(iii) the distribution F_U has moments of all orders (because of (i), L_U can be differentiated an arbitrary number of times at $\nu = 0$).

Because of (iii) the arguments leading from (35) to (38) are valid and (38) implies (35).

On the domain $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ we shall use the analytic extension

$$\bar{L}_U(z) = \frac{P_U(z)}{R_U(z)}$$

of L_U. This will of course not in general equal $\int_0^\infty e^{-zx}\, F_U(dx)$ since the integral need not (and for values of z with $\operatorname{Re} z$ too negative will not) converge. The function $\bar{L}_U$ has singularities precisely at the points where R_U has its roots.

With the structure (40) imposed on L_U, given $\theta > 0$ (39) must be solved for the γ_k and the r_k (recall that s_k is determined from γ_k, r_k by (36)), something achieved by identifying the numerator on the right of (39) with P_U, the denominator with R_U:

$$P_U(\nu) = -\sum_{k=1}^{m} r_k \gamma_k \pi_{\setminus k}^{\gamma}(\nu), \tag{41}$$

$$\frac{1}{\nu}\left(R_U(\nu) - P_U(\nu)\right) = \sum_{k=1}^{m} (r_k + s_k)\, \gamma_k \pi_{\setminus k}^{\gamma}(\nu), \tag{42}$$

using for (42) that the denominator in (39) due to (41) equals

$$\nu \sum_k (r_k + s_k)\, \gamma_k \pi_{\setminus k}^{\gamma}(\nu) + P_U(\nu).$$

(In order to identify the right hand side of (39) with $P_U(\nu)/R_U(\nu)$, one of course only needs proportionality (with the same factor) between the left and right hand side of (41) and (42). Demanding equality pinpoints the r_k which otherwise are determined only up to proportionality as was noted above after the derivation of (35)).

Because of (i) above both (41) and (42) are identities between polynomiums of degree $m-1$ and identifying the coefficients to the powers ν^j yields $2m$ equations with $2m$ unknowns, to be solved presently.

For the proof and statement of the main result we need two lemmas.

Lemma 4 *Suppose that $\mathcal{P}$ is a polynomium of degree $\leq m-1$ and that $\gamma_1, \ldots, \gamma_m$ are distinct complex numbers. Then*

$$\mathcal{P}(z) = \sum_{k=1}^{m} \frac{\mathcal{P}(\gamma_k)}{\pi_{\setminus k}^{\gamma}(\gamma_k)} \pi_{\setminus k}^{\gamma}(z) \quad (z \in \mathbb{C}), \tag{43}$$

and if a_{m-1} is the leading coefficient for $\mathcal{P}$ (so that $a_{m-1} = \frac{1}{(m-1)!} D^{m-1}\mathcal{P}(z)$ for all z),

$$a_{m-1} = \sum_{k=1}^{m} \frac{\mathcal{P}(\gamma_k)}{\pi_{\setminus k}^{\gamma}(\gamma_k)}. \tag{44}$$

Proof. The expression on the right of (43) is well defined precisely because the γ_k are distinct. Since $\pi_{\setminus k}^{\gamma}(\gamma_{k_0}) = \delta_{kk_0}$ it follows that the two sides of (43) agree at the m points γ_{k_0}, $1 \leq k_0 \leq m$. Since both are polynomiums of degree $\leq m-1$, they are identical. (44) follows directly from (43) when matching the coefficients to z^{m-1} on both sides. ■

Lemma 5 *Let $\gamma_0 \in \mathbb{C}$ and let ϕ be a function, analytic in an open set containing γ_0. Then*

$$\lim_{\underline{\gamma} \to \underline{\gamma}_0} \sum_{k=1}^{m} \frac{\phi(\gamma_k)}{\pi^{\underline{\gamma}}_{\setminus k}(\gamma_k)} = \frac{D^{m-1}\phi(\gamma_0)}{(m-1)!}. \tag{45}$$

Here $\underline{\gamma}_0 = (\gamma_0, \ldots, \gamma_0)$ and the limit is taken through $\underline{\gamma} = (\gamma_1, \ldots, \gamma_m)$ with all γ_k distinct.

Note. It is essential for the validity of (45) that all γ_k converge to γ_0 simultaneously. If ϕ is a polynomium of degree $\leq m-1$, (45) follows from (44) with the values of the sum on the left of (45) not depending on $\underline{\gamma}$ and γ_0. For $\phi = \mathcal{P}$ such a polynomium we may also use (45) in its general form to obtain a limiting expansion from (43): write the sum in (43) as

$$\pi^{\underline{\gamma}}(z) \sum_{k=1}^{m} \frac{\mathcal{P}(\gamma_k)/(z-\gamma_k)}{\pi^{\underline{\gamma}}_{\setminus k}(\gamma_k)}$$

and use (45) (for $z \neq \gamma_0$ fixed) with

$$\phi(z') = \frac{\mathcal{P}(z')}{z-z'}$$

to obtain

$$\mathcal{P}(z) = (z-\gamma_0)^m \frac{1}{(m-1)!} D^{m-1}_{z'} \frac{\mathcal{P}(z')}{z-z'}\Big|_{z'=\gamma_0}.$$

Using Leibniz' rule for the differentiation, one easily recovers the Taylor expansion

$$\mathcal{P}(z) = \sum_{k=1}^{m} \frac{D^{k-1}\mathcal{P}(\gamma_0)}{(k-1)!}(z-\gamma_0)^{k-1}.$$

Proof. (Due to Bo Markussen). From ϕ, extract the first m terms of the Taylor expansion around γ_0 and consider

$$\tilde{\phi}(z) = \phi(z) - \sum_{k=1}^{m} \frac{D^{k-1}\phi(z)}{(k-1)!}(z-\gamma_0)^{k-1}. \tag{46}$$

Using (44) with $\mathcal{P}$ the polynomium defined by the sum in (46), it is clear that the assertion of the lemma amounts to

$$\lim_{\underline{\gamma} \to \underline{\gamma}_0} \sum_{k=1}^{m} \frac{\tilde{\phi}(\gamma_k)}{\pi^{\underline{\gamma}}_{\setminus k}(\gamma_k)} = 0. \tag{47}$$

Because ϕ is analytic (in a neighborhood of γ_0) we may write

$$\tilde{\phi}(z) = \psi(z)\,(z-\gamma_0)^{m-1}$$

where ψ is analytic and $\psi(\gamma_0) = 0$.

For the next steps in the argument, fix $\gamma_1, \ldots, \gamma_m$ distinct.

The sum $\mathcal{S}_m$ in (47) we write as

$$\begin{aligned}\mathcal{S}_m &= \sum_{k=1}^{m-1} \left(\psi\left(\gamma_k\right) - \psi\left(\gamma_m\right)\right) \frac{\left(\gamma_k-\gamma_0\right)^{m-1}}{\pi_{\setminus k}^{\gamma}\left(\gamma_k\right)} \\ &\quad + \sum_{k=1}^{m} \psi\left(\gamma_m\right) \frac{\left(\gamma_k-\gamma_0\right)^{m-1}}{\pi_{\setminus k}^{\gamma}\left(\gamma_k\right)}.\end{aligned}$$

By (44) the second sum equals $\psi\left(\gamma_m\right)$. The first

$$= \sum_{k=1}^{m-1} \psi_m\left(\gamma_k\right) \frac{\left(\gamma_k-\gamma_0\right)^{m-2}}{\prod_{\ell=1,\ell\neq k}^{m-1}\left(\gamma_k-\gamma_\ell\right)} = \mathcal{S}_{m-1},$$

where

$$\psi_m\left(z\right) = \frac{\psi\left(z\right)-\psi\left(\gamma_m\right)}{z-\gamma_m}\left(z-\gamma_0\right).$$

The sum $\mathcal{S}_{m-1}$ has the same structure as $\mathcal{S}_m$, only m has been replaced by $m-1$ and ψ by ψ_m. $\mathcal{S}_{m-1}$ may therefore be handled exactly as $\mathcal{S}_m$, using the decomposition

$$\psi_m\left(\gamma_k\right) = \left(\psi_m\left(\gamma_k\right) - \psi_m\left(\gamma_{m-1}\right)\right) + \psi_m\left(\gamma_{m-1}\right)$$

to split $\mathcal{S}_{m-1}$ into two sums, in one of which k ranges from 1 to $m-2$ while the other reduces to $\psi_m\left(\gamma_{m-1}\right)$ by (44). Continuing and defining recursively (for $2 \leq k \leq m$ with $\psi_{m+1} \equiv \psi$)

$$\psi_k\left(z\right) = \frac{\psi_{k+1}\left(z\right)-\psi_{k+1}\left(\gamma_k\right)}{z-\gamma_k}\left(z-\gamma_0\right), \tag{48}$$

it is seen that each ψ_k is analytic and proceding by induction it follows that for $0 \leq m' \leq m-2$,

$$\mathcal{S}_m = \sum_{j=0}^{m'} \psi_{m+1-j}\left(\gamma_{m-j}\right) + \sum_{k=1}^{m-m'-1} \psi_{m-m'}\left(\gamma_k\right) \frac{\left(\gamma_k-\gamma_0\right)^{m-m'-2}}{\prod_{\ell=1,\ell\neq k}^{m-m'-1}\left(\gamma_k-\gamma_\ell\right)}.$$

Thus, for $m' = m-2$,

$$\mathcal{S}_m = \sum_{k=1}^{m} \psi_{k+1}\left(\gamma_k\right). \tag{49}$$

Note that here for $k \leq m-1$ the k'th term is

$$\psi_{k+1}(\gamma_k) = (\gamma_k - \gamma_0) D_z \psi_{k+2}(z) \Big|_{z=\gamma_k^*} \tag{50}$$

with γ_k^* a point on the line segment connecting γ_k and γ_{k+1}. We want this to $\to 0$ as $\underline{\gamma} \to \underline{\gamma}_0$ and for this it suffices to show that $D_z \psi_{k+2}$ stays bounded *uniformly* in the γ_j and z when all of these are close to γ_0. We therefore need to be able to control repeated derivatives of the ψ_k, uniformly in all arguments, in a neighborhood of γ_0. As the final step in the proof we therefore show by induction that for $2 \leq k \leq m+1$ and all $m' \geq 0$,

$$\sup_{\gamma_1, \ldots, \gamma_m, z \in \mathcal{O}} \left| D_z^{m'} \psi_k(z) \right| < \infty \tag{51}$$

where $\mathcal{O}$ is a small neighborhood of $\underline{\gamma}_0$: if true, by (50) all terms in (49) for $k \leq m-1$ will then $\to 0$ as $\underline{\gamma} \to \underline{\gamma}_0$ and since also the term for $k = m$, $\psi(\gamma_m)$, tends to $\psi(\gamma_0) = 0$, the desired conclusion $\mathcal{S}_m \to 0$ will follow.

For $k = m+1$ (51) is a trivial assertion about ψ and its derivatives. If (51) has been shown for $k+1$ and all m' we find from (48) first that $\psi_k(z)$ is bounded and then for $m' \geq 1$,

$$\begin{aligned} D_z^{m'} \psi_k(z) &= (z - \gamma_0) D_z^{m'} \frac{\psi_{k+1}(z) - \psi_{k+1}(\gamma_k)}{z - \gamma_k} \\ &\quad + m' D_z^{m'-1} \frac{\psi_{k+1}(z) - \psi_{k+1}(\gamma_k)}{z - \gamma_k}. \end{aligned} \tag{52}$$

But by elementary calculations

$$\begin{aligned} & D_z^{m'} \frac{\psi_{k+1}(z) - \psi_{k+1}(\gamma_k)}{z - \gamma_k} \\ = \; & \frac{(-1)^{m'} m'!}{(z - \gamma_k)^{m'+1}} \sum_{q=0}^{m'} \frac{(-1)^q}{q!} (z - \gamma_k)^q D_z^q (\psi_{k+1}(z) - \psi_{k+1}(\gamma_k)) \end{aligned} \tag{53}$$

and here the sum

$$= \sum_{q=0}^{m'} \frac{(\gamma_k - z)^q}{q!} D_z^q \psi_{k+1}(z) - \psi_{k+1}(\gamma_k) = \frac{(\gamma_k - z)^{m'+1}}{(m'+1)!} D_z^{m'+1} \psi_{k+1}(\gamma^*)$$

for some γ^* on the line segment between z and γ_k. It is now clear from the induction hypothesis that (53) has the desired boundedness properties, hence so has $D_z^{m'} \psi_k$ from (52) as desired. ■

We are now able to state and prove the main result of the paper. Recall from Section 2 that the $(V_n)_{n\geq 1}$ are independent and of phase-type so the Laplace transform for the V_n when $n \geq 2$ is given by (12), while the distribution of V_1 is either given by the survivor function (10) using the probability $\mathbb{P}^{x_0,i_0}$ or is the same as that of V_2 using the probability $\mathbb{P}^{x_0}$. Recall also the notation $Q_\delta = Q - \delta I$.

Theorem 6 *Suppose that the waiting times $(V_n)_{n\geq 1}$ are independent and of phase-type as just described, and the claims $(U_n)_{n\geq 1}$ are independent of (V_n) and iid with a distribution whose Laplace transform L_U is given by (40) with R_U a polynomium of degree $m \geq 1$ and P_U a polynomium of degree $\leq m-1$ and P_U and R_U normalized as described after (40). Then the Laplace transform of T_{ruin} is given as follows: for $\theta > 0$,*

$$\mathcal{L}(\theta) = \mathbb{E}^{x_0,i_0} e^{-\theta \mathrm{T}_{ruin}} = \frac{-\sum_{k=1}^m r_k \left(Q_{\theta-\alpha\gamma_k}^{-1} q\right)_{i_0} e^{\gamma_k x_0}}{\sum_{k=1}^m r_k} \tag{54}$$

where for $1 \leq k \leq m$, the $\gamma_k = \gamma_k(\theta)$ are the precisely m possibly complex valued solutions, counted with multiplicity, to the master equation

$$L_V(\theta - \alpha z) = \frac{R_U(z)}{P_U(z)} \left(= \frac{1}{\bar{L}_U(z)} \right) \tag{55}$$

that satisfy $\mathrm{Re}\, z < 0$, *and where*

$$r_k = r_k(\theta) = -\frac{P_U(\gamma_k)}{\gamma_k \pi_{\setminus k}^{\gamma_0}(\gamma_k)} \tag{56}$$

provided all the γ_k are distinct while if not, (54) is defined by continuous extension using Lemma 5.

In the case where V_1 has the same distribution as the V_n for $n \geq 2$,

$$\mathcal{L}(\theta) = \sum_{i_0 \in E} a_{i_0} \mathbb{E}^{x_0,i_0} e^{-\theta \mathrm{T}_{ruin}} = \frac{\sum_{k=1}^m r_k L_V(\theta - \alpha\gamma_k) e^{\gamma_k x_0}}{\sum_{k=1}^m r_k}. \tag{57}$$

Note. The master equation (55) also has the alternative and quite appealing form

$$\bar{L}_U(z) L_V(\theta - \alpha z) = 1,$$

where for $\mathrm{Re}\, z < 0$ it is of course essential that it is the analytic continuation $\bar{L}_U$ rather than the Laplace transform L_U itself that appears on the left.

Remark 1 *For most (we believe all but countably many) $\theta > 0$, the m solutions to (55) with $\operatorname{Re}\gamma_k < 0$ will be distinct, hence the r_k are well defined by (56) and so are the Laplace transforms (54) and (57). Since the Laplace transforms are continuous functions of θ it suffices to know them for a dense set of θ-values, however we find it useful to note that e.g. the expression on the right of (54) is well defined even if two or more γ_k are the same (in which case (56) does not make sense): the precise expression depends on how the γ_k are grouped according to the distinct coinciding values. If e.g. all $\gamma_k = \gamma_0$, the right hand side of (54) becomes by Lemma 5*

$$\frac{D_\gamma^{m-1} \frac{P_U(\gamma)}{\gamma} \left(Q_{\theta-\alpha\gamma}^{-1}\right)_{i_0} e^{\gamma x_0} \,|_{\gamma=\gamma_0}}{-D_\gamma^{m-1} \frac{P_U(\gamma)}{\gamma} \,|_{\gamma=\gamma_0}}.$$

And if $\gamma_1 = \gamma_2 = \gamma_0$ and the γ_k for $k \geq 3$ are distinct and $\neq \gamma_0$, then r_k is well defined by (56) if $k \geq 3$ and to compute the right hand side of (54) one further needs

$$-\sum_{k=1}^{2} r_k \left(Q_{\theta-\alpha\gamma_k}^{-1}\right)_{i_0} e^{\gamma_k x_0} = D_\gamma \frac{P_C(\gamma)}{\gamma \prod_{j\geq 3}(\gamma-\gamma_j)} \left(Q_{\theta-\alpha\gamma}^{-1}\right)_{i_0} e^{\gamma x_0} \,|_{\gamma=\gamma_0} \,,$$

$$\sum_{k=1}^{2} r_k = D_\gamma \frac{P_C(\gamma)}{\gamma \prod_{j\geq 3}(\gamma-\gamma_j)} \,|_{\gamma=\gamma_0}$$

as follows using Lemma 5 with $m = 2$.

Typically the case of concurring γ_k arises as follows: as θ increases (or decreases) through some critical value, two distinct real solutions of (55) collapse into one and then split into two complex conjugate solutions.

With concurring γ_k, the basic structure of the martingale determining function g also changes from the form (31): if e.g all $\gamma_k \to \gamma_0$ (with the γ_k distinct during the limit), the limiting g takes the form

$$g(t; x, i) = \begin{cases} -e^{-\theta t} \frac{1}{(m-1)!} D_\gamma^{m-1} \frac{P_U(\gamma)}{\gamma} \left(Q_{\theta-\alpha\gamma}^{-1}\right)_i e^{\gamma x} \,|_{\gamma=\gamma_0} & (x > 0) \\ e^{-\theta t} \frac{1}{(m-1)!} D_\gamma^{m-1} \frac{P_U(\gamma)}{\gamma} \,|_{\gamma=\gamma_0} & (x = 0) \end{cases}$$

using Lemma 5 as above. For $x > 0$ this yields a contribution of the form

$$e^{-\theta t + \gamma_0 x} \sum_{\ell=0}^{m-1} \tilde{c}_{i\ell} x^\ell,$$

i.e. the different exponentials $e^{\gamma_k x}$ from the original g in (31) are replaced by $e^{\gamma_0 x}$ times a polynomium in x of degree $\leq m-1$.

Proof. (Theorem 6). Let $\theta > 0$ be given. We first show that the master equation (55) has exactly m solutions γ_k (counted with multiplicity) with $\operatorname{Re}\gamma_k < 0$. Rewrite the equation as

$$P_U(z)\, L_V(\theta - \alpha z) = R_U(z), \tag{58}$$

an equivalent formulation since P_U and R_U do not have any common roots. Because $\theta > 0$ both sides of (58) are analytic in an open set containing the domain $\{z : \operatorname{Re} z \le 0\}$ of the complex plane. As noted above ((ii), p.15) all the m roots of R_U satisfy $\operatorname{Re} z < 0$. Now take $\rho > 0$ and let Γ_ρ denote the interior of the subset of $\mathbb{C}$ determined by the boundary

$$\partial\Gamma_\rho = \{z : |z| = \rho, \operatorname{Re} z < 0\} \cup \{z : z = iy, -\rho \le y \le \rho\}.$$

By Rouché's theorem from complex function theory, R_U and the difference $R_U - P_U L_V(\theta - \alpha\,\cdot)$ will have the same number of zeros (counted with multiplicity), i.e. m zeros, in Γ_ρ provided

$$|P_U(z) L_V(\theta - \alpha z)| < |R_U(z)| \quad (z \in \partial\Gamma_\rho). \tag{59}$$

But if $z \in \partial\Gamma_\rho$ with $\operatorname{Re} z < 0$, $|z| = \rho$, since$|L_V(\theta - \alpha z)| < 1$ we only need $|P_U(z)| \le |R_U(z)|$ in order to obtain (59), which is true for ρ large enough because R_U is a polynomium of higher degree than P_U. And if $z \in \partial\Gamma_\rho$ with $\operatorname{Re} z = 0$, $|z| \le \rho$ we still have $|L_V(\theta - \alpha z)| < 1$ but are now inside the domain where $L_U(z) = P_U(z)/R_U(z)$ is defined as a Laplace transform, hence $|L_U(z)| \le 1$ i.e. $|P_U(z)| \le |R_U(z)|$. Thus (59) is true and we see that for ρ large enough, (55) has precisely m solutions in Γ_ρ, hence precisely m solutions z satisfying $\operatorname{Re} z < 0$.

Assume now that the solutions γ_k are distinct. Recalling that (see (36)),

$$\begin{aligned} s_k &= r_k a^T Q^{-1}_{\theta - \alpha\gamma_k} q \\ &= -r_k L_V(\theta - \alpha\gamma_k) \\ &= -r_k \frac{R_U(\gamma_k)}{P_U(\gamma_k)} \end{aligned}$$

we see that the right hand side of (42) becomes

$$\sum_{k=1}^{m} r_k \left(1 - \frac{R_U(\gamma_k)}{P_U(\gamma_k)}\right) \pi^{\gamma_0}_{\setminus k}(\nu)$$

and inserting r_k given by (56) this

$$\begin{aligned} &= \sum_{k=1}^{m} \frac{R_U(\gamma_k) - P_U(\gamma_k)}{\gamma_k \pi^{\gamma_0}_{\setminus k}(\gamma_k)} \pi^{\gamma_0}_{\setminus k}(\nu) \\ &= \frac{1}{\nu}\left(R_U(\nu) - P_U(\nu)\right) \end{aligned}$$

by Lemma 4. This establishes (42) and (41) follows similarly inserting (56) on the right hand side of (41) and again using Lemma 4. It is more delicate to treat the case where two or more of the γ_k coincide – we don't give the details but refer to Remark 1. But in all cases now, by the discussion preceding the statement of the theorem, for $\theta > 0$ given and γ_k the m solutions to (55) with $\operatorname{Re}\gamma_k < 0$, the process $\left(g\left(\mathrm{T}_{\mathrm{ruin}} \wedge t; \tilde{X}_{\mathrm{T}_{\mathrm{ruin}}\wedge t}, \tilde{J}_{\mathrm{T}_{\mathrm{ruin}}\wedge t}\right)\right)_{t\geq 0}$ is a bounded martingale when g is as in (31), $K = -\sum_k r_k$ (see (37)), $c_{|k} = r_k Q^{-1}_{\theta-\alpha\gamma_k} q$ (see (34)) and r_k is given by (56) (if all the γ_k are distinct, otherwise the definition of g is obtained by continuous extension from Lemma 5). Thus by optional sampling, cf. (15),

$$\mathbb{E}^{x_0,i_0} 1_{(\mathrm{T}_{\mathrm{ruin}}\leq t)} K e^{-\theta \mathrm{T}_{\mathrm{ruin}}} + \mathbb{E}^{x_0,i_0} 1_{(\mathrm{T}_{\mathrm{ruin}}>t)} e^{-\theta t} \sum_{k=1}^{m} c_{\tilde{J}_t,k} e^{\gamma_k \tilde{X}_t} = g\left(0; x_0, i_0\right) \quad (t \geq 0). \tag{60}$$

Since $\operatorname{Re}\gamma_k < 0$ and $\tilde{X}_t = X_t > 0$ on the set $(\mathrm{T}_{\mathrm{ruin}} > t)$, the last term is dominated by a constant times $e^{-\theta t}$ which $\to 0$ as $t \to \infty$. Thus, for $\theta > 0$,

$$\mathbb{E}^{x_0,i_0} e^{-\theta \mathrm{T}_{\mathrm{ruin}}} = \frac{1}{K} g\left(0; x_0, i_0\right)$$

and (54) follows. Multiplying by a_{i_0} and summing on i_0 gives (57) since

$$\sum_{i_0\in E} a_{i_0} \left(Q^{-1}_{\theta-\alpha\gamma_k} q\right)_{i_0} = a^T Q^{-1}_{\theta-\alpha\gamma_k} q = -L_V\left(\theta - \alpha\gamma_k\right),$$

see (12). ∎

Although Theorem 6 does not give the Laplace transform $\mathcal{L}(\theta)$ in closed analytic form, the result is good enough to make numerical computation in concrete models easy: the master equation (55) can be rewritten as $\mathrm{P}(z) = 0$ with P a polynomium, hence the problem of determining the $\gamma_k(\theta)$ for each θ is reduced to that of locating the relevant roots of P. After that, finding the $r_k(\theta)$ and $c_{ik}(\theta)$ causes no problems as long as the γ_k are distinct. Note though that for values of θ where the γ_k are distinct, but some of them close together, the formula (56) for the r_k becomes numerically unstable.

For $m = 1$ with the U_n exponential so that $R_U(z)/P_U(z) = (\beta + z)/\beta$, using (12), (55) reduces to (17) from Theorem 2.

We shall conclude with a discussion on how to determine the ultimate ruin probability p_{ruin}. This may be done using (54) and taking limits as $\theta \downarrow 0$, cf. (4), but we also have

Corollary 7 *Suppose that $\alpha\xi > \mathbb{E}U_1$ so that $p_{\text{ruin}} < 1$. Then the equation*

$$L_V(-\alpha z) = \frac{R_U(z)}{P_U(z)} \tag{61}$$

has precisely m possibly complex valued solutions $\gamma_k = \gamma_k(0)$, $1 \le k \le m$, counted with multiplicity, that satisfy $\operatorname{Re}\gamma_k < 0$, and the ruin probability is given by the expression

$$p_{\text{ruin}} = \mathbb{P}^{x_0,i_0}(\mathrm{T}_{\text{ruin}} < \infty) = \frac{-\sum_{k=1}^m r_k(0)\left(Q^{-1}_{-\alpha\gamma_k(0)}q\right)_{i_0} e^{\gamma_k(0)x_0}}{\sum_{k=1}^m r_k(0)} \tag{62}$$

where

$$r_k(0) = -\frac{P_U(\gamma_k(0))}{\gamma_k(0)\pi^{\gamma_0(0)}_{\setminus k}(\gamma_k(0))} \tag{63}$$

provided all the $\gamma_k(0)$ are distinct while if not, (62) is defined by continuous extension using Lemma 5.

In the case where V_1 has the same distribution as the V_n for $n \ge 2$,

$$p_{\text{ruin}} = \sum_{i_0 \in E} a_{i_0}\mathbb{P}^{x_0,i_0}(\mathrm{T}_{\text{ruin}} < \infty) = \frac{\sum_{k=1}^m r_k(0) L_V(-\alpha\gamma_k(0)) e^{\gamma_k(0)x_0}}{\sum_{k=1}^m r_k(0)}.$$

Note. (61) is obtained from (55) by formally taking $\theta = 0$. Obviously (61) always has the solution $z = 0$.

Proof. We imitate the proof of Theorem 6 with the precaution that all action must take place in a part of the complex plane bounded away from 0, and therefore proceed as follows: choose $\varepsilon > 0$ so small that $\mathbb{E}e^{\varepsilon U_1} < \infty$ (i.e. $R_U(z)$ does not have any roots z with $|z| \le \varepsilon$), and replace Γ_ρ from p.22 with the open region Γ^ε_ρ determined as interior to the boundary

$$\partial\Gamma^\varepsilon_\rho = \{z : |z| = \rho, \operatorname{Re} z < -\varepsilon\} \cup \{z : z = -\varepsilon + iy, -\rho \le y \le \rho\}.$$

To conclude by Rouché's theorem that (61) has precisely m solutions for ρ sufficiently large and ε sufficiently small, we need to show (cf. (59)) that

$$\left|\bar{L}_U(z)\right| |L_V(-\alpha z)| < 1 \quad \left(z \in \partial\Gamma^\varepsilon_\rho\right). \tag{64}$$

If $|z| = \rho$ this is argued exactly as in the proof of Theorem 6. If $z = -\varepsilon + iy$ with $-\rho \le y \le \rho$, note that for $\varepsilon > 0$ small enough in fact

$$\left|\bar{L}_U(z)\right| = \left|\int_0^\infty e^{(\varepsilon - iyx)} F_U(dx)\right| \le \bar{L}_U(-\varepsilon)$$

while

$$|L_V(-\alpha z)| = \left|\int_0^\infty e^{-\alpha(\varepsilon - iyx)} F_V(dx)\right| \leq L_V(\alpha\varepsilon).$$

To complete the proof of (64) it suffices to observe that $\varepsilon \mapsto \bar{L}_U(-\varepsilon)L_V(\alpha\varepsilon)$ is continuously differentiable in a neighborhood of $\varepsilon = 0$ with a derivative that satisfies

$$D_\varepsilon\left(\bar{L}_U(-\varepsilon)L_V(\alpha\varepsilon)\right)|_{\varepsilon=0} = \mathbb{E}U_1 - \alpha\xi < 0$$

because of the basic assumption made in the statement of the corollary, and thus, for $\varepsilon > 0$ small enough,

$$\bar{L}_U(-\varepsilon)L_V(\alpha\varepsilon) < \bar{L}_U(0)L_V(0) = 1.$$

Proceeding now exactly as in the proof of Theorem 6, consider the m solutions $\gamma_k(0)$ to (61) and define (being lazy and assuming that the solutions are distinct) $r_k(0)$ by (63), $K(0) = -\sum_1^m r_k(0)$ and $c_{ik}(0) = r_k(0)\left(Q^{-1}_{-\alpha\gamma_k(0)}q\right)_i$. Then $\left(g\left(\mathrm{T}\wedge t; \tilde{X}_{\mathrm{T}\wedge t}, \tilde{J}_{\mathrm{T}\wedge t}\right)\right)_{t\geq 0}$ is a $\mathbb{P}^{x_0,i_0}$-martingale where

$$g(t; x, i) = \begin{cases} \sum_1^m c_{ik}(0)e^{\gamma_k(0)x} & (x > 0) \\ -\sum_1^m r_k(0) & (x = 0), \end{cases}$$

and the analogue of (60) becomes

$$K(0)\mathbb{P}^{x_0,i_0}(\mathrm{T}_{\mathrm{ruin}} \leq t) + \mathbb{E}^{x_0,i_0}1_{(\mathrm{T}_{\mathrm{ruin}}>t)}\sum_{k=1}^m c_{\tilde{J}_t,k}(0)e^{\gamma_k(0)\tilde{X}_t} = g(0; x_0, i_0) \quad (t \geq 0).$$

The last term is dominated by (using that $\tilde{X}_t = X_t$ on $(\mathrm{T}_{\mathrm{ruin}} > t)$)

$$\mathbb{E}^{x_0,i_0}1_{(\mathrm{T}_{\mathrm{ruin}}>t)}\sum_{k=1}^m \left|c_{\tilde{J}_t,k}(0)\right| e^{X_t \operatorname{Re}\gamma_k(0)}. \tag{65}$$

But the assumption $\alpha\xi > \mathbb{E}U_1$ ensures that $\lim_{t\to\infty} X_t = \infty$ a.s. so because $\operatorname{Re}\gamma_k(0) < 0$, (65) vanishes as $t \to \infty$, hence (60) in the limit yields the desired equation

$$K(0)\mathbb{P}^{x_0,i_0}(\mathrm{T}_{\mathrm{ruin}} < \infty) = g(0; x_0, i_0).$$

■

Corollary 7 applies only under the assumption $\alpha\xi > \mathbb{E}U_1$, but in all cases the ruin probability may be found from the Laplace transform (54), see (4). It is of some interest to verify why, in case $\alpha\xi \leq \mathbb{E}U_1$, letting $\theta \downarrow 0$ in (54) yields the limit 1, and this we now argue assuming the sharp inequality $\alpha\xi < \mathbb{E}U_1$ to hold.

If formally we put $\theta = 0$ in (55), we obtain the equation (61) which as already noted always has $z = 0$ as a solution. Furthermore, differentiating at $z = 0$ gives for the left hand side the derivative $\alpha\xi$ and for the right, the derivative $\mathbb{E}U_1$. Thus (13) is simply an inequality between the derivatives for $z = 0$ on the left and right of (61).

Returning to (55) as it is used in Theorem 6, fix $\theta > 0$ close to 0 and suppose that (13) holds in the sharp form $\alpha\xi < \mathbb{E}U_1$. Let $z = x$ be real and ≤ 0. Then $L_V(\theta - \alpha x)$ for $x = 0$ is

$$L_V(\theta) < 1 = \frac{R_U(0)}{P_U(0)}.$$

Furthermore the derivative

$$D_x L_V(\theta - \alpha x)|_{x=0} \leq \alpha\xi < \mathbb{E}U_1 = D_x \frac{R_U(x)}{P_U(x)}|_{x=0}$$

which is enough (draw a picture) to conclude that *if* $\alpha\xi < \mathbb{E}U_1$, *then the master equation (55) has for* $\theta > 0$ *sufficiently small at least one real solution* $\gamma(\theta) < 0$ *such that* $\lim_{\theta\to 0}\gamma(\theta) = 0$ which is in agreement with Theorem 6 in the following sense: suppose that as $\theta \downarrow 0$, (55) has m distinct solutions $\gamma_1(\theta), \ldots, \gamma_m(\theta)$ such that $\gamma_1(\theta) < 0$ and converges to 0, while $\gamma_k(\theta)$ converges to a non-zero limit $\gamma_k(0)$ for $k \geq 2$. Then, by (56),

$$\lim_{\theta\to 0}\gamma_1(\theta) r_1(\theta) = -\frac{\mathcal{P}_U(0)}{\prod_2^m(-\gamma_k(0))} \tag{66}$$

while for $k \geq 2$, $\lim_{\theta\to 0}\gamma_1(\theta) r_k(\theta) = 0$. Consequently (54) implies, when multiplying by $\gamma_1(\theta)$ in numerator and denominator, that with κ the limit (66),

$$p_{\text{ruin}} = \frac{-\kappa\left(Q^{-1}q\right)_{i_0}}{\kappa}$$

which equals 1 because of (7).

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