MARKOV PROPERTY AND OPERADS

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I. INTRODUCTION

In conformal field theory, people look at a Riemann surface Σ with boundary $\partial\Sigma$, and the set of maps from Σ into a Riemannian manifold M. The case which will be of interest for us in this present work is when the genus of the Riemann surface is 0. This corresponds to a punctured sphere. We suppose that there are one input loop and n output loop. The map from Σ into M are chosen at random, with the formal probability law:

$$d\mu(t\psi) = 1/Z \exp[-I(\psi)]dD(\psi) \tag{1.1}$$

where dD is the formal Lebesgue measure, $I(\psi)$ the energy of the map and Z a normalizing constant called the partition function destinated to get a probability law. Segal ([Se]) has given a series of axioms which should be satisfied by this theory. In particular, conformal field theory predicts the existence of an Hilbert space H associated to each loop space such that the surface Σ realizes a map from $H^{\otimes n}$ into H, if we consider the case of the $n+1$-punctured sphere. $Hom(H^{\otimes n}, H)$ is the archetype of an operad. Namely, if we consider n elements of $Hom(H^{\otimes n_i}, H)$ and an element of $Hom(H^{\otimes n}, H)$, we deduce by composition an element of $Hom(H^{\otimes \sum n_i}, H)$. This composition operation will correspond to the operation of glueing n $1+n_i$ punctures spheres in a sphere with $1+\sum n_i$ punctured points. For the litterature about this statement, we refer to [H.L], [K.S.V], [H_1], [H_2], [Ts]. For material about operads, we refer to the proceedings of Loday, Stasheff and Voronov ([L.S.V]).

The problem of the measure $d\mu$ is that it is purely hypothetical: in the case when the manifold M is replaced by R, it is a Gaussian measure, which gives random distributions (See [Ne], [Sy], [G.J]). But it is difficult to say what are distributions who lives on a manifolds.

Our statement is the following:

-)Define a measure over the space of spheres with $1+n$ punctured points.

-)Define an Hilbert space H associated to each loop space given the punctured points on the sphere.

-)Define associated to the sphere withg $1+n$ punctured points an element of $Hom(H^{\otimes n}, H)$, such that the application is compatible with the action of sewing spheres along their boundary.

For that, we use the theory of infinite dimensional process, especially the theory of Brownian motion over a loop group of Airault-Malliavin ([A.M]) and Brzezniak-Elworthy ([B.E]). Let us recall that the theory of infinite dimensional processes over infinite dimensional manifolds has a lot of aspects. The first who have studied Brownian motion over infinite dimensional manifolds is Kuo ([Ku]). The Russian school has its own version ([B.D], [D], [B.G]). The theory of Dirichlet forms allow to study Ornstein-Uhlenbeck processes over some loop spaces ([Dr.R], [A.L.R]). Our study is related to the theory of Airault-Malliavin, but in order to produce random cylinders, Airault-Malliavin have look a 1+1 dimensional theory: the first 1 is related to the dimension of the propagation time of the dynamic and the second 1 is involved with the internal time of the theory (The loop space). Our theory is 1+2 dimensional, because the internal time of the theory is 2 dimensional.

1+2 dimensional theories were already studied by Léandre in [L_4] in order to study the Wess-Zumino-Novikov-Witten model on the torus, in [L_5] in order to study Brownian cylinders attached to branes and in [L_4] in order to study one of the concretisation of Segal's axiom by using C^k random fields. In [L_3] and in [L_4], stochastic line bundle are used. In [L_2], we give a general construction of $1+n$ dimensional theory, and we perform a theory of large deviation, in order to compute the action of the theory. In [L_6], we study stochastic cohomology of the space of random spheres, related to operads (For the aspect of operads related to n-fold loop space, we refer to the proceeding of Loday-Stasheff-Voronov ([L.S.V]). The problem in [L_8] is that there is no Markov property of the random field, such that we cannot realize an operad by sewing pnctured spheres.

Our goal is to construct a $1+2$ dimensional Wess-Zumino-Novikov-Witten model on the punctured sphere, which is Markovian on the boundary on the sphere. This Markov property will allow to realize an operad, by sewing random spheres along their boundary. For the material of sewing surface, by using the formal measure of physicist, we refer to the surveys of Gawedzki ([Ga$_1$], [Ga$_2$], [Ga$_3$]).

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II. PUNCTURED RANDOM SPHERES AND MARKOV PROPERTY

In order to construct a sphere with $1+n$ punctured points, we define first a sphere with $1+2$ punctured points (a pant), and we sew the pants along their boundary.

We consider a compact connected Lie group G of dimension d, equipped of its bi-invariant metric. We can imbedd it isometrically in a special orthogonal group.

We consider the Hilbert space H of maps from $S^1 \times [0,1]$ endowed with the following Hilbert structure:

$$\|h\|^2_{S^1\times[0,1]} = \int_{S^1\times[0,1]} |h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial/\partial s h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial/\partial t h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial^2/\partial s\partial t h(S)|^2 dS \tag{2.1}$$

where $S=(s,t)$ belongs to $S^1\times[0,1]$ We can consider the free loop space of maps from S^1 into R with the Hilbert structure:

$$\|h\|^2_{S^1\times[0,1]} = \int_0^1 |h(s)|^2 ds + \int_0^1 |h'(s)|^2 ds \tag{2.2}$$

We can find an element $e(s)$ of this Hilbert space such that

$$h(0) = < h, e > \tag{2.3}$$

wher $e(s) = \lambda \exp[-s] + \mu \exp[s]$ for $0 \le s \le 1$ such that $e(0)=e(1)$ but $e'(0) \neq e'(1)$.

We add in (2.1) the Neumann boundary condition:

$$\partial/\partial t h(s,0) = \partial/\partial t h(s,1) = 0 \tag{2.4}$$

Let us recall that the Green kernel over $[0,1]$ associated to the Hilbert space of functions from $[0,1]$ into R with Neumann boundary condition, associated to the Hilbert structure:

$$\int_0^1 |h(t)|^2 dt + \int_0^1 |h'(t)|^2 dt \tag{2.5}$$

satisfy to

$$e_t(t') = (\mu_t^- \exp[-t'] + \lambda_t^- \exp[t'])1_{t'\le t} + (\mu_t^+ \exp[-t'] + \lambda_t^+ \exp[t'])1_{t'>t} \tag{2.6}$$

where $\mu_t^-, \lambda_t^-, \mu_t^+, \lambda_t^+$ depend smoothly on t. The Green kernel associated to the Hilbert structure (2.1) are the product of the one dimensional Green kernel $e_s(s')e_t(t') = E_{s,t}(s',t')$.

We would like to consider the same Hilbert space with the constrain $h(s^1,1) = h(s^2,1) = 0$ for two given times $s^1 < s^2$ (We can choose another condition, but we choose the simplest condition for the sake of simplicity). When we add this condition, we get another Hilbert space H^1 which is a finite codimensional subspace of the initial Hilbert space H.

We can find an orthonormal of the orthogonal complement of H^1 constituted from two maps $h^1(s,t)$ and $h^2(s,t)$ which are smooth in (s,t). Let us consider the Brownian motion with values in H. It is a 3 dimensional Gaussian process $B_u(s,t)$ where u denotes the propagation time and (s,t) the internal time. The Brownian motion with values in H^1 can be seen as

$$B_{1,u}(s,t) = \alpha^1 B_u(s,t) + \beta^1 B_u^1 h^1(s,t) + \gamma^1 B_u^2 h^2(s,t) \tag{2.7}$$

where $(\alpha^1, \beta^1, \gamma^1)$ are deterministic constants and B_u^1 and B_u^2 are two R-valued independent Brownian motion. In the sequel, we will choose this procedure in order to construct the Brownian motion $B_{1,u}(S)$ with values in H_1.

Let us consider the time $t = 1$ where the loop splitts in two loops given by s_1 and s_2. We get after this splitting two circles. We consider the Hilbert space H^2 of maps from $S^1 \times [0,1]$ into R submitted to the boundary conditions $h(s,0) = h(s,1) = 0$ with the Hilbert structure:

$$\int_{S^1\times[0,1]} |\partial^2/\partial s\partial t h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial/\partial t h(S)|^2 dS \tag{2.8}$$

In fact we should act some normalizing constant due to the fact that we do not consider the normalized Lebesgue measure over each circles giving by splitting the circle into 2 circles. The Green kernel associated to this problem are the product of the Green kernel associated to (2.2) and the Green kernel associated to the Hilbert space of functions from $[0,1]$ into R equal to 0 in $t = 0$ and $t = 1$ associated to the Hilbert structure $\int_0^1 |h'(t)|^2 dt$. The Green kernel associated to this Hilbert space are of the type

$$e_t^2(t') = a_t t' 1_{t\geq t'} + b_t(t'-1)1_{t<t'} \tag{2.9}$$

where a_t and b_t are smooth. Therefore the Green kernel associated to the Hilbert space H^2 $E_{s,t}^2(s',t')$ satisfy to

$$E_{s,t}^2(s',t') = e_s(s')e_t^2(t') \tag{2.10}$$

We consider an analoguous Hilbert space H^3 with the Hilbert structure (2.8) and the boundary condition $h(s,0) = 0$ (without the boundary condition $h(s,1) = 0$). The Green kernel in t are of the type

$$e_t^3(t') = a_t t' 1_{t'\leq t} \tag{2.11}$$

and the global Green kernel satisfy to

$$E_{s,t}^3(s',t') = e_s(s')e_t^3(t') \tag{2.12}$$

Over each Hilbert space, we consider the Brownian motion $B_{i,.}(.,.)$. Let Σ be a pant (The elementary surface). Its boundary is constituted of circles, and we get tubes near the output boundary $S^1 \times [0,1/2]$ and tube near the input boundary $S^1 \times [1/2,1]$. Near the boundary, we consider the Brownian motion with values in H^3, by taking care that the starting condition $h(s,0) = 0$ is inside Σ for an output boundary and this condition is outside Σ for an input boundary. We choose 3 independents Brownian motion $B_{3,,.}(.)$ over H^3. We multiply thes Brownian motion by a constant $g(t)$ equal to 0 only in 0 and 1 such that $g(1/2)B_{3,.}(.,1/2)$ corresponds to a normalized circle of length 1. Outside these boundary tubes, we consider over the cylinder with constrain $h(s_1,1) = h(s_2,1) = 0$, a Brownian motion with values in H^1, chosen indepently of the others, but which intersect the input boundary tube on the cylinder $S^1 \times [1-\epsilon, 1]$: we multiply by a smooth function $g(t) > 0$ which is 0 only in $1-\epsilon$. When the loop $s \to h(s,t)$ splitts in two loops, we get two loops: we **add** the Brownian with values in H^2 over each (Two independent one modulo some normalizing constants), and we get two cylinders which intersect the exit tube $S^1 \times [0,1/2]$ over the tube $S^1 \times [0,\epsilon]$. We mutiply these Brownian motion by a smooth function $g(t) > 0$, and which is 0 on ϵ.

After performing all these glueing operations, we get an infinite dimensional Gaussian process parametrized by $[0,1] \times \Sigma$ $u \to B_{tot,u}(.)$, which satisfies to the following properties:

-)For all $S \in \Sigma$, $u \to B_{tot,u}(S)$ is a Gaussian martingale.

-)$(u,S) \to B_{tot,u}(S)$ is almost surely Hoelder, and if $<,>$ denotes the right bracket of the martingale theory, we get for $u \leq 1$

$$< B_{tot,.}(S), B_{tot,.}(S') > \leq Cd(S,S')^{1/2} \tag{2.13}$$

over each elementary parts of the pant Σ where the construction is done. Moreover, over the pant Σ, $(u,S) \to B_{tot,u}(S)$ is almost surely continuous.

c)Over each boundary of the pant, $u \to B_{tot,u}(S)$ are independents.

In order to curve these Gaussian processes, we use the theory of Brownian motion over a loop group of Airault-Malliavin ([A.M]) and Brzezniak-Elworthy ([Br.E]).

Let e_i a basis of the Lie algebra of G. Lert $B^i_{tot,.}(.)$ be d independent copies of $B_{tot,..}(S)$. We writte $d_u B_{tot,u}(S) = \sum e_i d_u B^i_{tot,u}(S)$. We consider the equation in Stratonovitch sense:

$$d_u g_u(S) = g_u(S) d_u B_{tot,u}(S) \tag{2.14}$$

starting from e, the unit element in the group G..

We get (See [L_2], [L_4]) for proof in a closed context.

Theorem II.1: Over each elemantary part of the pant where the leading Brownian motion is constructed, the random field $S \to g_1(S)$ is almost surely $1/2 - \epsilon$ Hoelder. Moreover, the random field on Σ: $S \to g_1(S)$ is almost surely continous, and its restriction on each circle on the boundary are independents.

In order to get a general $1+n$ punctured sphere, we sew successively pants, which are independents, except on the boundary, with a glueing condition. This glueing condition is, when we sew an exit loop of a pant to an input loop of another pant, we choose the same Brownian motion on H^3. We can do that, because the restriction to $S^1 \times \{1/2\}$ are the same. We get by that a tree or a punctured sphere $\Sigma(1,n)$. We get:

Theorem II.2: Over each punctured sphere $\Sigma(1,n)$, the random field $S \to g_1(S)$ got after this sewing procedure is almost surely continuous.

By using this procedure, if we consider a $1+n$ punctured spheres $\Sigma(1,n)$ and n punctured spheres $\Sigma(1,n_i)$, we can glue the input loop to each $\Sigma(1,n_i)$ to the output loop of $\Sigma(1,n)$ and we get a sphere $\Sigma(1,\Sigma n_i)$. We suppose that all the data in this sewing procedure are independents, except for the Brownian motion in H^3 when we sew an output boundary in $\Sigma(1,n)$ to an input boundary in $\Sigma(1,n_i)$. Let us suppose that the random fields are sewed on the loops $(\partial\Sigma)_i$.

We get some thing like a Markov property along the sewing boundary:

Theorem II.3: The random fields $S \to g_1(S)$ over $\Sigma(1,\sum n_i)$ are conditionally independents over each $\Sigma(1,n_i)$ and over $\Sigma(1,n)$ conditionally to each $(\partial\Sigma)_i$.

Proof: We remark that for H^3

$$< B_{3,.}(s,t+1/2) - B_{3,.}(s,1/2), B_{3,.}(s',1/2) >= 0 \tag{2.15}$$

and that

$$B_{3,.}(s,t+1/2) - B_{3,.}(s,1/2), B_{2,.}(s',1/2-t') - B_{2,.}(s',1/2) >= 0 \tag{2.16}$$

because in the t direction in H^3 , we have the covariance of a Brownian motion. This shows that the process $B_{3,.}(.,t+1/2) - B_{3,.}(.,t)$ and $B_{3,.}(.,1/2-t') - B_{3,.}(.,1/2)$ are independents. The only problem in establishing the Markov property lies near the boundary. But if we we writte

$$g_1(S) = Id + \sum \int_{0<u_1..<u_n<1} dB_{u_1}(S)..dB_{u_n}(S) \tag{2.17}$$

we, after imbedding the group G in a matrix algebra

$$g_1(S) - g_1(S') = \sum \int_{0<u_1<..<u_n<1} (dB_{u_1}(S)..dB_{u_n}(S) - dB_{u_1}(S')....dB_{u_n}(S')) \tag{2.18}$$

and we writte $dB_u(S') = dB_u(S') - dB_u(S) + dB_u(s)$ and we distribute in (2.17). Let us choose two points on the same component of the boundary S_1, S_2 in the boundary, and two points S' and S" not on the side of the boundary. We get that $g_1(S') - g_1(S_1)$ and $g_1(s") - g_1(S_2)$ are conditionnaly independent when we suppose given the random field $g_1(S)$ on the boundary. Therefore the result.
$\diamondsuit$

III. LINE INTEGRALS

When we consider the random punctured sphere $\Sigma(1,n)$, we get vertical loops given by $s \to g_1(s,t)$. Since $\Sigma(1,n)$ is built from elementary pants $\Sigma(1,2)$, it is enough to look each vertical loop $s \to g_1(s,t)$ over each elementary pants.

They are of 4 types:

-)The loop near the input boundary (Hilbert space $H^1 \oplus H^2$).

-)The loops in the body of the pants (Hilbert space H^1).

-)The two loops which are created from a big loop (Hilbert space $H^1 \oplus H^2$).

-)The loops near the exit boundary (Hilbert space $H^2 \oplus H^3$.

Let us consider a one form ω over G. We would like to define for each type of this loop the stochastic Stratonovitch integral:

$$\int_0^1 < \omega(g_1(s,t)), d_s g_1(s,t) > \tag{3.1}$$

We extend conveniently the one form ω in a smooth form bounded as well as all its derivatives over the matrix algebra where the matrix group is imbeddded. The technics are very similar to the technics of [L$_4$], part III.

Let dB_u be a Brownian motion with values in the Lie algebra of G. We consider the solution of the stochastic differential equation which gives the Brownian motion from e in the Lie group G:

$$d_u g_u = g_u d_u B_u \tag{3.2}$$

The equation of the differential of the differential of the stochastic flow associated to (3.2) is given (See [I.W], [K], [Bi]) by

$$d_u \phi_u = \phi_u d_u B_u \tag{3.3}$$

and the inverse of the differential of the the flow is given by an analoguous equation. It can be identified to g_u.

Let us consider a finite dimensional family $B_u(\alpha)$ of Brownian motion in the Lie algebra of G depending smoothly of a finite dimensional parameter α where α lives in a finite dimensional family of Brownian motion. We consider the stochastic differential equation depending on a parameter:

$$dg_u(\alpha) = g_u(\alpha) d_u B_u(\alpha) \tag{3.4}$$

The solution of the equation (3.4) has a smootyh version in the finite dimensional parameter α.

$\partial/\partial\alpha g_u(\alpha)$ is for instance the solution of the linear stochastic differential equation with second member:

$$d_u \partial/\partial_u(\alpha) = \partial/\partial\alpha g_u(\alpha) d_u B_u(\alpha) + g_u(\alpha) d_u \partial/\partial\alpha B_u(\alpha) \tag{3.5}$$

This equation can be solved by the method of variation of the constant. We get:

$$\partial/\partial\alpha g_u(\alpha) = (\int_0^u g_v(\alpha) d_v \partial/\partial\alpha B_v(\alpha) g_v^{-1}(\alpha)) g_u(\alpha) \tag{3.6}$$

We will writte $s \to B_.(s,t) = B.(s)$ the leading Gaussian martingale which gives random loops:

$$d_u g_u(s) = g_u(s) d_u B_u(s) \tag{3.7}$$

Let us remark that $B_u(s)$ is a finite combination of $B_{1,u}(s,t), B_{2,u}(s,t)$ and $B_{3,u}(s,t)$ as well as B_u^1 and B_u^2. The key remark in [L$_4$] (3.7) is the following. Let us suppose that $0 \leq s \leq s + \Delta s \leq t \leq t + \Delta t \leq 1$, and let us compute the right bracket between $B_.(s+\Delta s) - B_.(s)$ and $B_.(t+\Delta t) - B_.(t)$. Let us introduce the correlators of $B_.(s)$ $e(s)$. We get since

$$e(s+\Delta s - t - \Delta t) - e(s - t - \Delta t) - e(s - t + \Delta s) + e(s-t) = e"(s-t)\Delta t \Delta s + O(\Delta t + \Delta s)^3 \tag{3.8}$$

an estimate of the right bracket $B_.(s+\Delta s) - B_.(s)$ and $B_.(t+\Delta t) - B_.(t)$ in $C\Delta s \Delta t) + O(\Delta t + \Delta s)^3$ where C is continuous. The only irregularity in e comes from 0 identified to 1 in the circle. Let us forget for simplicity the normalizing constants $g(t)$ which appear in the Gaussian random field which is parametrized by $S^1 \times [0,1]$.

This shows that we can diagonalize the four non independent Brownian motions $B_.(s)$, $B_.(s+\Delta s)$, $B_.(t)$, $B_.(t+\Delta t)$. We find two couples of independent Brownian motions $(w_.(1), w_.(2))$ and $(w_.(3), w_.(4))$ such that

$$\begin{aligned} B_.(s) &= w_.(1) \\ B_.(s+\Delta s) &= \alpha(s,\Delta s) w_.(1) + \beta(s,\Delta s) w_.(2) \\ B_.(t) &= w_.(3) \\ B_.(t+\Delta t) &= \alpha(t,\Delta t) w_.(3) + \beta(t,\Delta t) w_.(4) \end{aligned} \tag{3.9}$$

Moreover, if t does not belong to $]s, s+\Delta s[$, the covariance of $B_.(s+\Delta s) - B_.(s)$ and $B_.(t)$ behaves as Δs because $e(s+\Delta s - t) - e(s-t) = e'(s-t)\Delta s + O(\Delta)^2$.

Moreover,

$$\alpha(s,\Delta s) = 1 + C\Delta s + O(\Delta s)^{3/2} \tag{3.10}$$

$$B(s,\Delta s) = C\sqrt{\Delta s} + C\Delta s + O(\Delta s)^{3/2} \tag{3.11}$$

because $e(s+\Delta s - s) - e(0) = e'_+(o)\Delta s + O(\Delta s)^{3/2}$ because e has half derivatives in 0 and $\Delta s > 0$ and $B_.(s+\Delta s)$ has a covariance in $< B_.(s+\Delta s), B_.(s+\Delta s) >= C(s) + C^1(s)\Delta s$. From (3.7), we deduce that $< w_.(1), w_.(4) >= O(\sqrt{\Delta t})$, $< w_.(3), w_.(2) >= O(\sqrt{\Delta s})$ and the right bracket $< w_.(2), w_.(4) >= O(\sqrt{\Delta s \Delta t})$. We remark that $\partial / \partial \sqrt{\Delta s} \alpha(s, \Delta s d)_0 = 0$.

Let us remark that G is imbedded isometrically in a space of linear matrices. It follows from the previous considerations that in law:

$$g_.(s+\Delta s) = g_.(s) + \sqrt{\Delta s} g_.^1(s) + \Delta s g_.^2(s) + O(\Delta s)^{3/2} \tag{3.12}$$

where $g_.^1(s) = \int_0^. g_u(w_.(1)) \partial / \partial \sqrt{\Delta s} \beta(s,0) dw_u(2) g_u(w_.(1))^{-1} g_.(w_.(1))$. We don't writte the analoguous expression for $g_.^2(s)$. There is a double integral in $dw_.(2)$ where the simple derivatives of $\beta(s,0)$ in $\sqrt{\Delta s}$ appear and a simple integral when the second derivative in $\sqrt{\Delta s}$ of $\alpha(s,\Delta s)$ and $\beta(s,\Delta s)$ appear and polynomial expressions in $g_u(w_.(1))$ and $g_u^{-1}(w_.(1))$.

Moreover, in law

$$g_.(t+\Delta t) = g_.(t) + \sqrt{\Delta t} g_.^1(t) + \Delta t g_.^2(t) + O(\Delta t)^{3/2} \tag{3.13}$$

Let f and g be two smooth functions over the matrix space. We suppose that there are bounded as well as their derivatives of all orders. We have the main estimate which follows from the property listed after (3.9), (3.10), (3.11)

$$E[f(g_u(s)) g_u^1(s) h(g_v(t)) g_v^1(t)] = C(s,t)\sqrt{\Delta s \Delta t} + O(\sqrt{\Delta s} + \sqrt{\Delta t})^{3/2} \tag{3.14}$$

where $C(s,t)$ is continuous. Namely, we can fix first of all $w_.(2)$ and $w_.(4)$. The covariance $< w_.(2), w_.(4) >$ is in $O(\sqrt{\Delta s}\sqrt{\Delta t})$. We can perturb $w_.(1)$ by a term in $o(\sqrt{\Delta t})w_.(4)$ and by $O(\sqrt{\Delta s})w_.(2)$ and symmetrically $w_.(3)$ is perturbed by a term in $O(\Delta t)w_.(4)$ and $O(\sqrt{\Delta s})w_.(2)$ such that the perturbated process are independent of $w_.(2)$ and $w_.(4)$. We conditionate under $w_.(2)$ and $w_.(4)$, and each expression has an H-derivative in Malliavin sense in $w_.(2)$ bounded by $\sqrt{\Delta s}$ and an H-derivative in $w_.(4)$ bounded by $\sqrt{\Delta t}$. We apply after using this procedure the Clark-Ocone formula, and we get integral of the following type, when the H-derivative of A, B, C, D, have the same bound

$$E(\int_0^1 A\sqrt{\Delta t}\delta w_.(4) + \int_0^1 B\sqrt{\Delta s}\delta w_.(2)) \int_0^1 C\delta w_.(4) \int_0^1 D\delta w_.(2)] \tag{3.15}$$

Wa apply Itô formula. The only problem is when we concatenate twice the same $\delta w_.(4)$ or twice the same $\delta w_.(2)$. This leads to quantities of the type $\int_0^1 CA\sqrt{\Delta t} \int_0^u D\delta w_.(2)$. But the H-derivative of CA satisfy the same bound, and we deduce our result.

We consider a smooth 1-form ω_v in the space of matrices with bounded derivatives of all orders which depends smoothly on a finite dimensional parameter v. We suppose that the derivatives in v are bounded.

We consider 2^N, N being a big integer, and the dyadic subdivision of [0,1] associated to 2^N. We call the time of this subdivision s_i with $s_i < s_{i+1}$ such that $s_{i+1} - s_i = 2^{-N}$. If $s \in [s_i, s_{i+1}]$, we call

$$g_u^N(s) = g_u(s_i) + \frac{s - s_i}{s_{i+1} - s_i}(g_u(s_{i+1} - g_u(s_i)) \tag{3.16}$$

$s \to g_1^N(s)$ is piecewise differentiable. We consider the random variable

$$A_v^N = \int_0^1 < \omega_v(g_1^N(s)), d_s g_1^N(s) > \tag{3.17}$$

Proposiotion III.1: When $N \to \infty$, the sequence of random variables A_v^N tends in L^2 to a limit random variable called $\int_{S^1} < \omega_v(g_1 0(s)), d_s g_1(s) >= A_v$. Moreover, there exists a smooth version of the stochastic line integral A_v in v.

Proof: let us forget for the moment the parameter v. We writte

$$A^N = \sum_i \int_{[s_i, s_{i+1}]} < \omega(g_1^N(s)), d_s g_1^N(s) >= \sum (B_i^N + C_i^N) \tag{3.18}$$

where B_i^N is the Bracket term

$$B_i^N = \int_{[s_i, s_{i+1}]} < \omega(g_1^N(s)) - \omega(g_1^N(s_i)), d_s g_1^N(s) > \tag{3.19}$$

and C_i^N is the Itô term

$$C_i^N =< \omega(g_1(s_i)), \Delta_s d g_1(s_i) > \tag{3.20}$$

We writte

$$C_i^N = D_i^N + E_i^N + O(2^{-3N/2} \tag{3.21}$$

where

$$D_i^N = \sqrt{s_{i+1} - s_i} < \omega(g_1(s_i)), g_1^1(s_i) > \tag{3.22}$$

and

$$E_i^N = (s_{i+1} - s_i) < \omega(g_1(s_i)), g_1^2(s_i) > \tag{3.23}$$

First step: convergence of $\sum E_i^N$.

In $g_1^2(s_i)$ whose writting is derived from (3.5) by taking another derivative, there is a linear integral which come from the second derivative of $\alpha(s_i+\Delta s_i)$ and from the second derivative of $\beta(s_i+\Delta s_i)$ in $\sqrt{\Delta s}$ and a double integral which comes from the first derivative of $\beta(s,\Delta s)$. The term in the linear integral can be treated in the following way: we get $\sum E_{i,1}^N$. If $M > N$

$$(\sum E_{i,1}^N - \sum E_{j,1}^M)^2 = (\sum_i (\sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} E_{i,1}^N - E_{j,1}^M))^2 \tag{3.24}$$

In order to compute $\sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} E_{i,1}^N - E_{j,1}^M$, we writte $s_{i+1}-s_i = \sum s_{j+1}-s_j$ such that we have the sum to estimate

$$\sum \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} (s_{j+1}-s_j)(< \omega(g_1(s_i)), \tilde{g}_1(s_i) > - < \omega(g_1(s_j)), \tilde{g}_1(s_j) >) \tag{3.25}$$

$\tilde{g}_1(s_i)$ is the term in the simpler integral where we take two derivatives of $\alpha(s,\Delta s)$ and $\beta(s,\Delta s)$. The term which is integrated depends continously from s Therefore the contribution where we take two derivatives of $\alpha(s,\Delta s)$ does not put any problem. It remains to treat the contribution where we take two derivatives of $\beta(s,\Delta s)$. We can replace the term considered by

$$\sum_i \sum_{[s-j,s_{j+1}]\subseteq[s_i,s_{i+1}]} < \omega(g_1(s_i)), \overline{g}_1(s_i) > - < \omega(g_1(s_j), \overline{g}_1(s_j) > \tag{3.26}$$

where we have removed the term in two derivatives by $\sqrt{\Delta s_j}\Delta_{s_j} B_.(s_j))$. We will writte $B_.(s_i+\Delta_{s_i})-B_.(s_i) = \sum B_.(s_j+\Delta s_j) - B_.(s_j)$ and we see that $B_.(s_j+\Delta s_j)-B_.(s_j), B_.(s_{j'}+\Delta s_{j'})-B_.(s_{j'}) >= O(\Delta s_j \Delta s_{j'})$ if $j \neq j'$ and equal to $O(\Delta s_j$ if $j=j'$. This shows by proceeding as in (3.14) that the L^2 norm of

$$\sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} (< \omega(g_1(s_i)), \overline{g}_i(s_i) > - < \omega(g_1(s_j)), \overline{g}_1(s_j) >) \tag{3.27}$$

behaves as $O(1/N)\Delta s_i$ because $\omega(g_1(s))$ depends continuously of s and after using the desintegration argument used after (3.14).

The problem arises when we take the double integral. In order to study its sum, we remplace $\sqrt{\Delta s} w_.(2)$ in (3.8) by $B_.(s_i+\Delta s_i) - B_.(s_i)$ and take the double stochastic integral which is associated by taking the derivative of the flot $g_u(s_i)$. For the convergence of E_i^N, we can assimilate $(s_{i+1}-s_i)g_u^2(s_i)$ with a double integral $\alpha_u(s_i)$ after performing these replacement. We sum over $[s_j,s_{j+1}] \subseteq [s_i,s_{i+1}]$. We get:

$$\begin{aligned} &< \omega(g_t(s_i), \alpha_t(s_i) > - \sum_j < \omega(g_t(s_j)), \alpha_t(s_j) >= \sum (< \omega(g_t(s_i)) - \omega(g_t(s_j)), \alpha_t(s_j) > + \\ &< \omega(g_t(s_i)), \alpha_t(s_i) - \sum \alpha_t(s_j) >= \delta_i^N + \epsilon_i^N \end{aligned} \tag{3.28}$$

The sum of the first term tends clearly to 0 in L^2. The difficult term is to estimate the term in ϵ_i^N. In the double integral which compose $\alpha_t(s_i)$, we writte

$$B_.(s_i+\Delta s_i) - B_.(s_i) = \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} B_.(s_j+\Delta s_j) - B_.(s_j) \tag{3.29}$$

We distribute the integrands. Over each $dB_.(s_i+\Delta s_i) - dB_.(s_i)$, there is in the double integral a term $B_.(s_i)$ measurable,, which is adapted and which depends on a continuous way of s_i. Since it depends on a continuous way of s_i, we can replace it when we distribute by the corresponding terms in s_j in $\alpha_t(s_i)$. After distributing in $\alpha_t(s_i) - \sum \alpha_t(s_j)$, the diagonal terms are substractings, and it remains to study the process

$$\begin{aligned} \delta_t^{'N} = \sum_i < \omega(g_t(s_i)), & \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}],[s_{j'},s_{j'+1}]\subseteq[s_i,s_{i+1}],j\neq j'} \\ \int_{0<u<v<t} & r_u(s_j) d_u \Delta_{s_j} B_u(s_j) r_v(s_{j'}), d_v \Delta_{s_{j'}} B_v(s_{j'}) \end{aligned} \tag{3.30}$$

We replace $d_u\Delta_{s_j}B_u(s_j)$ by $\sqrt{\Delta s_j}d_uw_u(2,s_j)$ after using the operations of (3.9), and we get a process δ_t^N. We decompose the semi-martingale δ_t^N into a finite variational part which goes by using (3.14) to 0 and a martingale part M_t^N. We would like to show that this martingale tends to 0. For that, we compute its quadratic variat-ion. We get a sum over all quadruple $[s_{j_1},s_{j_1+1}]$, $[s_{j_2},s_{j_2+1}]$, $[s_{j_3},s_{j_3+1}]$ and $[s_{j_4},s_{j_4+1}]$.

First case: let us suppose that all elements of the quadruple are differents. We conditionate along $w_.(2,s_{j_k})$ by using the fact that the covariance of $B_.(s+\Delta s)-B_.(s)$ and of $B_.(t)$ is in Δs if t does not belong to $]s,s+\Delta s[$. It is possible to do that because the covariance matrix of $w_.(2,s_{j_k})$ is in in $I+O(\Delta s_j)$, and we can compute its inverse quite easily. We conclude as in (3.14) and (3.15). So the term which arises from this type of quadruple is in 2^{-4M}. There are at most $2^{2N}2^{4(M-N)}$ such quadruples. The total contribution is 2^{-N} which tends to 0 when $N\to\infty$.

-)**Second case**: there are 3 intervals $[s_j,s_{j+1}]$ differents. This can come from a concatenation of terms d_v for $u<v$ in the stochastic integral deduced from (3.31) for δ_t^N or a concatenation of the same d_u in this stochastic integral. We conditionate under the Brownian motion which give the increment, and we get a contribution of each term in 2^{-3M} by doing as in the first case. There are at most $2^N2^{M-N}2^{2(M-N)}=2^{3M}2^{-2N}$ such possibilities. The total contribution is in 2^{-2N} which tends to 0 when $N\to\infty$.

-)**Third case**: there are two intervals $[s_j,s_{j+1}]$ differents. The contribution of each element which appears after the concatenation done in order to get two intervals is 2^{-2M} by doing as in the first case. There are at most $2^N2^{2(M-N)}$ such contributions. The total contribution is in 2^{-N} which converges to 0 when $N\to\infty$.

This shows that $\sum E_i^N$ is a Cauchy sequence in L^2.

Second step: convergence of the Itô term $\sum D_i^N$.

We writte

$$\alpha_i^N = D_i^N - \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} D_j^M \tag{3.31}$$

and we would like to show that $\sum\alpha_i^N$ converges in 0 in L^2.

There are two terms to study:

-)The first term is the contribution of $E[\alpha_i^N\alpha_{i'}^N]$ for $i\neq i'$. By (3.14),

$$\sum_{i\neq i} E\alpha_i^N\alpha_{i'}^N] \to 2\int_{S^1\times S^1} C(s,t)dsdt - 2\int_{S^1\times S^1} C(s,t)dsdt = 0 \tag{3.32}$$

-)The second term is the contribution of $\sum E[(\alpha_i^N)^2]$. By using the consideration of the first step, we can writte modulo a term which vanishes writte that

$$\begin{aligned}\alpha_i^N &=< \omega(g_1(s_i)),\Delta_{s_i}g_1(s_i)> - \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} <\omega(g_1(s_j)),\Delta_{s_j}g_1(s_j)> \\ &= \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} <\omega(g_1(s_i))-\omega(g_1(s_j)),\Delta_{s_j}g_1(s_j)>=\sum\beta_j^N\end{aligned} \tag{3.33}$$

We replace $\Delta s_jg_1(s_j)$ by $\sqrt{\Delta s_j}g_1^1(s_j)$ and we use (3.9). We remark that $<B_.(s_i),w_.(2)>=O(\sqrt{\Delta s_j})$ as well as $<B_.(s_i),w_.(4)>$. We distribute in $(\sum\beta_j^N)^2$, and when we have distributed, we conditionate by $w_.(2)$ and $w_.(4)$ the Brownian motion $w_.(1)$, $w_.(2)$ and $B_.(s_i)$, in order to do the analoguous of (3.14) in this situation. We deduce that if $j\neq j''$ $E[\beta_j^N\beta_{j'}^N]=O(1/N)2^{-2M}$ and $E[(\beta_j^N)^2]=O(1/N)2^{-M}$. This concludes the second term. The Itô term converges.

Third step: study of the convergence of the Bracket term $\sum B_i^N$. We writte

$$\omega(g_1^N(s))-\omega(g_1(s_i)) = \frac{s-s_i}{\sqrt{s_{i+1}-s_i}}g<\nabla(\omega(g_1(s_i))),g_1^1(s)> + O(s-s_i) \tag{3.34}$$

and

$$d_sg_1^N(s) = \frac{ds}{\sqrt{s_{i+1}-s_i}}g_1^1(s_i)+dsg_1^2(s_i)+dso(s_{i+1}-s_i) \tag{3.35}$$

The more singular term in B_i^N is:

$$(3.36)\quad \alpha_i^N = \int_{s_i}^{s_{i+1}} \frac{s-s_i}{s_{i+1}-s_i} < g_1^1(s_i), \nabla\omega(g_1(s_i), g_1^1(s_i) > ds = (s_{i+1}-s_i) < g_1^1(s_i), \nabla\omega(g_1(s_i), g_1^1(s_i) >$$

This is as in the previous consideration a quadratic expression in $g_1^1(s_i)$. This expression can be treated exactly as in the first step for the convergence of $\sum E_i^N$, by writting a product of two $g_1^1(s_i)$ as a double integral and replacing $(s_{i+1}-s_i) < g_1^1(s_i), ., g_1^1(s_i) >$ by a double integral where we have removed $\sqrt{\Delta s_i} w_.(2)$ by $\Delta_{s_i} B_.(s_i)$. The sum of the others terms tend clearly to 0.

In order to show that $\int_{S^1} < \omega_v(g_1(s)), d_s g_1(s) >$ has a smooth version, we show that the derivative of A_v^N in the finite dimensional approximation of the stochastic integrals converge in L^2 as it was done previously. We conclude by using the Sobolev imbedding theorem as in [I.W].

$\diamondsuit$

We consider a more intrinsic approximation of the line integral. We use if $g_1(s_i, t)$ and $g_1(s_{i+1}, t)$ are close the approximation

$$(3.37)\qquad F_N(s, g_1(s_i,t), g_1(s_{i+1},t)) = \exp[\frac{s-s_i}{s_{i+1}-s_i}\log(g_1(s_{i+1},t)g_i(s_i,t)^{-1})]g(s_i,t)$$

conveniently extended on the set of all matrices. We put

$$(3.38)\qquad \tilde{g}_1^N(s,t) = F_N(s, g_1(s_i,t), g_1(s_{i+1},t))$$

We consider $\tilde{A}_v^N$ as in (3.19). If we look the asymptotic expansion of F_N near the diagonal, we see thay the more singular term in $d_s\tilde{g}_1^N(s,t)$ and $d_s g_1^N(s,t)$ coincide. this allows us to state the following theorem:

Theorem III.2: $\tilde{A}_v^N = \int_{S^1} < \omega_v(\tilde{g}_1^N(s,t)), d_s\tilde{g}_s^N(s,t) >$ tends in L^2 for the C^k topology over each compact of the finite dimensional parameter space to the Stratonovitch integral $\int_{S^1} < \omega_v(g_1(s,t)), d_s g_1(s,t) >$ which has a smooth version in v.

Remark: We don't know if the Stratonovitch integral of Theorem III.2 and of Proposition III.1 coincide. In the sequel, we will use the version of Theorem III.1, because the approximation is intrinsic.

Remark: instead of integrating over a circle, we can integrate over a segment.

IV.INTEGRAL OF A TWO FORM

We decompose the pant $\Sigma(1,2)$ in elementary cylinders $S^1 \times [0,1]$. Let $B_.(s,t)$ be the Brownian motion parametrized by these elementary cylinders. Each correlators check all the properties listed in the part IV of [L_4] such that each correlator is smooth outside the diagonals and its derivative has half limits on the diagonals, such that we can apply the technics of the part IV of [L_4]. The requested properties which come from the properties of the correlator are for elementary cylinders which constitute the pant:

Property H1

$$(4.1)\qquad < B_.(s+\Delta s, t) - B_.(s,t), B_.(u,v) >= O(\Delta s)$$

if u does not belong to $]s, s+\Delta s[$ and the symmetric property.

Property H2

$$(4.2)\qquad < B_.(s+\Delta s), t) - B_.(s,t), B_.(u, v+\Delta v) - B_.(u,v) >= O(\Delta s\Delta v)$$

if u does not belong to $]s, s+\Delta s[$ and t does not belong to $]v, v+\Delta v[$.

Property H3

$$(4.3)\qquad < B_.(s+\Delta s, t) - B_.(s,t), B_.(s'+\Delta s', u) - B_./(s', u) = O(\Delta s\Delta s')$$

if $]s', s'+\Delta s'[\bigcap]s, s+\Delta s[= \emptyset$ and the symmetric property.

Property H4: If $t' \geq t$,

$$< B_{.}(s+\Delta s,t') - B_{.}(s,t'), B_{.}(s+\Delta s,t) - B_{.}(s,t) >= C(t,t')\Delta s \tag{4.4}$$

where $C(t,t')$ is continuous, the same being true for the symmetric case.

We imbedd G into a matrix algebra isometrically. Let $g(s,t)$ be the random field parametrized by the torus with values in G. Let 2^N be an integer, and s_i be the associated dyadic subdivision of S^1 and t_j be the associated dyadic subdivision of S^1. We consider the polygonal approximation of $g(s,t)$, if $(s,t) \in [s_i,s_{i+1}] \times [t_j,t_{j+1}] = T_{i,j}$.

$$\begin{aligned} g^N(s,t) = g(s_i,t_j) &+ \frac{s-s_i}{s_{i+1}-s_i}(g(s_{i+1},t_j) - g(s_i,t_j)) + \frac{t-t_j}{t_{j+1}-t_j}(g(s_i,t_{j+1}) - g(s_i,t_j)) \\ &+ \frac{t-t_j}{t_{j+1}-t_j}\frac{s-s_i}{s_{i+1}-s_i}(g(s_{i+1},t_{j+1}) - g(s_i,t_{j+1}) - g(s_{i+1},t_j) + g(s_i,t_j)) \\ &= g(s_i,t_j) + \alpha_1^N(s) + \alpha_2^N(t) + \alpha_3^N(s,t) \end{aligned} \tag{4.5}$$

Let us consider a two form ω over G, conveniently extended in a two form ω over the Matrix algebra bounded with bounded derivatives of all orders. We suppose that the two form depends on a finite dimensional parameter v. We consider

$$A_v^N = \int_{S^1\times[0,1]} (g^N)^*\omega_v = \int_{S^1\times[0,1]} < \omega_v(g^N(s,t)), d_s g^N(s,t), d_t g^N(s,t) > \tag{4.6}$$

We have the following proposition:

Proposition IV.1: When $N \to \infty$, the traditional integral A_v^N tends for the C^k topology over each compact of the parameter space in L^2 to the stochastic integral in Stratonovirch sense:

$$\int_{T^2} g^*\omega_v = \int_{S^1\times[0,1]} < \omega(g(s,t)), d_s g(s,t), d_t g(s,t) > \tag{4.7}$$

where the stochastic integral $\int_{S^1\times[0,1]} g^*\omega_v$ has a smooth version in v.

Proof: We suppose first that there is no auxiliary parameter. We can writte:

$$\begin{aligned} A^N = &\int_{S^1\times[0,1]} < \omega(g^N(s,t)), d_s\alpha_1^N(s), d_t\alpha_2^N(t) > + \int_{S^1\times[0,1]} < \omega(g^N(s,t)), d_s\alpha_1^N(s), d_t\alpha_3^N(s,t) > \\ &+ \int_{S^1\times[0,1]} < \omega(g^N(s,t)), d_s\alpha_3^N(s,t), d_t\alpha_2^N(t) > \\ &+ \int_{S^1\times[0,1]} < \omega(g^N(s,t)), d_s\alpha_3^N(s,t), d_t\alpha_3^N(s,t) >= A_1^N + A_2^N + A_3^N + A_4^N \end{aligned} \tag{4.8}$$

.

STEP I: convergence of A_1^N. We repeat the considerations of the part III for $s \to B_{.}(s,t_j)$ and $t \to B_{.}(s_i,t)$. If we fix t_j, we get by (3.11) to an asymptotic expansion in order 3. We get expressions in the asymptotic expansion in $g_{.}^{1;\cdot}(s_i,t_j)$, $g_{.}^{2;\cdot}(s_i;t_j)$ and $g_{.}^{3;\cdot}(s_i,t_j)$. If we fix s_i, we go in (3.12) to an asymptotic expansion at order 3. We get derivatives in law $g_{.}^{\cdot;1}(si,t_j)$, $g_{.}^{\cdot;2}(s_i,t_j)$ and $g_{.}^{\cdot;3}(s_i,t_j)$.

We get:

$$\begin{aligned} A_1^N = &\sum_{i,j} < \omega(g(s_i,t_j)), g(s_{i+1},t_j) - g(s_i,t_j), g(s_i,t_{j+1}) - g(s_i,t_j) > \\ &+ \sum_{i,j}\int_{T_{i,j}} < \omega(g^N(s,t)) - \omega(g(s_i,t_j)), d_s\alpha_1^N(s), d_t\alpha_2^N(t) >= B_1^N + B_2^N \end{aligned} \tag{4.9}$$

Step I.1: convergence of B_1^N.

We writte

$$g(s_{i+1},t_j) - g(s_i,t_j) = \sqrt{s_{i+1}-s_i}g^{1;\cdot}(s_i,t_j) + (s_{i+1}-s_i)g^{2;\cdot}(s_i,t_j) + (s_{i+1}-s_i)^{3/2}g^{3;\cdot}(s_i,t_j) + O(s_{i+1}-s_i)^2 \tag{4.10}$$

and we writte

$$g(s_i,t_{j+1}) - g(s_i,t_j) = \sqrt{t_{j+1}-t_j}g^{\cdot;1}(s_i,t_j) + (t_{j+1}-t_j)g^{\cdot;2}(s_i,t_j) + (t_{j+1}-t_j)^{3/2}g^{\cdot;3}(s_i,t_j) + O(s_{i+1}-s_i)^2 \tag{4.11}$$

such that

$$B_1^N = C_1^N + C_2^N + C_3^N + C_4^N + C_5^N + error \tag{4.12}$$

with

$$C_1^N = \sum_{i,j} \sqrt{\Delta s_i}\sqrt{\Delta t_j} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_i,t_j), g^{\cdot;1}(s_i,t_j) > \tag{4.13}$$

$$C_2^N = \sum_{i,j} \sqrt{\Delta s_i}\Delta t_j < \omega(g(s_i,t_j)), g^{1\cdot\cdot}(s_i;t_j), g^{\cdot;2}(s_i;t_j) > \tag{4.14}$$

$$C_3^N = \sum_{i,j} \sqrt{\Delta s_i}\Delta t_j^{3/2} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_i,t_j), g^{\cdot;3}(s_i,t_j) > + \sum_{i,j}(\Delta s_i)^{3/2}\sqrt{\Delta t_j} < \omega(g(s_i,t_j)), g^{3;\cdot}(s_i,t_j), g^{\cdot;1}(s_i,t_j) > \tag{4.15}$$

$$C_4^N = \sum_{i,j} \Delta s_i\sqrt{\Delta t_j} < \omega(g(s_i,t_j)), g^{2;\cdot}(s_i,t_j), g^{\cdot;1}(s_i,t_j) > \tag{4.16}$$

$$C_5^N = \sum_{i,j} \Delta s_i\Delta t_j < \omega(g(s_i,t_j)), g^{2;\cdot}(s_i,t_j), g^{\cdot;2}(s_i,t_j) > \tag{4.17}$$

Step I.1.1: study of the convergence of $C_1^N = \sum_{i,j} C_{i,j,1}^N$.
We consider a bigger integer N' than N and we consider

$$D_{i,j,1}^{N'} = C_{i,j,1}^N - \sum_{T_{i',j'}\subseteq T_{i,j}} C_{i',j',1}^{N'} \tag{4.18}$$

Let us consider first the case where $0 \le s + \Delta s \le s' \le s' + \Delta s' \le 1$ and $0 \le t + \Delta t \le t' \le t' + \Delta t' \le 1$. We get if f and g are smooth functions with bounded derivatives of all orders:

$$E[f(g(s,t))h(g(s',t'))g^{1;\cdot}(s,t)g^{\cdot;1}(s,t)g^{1;\cdot}(s',t')g^{\cdot;1}(s',t')] = C(s,t,s',t')\sqrt{\Delta s}\sqrt{\Delta t}\sqrt{\Delta s}\sqrt{\Delta t'} + error \tag{4.19}$$

In order to see that, we begin by diagonalize $B_.(s,t)$ and $B_.(s',t')$.

$$B_.(s,t) = w_.(1) \tag{4.20}$$

We writte:

$$B_.(s+\Delta s,t) = \alpha(s,t,\Delta s)w_.(1) + \beta(s,t,\Delta s)w_.(3)$$
$$B_.(s,t+\Delta t) = \alpha(s,t,\Delta t)w_.(1) + \beta(s,t,\Delta t)w_.(4) \tag{4.21}$$

and the analoguous formulas for $B_.(s'+\Delta s', t')$ and $B_.(s', t'+\Delta t')$ with some others new auxiliary Brownian motions $w_.(5)$ and $w_.(6)$. Moreover

$$\alpha(s,t,\Delta s) = C + C\sqrt{\Delta s} + C\Delta s^{3/2} + O(\Delta s)^2 \tag{4.22}$$

and

$$\beta(s,t,\Delta s) = C\sqrt{\Delta s} + C\Delta s + C(\Delta s)^{3/2} + O(\Delta s)^2 \tag{4.23}$$

the same asymptotics result being true when we reverse the role of s, t.

The main result are the following:

$$< B_.(s+\Delta s, t) - B_.(s,t), B_.(u,v) >= O(\Delta s) \tag{4.24}$$

if u does not belong to $]s, s+\Delta s[$, the same equality being true if we reverse the role of s and t. We use for that that the Green kernel associated to the two dimensional problem are the product of the Green kernels associated to the one dimensional problem.

Moreover

$$< B_.(s+\Delta s, t) - B_.(s,t), B_.(u, v+\Delta v) - B_.(u,v) >= O(\Delta s \Delta v) \tag{4.25}$$

if u does not belong to $]s, s+\Delta s[$ and t does not belong to $]v, v+\Delta v[$ and

$$< B_.(s+\Delta s, t) - B_.(s,t), B_.(s'+\Delta s', u) - B_.(s',u) >= O(\Delta s \Delta s') \tag{4.26}$$

if $[s', s'+\Delta s'] \cap [s, s+\Delta s] = \emptyset$ by analoguous reasons, and using the fact that the Green kernel associated to $B_.(s,t)$ are the products of the one dimensional Green kernels..

In order to simplify the exposure, we writte $\Delta t = \Delta t' = \Delta s = \Delta s'$. We conditionate $B_.(s,t)$ and $B_.(s',t')$ by $w_.(3)$,, $w_.(4)$, $w_.(5)$, $w_.(6)$. We use the expansion of $g(s,t)$ and $g(s',t')$ in iterated integrals in order to compuite $g(s,t)$ and $g(s',t')$ in terms of $w_.(3), w_.(4), W_.(5), W_.(6)$ and its orthogonal complement, which can be easily be computed because the covariance matrix of $w_.(3), w_.(4), w_.(5), w_.(6)$ has a behaviour in $Id = error$. We useafter the Clark-Ocone formula (See [N]) in order to compute the conditional of $h(g(s,t))$ as an Itô integral in $w_.(3)$, $w_.(4)$, $w_.(5)$ and $w_.(5)$ with term bounded by $\sqrt{\Delta s}$ by (4.24). We get to take the expectation of the product of four Itô integral or 5 or 6, whose expectation can be computed by using the Itô formula and (4.25), (4.26) by applying iteratively the Itô formula and the Clark-Ocone formula.The same result holds by the same arguments for:

$$E[f(g(s,t'))h(g(s',t))g^{1;\cdot}(s,t')g^{\cdot;1}(s,t')g^{1;\cdot}(s',t)g^{\cdot;1}(s',t)] = C(s,t,s',t')\sqrt{\Delta s}\sqrt{\Delta t}\sqrt{\Delta s'}\sqrt{\Delta t'} + error \tag{4.27}$$

if we suppose that $\Delta s = \Delta s' = \Delta t = \Delta t'$.

We deduce from the previous considerations that:

$$\begin{aligned} E[\sum_{i \neq : i'; j \neq j'} D^{N'}_{i,j,1} D^{N'}_{i',j',1}] \to 2\int_{S^1\times[0,1]\times S^1\times[0,1]} C(s,t,s',t')dsdtds'dt' - \\ 2\int_{S^1\times[0,1]\times S^1\times[0,1]} C(s,t,s',t')dsdtds'dt' = 0 \end{aligned} \tag{4.28}$$

Let us now study the behaviour of

$$E[\sum_{i, j\neq j'} D^{N'}_{i,j,1} D^{N'}_{i,j',1}] \tag{4.29}$$

when $N' \to \infty$.

By the previous considerations, the contribution of the $T_{k,l}$ strictly interior to $T_{i,j}$ and of the $T_{k',l'}$ strictly interior to $T_{i,j'}$ vanish. Therefore, it is enough to study the contribution of

$$\begin{aligned} C^{1,N'}_{i,j,1} &= \sqrt{\Delta s_i}\sqrt{\Delta t_j}, < \omega(g(s_i,t_j), g^{1;\cdot}(s_i,t_j), g^{\cdot;1}(s_i;t_j) > \\ -\sum_{i'} &\sqrt{\Delta s_{i'}}\sqrt{\Delta t_j} < \omega(g(s_{i'},t_j)), g^{1;\cdot}(s_{i'},t_j), g^{\cdot;1}(s_{i'},t_j) > \end{aligned} \tag{4.30}$$

for $[s_{i'}, s_{i'+1}] \subseteq [s_i, s_{i+1}]$. We would like to show that $E[\sum_{i,j\neq j'} C^{1,N'}_{i,j,1} C^{1,N'}_{i,j',1}]$ tends to 0 when $N' \to \infty$. We will see later (See Step I.1.2 , Step I.1.3 and Step I.1.4) that we can replace $\sqrt{\Delta s_i} g^{1;\cdot}(s_i,t_j)$ by $\Delta_{s_i} g(s_i,t_j)$ and $\sqrt{\Delta t_j} g(s_i,t_j)$ by $\Delta_{t_j} g(s_i,t_j)$. iI is enough thetefore to consider the behaviour of

$$C^{2,N'}_{i,j,1} =< \omega(g(s_i,t_j)), \Delta_{s_i} g(s_i,t_j), \Delta_{t_j} g(s_i,t_j) > - \sum_{i'} < \omega(g(s_{i'},t_j)), \Delta_{s_{i'}} g(s_{i'},t_j), \Delta_{t_j} g(s_{i'},t_j) > \tag{4.31}$$

and to show that $E[\sum_{i,j\neq j'} C^{2,N'}_{i,j,1} C^{2,N'}_{i,j',1}]$ tends to 0.

But

$$\sum \Delta_{s_{i'}} g(s_{i'},t_j) = \Delta_{s_i} g(s_i,t_j) \tag{4.32}$$

Therefore

$$\begin{aligned} C^{2,N'}_{i,j,1} &= \sum < \omega(g(s_i,t_j)) - \omega(g(s_{i'},t_j)), \Delta_{s_{i'}} g(s_{i'},t_j), \Delta_{t_j} g(s_i,t_j) > \\ + \sum &< \omega(g(s_{i'},t_j), \Delta_{s_{i'}} g(s_{i'},t_j), \Delta_{t_j} g(s_i,t_j) - \Delta_{t_j} g(s_{i'},t_j) >= C^{3,N'}_{i,j,1} + C^{4,N'}_{i,j,1} \end{aligned} \tag{4.33}$$

By using the technics of the next steps, we can replace $\Delta_{s_{i'}} g(s_{i'},t_j)$ by $\sqrt{\Delta s_{i'}} g^{1;\cdot}(s_{i'},t_j)$ and $\Delta_{t_j} g(s_{i'},t_j)$ by $\sqrt{\Delta t_j} g^{\cdot;1}(s_{i'},t_j)$ and $\Delta_{t_j} g(s_{i'},t_j)$ by $\sqrt{\Delta t_j} g^{\cdot;1}(s_{i'},t_j)$ and $\Delta_{t_j} g(s_i,t_j)$ by $\sqrt{\Delta t_j} g^{\cdot;1}(s_i,t_j)$. We get two quantities $C^{5,N'}_{i,j,1}$ and $C^{6,N'}_{i,j,1}$

We compute $\sum_{i,j\neq j'} E[(C^{5,N'}_{i,j,1} C^{5,N'}_{i,j',1}$. There are two contributions. The first one is when we consider twice the same $s_{i'}$. There are 4 times of increments which appear $(s_i,t_j), (s_{i'},t_j), (s_i,t_{j'}$ and $(s'_i, t_j j')$. We take the conditional expectation along $\Delta_{s_{i'}} B_\cdot(s_{i'}), t_j)$, $\Delta_{t_j} B_\cdot(s_i,t_j)$, $\Delta_{s_{i'}} B(s_{i'},t_{j'})$ and $\Delta_{t_{j'}} B_\cdot(s_i,t_{j'})$ or more precisely olong the Brownian motion which arise from the diagonolisation (4.17) of the Brownian motions $B_\cdot(s_i,t_j)$, $B_\cdot(s_{i'},t_j)$, $B_\cdot(s_i,t_{j'})$ and $B_\cdot(s_{i'},t_{j'})$. Thje Stratonovitch integrals $g^{1;\cdot}(s,t)$ and $g^{\cdot;1}(s,t)$ are in fact Itô integrals. Moreover we can compute the conditional law of $g(s_i,t_j), g(s_{i'},t_j), g(s_i,t_{j'})$ $g(s_{i'},t_{j'})$ by writting them as iterated integrals and the Clark -Ocone formula to express the quantities which appear in this way as stochastic integral which are martingales and whose Bracket can be estimated with the others tems can be estimated by (4.21). There are a product of Martingale Itô integrals, whose expectation can be estimated by using succesivly the Itô formula and the Clark Ocone formula. We conclude by using (4.24), (4.25) and (4.26). We get that the co,ntribution when there is a coincidence leads to a term in $o(1/N)\Delta s_{i'} \Delta t_j \Delta t_{j'}$. When there is no coincidence, we conditionate by $\Delta_{s_{i'}} B_\cdot(s_{i'},t_j)$, $\Delta_{t_j} B_\cdot(s_i,t_j)$, $\Delta_{s_{i"}} B_\cdot(s_{i"},t_j)$ and $\Delta_{t_j} B_\cdot(s_i,t_{j'})$, or more precisely by the Brownian motions arising from the diagonalisation (4.17). We perform as before, and we get a contribution in $o(1/N)\Delta_{s_{i'}} \Delta_{s_{i"}} \Delta_{t_j} \Delta_{t_{j'}}$ Therefore $E[\sum_{i,j\neq j'} C^{5,N'}_{i,j,1} C^{5,N'}_{i,j',1}] \to 0$.

By the same type of trick and performing the conditional expectation along the increment $\Delta_s B_\cdot(s,t)$ and $\Delta_t B_\cdot(s,t)$ or more precisley by conditionating along the Brownian motions which appears in the diagonalisation (4.17) in $C^{6,N'}_{i,j,1} C^{6+,N'}_{i,j',1}$ and after using the Clark-Ocone formula, we see that $\sum_{i,j\neq j'} E[C^{6,N'}_{i,j,1} C^{6,N'}_{i,j',1}] \to 0$. The same holds for $E[\sum_{i,j\neq j'} C^{5,N'}_{i,j,1} C^{6,N'}_{i,j',1}]$.

Let us study the behaviour of $E[\sum_{i,j} (D^{N'}_{i,j,1})^2]$. By the considerations which will follow in the next step, it is enough to study the behaviour of

$$
\begin{aligned}
&< \omega(g(s_i,t_j)), \sum \Delta_{s_{i'}} g(s_{i'},t_j), \sum \Delta_{t_{j'}} g(s_i,t_{j'}) > \\
&- \sum < \omega(g(s_{i'},t_{j'}), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_{i'},t_{j'}) >= \\
&\{\sum_{i',j'} < \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_i,t_{j'}) > \\
&- \sum_{i',j'} < \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_{i'},t_{j'}) >\} \\
&+ \sum < \omega(g(s_{i'},t_{j'}) - \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_{i'},t_{j'}) = \tilde{G}^{N'}_{i,j,1} + G^{3,N'}_{i,j,1}
\end{aligned}
\tag{4.34}
$$

In $\tilde{G}^{N'}_{i,j,1}$, we writte:

$$
\begin{aligned}
&\Delta_{s_{i'}} g(s_{i'},t_j)\Delta_{t_{j'}}(g(s_i,t_{j'}) - \Delta_{s_{i'}} g(s_{i'},t_{j'})\Delta_{t_{j'}} g(s_{i'},t_{j'}) \\
&= (\Delta_{s_i} g(s_i,t_j) - \Delta_{s_i} g(s_i,t_{j'}))\Delta_{t_{j'}} g(s_i,t_{j'}) + \Delta_{s_{i'}} g(s_{i'},t_{j'})(\Delta_{t_{j'}} g(s_i,t_{j'}) - \Delta_{t_{j'}} g(s_{i'},t_{j'}))
\end{aligned}
\tag{4.35}
$$

and we deduce a decomposition of $\tilde{G}^{N'}_{i,j,1}$ into $G^{1,N'}_{i,j,1} + G^{2,N'}_{i,j,1}$ In $G^{1,N'}_{i,j,1}$, $G^{2,N'}_{i,j,1}$ and $G^{2,N'}_{i,j,1}$, we can replace $\Delta_{s_{i'}} g(_{i'},t_j)$, $\Delta_{t_{j'}} g(s_i,t_{j'})$ by $\sqrt{\Delta s_{i'}} g^{1;\cdot}(s_{i'},t_j)$ and $\sqrt{\Delta t_{j'}} g^{\cdot;1} g(s_i,t_{j'})$ and $\Delta_{s_{i'}} g(s_{i'},t_{j'})$ by $\sqrt{\Delta s_{i'}} g^{1;\cdot}(s_{i'},t_{j'})$ and $\Delta_{t_{j'}} g(s_{i'},t_{j'})$ by $\sqrt{\Delta t_{j'}} g^{\cdot;1}(s_{i'},t_{j'})$ by $\sqrt{\Delta t_{j'}} g^{\cdot;1}(s_{i'},t_{j'})$. We get $G^{3,N'}_{i,j,1}$ and $G^{4,N'}_{i,j,1}$.

If we compute the L^2 norm of $\sum_{i,j} G^{3,N'}_{i,j,1}$, we can distibute term by term in the product which appear. In each term, we distribute another time. There are 4 term where two expressions in $g^{1;\cdot}$ and $g^{\cdot;1}$ appear. We conditionate by the set of increments in the leading Brownian motion which appears in theses expressions, or more precisely of the terms which appear after the diagonalisation (4.21) in $\Delta_s B(s,t)$ and $\Delta_t B(s,t)$. We writte the solution of the differential equation in iterated integrals and we use the Clark-Ocone formula (See [N]). We use (4.24), (4.25) and (4.226). When we developp, there is the possibility that we get exactly 4 times $s_{i'}$, $s_{i"}$, $t_{j'}$ and $t_{j"}$, which leads to a contribution in $o(1/N)\sum_{i'\neq i",j'\neq j"} \Delta s_{i'}\Delta s_{i"}\Delta t_{j'}\Delta t_{j"}$. There is a contribution when there are 3 different time $s_i, t_{j'}, t_{j"}$ or $s_{i'}, s_{i"}, t_j$ which lead to a contribution in $\sum_{i,j'\neq j"} o(1/N)\Delta s_i \Delta t_{j'}\Delta t_{j"}$ or $\sum_{i'\neq i",j} o(1/N)\Delta s_{i'}\Delta s_{i"}\Delta t_j$ and a contribution where we get only two times s_i and t_j which leads to a contribution in $\sum_{i,j} o(1/N)\Delta s_i \Delta t_j$. Therefore, $\sum_{i,j} G^{3,N'}_{i,j,1}$ tends to 0 in L^2.

By the same argument, $\sum_{i,j} G^{1,N'}_{i,j,1}$ and $\sum^{2,N'}_{i,j}$ tend to 0 in L^2. By using this type of argument, we can get the requested limits.

Step I.1.2 Study of C^N_2 and C^N_4.

We writte

$$
C^N_2 = \sum_{i,j} C^N_{i,j,2}
\tag{4.36}
$$

We consider a bigger integer N' and we writte:

$$
D^{N'}_{i,j,2} = C^N_{i,j,2} - \sum_{T_{i',j'} \subseteq T_{i,j}} C^{N'}_{i',j',2}
\tag{4.37}
$$

We have the following behaviour:

$$
E[f(g(s,t))h(g(s',t'))g^{1;\cdot}(s,t)g^{\cdot;2}(s,t)g^{1;\cdot}(s',t')g^{\cdot;2}(s',t')] = C(s,t,s',t')\sqrt{\Delta s}\sqrt{\Delta s} + error
\tag{4.38}
$$

If $\Delta s = \Delta t$ and if $0 \le s \le s + \Delta s \le s' \le s' + \Delta s' \le 1$ and $0 \le t \le t + \Delta t \le t' \le t' + \Delta t' \le 1$. $C(s,t,s',t')$ is continuous. Namely, $g^{\cdot;2}(s,t)$ and $g^{\cdot;2}(s',t')$ are given by double stochastic integrals in the term $w_.(3)$ or $w_.(4)$ which appear in (4.21). It is the far the most complicated term, the terms in simple stochastic integrals can be treated as before. We conditionate after by the increments $\Delta_t B_.(s,t)$, $\Delta_{t'} B_.(s',t')$, $\Delta_s B_.(s,t)$ and $\Delta_{s'} B_.(s',t')$ or more precisely by the terms which arise from the diagonalisation in (4.21). We writte the double Stratonovitch integral which apperas in $g^{\cdot;2}(s,t)$ or $g^{\cdot;2}(s',t')$ as double Itô integral and a simple integral. After using the Clark-Ocone formula there are the expectation to compute of the product of at most 8 term and at least 2 Itô integrals. We use Itô formula successivly and Clark-Ocone formula successivly in order to get our estimate.

We have analoguous formulas we don't writte. Therefore:

$$E[\sum_{i\neq i';j\neq j'} D^{N'}_{i,j,2}D^{N'}_{i',j',2}] \to 2\int_{S^1\times[0,1]\times S^1\times[0,1]} C(s,t,s',t')dsds'dtdt' \\ -2\int_{S^1\times[0,1]\times S^1\times[0,1]} C(s,t,s',t')dsds'dtdt' = 0 \tag{4.39}$$

Let us study now the behaviour of

$$E[\sum_{i,j\neq j'} D^{N'}_{i,j,2}D^{N'}_{i,j',2}] \tag{4.40}$$

By the considerations which will follow, it is enough to study

$$C^{N'}_{i,j,2} = \Delta t_j < \omega(g(s_i,t_j)), \Delta_{s_i} g(s_i,t_j), g^{\cdot;2}(s_i,t_j) > \\ -\sum_{i',j'} \Delta t_{j'} < \omega(g(s_{i'},t_{j'}), \Delta_{s_{i'}} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'}) > \tag{4.41}$$

But we can writte:

$$\Delta_{s_i} g(s_i,t_j) = \sum \Delta_{s_{i'}} g(s_{i'},t_j) \tag{4.42}$$

such that:

$$\begin{aligned} C^{N'}_{i,j,2} &= \Delta t_j < \omega(g(s_i,t_j)), \sum \Delta_{s_{i'}} g(s_{i'},t_j), g^{\cdot,2}(s_i,t_j) > \\ &-\sum \Delta t_{j'} < \omega(g(s_{i'},t_{j'})), \Delta_{s_{i'}} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'}) > \\ &= \{\sum_{i',j'} \Delta t_{j'} < \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_j), g^{\cdot,2}(s_i,t_j) > \\ &-\Delta t_{j'} < \omega(g(s_i,t_j), \Delta_{s_{i'}} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'}) >\} \\ &+\sum_{i',j'} \Delta t_{j'}\{< \omega(g(s_i,t_j)) - \omega(g(s_{i'},t_{j'}), \Delta_{s_i} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'})) >\} = C^{1,N'}_{i,j,2} + C^{2,N'}_{i,j,2} \end{aligned} \tag{4.43}$$

In $C^{1,N'}_{i,j,2}$ and $C^{2,N'}_{i,j,2}$, we can replace by the considerations which will follow $\Delta_{s_{i'}}(g(s_{i'},t_{j'})$ by $\sqrt{\Delta s_{i'}} g^{1,\cdot}(s_{i'},t_{j'})$ and $\Delta_{s_{i'}}(g(s_{i'},t_j)$ by $\sqrt{\Delta s_{i'}} g^{1,\cdot}(s_{i'},t_j)$. We get expressions $C^{3,N'}_{i,j,2}$ and $C^{4,N'}_{i,j,2}$. $\sum C^{4,N'}_{i,j,2}$ tends in L^2 to 0 when $N' \to \infty$. Namely, if we distibute the term which appear in $(\sum C^{4,N'}_{i,j,2})^2$, there are 4 terms with increments $\sqrt{\Delta s_{i'}} g^{1;\cdot}(s_{i'},t_{j'})$ $\sqrt{\Delta s_{i"}} g^{1;\cdot}(s_{i"},t_{j"})$ and $\Delta t_{j'} g^{\cdot;2}(s_{i'},t_{j'})$ and $\Delta t_{j"} g^{\cdot;2}(s_{i"},t_{j"})$ which appear. We conditionate under the Brownian motions which are got after diagonalising the increments of the leadings Brownian motions which appear in these formulas and we get as before a norm in L^2 which tends to 0.

Therefore, only the behaviour of $E[\sum_{i,j\neq j'} C^{3,N'}_{i,j,2} C^{3,N'}_{i,j',21}]$ is important.

We writte:

$$\begin{aligned} C^{3,N'}_{i,j,2} &= \{\sum_{i',j'} \sqrt{\Delta s_{i'}} \Delta t_{j'} < \omega(g(s_i,t_j)), g^{1,\cdot}(s_{i'},t_j), g^{\cdot,2}(s_i,t_j) > - \\ &\sum_{i',j'} \sqrt{\Delta s_{i'}} \Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_j), g^{\cdot,2}(s_i,t_j) >\} \\ &+\{\sum_{i',j'} \sqrt{\Delta s_i} \Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_{j'}), g^{\cdot;2}(s_i,t_j) - g^{\cdot;2}(s_{i'},t_j) >\} \\ &+\{\sum_{i',j'} \sqrt{\Delta s_{i'}} \Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_j) - g^{\cdot;2}(s_{i'},t_{j'}) >\} = C^{5,N'}_{i,j,2} + C^{6,N'}_{i,j,2} + C^{7,N'}_{i,j,2} \end{aligned} \tag{4.44}$$

We have to handle with the convergence of $\sum_{i,j} C^{5,N'}_{i,j,2}$, $\sum_{i,j} C^{6,N'}_{i,j,2}$ and $\sum_{i,j} C^{7,N'}_{i,j,2}$ By the previous considerations, we have only to estimate $E[\sum_{i,j\neq j'} C^{5,N}_{i,j,2} C^{5,N'}_{i,j',2}]$, $E[\sum_{i,j\neq j'} C^{6,N'}_{i,j,2} C^{6,N'}_{i,j',2}]$ and $E[\sum_{i,j\neq j'} C^{7,N'}_{i,j,2} C^{7,N'}_{i,j',2}]$ as well as the sum where there exists others coincidences of indices i, i', j, j'. Clearly,

$$E[\sum_{i,jj'} C^{5,N'}_{i,j,2} C^{5,N'}_{i,j',2}] \to 0 \tag{4.42}$$

Namely, if we distribute, there are 6 increments which appear $\Delta_{s_{i'_1}} B(s_{i'_1}, t_{j_1})$, $\Delta_{s_{i'_1}} B(s_{i'_1}, t_{j'_1})$, $\Delta_{t_{j_1}} B(s_{i_1}, t_{j_1})$, $\Delta_{s_{i'_2}} B(s_{i'_2}, t_{j_2})$, $\Delta_{s_{i'_2}} B(s_{i'_2}, t_{j'_2})$ and $\Delta_{t_{j_2}} B(s_{i_2}, t_{j_2})$. Their mutual covariances satisfy to (4.20), (4.21) and (4.23)because $j_1 \neq j_2$ and because we dont have to consider when we distibute to consider the interaction between $\Delta_{s_{i'_1}}(s_{i'_1}, t_{j_1})$ and $\Delta_{s_{i'_1}} B(s_{i'_1}, t_{j'_1})$ and the interaction between $\Delta_{s_{i'_2}} B(s_{i'_2}, t_{j'_2})$ and $\Delta_{s_{i'_2}} B(s_{i'_2}, t_{j_2})$. We conclude after conditionating along these increments, or more precisely the Brownian motions which appear when we use the diagonalization (4.21). This allows us to show (4.45).

Moreover,

$$E[\sum_{i,j\neq j'} C^{6,N'}_{i,j,2} C^{6,N'}_{i,j',2}] \to 0 \tag{4.46}$$

Namely, when we distribute, there are 6 increments which appear $\Delta_{s_{i'_1}} B(s_{i'_1}, t_{j'_1})$, $\Delta_{t_{j_1}} B(s_{i_1}, t_{j_1})$, $\Delta_{t_{j_1}} B(s_{i'_1}, t_{j_1})$, $\Delta_{t_{j_1}} B(s_{i_1}, t_{j_1})$, $\Delta_{s_{i'_2}} B(s_{i'_2}, t_{j'_2})$, $\Delta_{t_{j_2}} B(s_{i_2}, t_{j_2})$ and $\Delta_{t_{j_2}} B(s_{i'_2}, t_{j_2})$. We can apply (4.24), (4.25) and (4.26) to these increments because we don't have to take the covariance between $\Delta_{t_{j_1}} B(s_{i_1}, t_{j_1})$ and $\Delta_{t_{j_1}} B(s_{i'_1}, t_{j_1})$ and the covariance between $\Delta_{t_{j_2}} B(s_{i_2}, t_{j_2})$ and $\Delta_{t_{j_2}}(s_{i'_2}, t_{j_2})$.

Let us consider the most complicated term $C^{7,N'}_{i,j,2}$ because in $g^{\cdot;2}(s_{i'}, t_j)$ and in $g^{\cdot;2}(s_{i'}, t_{j'})$ in (4.45), it is not the same subdividion in t_j. But since we consider

$$E[\sum_{i,j\neq j'} C^{7,N'}_{i,j,2} C^{7,N'}_{i,j',2}] \tag{4.47}$$

there are 6 increments to consider. $\Delta_{s_{i'_1}} B(s_{i'_1}, t_{j'_1})$, $\Delta_{t_{j_1}} B(s_{i'_1}, t_{j_1})$, $\Delta_{t_{j'_1}} B(s_{i'_1}, t_{j'_1})$, $\Delta_{s_{i'_2}} B(s_{i'_2}, t_{j'_2})$, $\Delta_{t_{j_2}} B(s_{i'_2}, t_{j_2})$ and $\Delta_{t_{j'_2}} B(s_{i'_2}, t_{j'_2})$ and we don't have to consider the correlation between $\Delta_{t_{j_1}} B(s_{i'_1}, t_{j_1})$ and $\Delta_{t_{j'_1}} B(s_{i'_1}, t_{j'_1})$ and the correlation $\Delta_{t_{j_2}} B(s_{i'_2}, t_{j_2})$ and $\Delta_{t_{j'_2}}(s_{i'_2}, t_{j'_2})$.We can apply (4.24), (4.25), (4.26) for the correlations we consider, and we can deduce as previously.

By the same reason

$$\sum_{i\neq i',j} E[C^{5,N'}_{i,j,2} C^{5,N'}_{i',j,2}] \to 0 \tag{4.48}$$

$$\sum_{i\neq i',j} E[C^{6,N'}_{i,j,2} C^{6,N'}_{i',j,2}] \to 0 \tag{4.49}$$

The same arguments arise when we consider:

$$\sum_{i\neq i',j} E[C^{7,N'}_{i,j,2} C^{7,N'}_{i',j,2}] \tag{4.50}$$

It remains to treat the case where there are two coincidences, that is to treat the case of $\sum E[(C^{5,N'}_{i,j,2})^2]$, $\sum E[(C^{6,N'}_{i,j,2})^2]$ and $\sum E[(C^{7,N'}_{i,j,2})^2]$. But as a matter of fact, we can show simply that

$$\sum_{i,j} E[(C^{5,N'}_{i,j,2})^2] \to 0 \tag{4.51}$$

We have namely the correlators between the following increments to consider:$\Delta_{s_{i'_1}} B(s_{i'_1}, t_j)$,$\Delta_{s'_1} B(s_{i'_1}, t_{j'_1})$, $\Delta_{t_j} B(s_i, t_j)$, $\Delta_{s_{i'_2}}(s_{i'_2}, t_j)$ and $\Delta_{s'_{i_2}} B(s_{i'_2}, t_{j'_1})$. But we have $t_{j'_1} \geq t_j$ and $t_{j'_2} \geq t_j$. Therefore:

$$< \Delta_{s_{i'_1}} B(s_{i'_1}, t_{j'_1}), \Delta_{s'_1} B(s_{i'_1}, t_j >= e(t_{j'_1} - t_j)(e(-\Delta s_{i'_1}) + e(\Delta s_{i'_1}) - 2e(0)) = C\Delta s_{i'_1}(t_{j'_1} - t_j) \tag{4.52}$$

because $t_{j'_1} \geq t_j$ and because e has half derivatives in 0. This remarks allows us to repeat the previous considerations as well as to use (4.24), (4.25) and (4.26).

Moreover

$$\sum E[(C^{6,N'}_{i,j,2})^2] \to 0 \tag{4.53}$$

We have no difficulty to show that because we don't have to consider the covariance of a $g^{1;0}(s_{i'}, t_j)$ and a $g^{1;0}(s_{i'}, t_{j'})$ and because $< g^{1;\cdot}(s_{i'}, t_j), g^{1;\cdot}(s_{i"}, t_j) = CO(\sqrt{\Delta s_{i'} \Delta_{s_{i"}}})$.

The difficult part is to show thar $\sum E[(C^{7,N'}_{i,j,2})^2] \to 0$, because two different subdivision $[t_{j'}, t_{j'+1}]$ and $[t_j, t_{j+1}]$ appear and because $t_{j'} \in [t_j, t_{j+1}]$. We writte the details of this limit, because it is the most complicated, the others limits are simpler. We writte:

(4.54)

$$\begin{aligned} C^{7,N'}_{i,j,2} &= \sum \sqrt{\delta s_{i'}} \Delta t_{j'} < \omega(g(s_i, t_j)), g^{1;\cdot}(s_{i'}, t_j), g^{\cdot;2}(s_{i'}, t_j) - g^{\cdot;2}(s_{i'}, t_{j'}) > \\ &+ \sum \sqrt{\Delta s_{i'}} \Delta t_{j'} < \omega(g(s_i, t_j)), g^{1;\cdot}(s_{i'}, t_{j'}) - g^{1;\cdot}(s_{i'}, t_j), g^{\cdot;2}(s_{i'}, t_j) > \\ - \sum < \sqrt{\Delta s_{i'}} \Delta t_{j'} &< \omega(g(s_i, t_j)), g^{1;\cdot}(s_{i'}, t_{j'}) - g^{1;\cdot}(s_{i'}, t_j), g^{\cdot;2}(s_{i'}, t_{j'}) > = C^{8,N'}_{i,j,2} + C^{9,N'}_{i,j,2} + C^{10,N'}_{i,j,2} \end{aligned}$$

By the previous considerations, the term $\sum C^{9,N'}_{i,j,2}$ and $\sum C^{9,N'}_{i,j,2}$ tends to 0 in L^2. The main difficulty is to show that

$$E[\sum_{i,j} (C^{8,N'}_{i,j,2})^2] \to 0 \tag{4.55}$$

We proceed for that as it was done in the previous part. We remark, by the same considerations than in the first part, that it is enough to replace $\Delta t_j g^{\cdot;2} g(s_{i'}, t_j)$ by a double integral $\int_{0<u<v<1} \alpha_u(s_{i'}, t_j)(dB_u(s_{i'}, t_{j+1})$ $-dB_u(s_{i'}, t_j))\alpha_v(s_{i'})(dB_v(s_{i'}, t_{j+1}) - dB_v(s_{i'}, t_j))$ where α_u and α_v are $B(s_{i'}, t_j)$ measurable. By the same argument, we replace $\Delta t_{j'} g^{\cdot;2}(s_{i'}, t_{j'})$ by a double integral $\int_{0<u<v<1} \alpha_u(s_{i'}, t_{j'})(dB_u(s_{i'}, t_{j'+1}) - dB_u(s_{i'}, t_{j'}))$ $\alpha_v(s_{i'}, t_{j'})(dB_v(s_{i'}, t_{j'+1}) - dB_v(s_{i'}, t_{j'})$ where $\alpha_u(s_{i'}, t_{j'})$ and $\alpha_v(s_{i'}, t_{j'})$ are $B_.(s_{i'}, t_{j'})$ measurable. To study the behaviour when $N' \to \infty$, we can replace without difficulty in this last expression $\alpha_u(s_{i'}, t_{j'})$ by $\alpha_u(s_{i'}, t_j)$. We writte:

$$dB_.(s_{i'}, t_{j+1}) - dB_.(s_{i'}, t_j) = \sum dB_.(s_{i'}, t_{j'+1}) - dB_.(s_{i'}, t_{j'}) \tag{4.56}$$

and we distribute in the first term of (4.55). The diagonal terms cancel, and we have to estimate when $N \to \infty$ the behaviour of

$$\begin{aligned} C^{11,N'}_{i,j,2} &= \sum \sqrt{\Delta s_{i'}} < \omega(g(s_i, t_j)), g^{1,\cdot}(s_{i'}, t_j) \\ , \sum_{t_k \neq t_{k'}} \int_{0<u<v<1} &< \alpha(u)(dB_u(s_{i'}, t_{k+1}) - dB_u(s_{i'}, t_k))\alpha(v)(dB_v(s_{i'}, t_{k'+1}) - dB_v(s_{i'}, t_{k'}) > \end{aligned} \tag{4.57}$$

where we sum over $[t_k, t_{k+1}] \subseteq [t_j, t_{j+1}]$ and $[t_{k'}, t_{k'+1}] \subseteq [t_j, t_{j+1}]$ for the sharper dyadic subdivision associated to $2^{N'}$. Instead of taking the following expression in time 1, let us take it in time r. We get a process $\sum C^{11,N'}_{i,j,2,r}$ (We replace $g(s_i, t_j)$ by $g_r(s_i, t_j)$, $g^{1;\cdot}(s_{i'}, t_j)$ by $g^{1;\cdot}_r(s_{i'}, t_j)$ and the double integral between 0 and 1 by a double integral between 0 and r. Let us consider the finite variational part $V^{N'}_r = \sum V^{N'}_{i,j,2,r}$ and the martingale part $M^{N'}_r = \sum M^{N'}_{i,j,2,r}$ associated to this process.

Let us begin to study the finite variational part of this process $V_r^{N'}$. This can come from a contraction between $\omega(g(s_i,t_j))$ and $g^{1;\cdot}(s'_i,t_j)$ which leads to a term in $\sqrt{\Delta s'_i}$, which is multiplied by a term in $\sqrt{\Delta s'_i}$. But the L^2 norm of the sum $\sum_{t_k\neq t_{k'}}$ can be estimated. We decompose first $\sum_{t_k\neq t_k}$ in a martingale term and a finite variational term. There is first a contraction between α_v and $dB_v(s'_i,t_{k'+1})-dB_v(s'_i,t_{k'})$ which leads to a term in $t_{k'+1}-t_{k'}$ The stochastic integral before can be estimated. We see the martingale term. By Itô formula; $\|\sum_{t_k\neq t_{k'}}\int_0^v\alpha_u(\delta B_u(s'_i,t_{k+1})-\delta B_u(s'_i,t_k)\|^2_{L^2}$ can be estimated in $\sum(t_{k'+1}-t_{k'})(t_{k"+1}-t_{k"})+\sum(t_{k+1}-t_k)=(t_{j+1}-t_j)^2+(t_{j+1}-t_j)$. Therefore the L^2 norm of this term behaves in $\sqrt{t_{j+1}-t_j}$. But since there is $(t_{k'+1}-t_{k'})$ before, we have a behaviour of this contribution in $\Delta s_i(t_{j+1}-t_j)^{3/2}$ whose sum vanish when $N\to\infty$. The second term comes from a contraction between $dB_u(s'_i,t_{k+1})-dB_u(s'_i,t_k)$ and $dB_v(s'_i,t_{k'+1})-dB_v(s'_i,t_{k'})$ which leads to a term in $(t_{k+1}-t_k)(t_{k'+1}-t_{k'})$ and therefore in a contribution in $(t_{j+1}-t_j)^2$. Therefore the total contribution is in $\Delta s_i(t_{j+1}-t_j)^2$, whose sum vanish when $N\to\infty$, because $< g^{1;\cdot}(s'i',t_j),g^{1;\cdot}(s_{i"},t_j)>=O\sqrt{\Delta s_{i'}\Delta s_{i"}}$

There is a contraction between $\omega(g(s_i,t_j))$ and $dB_v(s'_i,t_{k'+1})-dB_v(s'_i,t_{k'})$ which is in $(t_{k'+1}-t_{k'})$. This term cancel, because when we take the square of the L^2 norm of the sum, it behaves in $\sum_{i',i"}\Delta_{s_{i'}}\Delta_{si"}I_{i',i"}$, where $I_{i',i"}$ where $I_{i',i"}$ is a sum of quadruple $t_{k'},t_{k"},t_{k^3},t_{k^4}$ which behaves in $O(t_{j+1}-t_j)^3$ and a sum $\sum_{i'}\Delta s_i I_{i'}$ where $I_{i'}$ has a bound in $(t_{j+1}-t_j)^{3/2}$. The sum of these terms vanish, when $N\to\infty$ (See part III for analoguous considerations).

Let us estimate the martingale term $M_{i,j,2,r}^{N'}$. Let us estimate the L^2 norm of $M_r^{N'}$. We use Itô formula. It behaves as $\sum_{i',i"}\Delta s_{i"}\Delta s_{i'}I_{i,i'}+\sum_{i'}\Delta s_{i'}I_{i'}$ where $I_{i',i"}$ has a bound in $(t_{j+1}-t_j)^{3/2}$ and $I_{i'}$ the same. Therefore the L^2 norm of $M_r^{N'}$ vanish when $N\to\infty$.

Step I.1.3: study of C_5^N.

We writte

$$C_5^N=\sum C_{i,j,5}^N=\sum\Delta s_i\Delta t_j<\omega(g(s_i,t_j)),g^{2;\cdot}(s_i,t_j),g^{\cdot;2}(s_i,t_j)> \tag{4.58}$$

We consider a bigger integer N' of N and we want to study:

$$D_{i,j,5}^{N'}=C_{i,j,5}^N-\sum_{T_{i',j'}\subseteq T_{i,j}}C_{i',j',5}^{N'} \tag{4.59}$$

We writte

$$D_{i,j,5}^{N'}=C_{i,j,5}^{2,N}+C_{i,j,5}^{3,N'} \tag{4.60}$$

with

$$C_{i,j,5}^{2,N'}=\sum_{T_{i',j'}\subseteq T_{i,j}}\Delta s_{i'}\Delta t_{j'}<\omega(g(s_i,t_j))-\omega(g(s_{i'},t_{j'}),g^{2;\cdot}(s_i,t_j).,g^{\cdot;2}(s_i,t_j)> \tag{4.61}$$

and

$$\begin{aligned}C_{i,j,5}^{3,N'}=&\sum_{T_{i',j'}\subseteq T_{i,j}}\Delta s_{i'}\Delta t_{j'}\{<\omega(g(s_{i'},t_{j'}),g^{2;\cdot}(s_i,t_j),g^{\cdot;2}(s_i,t_j)>\\&-<\omega(g(s_{i'},t_{j'}),g^{2;\cdot}(s'_i,t'_j),g^{\cdot;2}(s_{i'},t_{j'})>\end{aligned} \tag{4.62}$$

It is clear that $\sum C_{i,j,5}^{2,N'}\to 0$ in L^2.

In order to estimate $C_{i,j,5}^{3,N'}$, we writte

$$\begin{aligned}&(g^{2;\cdot}(s_i,t_j)g^{\cdot;2}(s_i,t_j)-g^{2;\cdot}(s_{i'},t_{j'})g^{\cdot;2}(s_{i'},t_{j'})=\\&(g^{2;\cdot}(s_i,t_j)-g^{2;\cdot}(s_{i'},t_{j'}))g^{\cdot;2}(s_i,t_j)+g^{2;\cdot}(s_{i'},t_{j'})(g^{\cdot;2}(s_i,t_j)-g^{\cdot;2}(s_{i'},t_{j'})\end{aligned} \tag{4.63}$$

and we writte

$$g^{2,\cdot}(s_i,t_j)-g^{2;\cdot}(s_{i'},t_{j'})=g^{2;\cdot}(s_i,t_j)-g^{2;\cdot}(s_{i'},t_j)+g^{2;\cdot}(s_{i'},t_j)-g^{2;\cdot}(s_{i'},t_{j'}) \tag{4.64}$$

and a similar formula for $g^{\cdot;2}(s_i,t_j) - g^{\cdot,2}(s_{i'},t_{j'})$ and we conclude as in the end of step I.1.2 , in order to show that these terms converge to 0.

Step I.1.4: study of C_3^N.

We have if $s_i \neq s_{i'}$, by using the previous technics

(4.65)
$$E[< \omega(g(s_i,t_j)), g^{1;\cdot}(s_i,t_j), g^{\cdot;3}(s_i,t_j) >< \omega(g(s_{i'},t_{j'})), g^{1,\cdot}(s_{i'},t_{j'}), g^{\cdot,3}(s_i,t_{j'}) >] = O(\sqrt{\Delta s_i}\sqrt{\Delta s_{i'}})$$

Therefore $E[(C_3^N)^1 2] \to 0$.

Step I.2: convergence of B_2^N.

We writte in probability:

$$\begin{aligned}\omega(g^N(s,t)) - \omega(g(s_i,t_j)) = \nabla\omega(g(s_i,t_j))(g^N(s,t)) - g(s_i,t_j))\\ +\nabla^2\omega(g(s_i,t_j))(g^N(s,t) - g(s_i,t_j))^2 + O(\Delta t_j^{3/2} + O\Delta s_i^{3/2}\end{aligned} \tag{4.66}$$

The residual term converges to 0 by the previous arguments. It remains to treat the main term. We recall:

$$\begin{aligned}g^N(s,t) - g(s_i,t_j) = \frac{s-s_i}{s_{i+1}-s_i}(g(s_{i+1},t_j) - g(s_i,t_j)) + \frac{t-t_j}{t_{j+1}-t_j}(g(s_i,t_{j+1}) - g(s_i,t_j))\\ +\frac{t-t_j}{t_{j+1}-t_j}\frac{s-s_i}{s_{i+1}-s_i}(g(s_{i+1},t_{j+1}) - g(s_i,t_{j+1}) - g(s_{i+1},t_j) + g(s_i,t_j))\end{aligned} \tag{4.67}$$

Moreover

$$\int_{s_i}^{s_{i+1}} \frac{s-s_i}{s_{i+1}-s_i} ds = s_{i+1} - s_i \tag{4.68}$$

The integral of the first term of (4.64) leads to to the convergence of the sum of random quantities of a type analoguous to already considered quantities $< \nabla\omega(g(s_i,t_j)).\Delta_{s_i}g(s_i,t_j), \Delta_{s_i}g(s_i,t_j), \Delta_{t_j}g(s_i,t_j) >$ which converges by the methods used before. We can treat by the same method the convergence of $< \nabla\omega(g(s_i,t_j))(g(s_i,t_{j+1}) - g(s_i,t_j)), \Delta_{s_i}g(s_i,t_j), \Delta_{t_j}g(s_j,t_j) >$ which converge by the same methods as before. The term in $\frac{(t-t_j)(s-s_i)}{(\Delta t_j \Delta s_i}$ lead to analoguous terms. If we consider the term where the quadrat of $g^N(s,t) - g(s_i,t_j)$ appear, there is a term in $< \nabla^2\omega(g(s_i,t_j)); \Delta_{s_i}g(s_i,t_j)^2, \Delta_{s_i}g(s_i,t_j), \Delta_{t_j}g(s_i,t_j) >$ whose sum vanishes in L^2 by the same considerations as in Step I.1.4. The only problem comes when we take sum of the type $\sum_{i,j} < \nabla^2\omega(g(s_i,t_j)).\Delta_{s_i}g(s_i,t_j).\Delta_{t_i}g(s_i,t_j), \Delta_{s_i}g(s_i,t_j), \Delta_{t_i}g(s_i,t_j) >$ whose treatment is similar to step I.1.3 by expending a product of integrals into iterated integrals of length 2.

Step II: convergence of A_2^N and A_3^N.

The treatment for A_2^N and A_3^N are similars. So we will treat only the case of A_2^N.

We writte:

$$\begin{aligned}A_2^N = \sum_{i,j} < \omega(g(s_i,t_j)), d_s\alpha_3^N(s,t), d_t\alpha_2^N(t) >\\ = \sum_{i,j}\int_{T_{i,j}} < \omega(g^N(s,t)) - \omega(g(s_i,t_j)), d_s\alpha_3^N(s,t), d_t\alpha_2^N(t) >= B_1^N + B_2^N\end{aligned} \tag{4.69}$$

Step II.1: convergence of B_1^N.

$$\begin{aligned}\int_{T_{i,j}} < \omega(g(s_i,t_j)), df_s\alpha_3^N(s,t), d_t\alpha_2^N(t), >= \int_{T_{i,j}} \frac{ds}{s_{i+1}-s_i}\frac{(t-t_j)dt}{(t_{j+1}-t_j)^2}\\ < \omega(g(s_i,t_j)), g(s_{i+1},t_{j+1}) - g(s_i,t_{j+1}) - g(si+1,t_j) + g(s_i,t_j), g(s_i,t_{j+1}) - g(s_i,t_j) >\end{aligned} \tag{4.70}$$

The integral over $T_{i,j}$ is constant.

We writte:

$$g(s_{i+1},t_{j+1})-g(s_i,t_{j+1})-g(s_{i+1},t_j)+g(s_i,t_j)$$
$$(4.71)\qquad =\{g(s_{i+1},t_{j+1})-g(s_i,t_{j+1})\}-\{g(s_{i+1},t_j)-g(s_i,t_j)\}=\gamma^1_{i,j}-\gamma^2_{i,j}$$

The term in $\gamma^2_{i,j}$ can be treated as in step I.1. The term in $\gamma^1_{i,j}$ can be treated as in step I.1, because the increments between $\Delta_{s_i}B(s_i,t_j)$ and $\Delta_{s_i}B(s_i,t_{j+1}) >$ satisfy to (4.52), and we can do as in the treatment of (4.52)

Step II.2: convergence of B_2^N.

We use (4.66) and we conclude as in step I.2.

Step III: convergence of A_4^N.

We writte:

$$A_4^N=\sum_{i,j}\int_{T_{i,j}} < \omega(g(s_i,t_j)),d_s\alpha_3^N(s,t),d_t\alpha_3^N(s,t) >$$
$$(4.72)\qquad +\sum_{i,j}\int_{T_{i,j}} < \omega(g^N(s,t))-\omega(g(s_i,t_j)),d_s\alpha_3^N(s,t),d_t\alpha_3^N(s,t) >= B_1^N+B_2^N$$

.

Step III.1: convergence of B_1^N.

We writte with the notations of (4.71):

$$\int_{T_{i,j}} < \omega(g(s_i,t_j)),d_s\alpha_s^N(s,t),d_t\alpha_3^N(s,t)) >$$
$$(4.73)\qquad =2\int_{T_{i,j}}\frac{(t-t_j)dt}{t_{j+1}-t_j}\frac{ds}{s_{i+1}-s_i} < \omega(g(s_i,t_j)),\gamma^1_{i,j}+\gamma^2_{i,j},\gamma^1_{i,j}+\gamma^2_{i,j} >$$

The integral over $T_{i,j}$ is constant. In order to treat the sum, we writte the second $\gamma^1_{i,j}+\gamma^2_{i,j}$ as $\delta^1_{i,j}+\delta^2_{i,j}$ where

$$(4.74)\qquad \delta^1_{i,j}=g(s_{i+1},t_{j+1})-g(s_{i+1},t_j)$$

and

$$(4.75)\qquad \delta^2_{i,j}=-g(s_i,t_{j+1})+g(s_i,t_j)$$

and we perform the limit as in the previous considerations.

Step III.2: convergence of B_2^N.

We writte

$$(4.76)\qquad \int_{T_{i,j}}\alpha^N(s,t) < \omega(g^N(s,t))-\omega(g(s,t)),\gamma^1_{i,j}+\gamma^2_{i,j},\delta^1_{i,j}+\delta^2_{i,j} > dsdt$$

and we use (4.71) for $\alpha^N(s,t)$ a suitable function of (s,t).

When the form depends on a finite dimensional parameter, we show that the approximation of the stochastic integrals converge for all the derivatives of ω and we conclude by using the Sobolev imbedding theorem as in [I.W]. That is we consider the integrals

$$(4.77)\qquad \int_{S^1\times[0,1]} < \nabla_u^\alpha\omega(g^N(s,t)),d_sg^N(s,t),d_tg^N(s,t) >$$

which converges in L^2 for all multiindices α.

$\diamondsuit$

We would like to get the same theorem with a more intrinsic approximation of the random field $g(s,t)$ $\tilde{g}^N(s,t)$. As in the part III, the finite dimensional approximations of the integral $\int_{S^1\times[0,1]}\tilde{g}^{N,*}\omega$ will converge in L^2, but we don't know if they will converge to the same limit integral of $\int_{S^1\times[0,1]} g^{N,*}\omega$.

For that if $g(s,t_j)$ and $g(s,t_{j+1})$ are close, we use the functions:

$$F^N(t,g(s,t_j),g(s,t_{j+1}) = \exp[\frac{t-t_j}{t_{j+1}-t_j}\log(g(s,t_{j+1})g^{-1}(s,t_j)]g(s,t_j) \tag{4.78}$$

conveniently extended to the whole sets of matrices.

We approximate $g(s,t_{j+1})$, $g(s,t_j)$ as follows:

$$F_N(s,g(s_i,t_{j+1}),g(s_{i+1},t_{j+1})) = \exp[\frac{s-s_i}{s_{i+1}-s_i}\log(g(s_{i+1},t_{j+1})g^{-1}(s_i,t_{j+1}))]g(s_i,t_{j+1}) \tag{4.79}$$

conveniently extended over the whole matrix algebras as well as its inverse. Moreover,

$$F^N(s,g(s_i,t_j),g(s_{i+1},t_j)] = \exp[\frac{s-s_i}{s_{i+1}-s_i}\log(g(s_{i+1},t_j)g^{-1}(s_i,t_j))]g(s_i,t_j) \tag{4.80}$$

conveniently exrended as well as its inverse to the set of all matrices.

We take as approximation:

$$\begin{aligned}\tilde{g}^N(s,t) = \exp[\frac{t-t_j}{t_{j+1}-t_j}&\log(F^N(s,g(s_i,t_{j+1},g(s_{i+1},t_{j+1}))\\ &(F^N)^{-1}(s,g(s_i,t_j),g(s_{i+1},t_j))))]F^N(s,g(s_i,t_j),g(s_{i+1},t_j))\end{aligned} \tag{4.81}$$

We have the asymptotic expansion:

$$\begin{aligned}&F^N(t,g(s,t_j),g(s,t_{j+1}) = g(s,t_j)\\ &+\frac{t-t_j}{t_{j+1}-t_j}(g(s,t_{j+1})-g(s,t_j))+O((\frac{t-t_j}{t_{j+1}-t_j})^2(g(s,t_{j+1}-g(s,t_j))^2)\end{aligned} \tag{4.82}$$

We imbedd in this express the approximation of $g(s,t_{j+1})$ and of $g(s,t_j)$. this shows that in the expansion of $\tilde{g}^N(s,t)$, the more singular term is the same than in (4.5), modulo some more regular terms which converge. The main Itô integral is the same, but we don't know if the correcting terms are the same.

We get the main result of this part:

Theorem IV.2: when $N\to\infty$, the traditional integral integral $\tilde{A}_v^N = \int_{S^1\times[0,1]}(\tilde{g}^N)^*\omega_v$ converges in L^2 to the stochastic Stratonovitch integral:

$$\int_{S^1\times[0,1]} g^*\omega_v = \int_{S^1\times[0,1]} <\omega(g(s,t)),d_sg(s,t),d_tg(s,t)> \tag{4.83}$$

Moreover, $\int_{S^1\times[0,1]} g^*\omega_v$ has a smooth version in v.

Remark: we ignore if the the stochastic integral of Theorem IV.2 is equal to the stochastic integral of Proposition IV.1. In the rest of this paper, we will use the version of Theorem IV.2.

Remark: we can consider in the previous theorem a 2-tensor which is not obligatory a 2-form.

Since we have decomposed the $1+n$ punctured spher $\Sigma(1,n)$ in elementary cylinders C_i, we can get:

Theorem IV.3: When $N\to\infty$, the traditional integral $\tilde{A}_v^N = \int_{\Sigma(1,n)}(\tilde{g}^N)^*\omega_v = \sum\int_{C_i}(\tilde{g}^N)^*\omega_v$ converges in L^2 to the stochastic Stratonovitch integral

$$\int_{\Sigma(1,n)} g^*\omega_v = \sum_i\int_{C_i} g^*\omega_v = \sum_i\int_{S^1\times[0,1]} <\omega_v(g^i(s,t)),d_sg^i(s,t),d_tg^i(s,t)> \tag{4.84}$$

where g^i is the random field parametrized by each cylinder. Moreover the stochastic integral depend almost surely in a smooth way from the finite dimensional parameter v.

V. STOCHASTIC W.Z.N.W. MODEL ON THE PUNCTURED SPHERE

Let us consider the 3-form closed Z-valued ω over G which is supposed simple simply connected, which at the level of the Lie algebra of G is equal to

(5.1) $$\omega(X,Y,Z) = K < [X,Y], Z >$$

We extend ω in a s-form over the whole matrix algebra bounded with bounded derivatives of all orders. We can suppose that ω is Z-valued.

Let $\Sigma(1,n)$ be a $1+n$ punctured sphere. We deduce a family of loops $s \to g(s,t)$. Let $s \to g(s,t)$ such a loop. We repeat the considerations of [L$_1$] and [L$_4$] in order to define over such loop space $L_t(G)$ the stochastic 2-form:

(5.2) $$\tau_{st}(\omega) = \int_{S^1} < \omega(g(s,t)), d_s g(s,t), . >$$

We can define for that the following poor stochastic differology (see [Ch], [So] for the introduction of this notion in the deterministic case). Let Ω be the probability space where the random $1+n$ punctured sphere is defined:

Definition V.1: A stochastic plot of dimension m of $L(G)$ is given by a countable family (O, ϕ_i, Ω_i) where O is an open subset of R^m such that:

i)The Ω_i constitute a measurable partition of Ω.

ii) $\phi_i(u)(.) = \{s \to F_i(u, s, g(s,t))\}$ where F_i is a smooth function over $O \times S^1 \times R^N$ with bounded derivatives of all orders (R^N is the matrix algebra where we have imbedded G).

iii)Over Ω_i, for all $u \in U$, $\phi_i(u)(.)$ belongs to the loop group $L(G)$.

We identify two stochastic plots $(O, \phi_i^1, \Omega_i^1)$ and $(O, \phi_j^2, \Omega_j^2)$ if $\phi_i^1 = \phi_j^2$ almost surely over $\Omega_i^1 \cap \Omega_j^2$.

If $\phi_i(u)$ is a stochastic plot,

(5.3) $$\phi_i^* \tau_{st}(\omega)(X,Y) = \int_{S^1} < \omega(F_i(u,s,g(s,t)), d_s F_i(u,s,g(s,t)), \partial_X F_i(u,s,g(s,t)), \partial_Y F_i(u,s,g(s,t)) >$$

which defines a random smooth form over O by the rules of the Part III.

We can look at the apparatus of [L$_1$], [L$_4$], [L$_3$] to define a stochastic line bundle ξ_t^i over $L_t^i(G)$, with curvature $2\pi i k \tau_{st}(\omega)$ for k an integer. Let us recall how to do (See [L$_1$], p 463-464): let g_i be a countable system of finite energy loops in the group such that the ball of radius δ and center g_i for the uniform norm O_i determine an open cover of $L(G)$. we can suppose that δ is small. The loop g_i constitutes a distinghuished point in O_i. We construct if g belongs to O_i a distinguished curve joining g to g_i, called $l(g_i, g)$: since δ is small, $g_i(s)$ and $g(s)$ are joined by a unique geodesic for the group structure. $l_u(g_i, g)$ is the loop $s \to \exp_{g_i(s)}[u(g(s) - g_i(s))]$ where $g(s) - g_i(s)$ is the vector over the unique geodesic joining $g_i(s)$ to $g(s)$ and exp the exponential of the Lie group associated to the canonical Riemannian structure over the Lie group. This allows to define over O_i a distinghuised path joining $g(.)$ to $g_i(.)$. We choose a deterministic path joining the unit loop $e(.)$ to $g_i(.)$ $l_i(e(.), g_i(.))$, and by concatenation of the two paths, we get a distinguished path joining $g(.)$ to $e(.)$ $l_i(g(.), g_i(.))$ over O_i.

The second step is to specify a distinguished surface bounded by $l_i(e(.), g(.))$ and $l_j(e(.), g(.))$, where $g(.)$ belongs to $O_i \cap O_j$. Since δ is small, there is a path $u \to \exp_{g_i(.)}[u(g_j(.) - g_i(.))]$ joining $g_i(.)$ to $g_j(.)$. Because $L(G)$ is simply connected, because G is two-connected, the loop constituted of the path joining $e(.)$ to $g_i(.)$, the path joining $g_i(.)$ to $g_j(.)$ and the path joining $g_j(.)$ to $e(.)$ can be filled by a deterministic surface. We can moreover fill the small stochastic triangle onstituted of $l_.(g_i(.), g(.))$, $l_.(g_j(.), g(.))$ and the the exponential curve joining $g_i(.)$ to $g_j(.)$ by a small stochastic surface (See [L$_1$] for analoguous statements). We get a surface $B_{i,j}^t(g(.))$ which satisfies to our request and which is a stochastic plot. By pulling back (See [L$_1$], [L$_3$], [L$_4$]), we can consider the stochastic Z-valued form $\tau_{st}(\omega)$ and integrate it over the surface $B_{i,j}^t(g(.))$. We put

(5.4) $$\rho_{i,j}^t(g(.)) = \exp[-\sqrt{-1} 2\pi k \int_{B_{i,j}^t(g(.))} \tau_{st}(\omega)]$$

(See [L_3]).

Definition V.2: a measurable setion ϕ^t of the line bundle ξ_i^t associated to the stochastic transgression $\tau_{st}(2\pi\omega)$ over $L_t^i(G)$ is a collection of random variable $\alpha_j^{t,i}$ $L_t^i(G)$ measurables over O_j submitted to the rules

$$\alpha_{j'}^{t,i} = \alpha_j^{t,i}\rho_{j,j'}^t \tag{5.5}$$

almost surely over $O_i \cap O_j$. The Hilbert space of section Ξ_i^t of the line bundle ξ_t^i is the space of measurable sections of ξ_t^i such that

$$E[\|\phi^t\|^2] < \infty \tag{5.6}$$

where $\|\phi^t\| = |\alpha_j^{t,i}|$ over O_j, definition which is consistent, because $\rho_{j,j'}^t$ is almost surely of modulus 1 in (5.4).

Let us work in a loop space where the loop splitts in two loops. We get a splitting map $g_t^{tot} \to (g_t^1, g_t^2)$. Moreover,

$$\tau_{st}^{tot} = \tau_{st}^1(\omega) + \tau_{st}^2(\omega) \tag{5.7}$$

If we consider a couple of stochastic sections $(\phi^{1,t})$ and $\phi^{2,t}$ over the two small loop groups, this gives therefore a stochastic section $\phi^{tot,t}$ over the big loop group (See [L_3] for analoguous considerations), and the different operations are consistent with the glueing property of two loops, especially the notion of stochastic connection, we will define now ([L_1]).

Over O_i, the stochastic 1-form associated to the bundle ξ (we omitt to writte we work over $L_t(G)^i$ by writting only $L(G)$), is given by:

$$A_i(g(.)) = 2\pi k \int_0^1 \tau_{st} l_{i,t}(e(.), g(.))(\omega)(d/dt l_{i,t}(e(.), g(.)), \partial l_{i,t}(e(.), g(.)) \tag{5.8}$$

This gives the double integral:

$$2\pi k \int_0^1 \int_0^1 < \omega(l_{i,u}(e(.), g(.))(s), d_s l_{i,u}(e(.), g(.))(s), d_u l_{i,u}(e(.), g(.))(s), \partial l_{i,u}(e(.), g(.))(s) > \tag{5.9}$$

Let us consider a stochastic plot (O, ϕ_j, Ω_j) of dimension m. $\phi_j^* A_i$ is a random one form over O given if $u \in O$ by:

$$\begin{aligned} 2\pi k \int_0^1 \int_0^1 \omega(l_{i,t}(e(.), F_j(u, ., g(.))(s), d_s l_{i,t}(e(.), F_j(u, ., g(.))(s), \\ d_t l_{l_{i,t}(e(.), F_j(u,.,g(.))(s)}, \partial_X l_{i,t}(e(.), F_j(u, ., g(.))(s) >= \phi_j^* A_i(X) \end{aligned} \tag{5.10}$$

where X is a vector field over the parameter space O whose generic element is u. By the results of part II, this give a random smooth one form on O. This connection form are compatible with the application $g^{tot} \to (g^1, g^2)$ when the big loop splitts in one loop.

Let be an elementary cylinder in the $1+n$ punctured sphere. Let $\Omega_i, [t_i, t_{i+1}]$ where $\Omega_i \subseteq \Omega$ is a set of probability strictly positive and such over Ω_i $t \to \{s \to g(s,t)\}$ belongs to O_i. We suppose $t_{i+1} > t_i$ with the natural order which is inherited from the fact we consider over the $1+n$ punctured sphere n exit loop space and one input loop space. We can define the stochastic parallel transport from ξ^{t_i} to $\xi^{t_{i+1}}$ over Ω_i along the path $t \to \{s \to g(s,t)\}$ by the formula

$$\begin{aligned} \exp[-2\pi i k \int_{t_i}^{t_{i+1}} \int_0^1 \int_0^1 \omega(l_{i,u}(e(.), g(., t)(s), d_s l_{i,u}(e(.), g(., t)(s), d_u l_{i,u}(e(.) \\ , g(., t)(s), d_t l_{i,u}(e(.), g(., t)(s) > = \tau^{t_i, t_{i+1}} \end{aligned} \tag{5.11}$$

(See Part IV for the definition of the double stochastic integral). Let $\Sigma(1,n)$ be a $1+n$ punctured sphere. Let $L^i_{out}(G)$ the n output loop spaces and $L^1_{in}(G)$ the input loop space. We can define, by iterating, a generalization of the stochastic parallel transport, which applies a tensor product of sections ϕ^i_{out} over the output loop spaces to an element over the input loop space, because the different operations are compatible with the notion of glueing loops. We call this generalized parallel transport $\tilde{\tau}^{1,n}$. It is not measurable with respect of the σ-algebras given by the restriction to the random $1+n$ punctured to its boundary. Moreover, over each boundary, the laws of the loops are identical, and the Hilbert space of section of the bundle ξ_{out} and ξ_{in} are identical. We denote it by H. We consider the conditional expectation of $\tilde{\tau}^{1,n}$ with respect of the σ-fields generated by the loop groups at the boundary. We get:

Theorem V.3: $\tau^{1,n}$ associated to the $1+n$ punctured sphere defines an element of $Hom(H^{\otimes n}, H)$.

Moreover, when we give n random punctured spheres $\Sigma(1,n_i)$, and a punctured sphere $\Sigma(1,n)$, we can glue then in order to get a sphere $\Sigma(1+\sum n_i)$ according the rules of Part II. We get $\tau^{1,\sum n_i}$ which is got by Markov property of part II by composing over the input boundary of $\Sigma(,n_i)$ τ^{1,n_i} and $\tau^{1,n}$ along the output boundary of $\Sigma(1,n)$.

Let σ_i elements of $Hom(H^{\otimes n_i}, H)$. We deduce by composition an element of $Hom(H^{\otimes \sum n_i}, H)$. Moreover, it is natrural equivariant under the action of the symmetric groups over the n elements σ_i. We say that the collection of vector spaces $Hom(H^{\otimes n}, H)$ constitutes an operad (See [M], [Lo], [Lo.S.V])

We deduce form the Markov property of the random field parametrized by $\Sigma(1,\sum n_i)$ along the sewing boundary that:

Theorem V.4: $\tau^{1,n}$ realizes a morphism from the topological operad $\Sigma(1,n)$ got by sewing $1+n$ punctured spheres along their boundary into the operad $Hom(H^{\otimes n}, H)$.

We refer to [H] and [H.L] for the motivation of this part.

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