POSITIVITY THEOREM
FOR A STOCHASTIC DELAY EQUATION ON A MANIFOLD

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I INTRODUCTION

If Malliavin Calculus has a lot of precursors (See works of Hida, Albeverio, Fomin, Elworthy..), the rupture of Malliavin Calculus was to complete the differential operations on the Wiener space in all the L^p, such that the originality of Malliavin Calculus is to differentiate functional almost surely defined, because there is no Sobolev Imbedding in infinite dimension. The classical case of a functional almost surely defined is the case of a diffusion. It allowed to Malliavin to get a probabilistic proof of Hoermander's theorem ([Ma$_1$]), where the invertibility of Malliavin's matrix plays a big role.

Bismut considers Gram matrix ([Bi]), called in this situation deterministic Malliavin matrix, associated to the deterministic differential equation associated to the diffusion, and shows it plays a big role in the asymptotic of an hypoelliptic heat kernel. Ben Arous-Léandre ([B.L]) have shown that the non-degeneracy condition of Bismut plays a big role in order to know if an heat kernel is strictly positive or not. Later, Léandre ([L$_1$]) has generalized the positivity theorem for a density to the case of a jump process. Aida-Kusuoka-Stroock ([A.K.S]) have given an abstract version of this positivity theorem. Bally and Pardoux ([B.P]) have given a version of this theorem to the case of a stochastic heat equation and A. Millet and M. Sanz-Sole ([M.S]) have given a positivity theorem to a stochastic wave equation. Fournier ([F]) has generalized the theorem of Léandre ([L$_1$]) to the case of a non-linear jump process associated to Boltzmann equation. Léandre ([L$_1$]) has removed the traditional assumptions of boundedness of Malliavin Calculus in the positivity theorem of [B.L].

We are motivated in this work by another extension to this positivity theorem than diffusions, that is stochastic delay equation.

Bell and Mohammed ([B.M$_1$], [B.M$_2$]) have studied the stochastic delay equation

$$dx_t = \sum X_i(x_{t-\delta})dw_t^i \tag{1.1}$$

where w_t^i are flat Brownian motion and X_i are vector fields over R^d. They have shown, that we can apply to x_t the machinery of Malliavin Calculus, and have shown that the law of x_1 has a smooth density under suitable assumptions. We are interested in this theorem to an extension of this theorem to a compact Riemannian manifold M of dimension d'. Léandre and Mohammed have namely considered m vector fields over M and have studied the stochastic delay equation on the manifold:

$$dx_t = \sum \tau_{t,t-\delta} X_i(x_{t-\delta})dw_t^i \tag{1.2}$$

where $\tau_{t,t-\delta}$ is the stochastic parallel transport along the path $s \to x_s$, $s \in [t-\delta, t]$ from $x_{t-\delta}$ to x_t ([L.M]) for the Levi-Civita connection. Léandre and Mohammed have shown that the solution x_t of the stochastic delay equation is smooth in the stochastic Chen-Souriau sense ([L.M]), which is another Calculus, which can be applied as Malliavin Calculus, to functional almost surely defined.

Our goal is to apply Malliavin Calculus to the solution x_1 of (1.2).

Our theorem is the following:

Theorem: Suppose that the vector fields X_i spann $T_x(M)$ in all points of the manifold. Then x_1 has a smooth density q and $q > 0$.

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II. DIFFERENTIABILITY OF THE SOLUTION OF THE DELAY EQUATION

Let M be a compact Riemannian manifold. Let X_i $i = 1,..,m$ be some vector fields over the manifold M which spann in all x of the manifold the tangent space $T_x(M)$. We can imbedd M isometrically the manifold M in R^d and extend the vector fields X_i into vector fields over R^d with bounded derivatives of all orders. Let us consider the Levi-Civita connection over M, and let us consider a m-dimensional Brownian motion over R^m $B^i(t)$. Let x be a distinguished point over M and let $s \to \gamma(s)$ a curve from $[-\delta, 0]$ into M such that $\gamma(0) = x$, which is supposed smooth.

We consider the stochastic delay equation for $t \geq 0$ in Stratonovitch sense:

$$dx_{c,t} = \tau_{t,t-\delta} \sum X_i(x_{c,t-\delta}) dB_t^i \tag{2.1}$$

with initial condition $x_t = \gamma(t)$ over $[-\delta, 0]$.

Following the theory of Léandre-Mohammed ([L.M]), inspired by Driver's flow ([Dr]), there is by pulling back another functional diferential equation over $T_x(M)$:

$$\tau_{t,0}^{-1} dx_{c,t} = dx_{f,t} = \tau_{t-\delta,0}^{-1} \sum X_i(x_{c,t-\delta}) dB_t^i \tag{2.2}$$

where $\tau_{t,s}$ is the stochastic parallel transport between $x_{c,s}$ and $x_{c,t}$ along the curve $u \to x_{c,u}$. If $t < s$, it is $(\tau_{s,t})^{-1}$.

By using the argument of [L.M], the solution can be constructed step by step along $[k\delta, (k+1)\delta]$. It is clear for $[0, \delta]$, and if we suppose that $x_{c,t}$ is constructed over $[0, k\delta]$, $x_{f,t}$ is constructed over $[k\delta, (k+1)\delta]$ by (2.2), and therfore $x_{c,t}$.

We consider the polygonal approximation $dB_t^{n,i}$ of dB_t^i associated to the subdivision of [0,1] $[k/n, (k+1)/n]$. We can consider the family of solutions $x_{c,t}^n$ and $x_{f,t}^n$ associated to the stochastic differential equation:

$$dx_{c,t}^n = \tau_{t,t-\delta}^n \sum X_i(x_{c,t-\delta}^n) dB_t^{n,i} \tag{2.3}$$

$$dx_{f,t}^n = (\tau_{t-\delta,0}^n)^{-1} \sum X_i(x_{c,t-\delta}^n) dB_t^{n,i} \tag{2.4}$$

with the same initial data than $x_{c,t}$ and $x_{f,t}$.

Let us introduce the canonical vector fields Y_i over the frame bundle of M. Or, in an analoguous way, let us trivialize the frame bundle of M by adding an auxiliary bundle to the tangent bundle of M, such that the total bundle is trivial, and such that we get a global connection form A which extend the Levi-Civita connection on M.

Let us assume by induction that the stochastic gradient in the sense of Malliavin Calculus $(\nabla^j x_{f,t}^n, \nabla^j x_{c,t}^n)$, which are F_t measurable for the natural filtration of the Brownian motion B_t^i tends in all the L^p to the j^{th} stochastic gradient in the sense of Malliavin Calculus $(\nabla^j x_{f,t}, \nabla^j x_{c,t})$ over $[0, k\delta]$, uniformly in t. By (2.2), it is clearly true for $x_{f,t}$ over $[0, (k+1)\delta]$ and therefore, after a suitable trivialization of the bundle, $\tau_{t,k\delta}^n$ tends in all the Sobolev spaces of the Malliavin Calculuis to $\tau_{t,k\delta}$ over $[k\delta, (k+1)\delta]$ uniformly in t. We deduce that it is still true for $x_{c,t}$ over $[k\delta, (k+1)\delta]$.

Remark: In order to do this consideration, we have plungged M in R^d and trivialized the tangent bundle of M. It should be possible to do that more intrinsically, but we are not interested in this work by an intrinsic Calculus over $x_{c,t}$, but by a regularity result of the law of $x_{c,1}$.

Remark:: after imbedding the manifold in R^d, the stochastic differential equation of the derivative of $x_{c,t}^n$ is given by:

$$\begin{gathered} d\nabla_h x_{c,t}^n = \nabla_h \tau_{t,t-\delta}^n \sum X_i(x_{c,t-\delta}^n) dB_t^{n,i} + \\ \tau_{t,t-\delta}^n \sum DX_i(x_{c,t-\delta}^n) \nabla_h x_{c,t-\delta}^n dB_t^{n,i} dt + \tau_{t,t-\delta}^n \sum X_i(x_{c,t-\delta}^n) dh_t^{n,i} \end{gathered} \tag{2.5}$$

where we have done the perturbation $dB_t^{n,i}$ by $dB_t^{n,i} + \lambda dh_t^{n,i}$, where $h_t^{n,i}$ is the polygonal approximation of the deterministic element of the Cameron space h_t^i. Moreover, we have the following formula (See [Ar], [Bi])

$$\nabla_h \tau_{t,t-\delta}^n = \tau_{t,t-\delta}^n \int_{t-\delta}^t (\tau_{u,t-\delta}^n)^{-1} R(dx_s^n, \nabla_h x_s^n) \tau_{u,t-\delta}^n + A_{\nabla_h x_t^n} \tau_{t,t-\delta}^n - \tau_{t,t-\delta}^n A_{\nabla_h x_{t-\delta}^n} \tag{2.6}$$

(See [L$_2$] (3.84)) where R is the curvature tensor of the connection and A the connection form of the tangent bundle supposed trivialized.

This goes to the limit to the differential equation:

$$\tag{2.7} \begin{aligned} d\nabla_h x_{c,t} = (\tau_{t,t-\delta}\int_{t-\delta}^{t}(\tau_{u,t-\delta}^{-1}R(dx_{c,u},\nabla_h x_{c,u})\tau_{u,t-\delta} + A_{\nabla_h x_{c,t}}\tau_{t,t-\delta} - \tau_{t,t-\delta}A_{\nabla_h x_{c,t-\delta}}) \\ \sum X_i(x_{c,t-\delta})dB_t^i + \tau_{t,t-\delta}\sum(DX_i(x_{c,t-\delta})\nabla_h x_{c,t-\delta}dB_t^i + \tau_{t,t-\delta}\sum X_i(x_{c,t-\delta})dh_t^i \end{aligned}$$

It should be possible to get a more intrinsic version to this formula, but we choose a less intrinsic approach to be more accessible. For analoguous statement for Driver's flow, we refer to [L$_3$] and [Cr].

III. INVERTIBILITY OF THE MALLIAVIN MATRIX

In the previous part, we have shown that $x_{c,1}$ belongs to all the Sobolev spaces of Malliavin Calculus. Moreover, $\nabla_. x_{c,1}$ realizes a random map from the Cameron -Martin space into $T_{x_{c,1}}(M)$. We can compute its adjoint ${}^t\nabla x_{c,1}$, which realizes a map from $T_{x_{c,1}}(M)$ into the Cameron-Martin space. The Malliavin matrix is $C =< \nabla x_{c,1}, \nabla x_{c,1} >$. If C^{-1} belongs to all the L^p, the law of $x_{c,1}$ has got a smooth density, by Malliavin Calculus (See [Ma$_1$], [I.W], [Nu]). Classically, this theorem works over R^d, but we can come back to R^d by local trivialization and a suitable partition of unity over the manifold.

In this part, we will show:

Theorem III.1: The law of $x_{c,1}$ has got a smooth density $q(y) \geq 0$.

Proof: It is enough to show that C^{-1} belongs to all the L^p. We will use the methid of [N.S] to invert the quadratic Malliavin matrix and not the method of [B.M$_1$]. Let $Z_{t,s}$ be the solution of the stochastic differential equation equals to 0 before time s, equal to Id in time s:

$$\tag{3.1} \begin{aligned} d_t Z_{t,s} = (\tau_{t,t-\delta}\int_{t-\delta}^{t}(\tau_{u,t-\delta})^{-1}R(dx_{c,u},Z_{u,s})\tau_{u,t-\delta} + A_{Z_{t,s}}\tau_{t;t-\delta} - \tau_{t,t-\delta}A_{Z_{t-\delta,s}}) \\ \sum X_i(x_{c,t-\delta})dB_t^i + \tau_{t,t-\delta}\sum DX_i(x_{c,t-\delta})Z_{t-\delta,s}dB_t^i \end{aligned}$$

It is a linear delay equation in $Z_{t,s}$, which can be solved by the method of Picart iteration (See [L.M], [B.M]) after converting it in an Itô equation. $t \to Z_{t,s}$ is a semi-martingale, and we get by Kolmogorov lemma that

$$\tag{3.2} E[\sup \frac{|Z_{t,s} - Z_{t',s}|^p}{|t-t'|^{p\alpha}}] < \infty$$

for $\alpha < 1/2$.

Lemma III.2: We get that

$$\tag{3.3} \nabla_h x_{c,t} = \int_0^t Z_{t,s}\tau_{s,s-\delta}\sum X_i(x_{c,s-\delta})dh_s^i$$

Proof: Let us call call $A_t(h)$ the right hand side of (3.3). We get that

$$\tag{3.4} d_t A_t(h) = \tau_{t,t-\delta}\sum X_i(x_{c,t-\delta})dh^t + \int_0^t d_t Z_{t,s}\tau_{s,s-\delta}\sum X_i(x_{c,s-\delta})dh_s^i$$

We replace $d_t Z_{t,s}$ by its expression in (3.1), and we deduce, since $Z_{t,s}$ is solution of a linear equation that

$$\tag{3.5} \int_0^v Z_{u,s}\tau_{s,s-\delta}\sum X_i(x_{c,s-\delta})dh_s^i = A_v(h)$$

But in this last integral, we can reduce the integration time to $[0,s]$, because $Z_{u,s} = 0$ if $s > u$.

$\diamondsuit$

From the previous lemma, we deduce that the Malliavin matrix for the Brownian functional $x_{c,1}$ is

$$(3.6) \qquad \int_0^1 \sum < Z_{1,s}\tau_{s,s-\delta}X_i(x_{c,s-\delta}), . >^2 ds = C$$

We can use the method of Nualart-Sanz ([N.S]) in order to invert the Malliavin-Matrix. We get:

$$(3.7) \qquad P\{det(C) < \epsilon\} \leq P\{det \int_{1-\epsilon^\alpha}^1 \sum < Z_{1,s}\tau_{s,s-\delta}X_i(x_{c,s-\delta}), . >^2 ds < \epsilon\}$$

for some small $\alpha < 1/2$.

But $Z_{1,1} = I_d$. So this quantity is smaller that

$$(3.8) \qquad P\{\frac{\int_{1-\epsilon^\alpha}^1 |Z_{1,s} - I|^2 ds}{\epsilon^\alpha} \leq K\epsilon^{1-\alpha}\}$$

But the L^p norm by Jensen inequality of $\frac{\int_{1-\epsilon^\alpha}^1 |Z_{1,s}-I|^2 ds}{\epsilon^\alpha}$ is smaller than $\frac{\int_0^{\epsilon^\alpha} s^\delta ds}{\epsilon^\alpha} \leq K\epsilon^{1-\delta'}$ for δ arbitrarily closed from 1 and δ arbitrarily closed from 0. Therefore,

$$(3.9) \qquad P\{\int_{1-\epsilon^\alpha}^1 \frac{|Z_{1,s} - I|^2 ds}{\epsilon^\alpha} < K\epsilon^{1-\alpha}\} \leq K\epsilon^p$$

for all p.

Therefore C^{-1} belongs to all the L^p.

$\diamondsuit$

IV. POSITIVITY THEOREM

The goal of this part is to show the following theorem:

Theorem IV.1: The law of $x_{c,1}$ has got a strictly positive density.

For that, we introduce the deterministic delay equation:

$$(4.1) \qquad dx_{c,t}(h) = \tau_{t,t-\delta}(h) \sum X_i(x_{c,t-\delta}(h))h_t^i dt$$

where h_t^i is a a deterministic element of the Cameron-martin space:$\sum \int_0^1 |h_t^i|^2 dt < \infty$ and $\tau_{t,t-\delta}(h)$ the parallel transport along the deterministic curve $s \to x_{c,s}(h)$ between $t-\delta$ and t for the Levi-Civita connection on the manifold.

By a simple adaptation of the part II, we get:

Theorem IV.2: $h \to x_{c,1}(h)$ is Frechet differentiable and is in all h a submersion.

Proof: let us introduce the solution of the flat deterministic equation:

$$(4.2) \qquad dx_{f,t}(h) = \tau_{t-\delta,0}(h)^{-1} \sum X_i(x_{c,t-\delta}(h))h_t^i dt$$

By proceeding step by step as in the part II, we get that that $h \to x_{f,.}(h)$ is Frechet differentiable for the finite energy topology of paths in $T_x(M)$. Since the couple of $x_{c,.}(h)$ and $\tau_.(h)$ is a solution of a differential equation with bounded derivatives of all orders, we deduce that $h \to (x_{c,1}(h), \tau_1(h))$ is Frechet differentiable.

In order to compute the Gram matrix associated to $h \to x_{c,1}(h)$, called in this situation the deterministic Malliavin matrix ([Bi]), we consider the solution of the deterministic delay equation:

$$(4.3) \qquad \begin{aligned} d_t Z_{t,s}(h) = (\tau_{t,t-\delta}(h) \int_{t-\delta}^t (\tau_{u,t-\delta})^{-1} R(dx_{c,u}, Z_{u,s}(h))\tau_{u,t-\delta}(h) + A_{Z_{t,s}(h)}\tau_{t,t-\delta}(h) - \\ \tau_{t,t-\delta}(h) A_{Z_{t-\delta,s}(h)}) \sum X_i(x_{c,t-\delta}(h))h_t^i dt + \tau_{t,t-\delta}(h) \sum DX_i(x_{c,t-\delta}(h)) Z_{t-\delta,s} h_t^i dt \end{aligned}$$

with the following initial conditions: $Z_{u,s}(h) = 0$ if $u < s$ and $Z_{s,s}(h) = Id$. We get as in the part III (3.5), that the Gram matrix

$$C(h) =< Dx_{c,1}(h), Dx_{c,1}(h) >= \sum \int_0^1 < Z_{1,s}(h)\tau_{s,s-\delta}(h)X_i(x_{s-\delta}(h)),. >^2 ds \tag{4.4}$$

$C(h)$ is clearly invertible, because $s \to Z_{1,s}$ is of finite energy in s, by the next lemma, and because $Z_{1,1} = Id$.

Lemma IV.3: $s \to Z_{1,s}$ is of finite energy.

Proof: In order to state this lemma, we imbedd the manifold in a linear space and we trivialize the bundle. We get $Z_{t,s} = Id + \int_s^t \alpha_{u,s} du$ and $Z_{t,s'} = Id + \int_{s'}^t \alpha_{u,s'} du$ such that if $s' > s$

$$Z_{t,s} - Z_{t,s'} = \int_s^{s'} \alpha_{u,s} du + \int_{s'}^t (\alpha_{u,s} - \alpha_{u,s'}) du \tag{4.5}$$

But $\alpha_{u,s} - \alpha_{u,s'}$ is linear in $Z_{.,s} - Z_{.,s'}$ such that we can deduce from (4.3) that

$$Z_{t,s} - Z_{t,s'} = Z_{t,s} \int_s^{s'} \alpha_{u,s} du \tag{4.6}$$

because $Z_{t,s} - Z_{t,s'}$ solve the same delay linear equation than $Z_{t,s'}$, but with different initial data. This shows the property.

$\diamondsuit$

In particular

$$Dx_{c,1}(h).k = \int_0^1 Z_{1,s}\tau_{s,s-\delta} \sum X_i(x_{c,t-\delta}(h))k_s^i ds = \int_0^1 \tilde{Z}_{1,s}.k_s ds \tag{4.7}$$

Since $s \to Z_{1,s}$ is of finite enegy, we can as in [B.L] p 395 define

$$Dx_{c,1}.B = \int_0^1 \tilde{Z}_{1,s} ds.B_1 - \int_0^1 d_s\tilde{Z}_{1,s}.B_s ds \tag{4.8}$$

such that $Dx_{c,1}(h).B$ is a continuous function for the uniform norm.

Lemma IV.4: for all y in M, there exists an h such that $x_1(h) = y$.

Proof: let $\tilde{h}_t$ an element of the Cameron-Martin space in $T_x(M)$. By (4.1), and solving step by step the equation, we can find an h such that

$$x_{f,t}(h) = \tilde{h}_t \tag{4.9}$$

Now for a given y, we can find an $\tilde{h}$ such that the horizontal displacement along $\tilde{h}$ goes from x to y. We choose h later.

$\diamondsuit$

The beginning of the proof of Theorem IV.1 is exactly the same than the proof of theorem II.1 in [B.—) (which is an improvement of the lower Varadhan estimate of [L_1]).

Let h be a curve in the Cameron-Martin space such that $x_{c,1}(h) = y$. We have seen that the Gram matrix $< Dx_{c,1}(h), Dx_{c,1}(h) >$ is invertible. ${}^tDx_{c,1}(h)$ applies the tangent space in y of M to the Cameron-Martin space in $T_x(M)$. We study the equation:

$$dx_c(p)(t) = \tau_{t,t-\delta}(p)(\sum X_i(x_{c,t-\delta}(p)\{dB_t + h_t^i dt + ({}^tDx_{1,c}(h).p)_t^i\} \tag{4.10}$$

as in [B.L] (2.17).

Let us remark that if $B_t^i = 0$, we get an equation

$$dx'_c(p)(t) = \tau'_{t,t-\delta}(p) \sum X_i(x'_{c,t-s}(p))\{h_t^i dt + d({}^tDx_{1,c}(h)p)_t^i\} \tag{4.11}$$

and $p \to x'_c(p)(t)$ is a local diffeomorphism from a neighborhood of 0 in $T_y(M)$ into a neighborhood of y in M.

We will show later, by imbedding the manifold in R^d:

Theorem IV.4 (Stroock-Varadhan support theorem): For all multi-indices (α), all compact K, all η

$$\sup_{p\in K} E[\sup_{t\leq 1} |\partial^{(\alpha)} x_c(p)(t) - \partial^{(\alpha)} x'_c(p)(t)|^\eta| \sup_{t\in[k\delta,(k+1)\delta]} |B(t)-B(k\delta)| < \delta_k] \leq \epsilon \tag{4.12}$$

for a sequence of $\delta_k > 0$ depending only on ϵ, K α and η.

By (4.8), we get the analoguous of (2.20) in [B.L]

$$E[\sup_{t\leq 1} |\partial^{(\alpha)}(x_c(p - C(h)^{-1}Dx_{c,1}(h).B)(1))(t) - \partial^{\alpha}(x'_{c,1}(p)(t)|^n| \sup_{t\in[k\delta,(k+1)\delta]} |B(t)-B(k\delta)| < \delta_k] < \epsilon \tag{4.13}$$

We introduce following Bismut ([Bi]) and Léandre ([L$_1$], the Gaussian process $B_{1,t}$ defined by

$$B_{1,.} = B_. -^t Dx_{1,c}(h)C(h)^{-1}Dx_{c,1}(h).B \tag{4.14}$$

Folowing [Bi] and [L$_1$], $B_{1,.}$ and ${}^tDx_{c,1}(h)C(h)^{-1}Dx_{c,1}(h).B$ are two Gaussian processes which are independents. Moreover $x_{c,1}(p - C(h)^{-1}Dx_{c,1}(h).B)$ is $B_{1,.}$ measurable, as it can be seen by using the polygonal approximation of $x_{c,.}$ and $x_{f,.}$ as in in Part II.

Moreover, there exist a random set $B_{1,.}$ measurable of probability 0 containing a neighborhood of 0 in $T_y(M)$ such that the random map which to p associates $x_{c,1}(p - C(h)^{-1}Dx_{c,1}(h)(B))$ is a local diffeomorphism for an open subeset O of 0 in a neighborhood of y equal to y in $p(w_1)$.

We get, by applying the method of decomposition of the Wiener space in two pieces (See [Bi], [L$_1$] second part and [B.L] p 395) for $f \geq 0$:

$$E[f(x_{c,1}] = \int (2\pi)^{-d/2} dp \int \exp[-1/2 < p, C(h)p >] f(x_{c,1}(p - C(h)^{-1}Dx_{c,1}(h)(B)) dp_1(B_{1,.}) \tag{4.15}$$

$x_{c,1}(p - C(h)^{-1}Dx_{c,1}(h)(B))$ is $B_{1,.}$ measurable. We deduce as in [L$_1$] second part or in [B.L] p 396,

$$E[f(x_{c,1})] \geq \int_A dP_1(B_{1,.})(2\pi)^{-d/2} \int_0 \exp[-1/2 < p, C(h)p >] f(x_{c,1}(p - C(h)^{-1}Dx_{c,1}(h).B) dp \tag{4.16}$$

It remains to do the change of variable $z = x_{c,1}(p - C(h)^{-1}Dx_{c,1}(h)(B))$ on p in order to conclude, such that we get a lower bound of the density $q(y)$ which is strictly positive. We conclude by using the Cameron-Martin-Maruyama-Girsanov formula.

Let us prove the Stroock-Varadhan support theorem.

Let us recall that $(x_{c,t}, \tau_t)$ is the solution of a stochastic differential equation

$$dX_t = Y(X_t)dx_{f,t} \tag{4.17}$$

after imbedding the manifold in R^d such that Y is bounded with bounded derivatives of all orders. Let us prove Teorem IV.4 for the couple $(x_c(p)(t), \tau_t(p))$, by proceeding inductively over k for the interval $[0, k\delta]$. For $k = 0$, it is the classical Stroock-Varadhan theorem, because $s \to \gamma(s)$ is smooth for $s \in [-\delta, 0]$. Let us suppose that it is true for k, and let us show it is still true for $k+1$. Between $k\delta$ and $(k+1)\delta$, $dx_{f,t}(p)$ can be computed easily in term of $x_{c,t}(p)$ and $\tau_t(p)$ for t smaller than $k\delta$. We can choose $\delta_1, .., \delta_k$ small enough such that (4.12) is true for the differential elements appearing in $x_{f,t}(p)$ and in $x'_{f,t}(p)$. We can apply the classical Stoock-Varadhan of [B.L] Theorem A.1 to the flow between $[k\delta, (k+1)\delta]$ associated to $\tau'_{t-\delta,0}(p))^{-1} \sum X_i(x'_{c,t-\delta}(p))\{dB^i_t + h^i_t dt + d({}^tDx_{1,c}(h)p)^i_t\}$.

We get by this procedure 3 processes between $[k\delta, (k+1)\delta]$:

-)The original process $X_t(p)$.

-)The support process $X'_t(p)$.

-)The interpolating process $X'_t(p)$.

We consider the event: $\{\sup |\partial^{(\alpha)} X'_t(p) - \partial^{(\alpha)} X^1_t(p)| > K\}$. It is included in the union of the event $\{\sup_{t\leq \delta k} |\partial^{(\alpha)} X_t(p) - \partial^{(\alpha)} X'_t(p)| > K/2\} = A$ and the event $B = \cup_{|\beta|\leq|\alpha|+k'}\{\sup_{[\delta k,\delta(k+1)]} |\partial^{(\beta)} X'_t(p) - \partial^{(\beta)} X^1_t(p)| > K/2\}$. By the classical theorem of Stroock-Varadhan, we can find δ_{k+1} depending only on $K/2$ such that if we put $C(\delta_{k+1}) = \sup_{t\in[k\delta,(k+1)\delta]} |B(t) - B(k\delta)| \leq \delta_{k+1}$, we get

$$P\{B|C(\delta_{k+1}\} \leq \epsilon \tag{4.18}$$

Moreover $C(\delta_k + 1)$ is independent of $F(k\delta)$ the sigma algebra spanns by B_s for $s \leq \delta k$. Let us consider $C(\delta_1, .., \delta_k) = \cap_{j\leq k-1} \sup_{[0\delta,(j+1)\delta]} |B(t) - B(j\delta)| \leq \delta_{j+1}$. We get clearly

$$P\{\sup_{[o,(k+1)\delta]} |\partial^{(\alpha)} X'_t(p) - \partial^{(\alpha)} X^1_t(p)| > K|C(\delta_{k+1}) \cap C(\delta_1, .., \delta_k)\} \leq \epsilon + P\{A|C(\delta_1, .., \delta_k)\} \tag{4.19}$$

Let us consider the event $\{\sup_{[0,(k+1)\delta]} |\partial^{(\alpha)} X^1_t(p) - \partial^{(\alpha)} X_t(p)| > K\}$. It is included in the union of the set C $\{\sup_{[0,k\delta]} |\partial^{(\alpha)} X_t(p) - \partial^{(\alpha)} X'_t(p)| > \eta K\}$ and of the set $D = \{\sup_{[k\delta,(k+1)\delta]} |\partial^{(\alpha)} X^1_t(p) - \partial^{(\alpha)}(X_t(p))| > K\} \cap \{\sup_{[0,k\delta]} |\partial^{(\alpha)} X_t(p) - \partial^{(\alpha)} X'_t(p)| \leq \eta K\}$ Let us choose η small enough.Let us fix $C(\delta_1, ..\delta_k)$. By conditionating along $F(\delta k)$, we can chose η small enough δ_{k+1} chosen as before freely from $C(\delta_1, .., \delta_k)$ such that

$$P\{D|C(\delta_{k+1})C(\delta_1, ..\delta_{k-1})\} \leq \epsilon \tag{4.20}$$

On the other hand, we get by using the fact that the the σ-algebra $F(k\delta)$ is spanns by $\sigma(B(s) - B(k\delta)s \in [k\delta, (k + 1)\delta]$, we get that

$$P\{C|C(\delta_1, .., \delta_k)\} \geq P\{A|C(\delta_1, .., \delta_k\} \tag{4.21}$$

if η is small enough. It remains to use the recurrence hypothesis, and to find a $\delta_1, ..\delta_k$ such that the conditional expectation $P\{C|C(\delta_1, .., \delta_{k-1}\}$ is smaller than ϵ.

Remark: There is another way to get Theorem IV.1. We work over $[k\delta, (k+1)\delta]$, and we are interested by the law of $x_{c,(k+1)\delta}$. Between $[k\delta, (k + 1)\delta]$, conditionated by $F(k\delta)$, $x_{f,s} - x_{f,k\delta} = \tilde{B}_s$ is a Gaussian process. We perform Malliavin Calculus with respect of this process, and we perform classical Stroock-Varadhan theorem with respect of this process. That is we conditionate by $\sup_{[k\delta,(k+1)\delta]} |\tilde{B}_s| \leq \delta$ or not in order to conclude. But this argument works only in an elliptic situation. Namely, we have a more general theorem:

Theorem IV.5: Let us suppose that the inverse of the Malliavin matrix $< \nabla x_{c,1}, \nabla x_{c,1} >$ belongs to all the L^p such that the law of $x_{c,1}$ has a smooth density $q(y)$. Then $q(y) > 0$ if and only if there exists an h such that $x_{c,1}(h) = y$ and such that the Gram matrix $< Dx_{c,1}(h), Dx_{c,1}(h) >$ is invertible.

Proof:: the fact that the condition is sufficient is exactly proven as before. The fact that the condition is necessary has exactly the same proof than for diffusion, because $x_{c,1}$ is the limit in all the Sobolev spaces of Malliavin Calculus of $x_{c,1}(B^n)$ where B^n is the polygonal approximation of the flat Brownian motion B (See [B.L] and the part II of this work).
$\diamondsuit$.

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