# POSITIVITY THEOREM WITHOUT COMPACTNESS ASSUMPTION

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## I. INTRODUCTION

Let us consider a general manifold $M$ and some vector smooth fields $X_i, i = 0, .., m$. Let us suppose that in all points $x$, the Lie algebra spanns by the vector fields $X_i$, $X_0$ alone excluded is equal to $T_x(M)$. Let us consider an Hoermander's type operator $\Delta = X_0 + 1/2 \sum_{i>0} X_i^2$. Hoermander's theorem ([H]) states that under this assumption, the semi-group generated by $\Delta$ has a smooth density $p_t(x, y)$. We are concerned by showing when $p_1(x, y) > 0$. For that, we consider the control equation:

$$dx_t(h) = X_0(x_t(h))dt + \sum_{i>0} X_i(x_t(h))h_t^i dt \tag{1.1}$$

starting from $x$, where $h_t^i$ belongs to $L^2([0, 1])$. In $R^d$, when Hoermander's condition is given in $x$ and when the vector fields $X_i$ are bounded with bounded derivatives of all orders, Ben Arous and Léandre ([B.L]) have given the following criterium: $p_1(x, y) > 0$ if and only if there exists a $h$ such that $x_1(h) = y$ and such that $h' \to x_1(h')$ is a submersion in $h'$.

This last condition is called Bismut condition ([Bi]).

Tools used by Ben Arous and Léandre were Malliavin Calculus. This theorem was generalized by Léandre ([$L_1$]) for jump process. An abstract version for diffusion was given by Aida, Kusuoka and Stroock ([A.K.S]). Bally and Pardoux ([B.P]) has given a version of this theorem to the case of a stochastic heat equation. A. Millet and M. Sanz Sole ([M.S]) have given a positivity theorem to the case of a stochastic wave equation. Fournier ([F]) has generalized the theorem of Léandre ([$L_1$] to the case of a non-linear jump process associated to the Boltzmann equation. Léandre ([$L_3$]) has studied the case of a delay equation on a manifold.

By using the mollifer in Malliavin sense introduced by Jones-Léandre ([J.L]) and Léandre ([$L_2$], our goal is to remove the boundedness assumption in the theorem of Ben Arous and Léandre, and to generalize it to a general manifold $M$ not obligatory complete. Our theorem is the following:

**Theorem**: Let us suppose that in all points $x$ of the manifold $M$, the Lie algebra spanns by the $X_i$ , $X_0$ alone excluded, is equal to $T_x(M)$. Then $p_1(x, y) > 0$ if and only if there exists an $h$ such that $x_1(h) = y$ and such that $h' \to x_1(h')$ is a submersion in $h$.

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## PROOF OF THE THEOREM

Let us show that the condition is suffisant.

Let us introduce the solution of the stochastic differential equation in Stratonovitch sense, where $B_t^i$ are some independent Brownian motions:

$$dx_t(x) = X_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x))dB_t^i \tag{2.1}$$

starting from $x$. Let us introduce the exit time $\tau$ of the manifold. If $f$ is a smooth function on $M$, we have classically (See [I.W], [Nu]):

$$\int p_1(x,y)f(y)dy = E[f(x_1(x))1_{\tau>1}] \tag{2.2}$$

In general, we cannot apply Malliavin Calculus to the diffusion $x_t(x)$. In order to be able to apply Malliavin Calculus, we introduce the mollifers of Jones-Léandre ([J.L]) and Léandre ([$L_2$]). We consider a smooth function from $M$ into $R^+$, equal to 0 only in $x$ and which tends to $\infty$ when $y$ tends to infinity, the one compactification point of $M$. We consider a smooth function over $]-k,k[$ $(k \in R^+)$, equals to 1 over $[-k/2,k/2]$ and which behaves as $\frac{1}{(k-y)^{+r}}$ when $k \to k_-$. Outside $]-k,k[$, this function, called $g_k(y)$ is equals to $= \infty$. We suppose that $g_k \geq 1$.

We choose a big integer $r$. We choose a smooth function from $[1,\infty[$ into $[0,1]$, with compact support, equals to 1 in 1 and which decreases.

The mollifer functional of Jones-Léandre ([J.L]) is

$$F_k = h(\int_0^1 g_k(d(x_s(x)))ds) \tag{2.3}$$

$F_k$ belongs to all the Sobolev spaces in the sense of Malliavin Calculus if $r$ is big enough (See [J.L], [$L_2$], [Nu], [I.W], [Ma]), and is equal to 1 if $\sup_s d(x_s(x)) \leq k/2$, is smaller to 1 if $\sup_s d(x_s(x)) > k/2$ and is equal to 0 almost surely if $\sup_s d(x_s(x)) \geq k$. Moreover, $F_k \geq 0$.

We introduce the auxiliary measure $\mu_k$:

$$\mu_k : f \to E[F_k f(x_1(x))] \tag{2.4}$$

To the measure $\mu_k$, we can apply Malliavin Calculus. In particular, $\mu_k$ has a smaller density $q_k$ smaller than $p_1(x,y)$. In particular, if there exists a $h$ such that $x_1(h) = y$ and $h' \to x_1(h')$ is a submersion in $h$, we can find an enough big $k$ such that $q_k(y) > 0$, by the positivity theorem of Ben Arous and Léandre ([B.L]). This shows that the condition is suffisant.

In order to show that the condition is necessary, we remark that if $p_1(x,y) > 0$ in $y$, $q_k(y)$ is still strictly positive for $k$ enough big, because for $k$ big enough, for $\epsilon$ small

$$|E[(1_{\tau>1} - F_k)f(x_1(x))]| \leq \epsilon \|f\|_\infty \tag{2.5}$$

where $\|f\|_\infty$ denotes the uniform norm of $f$.

Therefore, it is enough to apply Ben Arous-Léandre result in the other sense.

**Remark**: Let us suppose that Hoermander's condition is checked only in $x$. We can suppose that $h$ is decreasing and that $g_k$ decreases to 1, such that $F_k$ increases to $1_{\tau>1}$. By Malliavin Calculus, $\mu_k$ has a density $q_k$, which increases. Let us consider the indicatrice functional of a set of measure 0 for the Lebesgue measure over $M$. We have:

$$\mu_k[f] = 0 \tag{2.6}$$

But

$$\mu_k[f] = E[F_k f(x_1(x))] = \tag{2.7}$$

anf $F_k f(x_1(x))$ increases and tends to $1_{\tau>1} f(x_1(x))$, which is in $L^1$. We deduce that

$$E[1_{\tau>1} f(x_1(x))] = 0 \tag{2.8}$$

This means that the the law of $x_1(x)$ has a density without to suppose that Hoemander's hypothesis is checked in all points.

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