GEOMETRIC TOMOGRAPHY AND LOCAL STEREOLOGY

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Abstract. A substantial portion of E. Lutwak's dual Brunn-Minkowski theory, originally applicable only to star-shaped sets, is extended to the class of bounded Borel sets. The extension is motivated by an important application to local stereology, a collection of stereological designs based on sections through a fixed reference point that has achieved significant medical results in neuroscience and cancer grading.

1. Introduction

The classical Brunn-Minkowski theory, born just over a century ago, provides the techniques for solving many problems in geometry concerning metric quantities such as volume, surface area, and mean width. The usual framework is the class of convex bodies in $\mathbb{R}^n$. The theory employs quantities called mixed volumes, of which volume, surface area, and mean width are examples. In fact, these are special mixed volumes called intrinsic volumes. It turns out that any intrinsic volume of a convex body can be represented as an average of volumes of its projections onto subspaces. This fact (called the Kubota integral recursion; see, for example, Schneider's book [33, p. 295] for this and a wealth of information about the Brunn-Minkowski theory) is one of many integral formulas that also form part of integral geometry. Such formulas have found an important application in *stereology*, defined in 1961 by H. Elias as the exploration of three-dimensional space from two-dimensional sections or projections of solid bodies. Applications of stereology include metallurgy and biology, where inferences about the structure of a three-dimensional mineral sample or biological tissue can be made via appropriate measurements of a sample of their two-dimensional slices.

In 1975, E. Lutwak [27] initiated the dual Brunn-Minkowski theory, in which the intersections of star bodies with subspaces replace the projections of convex bodies onto subspaces in the classical theory. Lutwak discovered that integrals over S^{n-1} of products of radial functions (see Section 2 for definitions and notation) behave like mixed volumes, and called them dual mixed volumes. Special cases of dual mixed volumes analogous to the intrinsic volumes are called dual volumes, and it can be shown that a dual Kubota integral recursion holds for these; instead of averaging volumes of projections, this involves averaging volumes of intersections

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with subspaces. In 1990, one of the authors (R.G.) introduced the term *geometric tomography* for the area of mathematics concerning the retrieval of information about a geometric object from data concerning its sections by subspaces or projections onto subspaces. Both the Brunn-Minkowski theory and its dual are useful in geometric tomography, and [11] also explains the nature of the duality between the two (insofar as it is understood).

In the late 1980's, a new branch of stereology called *local stereology* was pioneered by one of the authors (E.V.J.) and H. J. G. Gundersen, and has already achieved significant medical results in neuroscience and cancer grading. Local stereology, surveyed in [17], is a collection of stereological designs based on sections through a fixed reference point. As such, it relates especially with the part of geometric tomography that concerns intersections with subspaces, and in particular, with the dual Brunn-Minkowski theory. The first Summer School on Stereology and Geometric Tomography, held at Sandbjerg Manor, Denmark on May 20–25, 2000, was devoted to the interplay between geometric tomography and local stereology.

Many of the biological structures encountered in local stereology are far from being star shaped. (See Section 8 below for specific examples and an introduction to the methodology of local stereology.) This is the principal motivation for the first part of this paper, which provides a significant extension of the dual Brunn-Minkowski theory. In fact, it was always clear that the star bodies considered by Lutwak, bodies star-shaped at the origin and with a continuous radial function (and hence containing the origin), is unnaturally restrictive; for example, a convex body not containing the origin is not a star body according to this definition. Two of the authors (R.G. and A.V.) gave a more general definition of the term star body (the one used below), and in [12] extended part of Lutwak's theory to the wider class. However, even this class is much too small for the application to local stereology. The present paper finally gives a fully satisfactory extension of the main part of the dual Brunn-Minkowski theory, that involving dual volumes, to the class of bounded Borel sets, the largest class of sets for which measurability and convergence issues do not arise.

Though one theme of the paper points towards the application to local stereology, our extension of the dual Brunn-Minkowski theory includes other concepts and results. For example, we define the intersection body of a bounded Borel set and give the corresponding extension of Lutwak's theorem that pertains to the celebrated Busemann-Petty problem: If the central hyperplane sections of an origin-symmetric convex body in $\mathbb{R}^n$ are always smaller in volume than those of another such body, is its volume also smaller? The problem was stated in 1956, and solved in [9], [10], [38], and [39] only after the crucial notion of the intersection body of a star body was introduced by Lutwak [28]. (The answer is affirmative if $n \leq 4$ and negative otherwise.) Lutwak's theorem says that the answer to the Busemann-Petty problem is affirmative for any n if the body with the smaller sections is an intersection body.

The paper is organized as follows. After some basics and a summary of Lutwak's dual Brunn-Minkowski theory, we extend the part concerning dual volumes to the class of bounded Borel sets in Section 4. Two key ingredients are an integral transform called the point X-ray of order i and the Blaschke-Petkantschin formula from integral geometry. Once these are used to supply the correct definitions, some of the proofs follow quite closely those from the original theory. For certain inequalities and Lutwak's theorem on intersection bodies, however, more is needed. We require variants of Jensen's inequality for means that apply to Lebesgue-Stieltjes measures (Lemmas 4.7 and 4.8) and which may be of independent interest. Section 5 represents the first systematic effort to perform a similar extension of general dual mixed volumes. The results are rather inconclusive, not surprising since attempts to generalize mixed volumes in the classical Brunn-Minkowski theory much beyond the class of convex bodies have also been less than satisfactory. The remainder of the paper outlines the application to local stereology. This focuses on the local stereological volume estimators, which are defined in Section 6. In Section 7 we discuss various classes of sets that might be used as models for the objects encountered in practise, and derive corresponding practical formulas for the volume estimators. The final Section 8 is a brief overview of local stereology as it is practised today.

2. Definitions and notation

As usual, S^{n-1} denotes the unit sphere, B the unit ball, and o the origin in Euclidean n-space $\mathbb{R}^n$. By a *direction*, we mean a unit vector, that is, an element of S^{n-1}. If u is a direction, we denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u and by l_u the line through the origin parallel to u.

The *characteristic function* of a set A is denoted by 1_A.

We write V_k for k-dimensional Lebesgue measure in $\mathbb{R}^n$, where $k \in \{0, \dots, n\}$, and where we identify V_k with k-dimensional Hausdorff measure (V_0 is the counting measure). We also generally write V instead of V_n. We let $\kappa_n = V(B)$ and note that $V_{n-1}(S^{n-1}) = \omega_n = n\kappa_n$. The notation dz will always mean $dV_k(z)$ for the appropriate k with $k \in \{0, \dots, n\}$. The notation dS will denote integration on the Grassmannian $\mathcal{G}(n,k)$ of k-dimensional subspaces in $\mathbb{R}^n$ with respect to the canonical invariant probability measure, usually referred to as Haar measure in $\mathcal{G}(n,k)$.

We say that a set is *o-symmetric* if it is centrally symmetric, with center at the origin.

A set L is *star-shaped at o* if $L \cap l_u$ is either empty or a (possibly degenerate) closed line segment for each $u \in S^{n-1}$. If L is star-shaped at o, we define its *radial function* ρ_L by

$$\rho_L(u) = \begin{cases} \max\{c : cu \in L\} & \text{if } L \cap l_u \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

This definition is a slight modification of [11, (0.28)]; as defined here, the domain of ρ_L is always S^{n-1}.

A *body* is a compact set equal to the closure of its interior. By a *star body* in $\mathbb{R}^n$ we mean a body L star-shaped at o such that ρ_L, restricted to its support, is continuous. This definition, introduced in [12] (see also [11, Section 0.7]), allows bodies not containing o, unlike previous definitions; in particular, every convex body is a star body in this sense. (Other definitions, for example that of Klain [24], [25] are not relevant for our purposes, since we only require bounded sets.) We denote the class of star bodies in $\mathbb{R}^n$ by $\mathcal{L}^n$, and the subclass of star bodies containing o by $\mathcal{L}^n_o$. We write $\mathcal{B}^n$ for the class of bounded Borel sets in $\mathbb{R}^n$, $\mathcal{B}^n_s$ for the class of sets in $\mathcal{B}^n$ that are star-shaped at o, and $\mathcal{B}^n_{so}$ for the members of $\mathcal{B}^n_s$ that also contain o.

We denote by R the *spherical Radon transform*, defined by

$$(Rf)(u) = \int_{S^{n-1}\cap u^\perp} f(v)dv,$$

for bounded Borel functions f on S^{n-1}. The transform R is self-adjoint, that is,

$$\int_{S^{n-1}} f(u)(Rg)(u)\,du = \int_{S^{n-1}} (Rf)(u)g(u)\,du \tag{1}$$

for bounded Borel functions f and g on S^{n-1}; see, for example, [11, Theorem C.2.6]. On the right-hand side of (1), Rf is integrated with respect to the finite Borel measure in S^{n-1} defined for Borel subsets of S^{n-1} by

$$\mu(E) = \int_E g(u)\,du.$$

This suggests (see, for example, [13, p. 304]) the extension of R to a linear mapping from the space $\mathcal{M}(S^{n-1})$ of signed finite Borel measures in S^{n-1} into itself by

$$\int_{S^{n-1}} f(u)\,d(R\mu)(u) = \int_{S^{n-1}} (Rf)(u)\,d\mu(u) = \int_{S^{n-1}} \int_{S^{n-1}\cap u^\perp} f(v)\,dv\,d\mu(u), \tag{2}$$

for each bounded Borel function f on S^{n-1}. This definition preserves the self-adjoint property of R.

We shall need the following version of the Blaschke-Petkantschin formula; see, for example, [17, Proposition 4.5], with $p = k$, $q = 1$, and $r = 0$.

Proposition 2.1. *Let $k \in \{1, \dots, n-1\}$ and let g be a nonnegative bounded Borel function on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} g(x)dx = \frac{\omega_n}{\omega_k} \int_{\mathcal{G}(n,k)} \int_S g(x)\|x\|^{n-k}\,dx\,dS. \tag{3}$$

3. Lutwak's dual Brunn-Minkowski theory for the class $\mathcal{B}^n_{so}$

In this section we recall the basics of Lutwak's dual Brunn-Minkowski theory. Lutwak [27] worked with star bodies containing o in their interiors, but we note here that with appropriate minor modifications, his results extend immediately to the class $\mathcal{B}^n_{so}$.

The *dual mixed volume* $\widetilde{V}(L_1, \dots, L_n)$ of sets $L_1, \dots, L_n \in \mathcal{B}^n_{so}$ is defined by

$$\widetilde{V}(L_1, \dots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}(u)\rho_{L_2}(u) \cdots \rho_{L_n}(u)\,du. \tag{4}$$

For $i \in \{1, \dots, n\}$, the *dual volume* $\widetilde{V}_i(L)$ is

$$\widetilde{V}_i(L) = \widetilde{V}(L, i; B, n-i) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^i\,du, \tag{5}$$

the dual mixed volume of i copies of L and $(n-i)$ copies of B. In particular, $\widetilde{V}_n(L) = V(L)$. Lutwak observed that dual volumes have properties analogous to the intrinsic volumes of the Brunn-Minkowski theory.

If $x, y \in \mathbb{R}^n$, then the *radial sum* $x\widetilde{+}y$ of x and y is defined to be the usual vector sum $x+y$ if x and y are contained in a line through o, and o otherwise. If $L, M \in \mathcal{B}^n_{so}$ and $s, t \geq 0$, then the *radial linear combination* $sL\widetilde{+}tM$ can be defined by

$$sL\widetilde{+}tM = \{sx\widetilde{+}ty : x \in L, y \in M\},$$

or, equivalently, by

$$\rho_{sL\widetilde{+}tM} = s\rho_L + t\rho_M. \tag{6}$$

Lutwak [27] (see also [11, Theorem A.6.1]) found the following analogue of Minkowski's theorem on mixed volumes.

Proposition 3.1. *Let $L_j \in \mathcal{B}^n_{so}$, $j \in \{1, \dots, m\}$. The volume of the radial linear combination*

$$L = t_1 L_1 \widetilde{+} \cdots \widetilde{+} t_m L_m,$$

where $t_j \geq 0$, is a homogeneous polynomial of degree n in the variables t_j, whose coefficients are dual mixed volumes. Specifically,

$$V(L) = \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \widetilde{V}(L_{j_1}, \dots, L_{j_n}) t_{j_1} \cdots t_{j_n}.$$

Of course, Lutwak's definition (4) of the dual mixed volume $\widetilde{V}(L_1, \dots, L_n)$ is compatible with the previous theorem, and in particular

$$\widetilde{V}(L, \dots, L) = V(L). \tag{7}$$

Lutwak noted that dual mixed volumes enjoy basic properties analogous to those of mixed volumes. They are (see [11, Section A.6]) nonnegative, invariant under volume-preserving linear transformations, monotonic, and positively multilinear; the latter property means that

$$\widetilde{V}(sL_1\widetilde{+}tL_1', L_2, \dots, L_n) = s\widetilde{V}(L_1, L_2, \dots, L_n) + t\widetilde{V}(L_1', L_2, \dots, L_n) \tag{8}$$

when $s, t \geq 0$.

Let $L_j \in \mathcal{B}^n_{so}$, $j \in \{1, \dots, n\}$, and let $i \in \{1, \dots, n\}$. Lutwak proved the *dual Aleksandrov-Fenchel inequality* (see [11, Section B.4]):

$$\widetilde{V}(L_1, L_2, \dots, L_n)^i \leq \prod_{j=1}^{i} \widetilde{V}(L_j, i; L_{i+1}, \dots, L_n), \tag{9}$$

with equality if and only if $L_1, \dots, L_n$ are dilatates of each other, modulo sets of measure zero. The inequality has the same form as the classical Aleksandrov-Fenchel inequality. Two special cases of (9) are worthy of note. For $L, M \in \mathcal{B}^n_{so}$, define

$$\widetilde{V}_1(L, M) = \widetilde{V}(L, n-1; M) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^{n-1} \rho_M(u)\, du. \tag{10}$$

Note that

$$\widetilde{V}_1(L, L) = V(L) \tag{11}$$

for $L \in \mathcal{B}^n_{so}$.

The *dual Minkowski inequality* (see [11, (B.23)]) states that

$$\widetilde{V}_1(L, M)^n \le V(L)^{n-1} V(M), \tag{12}$$

with equality if and only if L is a dilatate of M, modulo a set of measure zero. Let $i \in \{1, \dots, n-1\}$. The extended dual isoperimetric inequality (see [11, (B.26)]) is

$$\left(\frac{\widetilde{V}_i(L)}{\widetilde{V}_i(B)}\right)^n \le \left(\frac{V_n(L)}{V_n(B)}\right)^i, \tag{13}$$

with equality if and only if L is an o-symmetric ball, modulo a set of measure zero.

4. Dual volumes for bounded Borel sets

Gardner and Volčič [12] (see also [11, Section A.6]) extended the definition of the dual volumes $\widetilde{V}_i(L)$ to the class $\mathcal{L}^n$ by replacing the integrand in (5) by half the *i-chord function* $\rho_{i,L}$ of L, defined for *real* $i > 0$ and $u \in S^{n-1}$ by

$$\rho_{i,L}(u) = \begin{cases} \rho_L(u)^i + \rho_L(-u)^i & \text{if } o \in L, \\ \big||\rho_L(u)|^i - |\rho_L(-u)|^i\big| & \text{if } o \notin L. \end{cases}$$

Note that it remains true that $\widetilde{V}_n(L) = V(L)$, for example. Clearly the same definition can be used for sets in the larger class $\mathcal{B}^n_s$; the paper [12] focused on the class $\mathcal{L}^n$ because it is more amenable to uniqueness results.

In this section we further extend a significant part of the dual Brunn-Minkowski theory to the class $\mathcal{B}^n$. A key ingredient is the following generalization of the i-chord function.

Let $C \in \mathcal{B}^n$ and let $i > 0$. The *point X-ray of C of order i at o* is defined by

$$X_{i,o}C(u) = \int_{\mathbb{R}} 1_C(tu)|t|^{i-1}\, dt. \tag{14}$$

If $C \in \mathcal{B}^n_s$, it is easy to see that

$$X_{i,o}C = \frac{1}{i}\rho_{i,C}; \tag{15}$$

the proof is the same as in [11, Lemma 5.2.2], where the more restrictive assumption that $C \in \mathcal{L}^n$ is not necessary.

Let $k \in \{1, \dots, n\}$ and let $C \in \mathcal{B}^n$ be a subset of $S \in \mathcal{G}(n, k)$. We define the *dual volume* $\widetilde{V}_{i,k}(C)$ by

$$\widetilde{V}_{i,k}(C) = \frac{i}{2k}\int_{S^{n-1}\cap S} X_{i,o}C(u)\, du. \tag{16}$$

When $k = n$, we call $\widetilde{V}_{i,n}(C)$ the *ith dual volume* of C and denote it by $\widetilde{V}_i(C)$. When $C \in \mathcal{L}^n$, these definitions coincide with the ones given in [11, (A.55), (A.57)]. Note also that

$$\widetilde{V}_{i,1}(C \cap l_u) = iX_{i,o}C(u),$$

for all $u \in S^{n-1}$.

Theorem 4.1. *Let $i > 0$, let $k \in \{1, \dots, n\}$, and let $C \in \mathcal{B}^n$ be a subset of $S \in \mathcal{G}(n,k)$. Then*

$$\widetilde{V}_{i,k}(C) = \frac{i}{k} \int_C \|x\|^{i-k} \, dx.$$

Proof. Using (14) and (16), we obtain

$$\begin{aligned} \widetilde{V}_{i,k}(C) &= \frac{i}{2k} \int_{S^{n-1}\cap S} X_{i,o}C(u) \, du \\ &= \frac{i}{2k} \int_{S^{n-1}\cap S} \int_{\mathbb{R}} 1_C(tu)|t|^{i-1} \, dt \, du \\ &= \frac{i}{2k} \int_{S^{n-1}\cap S} \int_{l_u} 1_C(x)\|x\|^{i-k}\|x\|^{k-1} \, dx \, du \\ &= \frac{i}{k} \int_C \|x\|^{i-k} \, dx, \end{aligned}$$

the final equality following from the Blaschke-Petkantschin formula (3) with n replaced by k and k replaced by 1 (or [11, Lemma 9.4.1] with n replaced by k, S identified with $\mathbb{R}^k$, i replaced by 1, and $f(x) = 1_C(x)\|x\|^{i-k}$). □

Corollary 4.2. *Let $i \in \{1, \dots, n\}$ and let $C \in \mathcal{B}^n$ be a subset of $S \in \mathcal{G}(n,i)$. Then*

$$\widetilde{V}_{i,i}(C) = V_i(C).$$

Proof. Set $i = k$ in Theorem 4.1. □

Many of the results that follow in this section were previously proved by various authors in varying degrees of generality. We generally confine references to the relevant results in [11], where detailed historical remarks may be found.

The following theorem is a generalization of the dual Kubota integral recursion (see [11, Theorem A.6.2]).

Theorem 4.3. *Let $C \in \mathcal{B}^n$, let $i > 0$, and let $k_1, k_2 \in \{1, \dots, n\}$ with $k_1 \le k_2$. If $S \in \mathcal{G}(n, k_2)$, then*

$$\widetilde{V}_{i,k_2}(C \cap S) = \frac{\kappa_{k_2}}{\kappa_{k_1}} \int_{\mathcal{G}(k_2,k_1)} \widetilde{V}_{i,k_1}(C \cap T) \, dT.$$

Proof. Using the fact that V_{n-1} is the unique Borel-regular, rotation-invariant measure in S^{n-1} such that S^{n-1} has measure $n\kappa_n$, we see that for any bounded Borel function f on S^{n-1},

$$\int_{S^{n-1}\cap S} f(u) \, du = \frac{k_2\kappa_{k_2}}{k_1\kappa_{k_1}} \int_{\mathcal{G}(k_2,k_1)} \int_{S^{n-1}\cap T} f(u) \, du \, dT.$$

We apply this with $f = X_{i,o}C$ to obtain

$$\begin{aligned}\widetilde{V}_{i,k_2}(C\cap S) &= \frac{i}{2k_2}\int_{S^{n-1}\cap S} X_{i,o}C(u)\,du\\ &= \frac{i\kappa_{k_2}}{2k_1\kappa_{k_1}}\int_{\mathcal{G}(k_2,k_1)}\int_{S^{n-1}\cap T} X_{i,o}C(u)\,du\,dT\\ &= \frac{\kappa_{k_2}}{\kappa_{k_1}}\int_{\mathcal{G}(k_2,k_1)} \widetilde{V}_{i,k_1}(C\cap T)\,dT.\end{aligned}$$

□

Taking $k_2 = n$ in the previous theorem, we see that if $C \in \mathcal{B}^n$, the ith dual volume $\widetilde{V}_i(C)$ is an average of dual volumes of its sections by subspaces of a fixed dimension.

Lemma 4.4. *Let f be a bounded even Borel function on S^{n-1} such that $(Rf)(u) = 0$ for almost all $u \in S^{n-1}$. Then $f(u) = 0$ for almost all $u \in S^{n-1}$.*

Proof. Let g be an arbitrary even function in $C^\infty(S^{n-1})$. Then (see, for example, [11, Theorem C.2.5]) there is an even function h in $C^\infty(S^{n-1})$ such that $g = Rh$. By (1),

$$\int_{S^{n-1}} f(u)g(u)\,du = \int_{S^{n-1}} f(u)(Rh)(u)\,du = \int_{S^{n-1}} (Rf)(u)h(u)\,du = 0.$$

Since g is arbitrary, $f(u) = 0$ for almost all $u \in S^{n-1}$. □

The next result extends the case $i > 0$ of [11, Theorem 7.2.3], whose statement contains a hypothesis on the sets that allows it to hold for all nonzero real i. An analogous extension for negative values of i, again containing an appropriate extra hypothesis on the sets, would be possible, but we do not need it here.

Theorem 4.5. *Let $C, D \in \mathcal{B}^n$, let $i > 0$, and let $k \in \{1, \dots, n-1\}$. Then*

$$\widetilde{V}_{i,k}(C\cap S) = \widetilde{V}_{i,k}(D\cap S)$$

for almost all $S \in \mathcal{G}(n,k)$ if and only if

$$X_{i,o}C(u) = X_{i,o}D(u)$$

for almost all $u \in S^{n-1}$.

Proof. If the second equation holds, the first follows directly from the definition of $\widetilde{V}_{i,k}$.

Assume that the first equation holds for some $k \in \{1, \dots, n-1\}$. If $k < n-1$, then the dual Kubota recursion, Theorem 4.3, implies that it also holds for $k = n-1$. In every case, therefore, we have

$$\widetilde{V}_{i,n-1}(C\cap u^\perp) = \widetilde{V}_{i,n-1}(D\cap u^\perp)$$

for almost all $u \in S^{n-1}$. Let $f = X_{i,o}C - X_{i,o}D$, and note that f is a bounded even Borel function on S^{n-1} such that

$$\int_{S^{n-1}\cap u^\perp} f(v)\,dv = 0$$

for almost all $u \in S^{n-1}$. By Lemma 4.4, $f = 0$ for almost all $u \in S^{n-1}$, and hence the second equation holds for such u. □

Let $C \in \mathcal{B}^n$ and let $i > 0$. We define the *i-chordal symmetral* $\widetilde{\nabla}_i C$ of C by

$$\rho_{\widetilde{\nabla}_i C}(u)^i = \frac{i}{2} X_{i,o}C(u), \tag{17}$$

for all $u \in S^{n-1}$. We also define the *intersection body* IC *of the bounded Borel set* C by

$$\rho_{IC}(u) = V_{n-1}(C \cap u^\perp), \tag{18}$$

for all $u \in S^{n-1}$. (There is a slight abuse of terminology here, since IC need not be a body.) Both $\widetilde{\nabla}_i C$ and IC are o-symmetric sets in $\mathcal{B}^n_{so}$. When $C \in \mathcal{L}^n$, definition (17) of the i-chordal symmetral coincides with [11, Definition 6.1.2], and when $C \in \mathcal{L}^n_o$ has a continuous radial function, definition (18) of the intersection body agrees with [11, Definition 8.1.1].

In [11, Theorem 8.1.16], it is shown that an origin-symmetric cylinder in $\mathbb{R}^4$ is not the intersection body of a star body with a continuous radial function, but it is clear from the argument presented there that it is the intersection body of a bounded Borel set. This shows that the notion we introduce here is genuinely different, even in the class of origin-symmetric convex bodies.

From (17) we see that if $S \in \mathcal{G}(n,k)$, then

$$\widetilde{V}_{i,k}(C \cap S) = \widetilde{V}_{i,k}(\widetilde{\nabla}_i C \cap S). \tag{19}$$

If K is a convex body in $\mathbb{R}^n$ containing o in its interior, IK need not be convex (see [11, Theorem 8.1.8]), but an important theorem of Busemann [11, Theorem 8.1.10] implies that IK is convex if K is also o-symmetric. While IK is clearly not convex if $o \notin K$, it is true that for each $S \in \mathcal{G}(n,2)$, $IK \cap S = L \cup (-L)$, where L is a convex body in S such that $L \cap (-L) = \{o\}$. We omit the proof, but note that this is a straightforward consequence of a generalization of Busemann's theorem called the Busemann-Barthel-Franz inequality (see [11, p. 303]).

Let $C \in \mathcal{B}^n$ and let D be an o-symmetric set in $\mathcal{B}^n_{so}$. Define

$$\widetilde{V}_1(C,D) = \frac{n-1}{2n} \int_{S^{n-1}} X_{n-1,o}C(u)\rho_D(u)\,du. \tag{20}$$

When $C = L \in \mathcal{B}^n_{so}$ and $D = M$ is an o-symmetric set in $\mathcal{B}^n_{so}$, definition (20) agrees with (10), by (15) with $i = n-1$. Also, when in addition $C, D \in \mathcal{L}^n_o$, (20) agrees with [11, (A.54)], for $i = 1$; it would be possible to extend the definition to other values of i, but we shall not do this here. Note that $\widetilde{V}_1(C,B) = \widetilde{V}_{n-1}(C)$ and that

$$\widetilde{V}_1(C,D) = \widetilde{V}_1(\widetilde{\nabla}_{n-1}C, D). \tag{21}$$

The next theorem is a generalization of [11, Theorem 8.1.3].

Theorem 4.6. *Let $C, D \in \mathcal{B}^n$. The following are equivalent:*
(i) *$\rho_{IC}(u) = \rho_{ID}(u)$ for almost all $u \in S^{n-1}$.*
(ii) *$\rho_{\widetilde{\nabla}_{n-1}C}(u) = \rho_{\widetilde{\nabla}_{n-1}D}(u)$ for almost all $u \in S^{n-1}$.*

(iii) $\widetilde{V}_1(C,E) = \widetilde{V}_1(D,E)$ *for all o-symmetric sets* $E \in \mathcal{B}^n_{so}$.

Proof. Theorem 4.5 immediately yields (i)⇔(ii). If (ii) holds, then (iii) follows from (21). Suppose that (iii) holds, let $f \in C(S^{n-1})$ be nonnegative, and let E be the o-symmetric set in $\mathcal{L}^n_o$ such that $\rho_E = (Rf)/(n-1)$. Then, using (20) and (1),

$$\begin{aligned}\widetilde{V}_1(C,E) &= \frac{n-1}{2n}\int_{S^{n-1}} X_{n-1,o}C(u)\rho_E(u)\,du\\ &= \frac{1}{2n}\int_{S^{n-1}} X_{n-1,o}C(u)(Rf)(u)\,du\\ &= \frac{1}{2n}\int_{S^{n-1}} (RX_{n-1,o}C)(u)f(u)\,du.\end{aligned}$$

Since f was arbitrary, (iii) implies that $(RX_{n-1,o}C)(u) = (RX_{n-1,o}D)(u)$ for almost all $u \in S^{n-1}$, and the injectivity of R on even functions then gives $X_{n-1,o}C(u) = X_{n-1,o}D(u)$ for almost all $u \in S^{n-1}$. Then (ii) follows from (17) with $i = n-1$. □

We now prove a strengthening of [11, Theorem 7.2.2]. We need a result related to Jensen's inequality for means that we shall derive from the following lemma.

Lemma 4.7. *Let E be a bounded Borel subset of $[0,\infty)$, and for $i > 0$, let μ_i denote the Lebesgue-Stieltjes measure induced by the function $f(t) = t^i$. Then $\mu_i(E)^{1/i}$ increases with i. Moreover, it increases strictly unless $E = [0,a]$ for some $a \geq 0$, modulo a set of Lebesgue measure zero.*

Proof. Suppose that $V_1(E) = a > 0$. We shall assume that $V_1(E \setminus [0,a]) > 0$ and prove that

$$F(i) = \mu_i(E)^{1/i} = \left(i\int_E t^{i-1}\,dt\right)^{1/i}$$

is strictly increasing for $i > 0$. Let $0 < i < j$, let $f(t) = t^i$, and let $f(E)$ denote the image of E under the map f. If $V_1(f(E)) = b$, then since f is strictly increasing, we have $V_1\left(f(E) \setminus [0,b]\right) > 0$. With $s = t^i$ below, we obtain

$$\begin{aligned}F(j)^j - F(i)^j &= j\int_E t^{j-1}\,dt - \left(i\int_E t^{i-1}\,dt\right)^{j/i}\\ &= \frac{j}{i}\int_{f(E)} s^{j/i-1}\,ds - \left(\int_{f(E)} ds\right)^{j/i}\\ &= \frac{j}{i}\int_{f(E)} s^{j/i-1}\,ds - b^{j/i}\\ &= \frac{j}{i}\int_{f(E)} s^{j/i-1}\,ds - \frac{j}{i}\int_0^b s^{j/i-1}\,ds\\ &= \frac{j}{i}\int_{f(E)\setminus[0,b]} s^{j/i-1}\,ds - \frac{j}{i}\int_{[0,b]\setminus f(E)} s^{j/i-1}\,ds.\end{aligned}$$

Now the last expression is positive, since the integrand $s^{j/i-1}$ is strictly increasing for $s > 0$, and

$$V_1\left(f(E) \setminus [0,b]\right) = V_1\left([0,b] \setminus f(E)\right) > 0.$$

□

Lemma 4.8. *Let $E \in \mathcal{B}^1$ and for $i > 0$, let*

$$G(i) = \left(\frac{i}{2}\int_E |t|^{i-1}\,dt\right)^{1/i}.$$

Then G is an increasing function, strictly increasing unless $E = [-a,a]$ for some $a \geq 0$, modulo a set of measure zero.

Proof. Let $E \in \mathcal{B}^1$, and let

$$E^+ = E \cap [0,\infty) \text{ and } E^- = (-E) \cap [0,\infty).$$

Then

$$G(i) = \left(\frac{\mu_i(E^+) + \mu_i(E^-)}{2}\right)^{1/i} = \left(\frac{F^+(i)^i + F^-(i)^i}{2}\right)^{1/i},$$

where $F^+(i) = \mu_i(E^+)^{1/i}$, $F^-(i) = \mu_i(E^-)^{1/i}$, and μ_i is the Lebesgue-Stieltjes measure induced by the function $f(t) = t^i$. By Lemma 4.7, for $0 < i < j$ we have

$$\left(\frac{F^+(i)^i + F^-(i)^i}{2}\right)^{1/i} \leq \left(\frac{F^+(j)^i + F^-(j)^i}{2}\right)^{1/i} \leq \left(\frac{F^+(j)^j + F^-(j)^j}{2}\right)^{1/j},$$

the last inequality following from Jensen's inequality for means (see, for example, [11, (B.3)]). If equality holds in the previous inequality, then the final statement of Lemma 4.7 shows that $E^+ = [0,a]$ and $E^- = [0,b]$ for some $a, b \geq 0$, modulo sets of measure zero. However, we must also have equality in Jensen's inequality for means, from which we conclude that $F^+(j) = F^-(j)$ and hence that $a = b$ and $E = [-a,a]$, modulo a set of measure zero. □

The following result generalizes [11, Theorem 7.2.2].

Theorem 4.9. *Let $C \in \mathcal{B}^n$ and let $i, j > 0$. If $i \leq j$, then*

$$\widetilde{V}_j(\widetilde{\nabla}_i C) \leq \widetilde{V}_j(C),$$

whereas the reverse inequality holds when $i \geq j$. Equality holds when $i \neq j$ if and only if $C = \widetilde{\nabla}_i C$, modulo a set of measure zero.

Proof. Suppose that $0 < i \leq j$. We have

$$\widetilde{V}_j(\widetilde{\nabla}_i C) = \frac{1}{n}\int_{S^{n-1}} \rho_{\widetilde{\nabla}_i C}(u)^j\,du = \frac{1}{n}\int_{S^{n-1}} \left(\frac{i}{2}X_{i,o}C(u)\right)^{j/i}\,du$$

and

$$\widetilde{V}_j(C) = \frac{j}{2n}\int_{S^{n-1}} X_{j,o}C(u)\,du.$$

Therefore it suffices to show that for all $u \in S^{n-1}$,

$$\left(\frac{i}{2} X_{i,o} C(u)\right)^{j/i} \leq \frac{j}{2} X_{j,o} C(u).$$

The proof is completed by Lemma 4.8 with $E = C \cap l_u$ and l_u identified with $\mathbb{R}$, since this shows that strict inequality occurs unless $i = j$, $C \cap l_u = \emptyset$, or $C \cap l_u = [-a, a]$ for some $a(u) \geq 0$, modulo a set of V_1-measure zero. By Fubini's theorem, the latter condition implies that C is a o-symmetric set in $\mathcal{B}^n_{so}$, modulo a set of V_n-measure zero, and hence that $C = \widetilde{\nabla}_i C$, modulo a set of measure zero. The proof for $i \geq j$ is similar. □

For the next result, we shall need the following definition. An *intersection body* in $\mathbb{R}^n$ is an origin-symmetric set E in $\mathcal{B}^n_s$ such that $\rho_E = R\mu$ for some (positive) finite Borel measure μ in S^{n-1}. (This is a slight weakening of Lutwak's definition (see [11, p. 304]), which is restricted to star bodies with continuous radial functions.) In this definition, a function is identified with the measure generated by it via integration over S^{n-1}, so that

$$(R\mu)(D) = \int_D \rho_E(u)\, du, \tag{22}$$

for all $D \in \mathcal{B}^n$. Observe that if E is the intersection body of a bounded Borel set C, then E is an intersection body; indeed, by (16) with $i = k = n-1$, we then have

$$\rho_E(u) = V_{n-1}(C \cap u^{\perp}) = \left(R\left(\frac{1}{2} X_{n-1,o} C\right)\right)(u),$$

for all $u \in S^{n-1}$; this means that (22) is satisfied with μ defined by

$$\mu(D) = \frac{1}{2} \int_D X_{n-1,o} C(u)\, du,$$

for all $D \in \mathcal{B}^n$. On the other hand, there are intersection bodies that are not intersection bodies of any bounded Borel set. Any origin-symmetric convex polytope in $\mathbb{R}^3$ or $\mathbb{R}^4$ has these properties, since such polytopes have radial functions of the form Rf for some nonnegative unbounded integrable function f on S^{n-1}; see [5].

The next theorem generalizes the case $i = n-1$ of [11, Lemma 8.2.7]. (A full extension of [11, Lemma 8.2.7] along these lines would be routine.)

Theorem 4.10. *Let $C, D \in \mathcal{B}^n$ be such that*

$$V_{n-1}(C \cap u^{\perp}) \leq V_{n-1}(D \cap u^{\perp}),$$

for almost all $u \in S^{n-1}$, and let E be an intersection body in $\mathbb{R}^n$. Then

$$\widetilde{V}_1(C, E) \leq \widetilde{V}_1(D, E).$$

Proof. The first hypothesis of the theorem is equivalent to

$$(RX_{n-1,o}C)(u) \leq (RX_{n-1,o}D)(u),$$

for almost all $u \in S^{n-1}$. If E is an intersection body, then $\rho_E = R\mu$ for some finite Borel measure μ in S^{n-1}. Then, by (2), we have

$$\begin{aligned}\widetilde{V}_1(C,E) &= \frac{n-1}{2n}\int_{S^{n-1}} X_{n-1,o}C(u)\rho_E(u)\,du\\ &= \frac{n-1}{2n}\int_{S^{n-1}} X_{n-1,o}C(u)\,d(R\mu)(u)\\ &= \frac{n-1}{2n}\int_{S^{n-1}} (RX_{n-1,o}C)(u)\,d\mu(u)\\ &\le \frac{n-1}{2n}\int_{S^{n-1}} (RX_{n-1,o}D)(u)\,d\mu(u) = \widetilde{V}_1(D,E).\end{aligned}$$

□

The following result is a generalization of Lutwak's theorem (see [28] or [11, Theorem 8.2.8]).

Corollary 4.11. *Let $C, D \in \mathcal{B}^n$ be such that*

$$V_{n-1}(C \cap u^\perp) \le V_{n-1}(D \cap u^\perp),$$

for almost all $u \in S^{n-1}$. If C is an intersection body in $\mathbb{R}^n$, then $V(C) \le V(D)$. Equality holds if and only if $C = D$, modulo a set of measure zero.

Proof. Taking $E = C$ in Theorem 4.10 and applying (11), (21), the dual Minkowski inequality (12), and Theorem 4.9, we obtain

$$\begin{aligned}V(C) = \widetilde{V}_1(C,C) &\le \widetilde{V}_1(D,C)\\ &= \widetilde{V}_1(\widetilde{\nabla}_{n-1}D,C)\\ &\le V(\widetilde{\nabla}_{n-1}D)^{(n-1)/n}V(C)^{1/n}\\ &\le V(D)^{(n-1)/n}V(C)^{1/n}.\end{aligned}$$

This shows that $V(C) \le V(D)$. If $V(C) = V(D)$, then equality must hold in the previous displayed inequality, so either $V(C) = 0$ or $V(\widetilde{\nabla}_{n-1}D) = V(D)$. Equality must also hold in the dual Minkowski inequality, so C is a dilatate of D, modulo a set of measure zero. Finally, since we must also have $\widetilde{V}_1(C,C) = \widetilde{V}_1(D,C)$, the dilatation factor must be one, so $C = D$, modulo a set of measure zero. □

The next result was proved for convex bodies independently by Busemann and Straus and by Grinberg; see [11, Theorem 9.4.4] and the references given there. It relies on another inequality [11, Corollary 9.2.5] concerning certain averages of volumes of simplices, one of whose vertices is at the origin and the others lie in the body. An inequality similar to the latter, but in which the simplices do not necessarily have one vertex fixed at the origin, was extended to compact sets by Pfiefer [32, Theorem 2]. In [31, p. 70], Pfiefer notes that the same methods prove the corresponding extension of [11, Corollary 9.2.5]. The extension goes routinely from compact sets to bounded Borel sets, and combining the equality conditions from Pfiefer's extension with those of [11, Theorem 9.4.4], we have the following result.

Proposition 4.12. *Let $C \in \mathcal{B}^n$ and let $i \in \{1, \ldots, n-1\}$. Then*

$$\frac{\kappa_n}{\kappa_i}\left(\int_{\mathcal{G}(n,i)} V_i(C\cap S)^n\,dS\right)^{1/n} \le \kappa_n^{(n-i)/n} V_n(C)^{i/n}.$$

Equality holds when $i > 1$ if and only if C is an o-symmetric ellipsoid, modulo a set of measure zero, and when $i = 1$ if and only if C is an o-symmetric convex body, modulo a set of measure zero.

The case $i = n-1$ of Proposition 4.12 gives a general form of the Busemann intersection inequality (see, for example, [11, Corollary 9.4.5]).

Corollary 4.13. *If $C \in \mathcal{B}^n$, then*

$$V(IC) \le \frac{\kappa_{n-1}^n}{\kappa_n^{n-2}} V(C)^{n-1}.$$

Equality holds if and only if C is an o-symmetric ellipsoid, modulo a set of measure zero.

We can now prove a general form of the extended dual isoperimetric inequality (13).

Corollary 4.14. *Let $C \in \mathcal{B}^n$ and let $i \in \{1, \ldots, n-1\}$. Then*

$$\left(\frac{\widetilde{V}_i(C)}{\widetilde{V}_i(B)}\right)^n \le \left(\frac{V_n(C)}{V_n(B)}\right)^i.$$

Equality holds if and only if C is an o-symmetric ball, modulo a set of measure zero.

Proof. By Theorem 4.3 with $k_1 = i$ and $k_2 = n$, Jensen's inequality for integrals (see, for example, [11, (B.8)]), and Proposition 4.12, we have

$$\begin{aligned}\widetilde{V}_i(C) &= \frac{\kappa_n}{\kappa_i}\int_{\mathcal{G}(n,i)} V_i(C\cap S)\,dS\\ &\le \frac{\kappa_n}{\kappa_i}\left(\int_{\mathcal{G}(n,i)} V_i(C\cap S)^n\,dS\right)^{1/n}\\ &\le \kappa_n^{(n-i)/n} V_n(C)^{i/n}.\end{aligned}$$

Noting that $\widetilde{V}_i(B) = \kappa_n$ for all $i > 0$, we see that the required inequality is just a rearrangement of the previous one.

Suppose that equality holds. Then equality holds in Proposition 4.12, so C must be an o-symmetric convex body, K, say, modulo a set of measure zero. Since equality also holds in Jensen's inequality for integrals, the integrand $V_i(C\cap S) = V_i(K\cap S)$ is constant for almost all $S \in \mathcal{G}(n,i)$. By Theorem 4.5 with $k = i$ and D an o-symmetric ball of suitable radius, we conclude that $X_{i,o}K = \rho_{i,K}/i$ is constant almost everywhere in S^{n-1}. The symmetry of K implies that ρ_K is also constant almost everywhere in S^{n-1} and so C is an o-symmetric ball, modulo a set of measure zero. □

5. Extending dual mixed volumes

With the results of the previous section in hand, it is natural to attempt a similar extension of other parts of the dual Brunn-Minkowski theory. The first result is a negative one.

Theorem 5.1. *Let radial addition* $\widetilde{+}$ *be defined for the class* $\mathcal{L}^n$ *by (6). There is no function* $\widetilde{V}:(\mathcal{L}^n)^n\to\mathbb{R}$ *that satisfies (7) and (8).*

Proof. Let $L_j\in\mathcal{L}^n$, $j\in\{1,\ldots,n\}$, and let $L=t_1L_1\widetilde{+}\cdots\widetilde{+}t_nL_n$, where $t_j\geq 0$. Suppose that $o\in L$. Then, by (6),

$$V(L)=\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^n\,du=\frac{1}{n}\int_{S^{n-1}}\left(t_1\rho_{L_1}(u)+\cdots+t_n\rho_{L_n}(u)\right)^n\,du.$$

On the other hand, by (7) and (8),

$$V(L)=\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\widetilde{V}(L_{j_1},\ldots,L_{j_n})t_{j_1}\cdots t_{j_n}.$$

Comparing coefficients of $t_1\cdots t_n$ in these two expressions for $V(L)$, we conclude that (4) must hold under our assumptions. Let $n=2$, and suppose that $L_1,L_2\in\mathcal{L}^2$ are such that $o\notin L_1$, $o\in L_2$, and $o\in L_1\widetilde{+}L_2$. Then, by (4),

$$\widetilde{V}(L_1,L_1)=\frac{1}{2}\int_{S^1}\rho_{L_1}(u)^2\,du\neq V(L_1),$$

since $o\notin L_1$. Therefore (7) cannot hold, and this contradiction proves the theorem. □

Let $\mathcal{B}^n_s(+)$ be the class of sets in $\mathcal{B}^n_s$ not containing o and contained in the half-space $H_n=\{x_n\geq 0\}$ in $\mathbb{R}^n$.

If $L_1,\ldots,L_n\in\mathcal{B}^n_s(+)$, define the dual mixed volume

$$\widetilde{V}(L_1,\ldots,L_n)=\frac{1}{2n}\int_{S^{n-1}}\big||\rho_{L_1}(u)\cdots\rho_{L_n}(u)|-|\rho_{L_1}(-u)\cdots\rho_{L_n}(-u)|\big|\,du. \tag{23}$$

Note that these quantities are nonnegative. Note also that

$$\widetilde{V}(L_1,\ldots,L_n)=\frac{1}{n}\int_{S^{n-1}\cap H_n}\rho_{L_1}(u)\cdots\rho_{L_n}(u)-(-1)^n\left(\rho_{L_1}(-u)\cdots\rho_{L_n}(-u)\right)\,du. \tag{24}$$

Using (24), it is easy to see that the other basic properties of dual mixed volumes—invariance under volume-preserving linear transformations ϕ such that $\phi L_j\in H_n$ for $j\in\{1,\ldots,n\}$, monotonicity, and positive multilinearity—are retained. If $L\in\mathcal{B}^n_s(+)$ and $i\in\{1,\ldots,n-1\}$, the usual dual volume

$$\widetilde{V}_i(L)=\frac{1}{2n}\int_{S^{n-1}}\left||\rho_L(u)|^i-|\rho_L(-u)|^i\right|\,du$$

can be obtained from (23) by setting $L_1=\cdots=L_i=L$ and $\rho_{L_j}(u)=\rho_{L_j}(-u)=1$ for all $u\in S^{n-1}$ and $j\in\{i+1,\ldots,n\}$. We can achieve this by taking $L_j=B'=S^{n-1}\cap H_n$ for $j\in\{i+1,\ldots,n\}$; so B' plays the role of the unit ball for the class $\mathcal{B}^n_s(+)$.

Theorem 5.2. *Let $L_j \in \mathcal{B}_s^n(+)$, $j \in \{1, \dots, m\}$ and let $L = t_1 L_1 \widetilde{+} \cdots \widetilde{+} t_m L_m$, where $t_j \geq 0$ and where $\widetilde{+}$ is defined by (6). Then*

$$V(L) = \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \widetilde{V}(L_{j_1}, \dots, L_{j_n}) t_{j_1} \cdots t_{j_n},$$

where the dual mixed volumes are defined by (23).

Proof. We have

$$\begin{aligned}
V(L) &= \frac{1}{2n} \int_{S^{n-1}} \big| |\rho_L(u)^n| - |\rho_L(-u)^n| \big| \, du \\
&= \frac{1}{n} \int_{S^{n-1} \cap H_n} \rho_L(u)^n - (-1)^n \rho_L(-u)^n \, du \\
&= \frac{1}{n} \int_{S^{n-1} \cap H_n} (t_1 \rho_{L_1}(u) + \cdots + t_m \rho_{L_m}(u))^n - (-1)^n (t_1 \rho_{L_1}(-u) + \cdots + t_m \rho_{L_m}(-u))^n \, du \\
&= \frac{1}{n} \int_{S^{n-1} \cap H_n} \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \big(\rho_{L_{j_1}}(u) \cdots \rho_{L_{j_n}}(u) - (-1)^n \rho_{L_{j_1}}(-u) \cdots \rho_{L_{j_n}}(-u) \big) t_{j_1} \cdots t_{j_n} \, du \\
&= \frac{1}{2n} \int_{S^{n-1}} \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \big| |\rho_{L_{j_1}}(u) \cdots \rho_{L_{j_n}}(u)| - |\rho_{L_{j_1}}(-u) \cdots \rho_{L_{j_n}}(-u)| \big| t_{j_1} \cdots t_{j_n} \, du \\
&= \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \widetilde{V}(L_{j_1}, \dots, L_{j_n}) t_{j_1} \cdots t_{j_n}.
\end{aligned}$$

□

While the previous theorem appears encouraging, we note that the basic inequalities no longer hold. Consider, for example, the dual Minkowski inequality (12) for $n = 2$:

$$\widetilde{V}(L_1, L_2)^2 \leq V(L_1) V(L_2), \tag{25}$$

which holds for $L_1, L_2 \in \mathcal{B}_{so}^2$. Let $L_j \in \mathcal{B}_s^2(+)$, $j = 1, 2$ be the sectors of annuli defined by $\rho_{L_j}(\theta) = a_j$ and $-\rho_{L_j}(-\theta) = b_j$, where $0 < b_j < a_j$, $0 \leq \theta \leq \pi/4$, and $\rho_{L_j}(\theta) = 0$, otherwise. Then, by (24), (25) becomes

$$(a_1 a_2 - b_1 b_2)^2 \leq (a_1^2 - b_1^2)(a_2^2 - b_2^2),$$

which is false unless $a_1/a_2 = b_1/b_2$. On the other hand one can also see that the reverse of inequality (25) does not generally hold either. For if we let $\varepsilon > 0$, $\rho_{L_j}(\theta) = f_j(\theta) > \varepsilon$ and $-\rho_{L_j}(-\theta) = \varepsilon$, $0 \leq \theta \leq \pi/4$, and $\rho_{L_j}(\theta) = 0$, otherwise, $j = 1, 2$, then as $\varepsilon \to 0$ the reverse inequality reads

$$\left(\int_0^{\pi/4} f_1(\theta) f_2(\theta) \, d\theta \right)^2 \geq \int_0^{\pi/4} f_1(\theta)^2 \, d\theta \int_0^{\pi/4} f_2(\theta)^2 \, d\theta,$$

which, by Hölder's inequality, is false unless $f_1 = cf_2$ for some constant c.

Finally, we observe that a modified notion of radial addition does permit an extension of dual mixed volumes to the class $\mathcal{B}_s^n$. Denote by l_u^+ the ray (half-infinite line) extending from o in the direction u, and for $L, M \in \mathcal{B}_s^n$, define $L\hat{+}M$ by

$$(L\hat{+}M) \cap l_u^+ = \left(L \cap l_u^+\right) + \left(M \cap l_u^+\right),$$

for each $u \in S^{n-1}$. This new addition may also be defined as follows. For $u \in S^{n-1}$, let

$$\rho_L^+(u) = \max\{\rho_L(u), 0\} \text{ and } \rho_L^-(u) = \max\{-\rho_L(-u), 0\},$$

and for $s, t \geq 0$, let

$$\rho^{\pm}_{sL\hat{+}tM} = s\rho_L^{\pm} + t\rho_M^{\pm}.$$

For any $L \in \mathcal{B}_s^n$ we have

$$V(L) = \frac{1}{n}\int_{S^{n-1}} \left(\rho_L^+(u)^n - \rho_L^-(u)^n\right)\, du.$$

Using this, it is easy to see that if we define

$$\widetilde{V}(L_1, \ldots, L_n) = \frac{1}{n}\int_{S^{n-1}} \left(\rho_{L_1}^+(u)\cdots\rho_{L_n}^+(u) - \rho_{L_1}^-(u)\cdots\rho_{L_n}^-(u)\right)\, du,$$

then Theorem 5.2 holds in the class $\mathcal{B}_s^n$ with $\widetilde{+}$ replaced by $\hat{+}$. However, this concept of dual mixed volume is incompatible with the ith dual volumes defined above. For example, with $n = 2$, $i = 1$, and $o \notin L$, the definition above gives

$$\widetilde{V}_1(L) = \frac{1}{2}\int_{S^1} \left(\rho_L^+(u) - \rho_L^-(u)\right)\, du.$$

But there is no set C such that $\widetilde{V}_1(L) = \widetilde{V}(L, C)$, since it is impossible that $\rho_C^+(u) = \rho_C^-(u) = 1$ for all $u \in S^1$.

In conclusion, the situation is reminiscent of that for the classical mixed volumes, in that all attempts to extend the definition to larger classes of sets lose some desirable property; compare, for example, the discussion in [4, Section 26].

6. Local stereological volume estimators

In this section, we present the local stereological volume estimators and establish the close connection to central concepts in the dual Brunn-Minkowski theory. Local stereological volume estimators are based on measurements in *random* sections through a fixed point which can be taken to be the origin o. We thus consider random subspaces in $\mathcal{G}(n, k)$ for some $k \in \{1, \ldots, n-1\}$. The random subspaces are assumed to be isotropic, that is, their common probability distribution is the unique rotation invariant probability measure (Haar measure) in $\mathcal{G}(n, k)$.

Local stereological volume estimators can be derived by using the Horvitz-Thompson procedure from sampling theory; see [35]. The key step is to determine the so-called sampling probabilities. For $C \in \mathcal{B}^n$, this involves finding the probability that an isotropic subspace meets an arbitrary volume element of C. The calculation of these sampling probabilities can be done by using the Blaschke-Petkantschin formula (3).

For $C \in \mathcal{B}^n$, this Horvitz-Thompson procedure leads to the following estimator of $V(C)$, based on an isotropic subspace $S \in \mathcal{G}(n,k)$ (see [17, (4.12)] with $p = k$ and $r = 0$):

$$\widehat{V}_{n,k}(C \cap S) = \frac{\omega_n}{\omega_k} \int_{C \cap S} \|x\|^{n-k}\, dx. \tag{26}$$

This is called the *local volume estimator of order* k. By (26) and Theorem 4.1 with $i = n$, an alternative formula for $\widehat{V}_{n,k}(C \cap S)$ is

$$\widehat{V}_{n,k}(C \cap S) = \frac{\kappa_n}{\kappa_k} \widetilde{V}_{n,k}(C \cap S), \tag{27}$$

so the local volume estimator is proportional to the corresponding dual volume. The local volume estimators are unbiased, that is, the mean value of $\widehat{V}_{n,k}(C \cap S)$ with respect to the distribution of S is equal to $V(C)$. This follows directly from the dual Kubota integral recursion, Theorem 4.3, with $i = k_2 = n$ and $k_1 = k$.

Local volume estimators based on subspaces of different dimensions are related. Indeed, the dual Kubota integral recursion, Theorem 4.3 shows that for $k_1 \le k_2$,

$$\widehat{V}_{n,k_2}(C \cap S) = \int_{\mathcal{G}(k_2,k_1)} \widehat{V}_{n,k_1}(C \cap T)\, dT. \tag{28}$$

If $k_1 \le k_2$, an isotropic subspace $T \in \mathcal{G}(n,k_1)$ can be generated by first generating an isotropic $S \in \mathcal{G}(n,k_2)$ and then an isotropic $T \in \mathcal{G}(n,k_1)$ with $T \subset S$ (see, for instance, [17, Proposition 3.15]). Therefore (28) can be interpreted as a conditional mean value result

$$\widehat{V}_{n,k_2}(C \cap S) = E(\widehat{V}_{n,k_1}(C \cap T)|S).$$

This implies the following relation for the variances (see [17, Proposition 4.8]):

$$\begin{aligned} \mathrm{Var}\widehat{V}_{n,k_1}(C \cap T) &= \mathrm{Var}E(\widehat{V}_{n,k_1}(C \cap T)|S) + E\mathrm{Var}(\widehat{V}_{n,k_1}(C \cap T)|S) \\ &= \mathrm{Var}\widehat{V}_{n,k_2}(C \cap S) + E\mathrm{Var}(\widehat{V}_{n,k_1}(C \cap T)|S) \\ &\ge \mathrm{Var}\widehat{V}_{n,k_2}(C \cap S). \end{aligned} \tag{29}$$

(The first equality in (29) is well known and easily proved from the definition of conditional variance; see, for example, [2, p. 217].) The variance thus decreases with increasing dimension of the subspace, an intuitively appealing property.

By (19), the probability distribution of $\widehat{V}_{n,k}(C \cap S)$ remains the same if C is replaced by the n-chordal symmetrical $\widetilde{\nabla}_n C$ of C. Therefore the shape of $\widetilde{\nabla}_n C$ determines the distribution of $\widehat{V}_{n,k}(C \cap S)$, up to a constant factor. In particular, if $\widetilde{\nabla}_n C$ is a ball then $\widehat{V}_{n,k}(C \cap S)$ is a constant multiple of $V(C)$ for all $S \in \mathcal{G}(n,k)$. Since $\widehat{V}_{n,k}(C \cap S)$ is unbiased we then have $\widehat{V}_{n,k}(C \cap S) = V(C)$ for all $S \in \mathcal{G}(n,k)$.

Let $j \in \{0, \ldots, n-1\}$ and let $T \in \mathcal{G}(n,j)$ be fixed. For $k \in \{1, \ldots, n-1\}$ with $k > j$ it is possible, using the Horvitz-Thompson procedure, to construct a local volume estimator based on an isotropic $S \in \mathcal{G}(n,k)$ containing T. This takes the form

$$\widehat{V}_{n,k(j)}(C \cap S) = \frac{\omega_{n-j}}{\omega_{k-j}} \int_{C \cap S} d(x,T)^{n-k}\, dx, \tag{30}$$

where $d(x,T)$ denotes the distance from x to T; see [17, (4.12)] with $p = k$ and $r = j$. Note that $\widehat{V}_{n,k(0)} = \widehat{V}_{n,k}$. Using a decomposition of Lebesgue measure, it is not difficult to see that

$$\widehat{V}_{n,k(j)}(C \cap S) = \int_T \widehat{V}_{n-j,k-j}((C-y) \cap S \cap T^{\perp})\, dy;$$

see [17, Proposition 4.6].

7. Mathematical set model and practical estimation

Standard stereology has employed two classes of compact sets to model the sets encountered in practise. The *convex ring* (sometimes called the *Hadwiger convexity ring*), introduced by Hadwiger in 1956, is the class of finite unions of convex bodies. Later, in 1959, Federer defined the sets of positive reach. A compact subset C of $\mathbb{R}^n$ is of *positive reach* if there is an $r > 0$ such that for each $x \in \mathbb{R}^n$ whose distance from C is less than r, there is a unique point in C that is nearest to x. Weil [36] discusses the two classes from the point of view of standard stereology.

It seems appropriate to expect any physical object viewed in the context of stereology (standard or local) to have the property that it is a body that meets any line in a bounded number of (possibly degenerate) line segments. Any member of the convex ring clearly has this property, but this class is too restrictive. A solid torus, for example, is a perfectly reasonable physical object that does not belong to the convex ring. On the other hand, a solid torus is a set of positive reach, and is also a member of the *star ring*, the class of finite unions of star bodies. The class of finite unions of bodies of positive reach and the star ring both seem general enough to include all objects of practical interest.

However, there are sets that are both star bodies and sets of positive reach, and yet do not meet every line in a finite number of line segments. Such a set can be obtained as follows. Fix $r > 0$ and consider a sequence of open disks of radius r in $\mathbb{R}^2$, situated so that they meet the top edge of the unit square $[0,1]^2$ and intersect it in a disjoint sequence of progressively (and sufficiently) small segments of the disks with a single limit point at $(1,1)$. The unit square with these segments removed is the required set; its intersection with the line $y = 1$ comprises an infinite union of disjoint line segments.

There are also star bodies that meet every line in a finite set of line segments and yet are not sets of positive reach. An example can be constructed as follows. Let D be a disk of radius less than 1, contained in the unit disk and containing the point $(0,1)$ in its boundary. Let D_n, $n \in \mathbb{N}$ be a sequence of disjoint nonempty open disks with a single limit point at $(0,1)$, each of which is disjoint from D and has center in the boundary of the unit disk. Let E be the unit disk with the set $\cup_n D_n$ removed. Since each line meets at most finitely many of the disks D_n, it meets E in a finite set of line segments, and since the radii of the disks D_n approach zero, E is not of positive reach.

Let us combine ideas from the previous two examples, and remove from the unit disk its nonempty, disjoint, and sufficiently small intersections with a sequence of open disks of fixed radius $r > 0$, where these intersections have a single limit point at $(0, 1)$. In this way we can obtain a star body that is also a set of positive reach and which meets every line in a bounded number of (possibly degenerate) line segments, yet which is not a physically reasonable object in the context of stereology.

In view of this situation, and since the dual Brunn-Minkowski theory above provides the mathematical tools for local stereology to consider bounded Borel sets, we shall simply consider here the class $\mathcal{I}^n$ of bodies in $\mathbb{R}^n$ that meet every line in a bounded number of (possibly degenerate) line segments, and revisit the previous section to obtain formulas useful in practise.

In $\mathbb{R}^3$, we have three different local volume estimators, namely $\widehat{V}_{3,1}$, $\widehat{V}_{3,2}$, and $\widehat{V}_{3,2(1)}$, with the notation of the previous section.

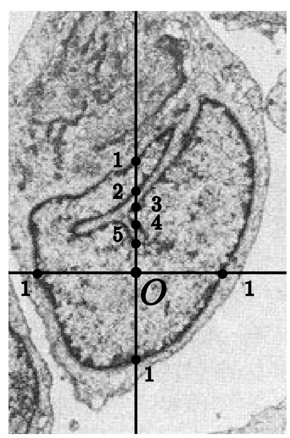

FIGURE 1. Numbering of intersection points in two perpendicular directions, used in the nucleator. Outline from an epithelial cell nucleus in a rat kidney glomerulus.

The estimator $\widehat{V}_{3,1}$ is based on information along an isotropic line l through o and by (26) with $n = 3$ and $k = 1$ is given by

$$\widehat{V}_{3,1}(C \cap l) = 2\pi \int_{C \cap l} \|x\|^2 \, dx.$$

Now suppose that $C \in \mathcal{I}^3$ and $u \in S^2$, and let E_u be the finite set of endpoints of the nondegenerate line segments in $C \cap l_u$. Order the points in $E_u \cap l_u^+$ according to decreasing distance from o, and let $\alpha(x) \in \mathbb{N}$ be the position of $x \in E_u \cap l_u^+$ in this order. See Figure 1. Similarly, order the points in $E_u \cap l_u^-$ according to decreasing distance from o, and let $\alpha(x) \in \mathbb{N}$ be the position of $x \in E_u \cap l_u^-$ in this order. Then (see also [17, Proposition 4.7]) the previous equation becomes

$$\widehat{V}_{3,1}(C \cap l_u) = \frac{2\pi}{3} \sum_{x \in E_u} (-1)^{\alpha(x)+1} \|x\|^3. \tag{31}$$

Often, measurements along two perpendicular directions in a section plane are combined. In that case, the estimator is called the *nucleator*; see [14].

The estimator $\widehat{V}_{3,2}$ is based on information in an isotropic plane S through o. From (26) with $n = 3$ and $k = 2$, we find

$$\widehat{V}_{3,2}(C \cap S) = 2 \int_{C \cap S} \|x\| dx.$$

For interactive collection of stereological measurements it is useful to discretize the planar integral using a line grid in the plane S. To be more specific, let l_0 be an arbitrarily chosen line in S through o, and let G be a grid of lines perpendicular to l_0 and spaced a distance h apart. See Figure 2.

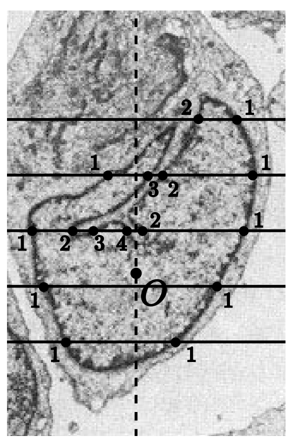

FIGURE 2. Numbering of intersection points with grid G, used in the rotator. The perpendicular dotted line is l_0.

Suppose that $C \in \mathcal{I}^3$; then $C \cap l$ consists of a finite number of line segments for any line l. Let E_G be the set of endpoints of the finite number of line segments of $C \cap G$. If l is a grid line in G and $x \in C \cap l$, we define $\alpha(x)$ as we did above for (31) but with o replaced by $l \cap l_0$; see Figure 1. A routine calculation shows that $\widehat{V}_{3,2}(C \cap S)$ may be approximated by

$$2h \sum_{x \in E_G} (-1)^{\alpha(x)+1} \left(\frac{1}{2} d(x, l_0) \|x\| + \frac{\|x\|^2 - d(x, l_0)^2}{2} \log \left(\frac{d(x, l_0) + \|x\|}{\sqrt{\|x\|^2 - d(x, l_0)^2}} \right) \right), \tag{32}$$

where $d(x, l_0)$ is the distance from x to l_0. See [21]; (32) is called the *isotropic rotator* in the stereological literature.

The estimator $\widehat{V}_{3,2(1)}$ is based on an isotropic plane S, containing a fixed line l_0 through o. From (30) with $n = 3$, $k = 2$, and $j = 1$, we obtain

$$\widehat{V}_{3,2(1)}(C \cap S) = \pi \int_{C \cap S} d(x, l_0)\, dx.$$

Assuming that $C \in \mathcal{I}^3$, a discretized version of $\widehat{V}_{3,2(1)}$ can be found as in the previous paragraph. With the notation introduced there, this takes the form

$$\frac{\pi}{2} h \sum_{x \in E_G} (-1)^{\alpha(x)+1} d(x, l_0)^2. \tag{33}$$

This is called the *vertical rotator*; see [21].

The three practical formulas (31), (32), and (33) have been implemented in a computer-assisted software package called the C.A.S.T.-GRID, developed for the interactive collection of stereological measurements; see [1]. We stress that this discussion of volume estimators represents only a fraction of the available techniques in local stereology. The next section supplies references that give an idea of the scope of the subject.

8. Methodology and applications of local stereology

Local stereological methods have been developed for the microscopical study of biological tissue in cases where the tissue is transparent and physical sections can be replaced by optical sections. Main parts of the local theory were presented in the early paper [20]. The procedure in the laboratory is typically as follows. The tissue sample of interest (for example, kidney, brain, or skin) is cut into a small number of blocks. Each block is subsequently cut isotropically into slabs of thickness 50-100 μm. A subset of the slabs is selected for microscopic analysis. When such a slab is transparent it is possible to focus down through the slab and thereby generate optical sections which can be displayed on a video screen. By moving the focal plane up and down in the slab, a whole continuum of optical sections is generated.

The general aim of local stereology is to estimate from optical sections quantitative properties of spatial structures which can be regarded as neighborhoods of points, called reference points. The model example is a cell population where each cell can be regarded as the neighborhood of its nucleus. Local stereological estimators of cell volume, surface area, etc. are based on optical sections through the cell nuclei, which are usually centrally placed in the cells. From a technical point of view, central sections are of better quality than sections from the peripheral part of the cell, where the optical section plane is often almost tangential to the cell boundary and accordingly the cell outline appears fuzzy; see [34]. That is why local methods are superior to global methods that require exhaustive sectioning of the cells. Prior to sections through fixed points, sections through uniform random points were also considered; see, for example, [6] and [18].

The main applied problem solved by local stereology is that of estimating moments in the cell-size distribution without specific assumptions of cell shape; see, for example, [16]. The emphasis has been on the estimation of mean size, where size is typically volume or surface area. Previous methods were based on shape assumptions such as that of spherical shape or ellipsoidal shape. (Note that if the cells are actually of spherical shape then with optical sectioning the diameters of the cells can be observed directly and a solution of the famous ill-posed problem of estimating the distribution of sphere diameters from the distribution of the diameters of circular disks in a section plane ([37]) is not needed anymore.) Cells have varying shape, however, and need not be convex or even star shaped with respect to their nucleus. Examples are endothel cells, epithelial cell nuclei, and podocytes; some extreme examples are shown in [15]. Also, most smooth muscle cells are far from star shaped. Another practical reason for developing the theory for general shapes is that one cannot judge from a section whether the cell is actually star shaped in $\mathbb{R}^3$.

Local stereological methods have been generalized in various directions. Surface area, length, and number can also be estimated using local techniques. For an early reference concerning surface area, see [19]. A more comprehensive account can be found in [17, Chapters 5 and 6]. Random slabs centered at the origin have been considered in [22] and [23]. Some of the measurements in the slabs are collected using spatial line grids and the estimators can in that case, under regularity conditions, be expressed in terms of alternating sums, as in (31).

A rich collection of local stereological methods has been developed for the estimation of cell sizes. Available are 14 local techniques (see [17, Tables 7.1-7.4]) of which we have only discussed in detail above three volume estimators.

The most significant medical results obtained by local stereological methods are in neuroscience and cancer grading. The structure of the human brain and its changes due to diseases such as Alzheimer's disease and HIV infection have been studied by local methods; see, for example, [3], [7], [8], and [29]. In particular, it has been possible by local methods to quantify the phenomenon called satellitosis where small glia cells are distributed around neurons in the brain; see [7]. In [8], the severe loss of neocortical neurons associated with HIV infection has been studied in detail by local methods. A preferential loss of large neocortical neurons was found. In [26] and [30], it was demonstrated that mean cell nuclear volume, estimated by local stereological methods at the time of diagnosis of cancer, has a significant prognostic value and may therefore be an important supplement to the subjective judgment of the pathologists.

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