The Moment Problem

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Summary: Recall that a probability measure μ on the real line with finite moments of all orders is called determinate if $\mu = \nu$ for any probability measure ν with the same moments as μ. There are three classical criteria for determinacy due to O. Perron, M. Riesz and T. Carleman. The Perron condition states that the Laplace transform of μ is finite in an open interval around 0 and it is the most commonly criteria used in probability theory. However, the Riesz and Carleman conditions are weaker than the Perron condition but difficult to apply due to the fact that they require precise estimates of the moments. The objective of this paper is to provide equivalent forms of the Riesz and Carleman conditions which are easier to apply. In particular, I shall show that each of the two conditions are equivalent to integrability of at least one function in a specified class of functions.

1. Introduction If $(T, \mathcal{B}, \mu)$ is a measure space, we let $L^q(\mu)$ denote the usual L^q-space with its usual L^q-norm $\|\cdot\|_q$ whenever $0 \le q \le \infty$ (see [8; (3.22) p.184–188]). If $f : T \to [0, \infty]$ is a non-negative function, we let $f \cdot d\mu$ denote the measure given by $B \curvearrowright \int_B f\, d\mu$ for $B \in \mathcal{B}$ and if $(S, \mathcal{A})$ is a measurable space and $\phi : T \to S$ is a μ-measurable function, we let $\mu_\phi(A) := \bar{\mu}(\phi^{-1}(A))$ for $A \in \mathcal{A}$ denote *the image measure*. If $t \in T$ is a given point, we let $\delta_t(B) := 1_B(t)$ denote *the Dirac measure* at t. Recall that μ is *discrete* if and only if $\mu = \sum_{n=1}^\infty p_n\, \delta_{t_n}$ for some and some sequences $(t_n) \subseteq T$ and $(p_n) \subseteq [0, \infty)$.

Let $\Pr(\mathbf{R})$ denote the set of all Borel probability measures on the real line $\mathbf{R}$. If $\mu \in \Pr(\mathbf{R})$, we let $M_\mu(q) := \int_{\mathbf{R}} |x|^q\, \mu(dx)$ denote its *absolute moment function* for all $q > 0$ and we let $\mathfrak{P}_\infty$ denote the set of all probability measures with *finite moment of all orders,* that is, the set of μ $\mu \in \Pr(\mathbf{R})$ satisfying $M_\mu(q) < \infty$ for all $q > 0$. If $\mu \in \mathfrak{P}_\infty$, we let $\mu[n] := \int_{\mathbf{R}} x^n\, \mu(dx)$ denote *the moments* of μ for all $n = 1, 2, \ldots$ and we say that μ is *determinate* if $\mu = \nu$ for any probability measure $\nu \in \mathfrak{P}_\infty$ satisfying $\nu[n] = \mu[n]$ for all $n \in \mathbf{N}$.

Recall that *the Hamburger moment problem* is the problem of finding necessary and/or sufficient conditions for determinacy of a given probability measure $\mu \in \mathfrak{P}_\infty$.

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Let $\mathfrak{P}_D$ denote the set of all determinate probability measures $\mu \in \mathfrak{P}_\infty$. Then we have the following classical sufficient condition for determinacy (see [7]):

(P) $\quad \limsup_{n\to\infty} \left(\frac{M_\mu(2n)}{(2n)!}\right)^{\frac{1}{2n}} < \infty \quad \Rightarrow \quad \mu \in \mathfrak{P}_D \quad$ (O. Perron)

(R) $\quad \liminf_{n\to\infty} \left(\frac{M_\mu(n)}{n!}\right)^{\frac{1}{n}} < \infty \quad \Rightarrow \quad \mu \in \mathfrak{P}_D \quad$ (M. Riesz)

(C) $\quad \sum_{n=1}^{\infty} M_\mu(2n)^{-\frac{1}{2n}} = \infty \quad \Rightarrow \quad \mu \in \mathfrak{P}_D \quad$ (T. Carleman)

Let $\mathfrak{P}_P$, $\mathfrak{P}_R$ and $\mathfrak{P}_C$ denote the set of all probability measures $\mu \in \mathrm{Pr}(\mathbf{R})$ satisfying the conditions (P), (R) and (C), respectively, and let $\mathfrak{P}_{bd}$ denote the set of all probability measures $\mu \in \mathrm{Pr}(\mathbf{R})$ with bounded support. Evidently, we have $\mathfrak{P}_P \subseteq \mathfrak{P}_R$ and since $M_\mu(q)^{1/q}$ is increasing on $(0, \infty)$, it follows easily that we have $\mathfrak{P}_R \subseteq \mathfrak{P}_C$. So by Carleman's theorem (see [5]) we have the following inclusions:

(1.1) $\quad \mathfrak{P}_{bd} \subseteq \mathfrak{P}_P \subseteq \mathfrak{P}_R \subseteq \mathfrak{P}_C \subseteq \mathfrak{P}_D \subseteq \mathfrak{P}_\infty$

Let $\mathbb{P}$ denote the of polynomials in the real variable x with real coefficients. Then $\mathbb{P} \subseteq L^q(\mu)$ for all $\mu \in \mathfrak{P}_\infty$ and all $0 \le q < \infty$ and we let $\mathfrak{P}_{L^q}$ denote the set all $\mu \in \mathfrak{P}_\infty$ for which $\mathbb{P}$ is dense in $(L^q(\mu), ||\cdot||_q)$. Evidently, we have

(1.2) $\quad \mathfrak{P}_{L^q} \subseteq \mathfrak{P}_{L^r} \subseteq \mathfrak{P}_{L^0} = \mathfrak{P}_\infty \quad \forall\, 0 \le r \le q < \infty$

and since $\frac{1}{2}\, e^{|x|} \le \cosh x \le e^{|x|}$ for all $x \in \mathbf{R}$, we have the following equivalent form of the Perron condition (P):

(1.3) $\quad \mu \in \mathfrak{P}_P$ if and only if $e^{\alpha|x|} \in L^1(\mu)$ for some $\alpha > 0$

The latter fact means that the Perron condition is easy to verify and it is the most commonly used criterion for determinacy used in probability theory. However, the Riesz and Carleman conditions are weaker than the Perron condition but they are difficult to apply to special cases. The objective of this paper is to provide equivalent forms of the Riesz and Carleman conditions which are easier to apply. In particular, I shall show that each of the two conditions are equivalent to a certain integrability condition of the form (1.3).

More precisely, let $\mathrm{B}_+(\mathbf{R})$ denote the set of all Borel functions $w : \mathbf{R} \to [0, \infty]$. If $w \in \mathrm{B}_+(\mathbf{R})$ is a given function, we let Pr_w denote the set of all probability measures $\mu \in \mathrm{Pr}(\mathbf{R})$ satisfying $w \in L^1(\mu)$, and if $\mathfrak{P} \subseteq \mathrm{Pr}(\mathbf{R})$ is a given set of probability measures, we say that w is a *test function* for $\mathfrak{P}$ if $\mathrm{Pr}_w \subseteq \mathfrak{P}$; that is, if $\mu \in \mathrm{Pr}(\mathbf{R})$ and $\int_{\mathbf{R}} w\, d\mu < \infty$ implies $\mu \in \mathfrak{P}$. Let $\mathfrak{P} \subseteq \mathrm{Pr}_w$ and $W \subseteq \mathrm{B}_+(\mathbf{R})$ be given sets. Then we let $\mathfrak{P}^*$ denote the set of all test functions for $\mathfrak{P}$; that is,

- $\quad \mathfrak{P}^* := \{\, w \in \mathrm{B}_+(\mathbf{R}) \mid \mathrm{Pr}_w \subseteq \mathfrak{P} \,\}$

and we say that W is a *complete set of test functions* for $\mathfrak{P}$ if $\mu \in \mathfrak{P} \Leftrightarrow \int_{\mathbf{R}} w\, d\mu < \infty$ for some $w \in W$. Note, that we have

(1.4) W is a complete set of test functions for $\mathfrak{P}$ if and only if $W \subseteq \mathfrak{P}^*$ and $W \cap L^1(\mu) \neq \emptyset$ for all $\mu \in \mathfrak{P}$ and if so, then $\mathfrak{P}^*$ is a complete set of test functions for $\mathfrak{P}$

Hence, if $\mathfrak{P}$ admits a complete set of test functions, then $\mathfrak{P}^*$ is the maximal complete set of test functions for $\mathcal{P}$.

Note that (1.3) states that $\mathrm{B}_P := \{e^{\alpha|x|} \mid \alpha > 0\}$ is a complete set of test functions for $\mathfrak{P}_P$ and the objective of this paper can now be restated as follows: Find decent complete set of test functions for the sets $\mathfrak{P}_R$ and $\mathfrak{P}_C$ and characterize the sets of all test function for $\mathfrak{P}_P$, $\mathfrak{P}_R$ and $\mathfrak{P}_C$. Of course there exists sets of probability measures which do *not* admit any complete set of test functions; for instance, the set $\mathfrak{P}_D$. In [2] it shown that there exists a probability measure $\mu \in \mathfrak{P}_\infty \setminus \mathfrak{P}_D$ of the form $\mu = \sum_{n\geq 0} p_n\, \delta_{x_n}$ where $x_0 = 0 < x_1 < x_2 < \cdots$ and such that the probability measures $\mu_k := \frac{1}{1-p_k} \sum_{n\neq k} p_n\, \delta_{x_n}$ belong to $\mathfrak{P}_D$ for all $k \geq 0$. Since $L^1(\mu) = L^1(\mu_k)$, we see that $\mathfrak{P}_D$ does *not* admit any complete set of test functions. However, in Section 3 I shall show that a large class of sets of probability measures (including $\mathfrak{P}_{bd}$, $\mathfrak{P}_P$, $\mathfrak{P}_R$, $\mathfrak{P}_C$ and $\mathfrak{P}_\infty$) do admit a complete set of test functions.

Let me at this point recall the following cardinal results from the current state of the moment problem:

Theorem 1.1: (Riesz & Krein & Berg; see [14], [9], [2], [4] and [1]**)** *Let μ be a given Borel probability measure on $\mathbf{R}$. If $\frac{1}{c} := \int_{\mathbf{R}} \frac{1}{1+x^2}\, \mu(dx)$ and $f := \frac{d\mu}{d\lambda}$ denotes the Radon-Nikodym derivative of μ with respect to the Lebesgue measure λ , then we have*

(1) $\mu \in \mathfrak{P}_D \;\Rightarrow\; \frac{c}{1+x^2} \cdot d\mu \in \mathfrak{P}_D \;\Leftrightarrow\; \mu \in \mathfrak{P}_{L^2} \;\Rightarrow\; \displaystyle\int_{-\infty}^{\infty} \frac{\log^- f(x)}{1+x^2}\, dx = \infty$

(2) $\mathfrak{P}_C \subseteq \mathfrak{P}_{L^q} \subseteq \mathfrak{P}_D \subseteq \mathfrak{P}_{L^r} \subseteq \mathfrak{P}_\infty \quad \forall\, 0 \leq r \leq 2 < q < \infty$

(3) *If μ is non-discrete, then $\mu \in \mathfrak{P}_D$ if and only if $\mu \in \mathfrak{P}_{L^2}$*

where $\log^- x := -\log(x \wedge 1)$ for all $x \geq 0$ with the convention $\log^- 0 := \infty$. The last condition in (1) goes under name Krein's condition.

2. Log-convex functions their dual functions Let $I \subseteq \mathbf{R}$ be a given interval and let $f : I \to [0, \infty]$ be a non-negative function. Then we let $D(f) := \{x \in I \mid f(x) < \infty\}$ denote *the domain* of f . Recall that f is *log-convex* if $f(\alpha\, x + (1-\alpha)\, y) \leq f(x)^\alpha\, f(y)^{1-\alpha}$ for all $x, y \in I$ and all $0 < \alpha < 1$ or equivalently, if $\log f$ is *convex*, where we use the following conventions:

$$0 \cdot \infty = \tfrac{a}{\infty} = \tfrac{0}{a} := 0 \ \ , \ \ a^0 := 1 \ \ \forall\, 0 \le a \le \infty \ \ , \ \ \log 0 := -\infty \ \ , \ \ \log \infty := \infty$$

$$\infty^\alpha := \infty \ \ \forall\, 0 < \alpha < \infty \ \ , \ \ \infty^\alpha := 0 \ \ \forall -\infty < \alpha < 0 \ \ , \ \ e^{-\infty} := 0 \ \ , \ \ e^{\infty} := \infty$$

and we define $\log^+ x := \log(x \vee 1)$ for all $x \in \bar{\mathbf{R}}$. If $I \subseteq (0,\infty)$, we say that f is *log-exp-convex* if $f(x^\alpha\, y^{1-\alpha}) \le f(x)^\alpha f(y)^{1-\alpha}$ for all $x, y \in I$ and all $0 < \alpha < 1$ or equivalently if the function $y \curvearrowright f(e^y)$ is log-convex on the interval $J := \{y \in \mathbf{R} \mid e^y \in I\}$. Since e^x is increasing and convex, we see that any log-convex function $f : I \to [0,\infty]$ is log-exp-convex on $I \cap (0,\infty)$.

We let $\mathcal{M}$ denote the set all lower semicontinuous, log-convex functions $M : (0,\infty) \to [0,\infty]$ such that $q \curvearrowright M(q)^{1/q}$ is increasing on $(0,\infty)$. Since x^α is concave on $[0,\infty)$ if $0 \le \alpha \le 1$ and convex on $[0,\infty)$ if $\alpha \ge 1$, it follows easily that we have

(2.1) An arbitrary supremum of log-convex (resp. log-exp-convex) functions is log-convex (resp. log-exp-convex) and if $f_1, f_2, \ldots, f_n : I \to [0,\infty]$ are log-convex (resp. log-exp-convex) functions and $\alpha_1, \ldots, \alpha_n \ge 0$ are given numbers, then the functions $\sum_{i=1}^n \alpha_i\, f_i(x)$ and $\prod_{i=1}^n f_i(x)^{\alpha_i}$ are log-convex (resp. log-exp-convex)

(2.2) If f is log-convex or log-exp-convex, then $D(f)$ is an interval and if $f(x_0) = 0$ for some $x_0 \in I$, then $f(x) = 0$ for all x in the interior of I

(2.3) An arbitrary supremum of functions belonging to $\mathcal{M}$ belongs to $\mathcal{M}$ and if $M_1, M_2, \ldots, M_n \in \mathcal{M}$ and $\alpha_1, \beta_1 \ldots, \alpha_n, \beta_n \ge 0$ are given non-negative numbers such that $\alpha_1 + \cdots + \alpha_n \le 1$, then the functions $\sum_{i=1}^n \alpha_i\, M_i(x)$ and $\prod_{i=1}^n M_i(x)^{\beta_i}$ belong to $\mathcal{M}$

If $\mu \in \Pr(\mathbf{R})$, we let $R_\mu(s) := \mu(x \in \mathbf{R} \mid |x| \ge s)$ denote *the tail distribution* of μ for all $s \ge 0$ and we let $\rho_\mu(s) := R_\mu(|s|)^{-1}$ denote *the inverse tail distribution* of μ for all $s \in \mathbf{R}$. Then we have

(2.4) $\rho_\mu : \mathbf{R} \to [0,\infty]$ is an even function such that ρ_μ is increasing and left continuous on $(0,\infty]$ with $\rho_\mu(0) = 1$ and $\lim_{x\to\infty} \rho_\mu(x) = \infty$. Conversely, if $\rho : \mathbf{R} \to [1,\infty]$ is an given even function such that ρ is increasing and left continuous on $(0,\infty)$ with $\lim_{x\to\infty} \rho_\mu(x) = \infty$, then exists a unique probability measure $\mu \in \Pr(\mathbf{R})$ satisfying $\mu([0,\infty)) = 1$ and $\rho_\mu(x) = \rho(|x|)$ for all $x \ne 0$

and if $M_\mu(q)$ denote the absolute moment function of μ for $q > 0$, it is well-known (and an easy consequence of Hölder's inequality) that $M_\mu \in \mathcal{M}$.

Let $w : \mathbf{R} \to [0,\infty]$ be a given function. Then we let $w_\epsilon(x) := w(x) \wedge w(-x)$ denote *the even envelope* of w . Note that w_ϵ is the largest even function on $\mathbf{R}$

dominated by w . We let $w^\diamond : \mathbf{R} \to [0,\infty]$ denote *the log-exp-convex envelope* of $w_\epsilon \vee 1$; that is:

(2.5) $\quad w^\diamond(0) := w(0) \vee 1 \;\;,\;\; w^\diamond(x) := \sup\{\, f(|x|) \mid f \in \mathcal{L}_w\} \;\; \forall\, x \neq 0$

where $\mathcal{L}_w$ denotes the set of all log-convex functions $f : (0,\infty) \to [0,\infty]$ satisfying $f(x) \leq w_\epsilon(x) \vee 1$ for all $x > 0$. We let $\widehat{w} : (0,\infty) \to [0,\infty]$ denote *the first log-dual function* of $w \vee 1$, and we let $w^\star : \mathbf{R} \to \mathbf{R}$ denote *the second log-dual function* of $w \vee 1$; that is:

(2.6) $\quad \widehat{w}(q) := \sup\limits_{y \in \mathbf{R}} \frac{|y|^q}{w(y) \vee 1} \;\; \forall\, q > 0$ and $w^\star(x) := \sup\limits_{q > 0} \frac{|x|^q}{\widehat{w}(q)} \;\; \forall\, x \in \mathbf{R}$

We let $\nabla w : \mathbf{R} \to [0,\infty]$ denote *the log-derivate* of $w_\epsilon \vee 1$; that is:

(2.7) $\quad \nabla w(x) := \inf\limits_{t > e^x} \frac{\log^+ w_\epsilon(t)}{\log t - x} \;\; \forall\, x \in \mathbf{R}$

Lemma 2.1: *Let* $\mu \in \mathrm{Pr}(\mathbf{R})$ *be a given probability measure and let* $w : \mathbf{R} \to [0,\infty]$ *be a non-negative Borel function. Then we have*

(1) $\quad |x|^q \leq \rho_\mu(x)\, M_\mu(q)$ *and* $\widehat{\rho}_\mu(q) \leq M_\mu(q) \;\; \forall\, x \in \mathbf{R}\; \forall\, q > 0$

(2) $\quad \displaystyle\int_{\mathbf{R}} \rho_\mu(x)^\alpha \,\mu(dx) \leq \frac{1}{1-\alpha} \quad \forall\, 0 < \alpha < 1$

(3) $\quad \displaystyle M_\mu(q) \leq \widehat{w}(q) \int_{\mathbf{R}} w^\star(x)\,\mu(dx) \quad \forall\, q > 0$

Proof: (1): Let $x \in \mathbf{R}$ and $q > 0$ be given. By Markov's inequality, we have $R_\mu(|x|) \leq |x|^{-q} M_\mu(q)$ and since $\rho_\mu(x) \geq 1$, we see that (1) follows from the definition of $\widehat{\rho}_\mu$.

(2): Let $0 < \alpha < 1$ be given, let $\tilde{\mu}$ denote the image measure of μ under the function $x \curvearrowright |x|$ and let $F(x) := \tilde{\mu}(-\infty, x]$ denote the distribution function of $\tilde{\mu}$. Since $R_\mu(x) = 1 - F(x-)$ for all $x \geq 0$ and $\tilde{\mu}(\mathbf{R}_+) = 1$, we have $\int_{\mathbf{R}} \rho_\mu(x)^\alpha\,\mu(dx) = \int_{\mathbf{R}} (1 - F(t-))^{-\alpha}\,\tilde{\mu}(dx)$ and since $t \curvearrowright (1-t)^{-\alpha}$ is increasing on $[0,1]$, we see that (2) follows from [8; (3.29.6) p.205].

(3): Since $\widehat{w}(q)\, w^\star(x) \geq |x|^q$, we see that (3) holds. □

Lemma 2.2: *Let* $w : \mathbf{R} \to [0,\infty]$ *and* $v : \mathbf{R} \to [0,\infty]$ *be non-negative functions. If* D_w *denote the set of all* $(\alpha,\beta) \in \mathbf{R}^2$ *satisfying* $\alpha > 0$ *,* $\beta > 0$ *and* $\beta\, x^\alpha \leq w_\epsilon(x) \vee 1$ *for all* $x > 0$ *, then we have*

(1) $\quad \widehat{w} \in \mathcal{M}$ *and* $w^\star$ *and* $w^\diamond$ *are even functions such that* $w^\diamond$ *is log-exp-convex on* $(0,\infty)$ *and* $w^\star$ *is increasing, log-exp-convex and left continuous on* $(0,\infty)$ *with* $w^\star(0) = 0$

(2) $w^\star(x) = \sup_{(\alpha,\beta)\in D_w} \beta\,|x|^\alpha \quad \forall\, x \in \mathbf{R} \quad (\,\sup\emptyset := 0)$

(3) $w^\star(x) \vee 1 \le w^\diamond(x) \le w_\epsilon(x) \vee 1 \le w(x) \vee 1 \quad \forall\, x \in \mathbf{R}$

(4) ∇w *is an increasing, upper semicontinuous and right continuous function from* $\mathbf{R}$ *into* $[0,\infty]$ *and if* $x \in \mathbf{R}$ *and* $y > 0$ *are given numbers, then we have*

(a) $\nabla w(x) \ge y \quad \Leftrightarrow \quad x \ge \frac{1}{y}\log\,\widehat{w}(y) \quad \Leftrightarrow \quad \widehat{w}(y)^{1/y} \le e^x$

(5) *If* $a \in \mathbf{R}$ *is a given number satisfying* $w^\star(x) \le v_\epsilon(x) \vee 1 \le w(x) \vee 1$ *for all* $|x| > a$ *, then we have*

(a) $\nabla w(x) = \nabla v(x) \quad \forall\, x \ge a$

(b) $\widehat{v}(y)^{1/y} \vee e^a = \widehat{w}(y)^{1/y} \vee e^a \quad \forall\, y > 0$

(6) $\nabla w(x) = \nabla w^\star(x) = \nabla w^\diamond(x) \;\; \forall\, x \in \mathbf{R} \;,\;\; \widehat{w}(q) = \widehat{w^\star}(q) = \widehat{w^\diamond}(q) \;\; \forall\, q > 0$

(7) $w^{\diamond\diamond}(x) = w^\diamond(x)$ *and* $w^{\star\star}(x) = w^\star(x) \quad \forall\, x \in \mathbf{R}$

(8) $\limsup_{|x|\to\infty} w^\diamond(x) = \infty \quad \Rightarrow \quad \exists\, c > 0$ *so that* $w^\diamond(x) \le c\, w^\star(x) \;\; \forall\, |x| \ge c$

Proof: (1): Since $q \curvearrowright |y|^q$ belong to $\mathcal{M}$ for all $y \in \mathbf{R}$ and $x \curvearrowright \alpha|x|^q$ is continuous on $\mathbf{R}\setminus\{0\}$ and increasing and log-exp-convex on $(0,\infty)$ for all $0 < q < \infty$ and all $0 \le \alpha \le \infty$, we see that (1) follows from (2.1) and (2.3).

(2): Let $\phi(x)$ denote the supremum on the left hand side of (2). If $D_w = \emptyset$, then $\phi(x) = 0$ by convention and by the definition of $\widehat{w}$ we see that $\widehat{w}(q) = \infty$ for all $q > 0$; that is, $w^\star(x) = 0 = \phi(x)$ for all $x \in \mathbf{R}$. So suppose that $D_w \ne \emptyset$ and let $(\alpha,\beta) \in D_w$ be given. Since w_ϵ is even and $w_\epsilon \le w$, we have $\beta|y|^\alpha \le w(y) \vee 1$ for all $y \in \mathbf{R}$. Hence, we see that $\widehat{w}(\alpha) \le \frac{1}{\beta}$ and so we have $\beta|x|^\alpha \le |x|^\alpha\,\widehat{w}(\alpha)^{-1}$ for all $x \in \mathbf{R}$. Thus, we see that $\phi(x) \le w^\star(x)$ for all $x \in \mathbf{R}$. Let $x \in \mathbf{R}$ be given. If $w^\star(x) = 0$, then we have $\phi(x) = w^\star(x)$. So suppose that $w^\star(x) > 0$ and let $0 < \gamma < w^\star w(x)$ be given. Since $w^\star(0) = 0 < w^\star(x)$, we have $x \ne 0$ and there exists $q > 0$ such that $|x|^q > \gamma\,\widehat{w}(q)$. Hence, by the definition of $\widehat{w}$, we have $\gamma\,|x|^{-q}\,y^q \le w(y) \vee 1$ for all $y > 0$; that is, $(q, \gamma\,|x|^{-q}) \in D_w$ and so we have $\phi(x) \ge \gamma\,|x|^{-q}\,|x|^q = \gamma$. Letting $\gamma \uparrow w^\star(x)$, we see that $\phi(x) \ge w^\star(x)$ which completes the proof of (2).

(3): By the definition of $w^\diamond$, we have $w^\diamond \le w_\epsilon \vee 1 \le w \vee 1$ and by (2), we see that $w^\star \vee 1 \le w_\epsilon \vee 1$. Since $w^\star \vee 1$ is even on $\mathbf{R}$ and log-exp-convex on $[0,\infty)$ with $w^\star(0) = 0$, we have $w^\star \vee 1 \le w^\diamond$.

(4+5): Let $t > 0$ be given and let us define $h_t(x) := \frac{\log^+ w_\epsilon(t)}{\log t - x}$ for $x < \log t$ and $h_t(x) := \infty$ for $x \ge \log t$. Then $h_t : \mathbf{R} \to [0,\infty]$ is an increasing continuous function for all $t > 0$ such that $\nabla w(x) = \inf_{t>0} h_t(x)$ for all $x \in \mathbf{R}$. Hence, we see that ∇w is an increasing, upper semicontinuous function from $\mathbf{R}$ into $[0,\infty]$ and consequently right continuous. Let $x \in \mathbf{R}$ and $y > 0$ be given. Since $\log^+ w_\epsilon \ge 0$

and $t^y e^{-xy} \le 1$ for all $0 < t \le e^x$, we have

$$x \ge \frac{1}{y} \log \widehat{w}(y) \Leftrightarrow e^{xy} \ge \frac{t^y}{w_\epsilon(t) \vee 1} \;\; \forall\, t > 0 \Leftrightarrow w_\epsilon(t) \vee 1 \ge t^y e^{-xy} \;\; \forall\, t > 0$$

$$\Leftrightarrow w_\epsilon(t) \ge t^y e^{-xy} \;\; \forall\, t > e^x \Leftrightarrow \frac{\log^+ w_\epsilon(t)}{\log t - x} \ge y \;\; \forall\, t > e^x \Leftrightarrow \nabla w(x) \ge y$$

which proves (4.a) and by (2) we have

$$\nabla w(x) \ge y \Leftrightarrow w_\epsilon(t) \vee 1 \ge t^y e^{-xy} \;\; \forall\, t > 0 \Leftrightarrow (y, e^{-xy}) \in D_w$$

$$\Leftrightarrow w^\star(t) \ge t^y e^{-xy} \;\; \forall\, t > 0 \Leftrightarrow \nabla w^\star(x) \ge y$$

Hence, we see that $\nabla w = \nabla w^\star$ and since $w^\star(x) \le v_\epsilon(x) \vee 1 \le w_\epsilon(x) \vee 1$ for all $x > a$, we see that $\nabla w(x) = \nabla w^\star(x) \le \nabla v(x)$ for all $x \ge a$. Thus, we see that (5.a) holds and (5.b) follows easily from (4.a) and (5.a).

(6) is an immediate consequence of (3) and (5). The first equality in (7) follows from (1) and the second equality in (7) follows from (6).

(8): By (1) and (2.2), we have that $J := \{x > 0 \mid w^\diamond(x) < \infty\}$ is an interval. Suppose that J is bounded. Then there exists $c > 0$ such that $w^\diamond(x) = \infty$ for all $x \ge c$. Hence, we have $w(x) = \infty$ for all $|x| \ge c$ and so we have $\widehat{w}(q) \le c^q$ and $|x|^q \, \widehat{w}(q)^{-1} \ge c^{-q} \, |x|^q$ for all $q > 0$ and all $x \in \mathbf{R}$. Hence, we see that $w^\star(x) = \infty$ for all $|x| > c$; that is (8) holds. Suppose that J is unbounded and let us define $f(s) := \log w^\diamond(e^s)$ for all $s \in \mathbf{R}$. Since $w^\diamond$ is log-exp-convex and J is an unbounded interval, then by (3) there exists $b \in \mathbf{R}$ such that f is finite and non-negative and convex on (b, ∞) . Let $f'(s)$ denote the right hand derivative of f for all $s > b$. Then f' is increasing on (b, ∞) and since $\limsup_{x \to \infty} w^\diamond(x) = \infty$, there exists $c > b$ such that $f(c) > 0$ and $f'(c) > 0$. Let $y \ge e^c$ be given and let us define $t := \log y$ and $\alpha := f'(t)$. Since $t > c$, we have $\alpha > 0$ and since f is finite and convex on (b, ∞) , we have $f(s) \ge \alpha \, (s - t) + f(t)$ for all $s > b$, In particular, we have $f(c) \ge \alpha \, (c - t) + f(t)$ and since $f \ge 0$ on $\mathbf{R}$, we have $f(s) \ge \alpha \, (s - t) + f(t) - f(c)$ for all $s \in \mathbf{R}$. Hence, if we define $\beta := \exp(f(t) - f(c) - \alpha t)$, then by (3) we have

$$w_\epsilon(x) \vee 1 \ge w^\diamond(x) = \exp(f(\log x)) \ge \exp(\alpha(\log x - t) + f(t) - f(c)) = \beta \, x^\alpha$$

for all $x > 0$; that is, $(\alpha, \beta) \in D_w$ and since $t = \log y$, then by (2) we have

$$w^\star(y) \ge \beta \, y^\alpha = \exp(\alpha \, t + f(t) - f(c) - \alpha t) = e^{-f(c)} \, w^\diamond(y)$$

for all $y > e^c$ which proves (8). □

Lemma 2.3: *Let* $w : \mathbf{R} \to [0, \infty]$ *be a non-negative function. Then we have*

(1) $$\int_\beta^\infty \widehat{w}(y)^{-\frac{1}{y}} \, dy = \int_{-\infty}^\infty e^{-x} \, (\nabla w(x) - \beta)^+ \, dx \quad \forall\, \beta \ge 0$$

(2) $$\liminf_{x \to \infty} e^{-x} \, \nabla w(x) = \liminf_{t \to \infty} t \, \widehat{w}(t)^{-1/t}$$

(3) $\limsup_{x\to\infty} e^{-x}\,\nabla w(x) = \limsup_{t\to\infty} t\,\widehat{w}(t)^{-1/t}$

(4) $\liminf_{x\to\infty} e^{-x}\,\nabla w(x) > 0 \quad\Leftrightarrow\quad \exists\,\alpha > 0$ *so that* $\liminf_{|s|\to\infty} e^{-\alpha|s|}\,w(s) > 0$

(5) $\limsup_{x\to\infty} e^{-x}\,\nabla w(x) > 0 \quad\Rightarrow\quad \exists\,\alpha > 0$ *so that* $\limsup_{s\to\infty} e^{-\alpha s}\,w_\epsilon(s) > 0$

Let $a \geq 0$ *is a given number and let* $\phi : (a,\infty) \to [0,\infty]$ *is a log-exp convex function such that* $\phi(x) \leq w_\epsilon(x) \vee 1$ *for all* $x > a$ *. Then we have*

(6) $\limsup_{s\to\infty} e^{-\alpha s}\,\phi(s) > 0$ *for some* $\alpha > 0$ *, then we have* $\limsup_{x\to\infty} e^{-x}\nabla w(x) > 0$

(7) $\int_\beta^\infty \frac{\log^+\phi(x)}{x^2}\,dx = \infty \;\;\forall\,\beta > a \quad\Rightarrow\quad \int_c^\infty e^{-x}\,\nabla w(x)\,dx = \infty \;\;\forall\,c \in \mathbf{R}$

Proof: (1): Let $\beta \geq 0$ be given and let us define $\phi(y) := \frac{1}{y}\log\widehat{w}(y)$ for all $y > 0$. Then $\widehat{w}(y)^{-1/y} = e^{-\phi(y)}$ for all $y > 0$ and by Lem.2.2.4 and Fubini's theorem we have

$$\int_\beta^\infty e^{-\phi(y)}\,dy = \int_\beta^\infty dy \int_{\phi(y)}^\infty e^{-x}\,dx = \int_{-\infty}^\infty dx \int_\beta^\infty e^{-x}\,1_{(0,\nabla w(x)]}(y)\,dy$$
$$= \int_{-\infty}^\infty e^{-x}\,(\nabla w(x) - \beta)^+\,dx$$

which proves (1).

(2+3): Let $x \in \mathbf{R}$ and $y > 0$ be given and let us define $t := y\,e^x$. Since $e^x = \frac{t}{y}$, then by Lem.2.1.4 we have $e^{-x}\,\nabla w(x) \geq y$ if and only if $t\,\widehat{w}(t)^{-1/t} \geq y$ and since $x \curvearrowright y\,e^x$ is continuous, strictly increasing and tends to ∞ as $x \to \infty$, we see that (2) and (3) hold.

(4): Let us define $\beta := \liminf_{x\to\infty} e^{-x}\,\nabla w(x)$. Suppose that $\beta > 0$ and let $0 < \gamma < \beta$ be given. Then there exists $x_0 > 0$ such that $\nabla w(x) \geq \gamma\,e^x$ for all $x \geq x_0$. Let $t > e^{x_0+1}$ be given. Since $x := \log t - 1 > x_0$, then by the definition of $\nabla w(x)$, we have

$$\log^+ w_\epsilon(t) \geq \gamma\,e^x\,(\log t - x) = \tfrac{\gamma}{e}\,t = \alpha t \;\text{ where }\; \alpha := \tfrac{\gamma}{e}$$

Hence, we see that $w(s) \geq e^{\alpha|s|}$ for all $|s| > e^{x_0+1}$; that is, the first condition in (4) implies the last condition. So suppose that the last condition in (4) holds. Then there exist positive numbers $\alpha,\beta > 0$ such that $w(s) \geq e^{\alpha|s|}$ for all $|s| > \beta$. Since $w_\epsilon(t) \geq e^{\alpha t} > 1$ for all $t > \beta$ and $t \curvearrowright \frac{\alpha t}{\log t - x}$ attains its minimum on (e^x,∞) at $t = e^{x+1}$, we have

$$\nabla w(x) = \inf_{t>e^x} \frac{\log^+ w_\epsilon(t)}{\log t - x} \geq \inf_{t>e^x} \frac{\alpha t}{\log t - x} = \alpha\,e^{x+1} \quad \forall\,x > \log\beta$$

and so we see that the last condition in (4) implies the first condition.

(5): By assumption, there exist positive numbers $\gamma > 0$ and $0 < x_1 < x_2 < \cdots$ such that $x_n \to \infty$ and $\nabla w(x_n) \geq \gamma\, e^{x_n}$ for all $n \geq 1$. Hence, if we define $t_n := e^{x_n+1}$ and $\alpha := \frac{\gamma}{e}$, then we have

$$\log^+ w_\epsilon(t_n) \geq \gamma\, e^{x_n} (\log t_n - x_n) = \tfrac{\gamma}{e}\, t_n = \alpha t_n$$

that is, $w_\epsilon(t_n) \geq e^{\alpha\, t_n}$ for all $n \geq 1$ and since $t_n \to \infty$, we see that the last condition in (5) holds.

(6): If there exists $c > a$ such that $w_\epsilon(x) = \infty$ for all $x > c$, then we have $\nabla w(x) = \infty$ for all $x \geq c$ and so we see that (6) holds. So suppose that the set $L := \{x > a \mid w_\epsilon(x) < \infty\}$ is unbounded and let us define $\psi(t) := \log^+ \phi(e^t)$ for all $t > b := \log a$. Since $\phi \vee 1$ is log-convex on (a, ∞) , we have that $\psi : (b, \infty) \to [0, \infty]$ is a convex function. Hence, we have that $\{x > b \mid \psi(x) < \infty\}$ is an interval and since L is unbounded and $\phi(x) \leq w_\epsilon(x) \vee 1$, there exists $c > b$ such that $0 \leq \psi(x) < \infty$ for all $x \geq c$. By assumption, there exist numbers $\alpha > 0$ and $e^c < s_1 < s_2 < \cdots$ such that $s_n \uparrow \infty$ and $\phi(s_n) \geq e^{\alpha\, s_n}$ for all $n \geq 1$ or equivalently $\psi(t_n) \geq \alpha\, e^{t_n}$ for all $n \geq 1$. Let us define $k_1 := 1$ and $u_1 := t_1$. Since $\alpha\, e^{\alpha u_1} \leq \psi(u_1) < \infty$ and $t_n \uparrow \infty$, there exists an integer $k_2 > k_1$ such that $\alpha\, e^{u_1} \leq \psi(u_1) \leq \alpha\, e^{u_2}$ where $u_2 := t_{k_1}$. Since $\alpha\, e^{\alpha u_2} \leq \psi(u_2) < \infty$ and $t_n \uparrow \infty$, there exists an integer $k_3 > k_2$ such that $\alpha\, e^{u_2} \leq \psi(u_2) \leq \alpha\, e^{u_3}$ where $u_3 := t_{k_3}$. Proceeding like this, we may inductively define a sequence $1 = k_1 < k_2 < k_3 < \cdots$ of integers satisfying $\alpha\, e^{u_n} \leq \psi(u_n) \leq \alpha\, e^{u_{n+1}}$ for all $n \geq 1$ where $u_n := t_{k_n}$. Since ψ is convex and finite on (c, ∞) , we have that ψ is absolutely continuous on (c, ∞) with a.e. derivative ψ' where $\psi'(x)$ denotes the right hand derivative of ψ at x for all $x > c$. Since $\phi(x) \leq w_\epsilon(x) \vee 1$ for all $x > a$, then by convexity of ψ we have

$$\nabla w(x) = \inf_{t > e^x} \tfrac{\log^+ w_\epsilon(t)}{\log t - x} \geq \inf_{u > x} \tfrac{\psi(u)}{u - x} \geq \inf_{u > x} \tfrac{\psi(u) - \psi(x)}{u - x} = \psi'(x) \;\; \forall\, x > c$$

and since $\alpha\, e^{u_n} \leq \psi(u_n) \leq \alpha\, e^{u_{n+1}}$, we have

$$\begin{aligned} \alpha \int_{u_n}^{u_{n+1}} e^t\, dt &= \alpha\, e^{u_{n+1}} - \alpha\, e^{u_n} \leq \psi(u_{n+1}) - \psi(u_n) \\ &= \int_{u_n}^{u_{n+1}} \psi'(t)\, dt \leq \int_{u_n}^{u_{n+1}} \nabla w(t)\, dt \end{aligned}$$

Hence, there exist numbers $\tau_n \in [u_n, u_{n+1}]$ such that $\nabla w(\tau_n) \geq \alpha\, e^{\tau_n}$ for all $n \geq 1$. Since $t_n \uparrow \infty$ and $u_n = t_{k_n}$, we see that $u_n \uparrow \infty$ and since $u_n \leq \tau_n \leq u_{n+1}$, we see that $\tau_n \uparrow \infty$. Thus we see that (6) holds.

(7): Let us define $b := \log a$ and $\psi(t) := \log \phi(e^t)$ for all $t > a$. As in the proof of (6), it suffices to consider the case where there exists $c > b$ such that $\psi(t) < \infty$ for all $t \geq c$. Let $\gamma > e^c$ be given and let us define $\beta := \log \gamma$. Then $\beta > c > a$ and as in the proof of (6), we have that $\psi'(x) \leq \nabla w(x)$ for all $x > c$ where ψ' denotes the right hand derivative of ψ . Since ψ is absolutely continuous on (c, ∞) with a.e. derivative ψ' , then by Fubini's theorem and the substitution $t = e^x$, we have

$$\int_\beta^\infty \frac{\log^+ \phi(t)}{t^2}\, dt = \int_\gamma^\infty e^{-x}\, \psi(x)\, dx = \frac{\psi(\gamma)}{\beta} + \int_\gamma^\infty dx \int_\gamma^x e^{-x}\, \psi'(y)\, dy$$

$$\leq \frac{\psi(\gamma)}{\beta} + \int_\gamma^\infty dx \int_\gamma^x e^{-x}\, \nabla w(y)\ dy = \frac{\psi(\gamma)}{\beta} + \int_\gamma^\infty dy \int_y^\infty e^{-x}\, \nabla w(y)\ dx$$

$$= \frac{\psi(\gamma)}{\beta} + \int_\gamma^\infty e^{-y}\, \nabla w(y)\ dy$$

By assumption, we have that the first integral is infinite and since $\psi(\gamma) < \infty$ and ∇w is increasing, we see that (7) holds. $\square$

3. Moment functions and test functions Recall that the set $\mathcal{M}$ of all lower semicontinuous, log-convex functions $M : (0,\infty) \to [0,\infty]$ such that $q \curvearrowright M(q)^{1/q}$ is increasing on $(0,\infty)$ contains every moment function M_μ and every dual function $\widehat{w}$. The following sets of "moment function" and "test functions" will play a crucial role in the succeeding discussion:

- $\mathcal{L}_{bd} := \left\{ M \in \mathcal{M} \;\middle|\; \limsup_{q\to\infty} M(q)^{\frac{1}{q}} < \infty \right\}$, $W_{bd} := \{\, w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}_{bd} \}$
- $\mathcal{L}_{P} := \left\{ M \in \mathcal{M} \;\middle|\; \limsup_{q\to\infty} \frac{1}{q}\, M(q)^{\frac{1}{q}} < \infty \right\}$, $W_{P} := \{\, w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}_{P} \}$
- $\mathcal{L}_{R} := \left\{ M \in \mathcal{M} \;\middle|\; \liminf_{q\to\infty} \frac{1}{q}\, M(q)^{\frac{1}{q}} < \infty \right\}$, $W_{R} := \{\, w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}_{R} \}$
- $\mathcal{L}_{C} := \left\{ M \in \mathcal{M} \;\middle|\; \int_1^\infty M(q)^{-\frac{1}{q}} = \infty \right\}$, $W_{C} := \{\, w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}_{C} \}$
- $\mathcal{L}_{\infty} := \{\, M \in \mathcal{M} \mid M(q) < \infty \quad \forall\, q > 0 \}$, $W_{\infty} := \{\, w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}_{\infty} \}$
- $W_K := \left\{ w \in \mathrm{B}_+(\mathbf{R}) \;\middle|\; \int_{-\infty}^\infty \frac{\log^+ w(x)}{1+x^2}\, dx = \infty \right\}$
- $W_K^s := \left\{ w \in \mathrm{B}_+(\mathbf{R}) \;\middle|\; \int_\alpha^\infty \frac{\log^+ w(x)}{x^2}\, dx = \infty \ \ \forall\, \alpha > 0 \right\}$
- $W_{\exp} := \left\{ w \in \mathrm{B}_+(\mathbf{R}) \;\middle|\; \exists\, \alpha > 0 \text{ so that } \limsup_{s\to\infty} e^{-\alpha s}\, w(s) > 0 \right\}$
- $W_{\exp}^s := \left\{ w \in \mathrm{B}_+(\mathbf{R}) \;\middle|\; \exists\, \alpha > 0 \text{ so that } \liminf_{|x|\to\infty} e^{-\alpha|x|}\, w(x) > 0 \right\}$

In the next section, we shall see that $\mathfrak{P}_x^* = W_x$ whenever "x" stands for any of the of the following five symbols: "bd" or "P" or "R" or "C" or "∞". Note that we have

(3.1) $\quad W_{\exp}^s \subseteq W_{\exp} \cap W_K^s \quad$ and $\quad W_K \subseteq W_K^s$

(3.2) If w is locally integrable or if w is even on $\mathbf{R}$ and increasing on $(0,\infty)$, then $w \in W_K$ if and only if $w \in W_K^s$

Let $\mathbf{C}$ denote *the complex plane* and let $F : \mathbf{C} \to \mathbf{C}$ be an *entire function*. Recall that F is of *finite exponential type* if $e^{-\alpha|z|}\,|F(z)|$ is bounded on $\mathbf{C}$ for some $\alpha > 0$, and that F of *minimal exponential type* if $e^{-\alpha|z|}\,|F(z)|$ is bounded on $\mathbf{C}$ for all $\alpha > 0$ (see [10] and [11]). We let $\mathrm{E}_{\mathrm{fin}}$ denote the set of entire functions of finite exponential type and we let $\mathrm{E}_{\mathrm{min}}$ denote the set of entire functions of minimal exponential type. Recall that *the Cartwright class*, which I shall denote Cw, is set of all entire functions $F \in \mathrm{E}_{\mathrm{fin}}$ satisfying $f \notin W_K$ where $f(x) := |F(x)|$ for all $x \in \mathbf{R}$.

Let $a_0, a_1, \ldots \geq 0$ be a given sequence. Then we say that (a_n) is *log-concave* if $0 < a_{n-1}\,a_{n+1} \leq a_n^2$ for all $n \geq 1$ or equivalently if there exists a convex function $\phi : [0,\infty) \to \mathbf{R}$ satisfying $a_n = e^{-\phi(n)}$ for all $n \geq 0$. Note that the sequence (a_n) is log-concave if and only if $a_n > 0$ for all $n \geq 0$ and the quotient sequence $q_n := \frac{a_n}{a_{n-1}}$ is decreasing and if and only if only if $a_0 > 0$ and there exists a decreasing sequence $q_1 \geq q_2 \geq \cdots > 0$ of positive numbers satisfying $a_n = a_0 \prod_{i=1}^n q_i$ for all $n \geq 1$. If $a_1, a_2, \ldots \geq 0$ is a an arbitrary sequence of non-negative numbers, we let (a_n°) denote *the log-concave hull* of (a_n); that is, $a_n^\circ := \inf_{(c_k)\in\mathcal{C}} c_n$ where $\mathcal{C}$ denotes the set of all log-concave sequences (c_k) satisfying $a_k \leq c_k$ for all $k \geq 1$

Theorem 3.1: *Let $M \in \mathcal{M}$ be a given function. If $0 < q_1 \leq q_2 \leq \cdots$ is any given increasing sequence of positive numbers satisfying $\sup_{n\geq 1} \frac{q_{n+1}}{q_n} < \infty$ and $q_n \to \infty$. Then we have the following characterizations of the sets $\mathcal{L}_P$ and $\mathcal{L}_R$:*

(1)
$$M \in \mathcal{L}_P \;\Leftrightarrow\; \limsup_{q\to\infty} \Big(\frac{M(q)}{q!}\Big)^{\frac{1}{q}} < \infty \;\Leftrightarrow\; \limsup_{n\to\infty} \Big(\frac{M(q_n)}{q_n!}\Big)^{\frac{1}{q_n}} < \infty$$
$$\Leftrightarrow\; \limsup_{n\to\infty} \frac{1}{q_n}\, M(q_n)^{\frac{1}{q_n}} < \infty$$

(2)
$$M \in \mathcal{L}_R \;\Leftrightarrow\; \liminf_{q\to\infty} \Big(\frac{M(q)}{q!}\Big)^{\frac{1}{q}} < \infty \;\Leftrightarrow\; \liminf_{n\to\infty} \Big(\frac{M(q_n)}{q_n!}\Big)^{\frac{1}{q_n}} < \infty$$
$$\Leftrightarrow\; \liminf_{n\to\infty} \frac{1}{q_n}\, M(q_n)^{\frac{1}{q_n}} < \infty$$

where $q! := \Gamma(q+1)$ for all $q > 0$ and Γ is the gamma function. If $\alpha, \beta > 0$ are given positive numbers and $0 < r_0 < r_1 < \cdots$ is any given increasing sequence of positive numbers satisfying $r_n \to \infty$ and $\sup_{n\geq 1} \frac{r_{n+1}-r_n}{r_n - r_{n-1}} < \infty$, then we have the following characterizations of the set $\mathcal{L}_C$:

(3)
$$M \in \mathcal{L}_C \;\Leftrightarrow\; \int_\beta^\infty M(\alpha q)^{-\frac{1}{\alpha q}}\, dq = \infty \;\Leftrightarrow\; \sum_{n=1}^\infty (r_n - r_{n-1})\, M(r_n)^{-\frac{1}{r_n}} = \infty$$

Proof: (1+2): Let us define $F(q) := \frac{1}{q} M(q)^{1/q}$ for all $q > 0$. By assumption, there exists a finite constant $C > 0$ such that $q_{n+1} \le C\, q_n$ for all $n \ge 1$. Let $n \ge 1$ and $q \in [q_n, q_{n+1}]$ be given. Since $y \curvearrowright M(y)^{1/y}$ is increasing on $(0,\infty)$, we have $F(q) \le \frac{q_{n+1}}{q} F(q_{n+1}) \le C\, F(q_{n+1})$ and $F(q_n) \le \frac{q}{q_n} F(q) \le C\, F(q)$ and since $q_n \uparrow \infty$, see that

$$\limsup_{q\to\infty} F(q) \le C \limsup_{n\to\infty} F(q_n) \;\;,\;\; \liminf_{n\to\infty} F(q_n) \le C \liminf_{q\to\infty} F(q)$$

Hence, we see that the first and last conditions in (1) or (2) are equivalent and by Stirling's formula, we have $q \cdot (q!)^{-1/q} \to e$ as $q \to \infty$. Hence, we see that the remaining equivalences in (1) and (2) holds.

(3): Let us define $f(q) := M(q)^{-1/q}$ for all $q > 0$. Since $M \in \mathcal{M}$, we see that f is decreasing and non-negative on $(0,\infty)$ and by (2.2) we see that f is either finite on $(0,\infty)$ or identically equal to ∞ on $(0,\infty)$. Hence, we see that the first equivalence in (3) holds and if $f \equiv \infty$, then all three statements in (3) hold trivially. So suppose that $f(y) < \infty$ for all $y > 0$. By assumption, there exists a finite constant $C > 0$ such that $r_{n+1} - r_n \le C\,(r_n - r_{n-1})$ for all $n \ge 1$ and since f is decreasing, we have

$$\sum_{n=1}^{\infty} (r_n - r_{n-1})\, f(r_n) \le \int_{r_0}^{\infty} f(q)\, dq \le \sum_{n=0}^{\infty} (r_{n+1} - r_n)\, f(r_n)$$
$$\le C \sum_{n=1}^{\infty} (r_n - r_{n-1})\, f(r_n)$$

and so we see that the last equivalence in (3) follows from the first equivalence. $\square$

Corollary 3.2: *Let $w, v \in \mathrm{B}_+(\mathbf{R})$ be given functions and let $a \ge 0$ be a given number satisfying $w^\star(x) \le v_\epsilon(x) \le w(x) \vee 1$ for all $|x| > a$. If "x" stands for one of the five symbols "bd" or "∞" or "P" or "R" or "C", then we have*

(1) $\mathfrak{P}_x = \{ \mu \in \mathrm{Pr}(\mathbf{R}) \mid M_\mu \in \mathcal{L}_x \}$

(2) $w \in W_x \;\Leftrightarrow\; v \in W_x \;\Leftrightarrow\; w^\diamond \in W_x \;\Leftrightarrow\; w^\star \in W_x$

(3) $\mathcal{L}_{bd} \subseteq \mathcal{L}_P \subseteq \mathcal{L}_R \subseteq \mathcal{L}_C \subseteq \mathcal{L}_\infty$

(4) $\mathfrak{P}_{bd} \subseteq \mathfrak{P}_P \subseteq \mathfrak{P}_R \subseteq \mathfrak{P}_C \subseteq \mathfrak{P}_D \subseteq \mathfrak{P}_\infty \;\;,\;\; W_{bd} \subseteq W_P \subseteq W_R \subseteq W_C \subseteq W_\infty$

Proof: (1): If "x" equals "P" or "R" or "C", then (1) follows from Thm.3.1 and if "x" equals "∞", then (1) is evident. If "x" equals "bd", then (1) follows from [8; Exc.3.16 p.235]. (2) is an immediate consequence of Lem.2.2.5+6. Let $M \in \mathcal{M}$ be given. Since $y \curvearrowright M(y)^{-1/y}$ is decreasing on $(0,\infty)$, we have $\int_q^{2q} M(y)^{-1/y}\, dy \ge q\, M(q)^{-1/q}$ for all $q > 0$. Hence, we see that $\mathcal{L}_R \subseteq \mathcal{L}_C$ and the remaining inclusions in (3) follow directly and the definition of $\mathcal{L}_x$. (4) is an immediate consequence of (3). $\square$

Theorem 3.3: *Let* $w \in \mathrm{B}_+(\mathbf{R})$ *be a given non-negative Borel function. Let* $a, p > 0$ *be a given numbers, let* $\phi : (a, \infty) \to [0, \infty]$ *be a log-exp-convex function and let* $F : (a, \infty) \to [0, \infty]$ *be a non-negative Borel function satisfying*

(1) $\quad \phi(x) \leq w(x) \vee 1 \ \ \forall\, |x| > a$ *and* $w_\epsilon(x) \leq F(x)\,(1 + \phi(x))^p \quad \forall\, x > a$

If $c \in \mathbf{R}$ *is any given number. Then we have*

(2) $\quad w \in W_{bd} \ \Leftrightarrow \ \exists\, c \in \mathbf{R}_+$ *so that* $w(x) = \infty \ \ \forall\, |x| \geq c$

(3) $\quad w \in W_\infty \ \Leftrightarrow \ \lim_{|x| \to \infty} |x|^{-q}\, w(x) = \infty \ \ \forall\, q > 0$

(4) $\quad w \in W_P \ \Leftrightarrow \ w \in W^s_{\exp} \ \Leftrightarrow \ \liminf_{x \to \infty} e^{-x}\, \nabla w(x) > 0$

(5) $\quad w \in W_R \Leftrightarrow \limsup_{x \to \infty} e^{-x}\, \nabla w(x) > 0 \Leftrightarrow w^\diamond \in W_{\exp} \Leftrightarrow w^\star \in W_{\exp} \Rightarrow w \in W_{\exp}$

(6) $\quad$ *If* $\lim_{x \to \infty} e^{-\alpha x}\, F(x) = 0 \ \ \forall\, \alpha > 0$ *, then* $w \in W_R$ *if and only if* $w_\epsilon \in W_{\exp}$

(7) $\quad w \in W_C \Leftrightarrow \int_c^\infty e^{-x}\, \nabla w(x)\, dx = \infty \Leftrightarrow w^\diamond \in W^s_K \Leftrightarrow w^\star \in W_K \Rightarrow w \in W^s_K$

(8) $\quad$ *If* $\int_a^\infty \frac{\log^+ F(x)}{x^2}\, dx < \infty$ *, then* $w \in W_C$ *if and only if* $w_\epsilon \in W^s_K$

Proof: (2): Suppose that $w \in W_{bd}$. Then there exists $c > 1$ such that $\widehat{w}(q) \leq c^q$ for all $q \geq c$. Hence, we have $w(x) \geq c^{-q}\, |x|^q$ for all $x \in \mathbf{R}$ and all $q \geq c$. Let $|x| \geq 2c$ be given. Since $w(x) \geq c^{-q}\, |x|^q \geq 2^q$ for all $q \geq c$, we see that $w(x) = \infty$ for all $|x| \geq 2c$. Conversely, if $c \in \mathbf{R}_+$ is a given number such that $w(x) = \infty$ for all $|x| \geq c$, then we have $\widehat{w}(q) \leq c^q$ for all $q > 0$. In particular, we see that $\widehat{w} \in \mathcal{L}_{bd}$ or equivalently $w \in W_{bd}$.

(3): Let $x \in \mathbf{R}$ and $q > 0$ be given. Since $|x|^q \leq \widehat{w}(q)\, (w(x) \vee 1)$ and $\widehat{w}(q) \leq c^q \vee \sup_{|y| \geq c} |y|^q\, (w(y) \vee 1)^{-1}$ for all $c > 0$, we see that (2) holds.

(4): Follows from Lem.2.3.2+4.

(5+6): By Lem.2.3.3, we see that the first equivalence in (5) holds and by Lem.2.3.5 we see that $w \in W_R$ implies $w \in W_R$ implies $w \in W_{\exp}$. Suppose that $w_\epsilon \in W_{\exp}$ and that $e^{-\alpha x}\, F(x) \to 0$ for all $\alpha > 0$. Then there exist $\alpha, b > 0$ such that $\limsup_{x \to \infty} e^{-\alpha x}\, w_\epsilon(x) > 0$ and $F(x) \leq e^{\beta x}$ for all $x \geq b$ where $\beta := \frac{\alpha}{p}$. So by (1) we have

$$0 < \limsup_{x \to \infty} e^{-\beta x}\, w_\epsilon(x)^{1/p} \leq \limsup_{x \to \infty} e^{-\beta x}\, (1 + \phi(x)) = \limsup_{x \to \infty} e^{-\beta x}\, \phi(x)$$

and since ϕ is log-exp-convex, then by (1), and Lem.2.3.6 we have $\limsup_{x \to \infty} e^{-x}\, \nabla w(x) > 0$ or equivalently $w \in W_R$. Since $\phi := w^\diamond$ and $\phi := w^\star$ satisfy condition (1) with $F \equiv 1$, we see that the remaining implications in (5) and (6) follows from Cor.3.2.2 and Lem.2.2.1.

(7+8): Since ∇w is increasing, we see that the first equivalence in (7) follows from Lem.2.3.1 and Thm.3.1.3. Suppose that the second condition in (7) holds and let $\beta > 0$ be given. Since ∇w is increasing, we see that the integral in (7) is infinite for all $c \in \mathbf{R}$; in particular for $c := (\log \beta) - 1$ and since$\nabla w(x) \le \log^+ w_\epsilon(e^{x+1})$, we have

$$\infty = \int_c^\infty e^{-x}\, \nabla w(x)\, dx \le \int_c^\infty e^{-x}\, \log^+\, w(e^{x+1})\, dx = e \int_\beta^\infty t^{-2}\, \log^+\, w(t)\, dt$$

Since $t^{-2} \le \frac{1+\beta^2}{\beta^2}\,(1+t^2)^{-1}$ for all $t \ge \beta$, we see that $w \in W_K^s$. Suppose that $w_\epsilon \in W_K^s$ and let $\beta > a$ be given. Since $1+\phi(x) \le 2\,(\phi(x) \vee 1)$, then by (1) we have

$$\begin{aligned}\infty = \int_\beta^\infty \tfrac{\log^+ w_\epsilon(x)}{x^2}\, dx &\le \int_\beta^\infty \tfrac{\log^+ F(x)}{x^2}\, dx + p\int_\beta^\infty \tfrac{\log(1+\phi(x))}{x^2}\, dx \\ &\le \int_a^\infty \tfrac{\log^+ F(x)}{x^2}\, dx + \tfrac{p\,\log 2}{\beta} + p\int_\beta^\infty \tfrac{\log^+ \phi(x)}{x^2}\, dx\end{aligned}$$

Hence, if the first integral in the last expression is finite, we see that the last integral is infinite for all $\beta > a$ and since ϕ is log-exp-convex, then by (1) and Lem.2.3.7 we see that $\int_c^\infty e^{-x} \nabla w(x)\, dx = \infty$ or equivalently $w \in W_C$. Since $\phi := w^\diamond$ and $\phi := w^\star$ satisfy condition (1) with $F \equiv 1$, we see that the remaining implications (7) and (8) follows from Cor.3.2.2, Lem.2.2.1 and (3.2). $\square$

Theorem 3.4: *Let* $a_0, a_1, \ldots \in \mathbf{R}_+$ *be non-negative numbers such that* $a_n^{1/n} \to 0$ *and* $a_n > 0$ *for infinitely many* $n \ge 0$. *Let* $f(z) := \sum_{n=0}^\infty a_n\, z^n$ *for* $z \in \mathbf{C}$ *denote the associated entire function and let us define*

$$w(x) := f(|x|) = \sum_{n=0}^{\infty} a_n\, |x|^n \;\; \textit{and} \;\; v(x) := \sup_{n \ge 1}\, a_n |x|^n \;\; \forall\, x \in \mathbf{R}$$

Then w *and* v *are finite and even functions on* $\mathbf{R}$ *such that* $w, v \in W_\infty$ *and* w *and* v *are log-exp-convex on* $(0, \infty)$. *Moreover, we have*

(1) $\quad w \in W_P \;\Leftrightarrow\; w \in W^s_{\text{exp}} \;\Leftrightarrow\; v \in W_P \;\Leftrightarrow\; v \in W^s_{\text{exp}}$

(2) $\quad w \in W_R \;\Leftrightarrow\; w \in W_{\text{exp}} \;\Leftrightarrow\; v \in W_R \;\Leftrightarrow\; v \in W_{\text{exp}} \;\Leftrightarrow\; f \notin \mathrm{E}_{\text{min}}$

(3) $\quad w \in W_C \;\Leftrightarrow\; w \in W_K \;\Leftrightarrow\; w \in W_K^s \;\Leftrightarrow\; v \in W_C \;\Leftrightarrow\; v \in W_K \;\Leftrightarrow\; v \in W_K^s$

(4) $\quad \int_0^\infty \frac{\log^+ |f(x)|}{1+x^2}\, dx < \infty \;\Leftrightarrow\; w \notin W_C \;\Leftrightarrow\; f \in \mathrm{Cw} \cap \mathrm{E}_{\text{min}} \;\Leftrightarrow\; f \in \mathrm{Cw}$

Let $(b_0, b_1, \ldots)$ *denote the log-concave hull of the sequence* $(1, a_1, a_2, \ldots)$ *and let us define* $c_n := \sup_{k \ge n} a_k^{1/k}$ *for all* $n \ge 1$. *Then we have*

(5) $\quad (b_n)$ *is a strictly positive log-concave sequence satisfying*

(a) $\quad b_0 = 1 \;\;,\;\; b_n = \widehat{v}(n)^{-1}$ *and* $a_n \le c_n^n \le b_n \le \prod_{i=1}^n c_i \quad \forall\, n \ge 1$

(b) $\quad \limsup_{n\to\infty}\, n^\alpha\, b_n^{1/n} = \limsup_{n\to\infty}\, n^\alpha\, a_n^{1/n} = \limsup_{n\to\infty}\, n^\alpha\, c_n \quad \forall\, \alpha \ge 0$

(6) $\quad w \in W_P \quad \Leftrightarrow \quad v \in W_P \quad \Leftrightarrow \quad \liminf_{n\to\infty} n\, b_n^{1/n} > 0$

(7) *If* $\limsup_{n\to\infty} n\, a_n^{1/n} < \infty$ *, then* $w \in W_P$ *if and only if* $\liminf_{n\to\infty} n\, c_n > 0$

(8) $\quad w \in W_R \quad \Leftrightarrow \quad v \in W_R \quad \Leftrightarrow \quad \limsup_{n\to\infty} n\, b_n^{1/n} > 0 \quad \Leftrightarrow \quad \limsup_{n\to\infty} n\, a_n^{1/n} > 0$

(9) $\quad w \in W_C \quad \Leftrightarrow \quad v \in W_C \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} b_n^{1/n} = \infty$

Proof: (1)–(3): Since $a_n^{1/n} \to 0$ and $a_n > 0$ for infinitely many $n \geq 0$, we see that v and w are finite, even functions belonging to W_∞ . Since $a_n\, x^n$ is increasing and log-exp-convex on $(0,\infty)$, we see that w and v are increasing and log-exp-convex on $(0,\infty)$. Since $a_n |x|^n \leq \beta^n\, v(\frac{x}{\beta})$ for all $\beta > 0$, all $x \in \mathbf{R}$ and all $n \geq 1$, we have

$$v(x) \leq w(x) \leq a_0 + \tfrac{\beta}{1-\beta}\, v\big(\tfrac{x}{\beta}\big) \quad \forall\, x \in \mathbf{R}\, \forall\, 0 < \beta < 1$$

Since $w(x) = f(x)$ for all $x \geq 0$ and $|f(z)| \leq w(|z|)$ for all $z \in \mathbf{C}$, we see that (1)–(3) follow from Thm.3.3.

(4): Let us number the four conditions in (4) by (a), (b), (c) and (d). Since w is even and $w(x) = f(x)$ for all $x \geq 0$, then by (3) we see that (a) implies (b). Suppose that (b) holds. By Cor.3.3.3 we have $W_R \subseteq W_C$. Hence, we see that $w \notin W_R$. So by (2) we have $f \in \mathrm{E}_{\min}$ and since $|f(x)| \leq w(x)$ for all $x \geq 0$, then by (3) we see that $f \in \mathrm{Cw} \cap \mathrm{E}_{\min}$; that is (b) implies (c). The implication: “(c) $\Rightarrow$ (d) $\Rightarrow$ (a)” are evident.

(5): Since $v \in W_\infty$ and v is finite, we have $0 < \widehat{v}(q) < \infty$ for all $q > 0$. Hence, if define $d_0 := 1$ and $d_n := \widehat{v}(n)^{-1}$ for $n = 1, 2, \ldots$, then $(d_0, d_1, \ldots)$ is a sequence of positive numbers. Since $q \curvearrowright \widehat{v}(q)^{1/q}$ is increasing, we have $\widehat{v}(1) \leq \widehat{v}(2)^{1/2}$ and since $\widehat{v}$ is log-convex, we have $\widehat{v}(n) \leq \widehat{v}(n-1)^{1/2}\, v(n+1)^{1/2}$ for all $n \geq 2$. Hence, we see that (d_n) is a log-concave sequence with $d_0 = 1$. Let $n \geq 1$ be a given integer. Since $|x|^{-n}\, v(x) \geq a_n$ for all $x \in \mathbf{R}$, we see that $\widehat{v}(n) \leq a_n^{-1}$ or equivalently $d_n \geq a_n$ for all $n \geq 1$. Thus, we conclude that $d_n \geq b_n$ for all $n \geq 0$. To prove the converse inequality, let (x_i) be a given log-concave sequence satisfying $x_0 \geq 1$ and $x_i \geq a_i$ for all $i \geq 1$. Then we have $x_n = x_0 \prod_{i=1}^n q_i$ for all $n \geq 1$ where $q_i := \frac{x_i}{x_{i-1}}$ is decreasing. Let $k, n \geq 0$ be given integers. Since (q_i) is decreasing, we have

$$k < n \quad \Rightarrow \quad x_k = x_0 \prod_{i=1}^{n} q_i \prod_{i=k+1}^{n} q_i^{-1} \leq x_n\, q_{n+1}^{k-n}$$

$$k > n \quad \Rightarrow \quad x_k = x_0 \prod_{i=1}^{n} q_i \prod_{i=n+1}^{k} q_i \leq x_n\, q_{n+1}^{k-n}$$

and since $x_0 \geq 1$ and $x_k \geq a_k$ for all $k \geq 1$, we have

$$v(q_{n+1}^{-1}) \vee 1 = 1 \vee \sup_{k \geq 1} a_k \, q_{n+1}^{-k} \leq \sup_{k \geq 0} x_k \, q_{n+1}^{-k} \leq x_n \, q_{n+1}^{-n}$$

$$\widehat{v}(n) \geq q_{n+1}^{-n} \, (v(q_{n+1}^{-1}) \vee 1)^{-1} \geq x_n^{-1}$$

for all $n \geq 1$ and since $d_0 = 1 \leq x_0$, we see that $d_n \leq x_n$ for all $n \geq 0$. Taking infimum over (x_i) and recalling that $d_n \geq b_n$, we see that $d_n = b_n$ for all $n \geq 0$; which proves of the first two equalities in (5.a) and since (d_n) is log-concave, then so is (b_n). Since $b_n \geq a_n$ for all $n \geq 1$ and $b_n^{1/n} = \widehat{v}(n)^{-1/n}$ is decreasing, we see that $a_n \leq {c_n}^n \leq b_n$. Let us define $y_0 := 1$ and $y_n := \prod_{i=1}^n c_i$ for $n \geq 1$. Since (c_i) is decreasing, we have that (y_n) is log-concave and since $c_i \geq c_n \geq a_n^{1/n}$ for all $1 \leq i \leq n$, we have $y_n \geq a_n$ for all $n \geq 1$. Hence, we see that $y_n \geq b_n$ for all $n \geq 1$ which completes of (5.a).

(6): Follows directly from (1), (5) and Thm.3.1.1.

(7): Suppose that $w \in W_P$. By (1) there exist positive numbers $\alpha, \beta > 0$ such that $v(x) \geq e^{-\alpha x}$ for all $x \geq \beta$ and since $\limsup_{n \to \infty} n \, a_n^{1/n} < \infty$, then by (5.b) there exists $\gamma > e\alpha$ such that $c_k \leq \frac{\gamma}{k}$ for all $k \geq 1$. Let us define $x_n := \frac{1}{c_n}$ and $f(0) := 0$ and $f(t) := t \, \log \frac{1}{t}$ for all $t > 0$. Observe that f is continuous and strictly increasing on the interval $[0, e^{-1}]$ with $f(0) = 0$ and $f(e^{-1}) = e^{-1}$. Since $0 < \frac{\alpha}{\gamma} < e^{-1}$, there exists a unique number $0 < \lambda < e^{-1}$ solving the equation $f(\lambda) = \frac{\alpha}{\gamma}$ and since $c_n \to 0$, there exists an integer $m > 1$ such that $x_n \geq \beta$ for all $n \geq m$. Let $n \geq m$ be a given integer satisfying $nc_n \leq \frac{\gamma}{e}$. If $k \geq n$, then we have $a_k \, x_n^k \leq c_k^k \, x_n^k \leq 1$ and if $1 \leq k < n$, we have $kc_n \leq nc_n \leq \frac{\gamma}{e}$ and

$$\log\bigl(a_k \, x_n^k\bigr) \leq k \, \log\bigl(\tfrac{c_k}{c_n}\bigr) \leq k \, \log\bigl(\tfrac{\gamma}{kc_n}\bigr) = \gamma x_n \, f\bigl(\tfrac{kc_n}{\gamma}\bigr) \leq \gamma x_n \, f\bigl(\tfrac{nc_n}{\gamma}\bigr)$$

Since $x_n \geq \beta$, we have

$$\alpha x_n \leq \log \, v(x_n) \leq \gamma x_n \max_{1 \leq k < n} f\bigl(\tfrac{kc_n}{\gamma}\bigr) \leq \gamma x_n \, f\bigl(\tfrac{nc_n}{\gamma}\bigr)$$

that is, $\frac{\alpha}{\gamma} = f(\lambda) \leq f(\frac{nc_n}{\gamma})$ and since $\frac{nc_n}{\gamma} \leq e^{-1}$ and f is strictly increasing on $(0, e^{-1}]$, we conclude that $nc_n \geq \lambda\gamma$ for all $n \geq m$ satisfying $nc_n \leq \frac{\gamma}{e}$. Since $\lambda\gamma \leq \frac{\gamma}{e}$, we conclude that $nc_n \geq \lambda\gamma$ for all $n \geq m$ and consequently, we have $\liminf \, nc_n \geq \lambda\gamma > 0$. The converse implication follows from (5.a) and (6).

(8) and (9) follow directly from (2), (3), (5) and Thm.3.1.2+3. □

□

Examples 3.5: (1): Let us define $w(x) := e^{\sqrt{|x|}}$ if $x \leq 0$ and $w(x) := e^{|x|}$ if $x \geq 0$. Then we have $w \in W_{\exp} \cap W_K^s$ and since $w^\diamond(x) \leq w_\epsilon(x) = e^{|x| \wedge \sqrt{|x|}}$, we have $w_\epsilon \notin W_{\exp} \cup W_K$, $w^\diamond \notin W_{\exp} \cup W_K$. So by Thm.3.3.5+7 we see that $w \notin W_R$ and $w \notin W_C$ but $w \in W_{\exp} \cap W_K^s$.

(2): Let us define $w(0) = 1$, $w(x) := e^{1/|x|}$ if $0 < |x| \leq 1$ and $w(x) = e$ if $|x| \geq 1$. Then w is an even and finite function such that w is log-convex and log-exp-convex on $(0,\infty)$. Hence, we have $w^\diamond = w$ and so we see that $w^\diamond \in W_K$ but $w \notin W_\infty$ and $w \notin W_C$

(3): Let $\alpha > 0$ and $p, q \in \mathbf{R}$ be given numbers and let us consider the function:

$$w(x) := \exp\left\{ \frac{\alpha\,|x|}{(1+\log^+|x|)^p\,(1+\log^+\log^+|x|)^q} \right\} \quad \forall\, x \in \mathbf{R}$$

Then a straight forward computation shows that w is increasing and log-exp-convex on $[a,\infty)$ for some sufficiently large $a > 1$. Hence, by Thm.3.3 we have

(1) $w \in W_C \;\Leftrightarrow\;$ either $p < 1$ or $p = 1$ and $q \leq 1$

(2) $w \in W_P \;\Leftrightarrow\; w \in W_R \;\Leftrightarrow\; p \leq 0$ and $q \leq 0$

(4): Let $L \subseteq \mathbf{N}_0$ be a given infinite set of non-negative integers and let $0 \leq n_1 < n_2 < \cdots$ denote the elements in L in increasing order. If we define

$$w(x) := \sum_{n\in L} \frac{|x|^n}{n!} \;\; \forall\, x \in \mathbf{R} \;\text{ and }\; \rho := \liminf_{n\to\infty} \frac{n_k}{n_{k+1}}$$

then we are in the setting of Thm.3.4 with $a_n = (n!)^{-1/n}$ if $n \in L$ and $a_n = 0$ if $n \notin L$. Since $(n!)^{-1/n}$ is decreasing in n we have $c_n = (n_k!)^{-1/n_k}$ for all $n_{k-1} < n \leq n_k$. So by Stirling's formula, we see that $\limsup n\, a_n^{1/n} = e$ and $\liminf n\, c_n = e\rho$. Hence, by Thm.3.4 we have

(3) If $\rho > 0$, then we have $w \in W_P \subseteq W_R \subseteq W_C$

(4) If $\rho = 0$, then we have $w \in W_R \setminus W_P$

4. Complete sets of test functions With the provision of the previous sections, we can now proceed to the general solution of specifying a class of sets of probability measures admitting a complete set of test functions. To do this we need the following preorderings on the sets $\mathcal{M}$, $\mathrm{B}_+(\mathbf{R})$ and $\Pr(\mathbf{R})$: If $L, M \in \mathcal{M}$ and $v, w \in \mathrm{B}_+(\mathbf{R})$ are given functions and $\mu, \nu \in \Pr(\mathbf{R})$ are given probability measures, we define

- $L \vdash M \;\Leftrightarrow\; \exists\, \alpha > 0$ so that $\limsup\limits_{q\to\infty} \left\{ \frac{L(\alpha q)}{1+M(q)^\alpha} \right\}^{\frac{1}{q}} < \infty$
- $v \models w \;\Leftrightarrow\; \widehat{w} \vdash \widehat{v}$
- $\nu \preceq \mu \;\Leftrightarrow\; M_\nu \vdash M_\mu$

Then the reader easily verifies the three relations are preorderings (i.e. transitive and reflexive relations) on the respective spaces. If $\mathcal{L} \subseteq \mathcal{M}$, $W \subseteq \mathrm{B}_+(\mathbf{R})$ and $\mathfrak{P} \subseteq \Pr(\mathbf{R})\mathcal{P} \subseteq \Pr(\mathbf{R})$ are given sets, we say that

- $\mathcal{L}$ is *lower* ($\vdash$)*-directed* if $M \in \mathcal{L}$ implies $L \in \mathcal{L} \ \forall L \in \mathcal{M}$ with $L \vdash M$
- W is *upper* ($\models$)*-directed* if $w \in W$ implies $v \in W \ \forall v \in \mathrm{B}_+(\mathbf{R})$ with $w \models v$
- $\mathfrak{P}$ is *lower* ($\preceq$)*-directed* if $\mu \in \mathfrak{P}$ implies $\nu \in \mathfrak{P} \ \forall \nu \in \Pr(\mathbf{R})$ with $\nu \preceq \mu$

Lemma 4.1: *Let $v, w \in \mathrm{B}_+(\mathbf{R})$ be given functions. Then we have*

(1) *If $w \in W_{bd}$ or if $v \notin W_\infty$, then $v \models w$*

(2) $w^\star \models w^\diamond \models w \models w^\star$

and the following three statements are equivalent

(3) $v \models w$

(4) *There exist positive numbers C , c , β , γ , $\delta > 0$ satisfying*

(a) $v^\diamond(\delta x) \leq C\big(1 + |x|^\beta + w^\star(x)^\gamma\big) \quad \forall |x| \geq c$

(5) *There exist non-negative numbers C , c , β , $\gamma \geq 0$ and functions $f : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \to [0,\infty]$ satisfying*

(a) $\liminf\limits_{|x|\to\infty} \frac{|f(x)|}{|x|} > 0$ *and* $\limsup\limits_{|x|\to\infty} |x|^{-\beta} h(x) < \infty$

(b) $v^\star(f(x)) \leq C\, h(x)\,\big(1 + w(x)^\gamma\big) \quad \forall |x| > c$

Proof: (1): Suppose that $w \in W_{bd}$. Since $\widehat{w} \in \mathcal{L}_{bd}$, there exists $c > 0$ such that $\widehat{w}(q) \leq c^q \leq c^q\,(1 + \widehat{v}(q))$ for all $q \geq c$. Hence, we see that $\widehat{w} \vdash \widehat{v}$ or equivalently $v \models w$. Suppose that $v \notin W_\infty$. Since $\widehat{v} \notin \mathcal{L}_\infty$, there exists $r > 0$ such that $\widehat{v}(r) = \infty$ and since $\widehat{v}(q)^{1/q}$ is increasing, we have $\widehat{v}(q) = \infty$ for all $q \geq r$. Hence, we see that $\widehat{w} \vdash \widehat{v}$ or equivalently $v \models w$.

(2): Immediate consequence of Lem.2.2.6.

(3) $\Rightarrow$ (4): Suppose that $v \models w$. Then there exist positive numbers $\kappa > 0$ and $\beta > 0$ such that $\widehat{w}(\kappa q) \leq \beta^q\,(1 + \widehat{v}(q)^\kappa)$ for all $q \geq \beta$. If $v \in W_{bd}$, there exists $a > 0$ such that $\widehat{v}(q) \leq a^q$ for all $q > 0$ and so we have $\widehat{w}(\kappa q) \leq \beta^q\,(1 + a^{\kappa q})$ for all $q \geq \beta$. Hence, we see that $\widehat{w} \in \mathcal{L}_{bd}$ or equivalently $w \in W_{bd}$. Hence, by Cor.3.2.2 we have $w^\star \in W_{bd}$ and so by Thm.3.3.2 we see that (4.a) holds trivially. So suppose that $v \notin W_{bd}$. By Thm.3.3.2 there exists $x_0 \in \mathbf{R}$ such that $|x_0| > 1$ and $v(x_0) < \infty$. Hence, if we define $\lambda := v(x_0) \vee 1$, then $1 \leq \lambda < \infty$ and we have $\widehat{v}(q) \geq |x_0|^q\,(v(x_0) \vee 1)^{-1} \geq \lambda^{-1}$ for all $q > 0$. In particular, we see that $1 + \widehat{v}(q)^\kappa \leq 2\lambda^\kappa\,\widehat{v}(q)^\kappa$ for all $q > 0$. Let $x \in \mathbf{R}$ be given. By Lem.2.2.6, we have

$$\tfrac{|x|^{\kappa q}}{w^\star(x) \vee 1} \leq \widehat{w}(\kappa q) \leq \beta^q\,(1 + \widehat{v}(q)^\kappa) \leq 2\beta^q\,\lambda^\kappa\,\widehat{v}(q)^\kappa \quad \forall q \geq \beta$$

Let us define $\gamma := \frac{1}{\kappa}$ and $\delta := \beta^{-\gamma}$. Since $\widehat{v}(q) \geq \lambda^{-1}$ for all $q > 0$, we have

$$\frac{|\delta x|^q}{\widehat{v}(q)} \leq 2^\gamma \lambda \, (1 \vee w^\star(x))^\gamma \leq 2^\gamma \lambda \, \left(1 + w^\star(x)^\gamma\right) \quad \forall\, q \geq \beta \;\forall\, x \in \mathbf{R}$$

$$\frac{|\delta x|^q}{\widehat{v}(q)} \leq \lambda \, |\delta x|^q \leq \lambda \, |\delta x|^\beta \quad \forall\, 0 < q \leq \beta \;\forall\, |x| \geq \frac{1}{\delta}$$

and so we have $v^*(\delta x) \leq A\,(1+|x|^\beta+w^*(x)^\gamma)$ for all $|x| \geq \frac{1}{\delta}$ where $A = 2^\gamma\,\lambda+\delta^q\,\lambda$. Thus, we see that (4.a) follows from Lem.2.2.8.

(4) $\Rightarrow$ (5): Take $f(x) := \delta x$ and $h(x) := 1 + |x|^\beta$.

(5) $\Rightarrow$ (3): Suppose that (5) holds. If $v \notin W_\infty$, then (3) follows from (1). So suppose that $v \in W_\infty$ and let $q > 0$ be given. By (5.a), there exist $\delta > 0$ and $b > c$ such that $|f(x)| \geq \delta |x|$ and $h(x) \leq b|x|^\beta$ for all $|x| \geq b$ and since $v \in W_\infty$, then by Lem.2.2.3 and Cor.3.2.2 we see that $v^\star \in W_\infty$. Let $q > 0$ be given. Then by (5.b) we have

$$|\delta x|^{-q-\beta} v^\star(\delta x) \leq C\,\delta^{-\beta-q}\,|x|^{-q-\beta}\,h(x)(1 + w(x)^\gamma) \leq bC\,\delta^{-\beta-q}\,|x|^{-q}(1 + w(x)^\gamma)$$

for all $|x| \geq b$ and since $v^\star \in W_\infty$, then by Thm.3.3.3 we see that $w \in W_\infty$. Hence, by Thm.3.3.3 there exists $a > b$ such that $w(x) \geq 1 + |x|^\beta$ for all $|x| \geq a$. Let $|x| \geq a$ and $q > 0$ be given. Since $w(x) \geq 1 + |x|^\beta$ and $w^\star$ is even on $\mathbf{R}$ and increasing on $[0, \infty)$, then by (5.b) we have

$$\frac{|\delta x|^q}{\widehat{v}(q)} \leq v^\star(\delta x) \leq bC\,|x|^\beta\,(1 + w(x)^\gamma) \leq 2bC\,|x|^\beta\,w(x)^\gamma \leq 2bC\,w(x)^{\gamma+1}$$

Hence, if we define $\alpha := \frac{1}{1+\gamma}$ and $C_0 := (2bc)^\alpha\,\delta^{-\alpha q}$, then we have

$$\frac{|x|^{\alpha q}}{w(x)\vee 1} = \frac{|x|^{\alpha q}}{w(x)} \leq C_0\,\widehat{v}(q)^{1/\gamma} \leq C_0\,(1 + \widehat{v}(q)^\alpha) \quad \forall\, |x| \geq a$$

$$\frac{|x|^{\alpha q}}{w(x)\vee 1} \leq a^{\alpha q} \leq a^{\alpha q}\,(1 + \widehat{v}(q)^\alpha) \quad \forall\, |x| \leq a$$

and so we see that $\widehat{w}(\alpha q) \leq (C_0 \vee a^{\alpha q})\,(1 + \widehat{v}(q)^\alpha)$ for all $q > 0$. Thus, we have $\widehat{w} \vdash \widehat{v}$ or equivalently $v \models w$. $\square$

Corollary 4.2: *Let* $W \subseteq \mathrm{B}_+(\mathbf{R})$ *be a non-empty upper* $(\models)$*-directed set of functions and let* $w \in \mathrm{B}_+(\mathbf{R})$ *be a given function. Then we have*

(1) $W_{bd} \subseteq W$ *and if* $W \neq \mathrm{B}_+(\mathbf{R})$ *, then we have* $W \subseteq W_\infty$

(2) $w \in W \quad \Leftrightarrow \quad w^\diamond \in W \quad \Leftrightarrow \quad w^\star \in W$

Let $f : \mathbf{R} \to \mathbf{R}$ *and* $h : [0, \infty] \to [0, \infty]$ *be given Borel functions satisfying*

(3) $\liminf\limits_{|x|\to\infty} \frac{|f(x)|}{|x|} > 0$ *and* $\liminf\limits_{x\to\infty} x^{-\beta}\,h(x) > 0$ *for some* $\beta > 0$

If $v, v_1, \dots v_n \in W$ *and* $w \in \mathrm{B}_+(\mathbf{R})$ *are given functions and* $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n > 0$ *are positive numbers, then we have*

(4) *If* $\limsup_{n\to\infty} \frac{v^\star(f(x))}{(1+|x|^p)\,(1+w(x)^q)} < \infty$ *for some numbers* $p, q > 0$ *, then* $w \in W$

(5) *If* $\liminf_{|x|\to\infty} w(x) > 0$ *, then* $v \cdot w \in W$

(6) $v(f(x)) \in W$, $h(v(x)) \in W$, $\sum_{i=1}^{n} \beta_i\, v_i(x)^{\alpha_i} \in W$, $\prod_{i=1}^{n} v_i(x)^{\alpha_i} \in W$

Proof: (1)–(5) are easy consequences of (3) and Lem.4.1. If $W = \mathrm{B}_+(\mathbf{R})$, then (6) holds trivially, and if $W \neq \mathrm{B}_+(\mathbf{R})$, then by (1) we have $W \subseteq W_\infty$. Hence, we see that (6) follows from (3)–(5) and Thm.3.3.3. $\square$

Lemma 4.3: *Let* $\mu, \nu \in \Pr(\mathbf{R})$ *be given probability measures and let* $f : \mathbf{R} \to \mathbf{R}$ *be a* μ*-measurable function. Then we have*

(1) $M_\mu \vdash \widehat{\rho}_\mu$ *and* $\widehat{\rho}_\mu \vdash M_\mu$

(2) $\nu \preceq \mu$ *if and only if* $\rho_\mu \models \rho_\nu$

(3) *If there exists* $\alpha, \beta > 0$ *such that* $|f(x)| \leq \alpha + \beta\,|x|$ μ*-a.s., then* $\mu_f \preceq \mu$

(4) *If there exists* $\alpha, \beta > 0$ *such that* $|x| \leq \alpha + \beta\,|f(x)|$ μ*-a.s., then* $\mu \preceq \mu_f$

Proof: (1): By Lem.2.1.1, we see that $\widehat{\rho}_\mu \vdash M_\mu$ and if we define $u(x) := \sqrt{\rho_\mu(x)}$, then by Lem.2.1.2+3 and Lem.2.2.3, we see that $M_\mu \vdash \widehat{u}$. By Lem.4.1, we have $\rho_\mu = u^2 \models u$ or equivalently $\widehat{u} \vdash \widehat{\rho}_\mu$ and since $M_\mu \vdash \widehat{u}$, we conclude that $M_\mu \vdash \widehat{\rho}_\mu$ and $\widehat{\rho}_\mu \vdash M_\mu$.

(2): Immediate consequence of (1).

(3): If $|f(x)| \leq \alpha + \beta|x|$ μ-a.s., then $\rho_\mu(\alpha + \beta\,x) \leq \rho_{\mu_f}(x)$ for all $x \in \mathbf{R}$. Hence, by Lem.4.1, we see that $\rho_\mu \models \rho_{\mu_f}$ and so by (2) we conclude that $\mu_f \preceq \mu$.

(4): If $|x| \leq \alpha + \beta|f(x)|$ μ-a.s., then $\rho_\mu(\alpha + \beta\,x) \geq \rho_{\mu_f}(x)$ for all $x \in \mathbf{R}$ and so by (2) and Lem.4.1 we conclude that $\mu_f \preceq \mu$. $\square$

Proposition 4.4: *Let* $w \in \mathrm{B}_+(\mathbf{R})$ *be a given function such that* $w(x_0) < \infty$ *for at least one* $x_0 \in \mathbf{R}$ *. Then there exist a probability measure* $\mu \in \Pr(\mathbf{R})$ *such that* $w \in L^1(\mu)$ *and* $\widehat{w} \vdash M_\mu$ *.*

Proof: Let us first suppose that $\lim_{|x|\to\infty} w(x) = \infty$ and let us define $h(x) := w_\epsilon(x) \vee 1$ for all $x > 0$ and

$$u_0(x) := \inf_{y \geq x} h(y) \;\; \forall\, x > 0 \;\; \text{and} \;\; u(x) := \sup_{0<y<x} u_0(y) = \lim_{y\uparrow x} u_0(y) \;\; \forall\, x > 0$$

Then $h:(0,\infty)\to[1,\infty]$ is a Borel function and u_0 and u are increasing functions on $(0,\infty)$ such that $1\le u(x)\le u_0(x)\le h(x)$ for all $x>0$ and u is left continuous on $(0,\infty)$. Since $\lim_{|x|\to\infty} w(x)=\infty$, we have $\lim_{|x|\to\infty} h(x)=\infty$ and so we see that $\lim_{|x|\to\infty} u(x)=\infty$. Hence, by (2.4) there exists a unique probability measure $\nu\in\Pr(\mathbf{R})$ such that $\nu(\mathbf{R}_+)=1$ and $\rho_\nu(x)=u(|x|)^2$ for all $x\ne 0$. Since u and h are Borel functions, we see that the set:

$$D:=\{\,(x,y)\mid y\ge \tfrac{x}{2}>0\ \text{ and }\ h(y)\le 1+u(x)\,\}$$

is a Borel subset of $\mathbf{R}^2$. Let $x>0$ be given. Then there exists $z\ge\frac{x}{2}$ such that $u_0(z)\le\frac{1}{2}+u(x)$ and since $u_0(z)=\inf_{y\ge z} h(y)$, there exists $y\ge z\ge\frac{x}{2}$ such that $h(y)\le\frac{1}{2}+u_0(z)\le 1+u(x)$. Hence, we see that the sections $D_x:=\{y\mid(x,y)\in D\}$ are non-empty for all $x>0$. So by the measurable selection theorem (see [15; Thm.2.2.11 p.348]) there exists a universally measurable function $\phi:(0,\infty)\to(0,\infty)$ such that $(x,\phi(x))\in D$ for all $x>0$ or equivalently $\phi(x)\ge\frac{x}{2}$ and $h(\phi(x))\le 1+u(x)$ for all $x>0$. Let $x_0\in\mathbf{R}$ be chosen such that $w(x_0)<\infty$ and let us define

$$\psi(x):=\begin{cases} x_0 & \text{if } x=0\\ \phi(|x|) & \text{if } x\ne 0 \text{ and } w(\phi(|x|))\le w(-\phi(|x|))\\ -\phi(|x|) & \text{if } x\ne 0 \text{ and } w(\phi(|x|))> w(-\phi(|x|))\end{cases}$$

Since w is a Borel function and ϕ is universally measurable, we see that $\psi:\mathbf{R}\to\mathbf{R}$ is universally measurable and by the definition of ϕ and ψ we have $|\psi(x)|=\phi(|x|)\ge\frac{|x|}{2}$ for all $x\ne 0$ and

$$w(\psi(x))=w_\epsilon(\phi(|x|))\le h(\phi(|x|))\le 1+u(|x|)=1+\sqrt{\rho_\nu(x)}\quad\forall\,x\ne 0$$

Since $w(\psi(0))=w(x_0)<\infty$, then by Lem.2.1.2, we have $w(\psi(x))\in L^1(\nu)$ or equivalently $w\in L^1(\mu)$ where $\mu:=\nu_\psi$ denotes the image measure of ν under the universally measurable function ψ . Since $|x|\le 2\,|\psi(x)|$ for all $x\in\mathbf{R}$, then by Lem.4.3.4 we have $\nu\preceq\mu$. So by Lem.4.3.1+2 we have $\widehat{\rho}_\nu\vdash M_\mu$ and since $\rho_\nu(x)=u(|x|)^2\le 1+w_\epsilon(x)^2\le 1+w(x)^2$ for all $x\ne 0$, then by Lem.4.1 we have $\rho_\nu\models w$ or equivalently $\widehat{w}\vdash\widehat{\rho}_\nu$. Recalling that $\widehat{\rho}_\nu\vdash M_\mu$, we conclude that $w\in L^1(\mu)$ and $\widehat{w}\vdash M_\mu$.

Suppose that $\liminf_{|x|\to\infty} w(x)<\infty$. Then there exist numbers $c>0$ and $x_1,x_2,\ldots\in\mathbf{R}$ such that $|x_n|\ge 2^n$ and $w(x_n)\le c$ for all $n=1,2,\ldots$. Let us define $\mu:=\frac{1}{a}\sum_{n=1}^\infty n^{-2}\,\delta_{x_n}$ where $a:=\sum_{n=1}^\infty n^{-2}$. Since $w(x_n)\le c$ and $|x_n|\ge 2^n$ for all $n\ge 1$, we have

$$\int_{\mathbf{R}} w\,d\mu\le\tfrac{c}{a}\ \text{ and }\ M_\mu(q)=\tfrac{1}{c}\sum_{n=1}^\infty n^{-2}\,|x_n|^q\ge\tfrac{1}{c}\sum_{n=1}^\infty n^{-2}\,2^{nq}=\infty$$

for all $q>0$. Hence, we see that $w\in L^1(\mu)$ and that $M_\mu(q)=\infty$ for all $q>0$ Thus, we have $\widehat{w}\vdash M_\mu$. $\square$

Theorem 4.5: *Let* $\mathfrak{P} \subseteq \Pr(\mathbf{R})$ *be a given non-empty set lower* $(\preceq)$*-directed of probability measures and let us define*

$$\mathcal{L}_{\mathfrak{P}} := \{ L \in \mathcal{M} \mid \exists \, \mu \in \mathfrak{P} \ : \ L \vdash M_\mu \} \ \ , \ \ W_{\mathfrak{P}} := \{ w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}_{\mathfrak{P}} \}$$

$$\mathrm{A}_{\mathfrak{P}} := \Big\{ w \in \mathrm{B}_+(\mathbf{R}) \ \Big| \ \exists \, \mu \in \mathfrak{P} \ : \ w(x) = \sum_{n=1}^{\infty} M_\mu(n)^{-1} \, |x|^n \ \ \forall \, x \in \mathbf{R} \Big\}$$

Then $\mathcal{L}_{\mathfrak{P}}$ *is lower* $(\preceq)$*-directed,* $W_{\mathfrak{P}}$ *is upper* $(\models)$*-directed and we have*

(1) $\quad \mathfrak{P}_{bd} \subseteq \mathfrak{P} = \{ \mu \in \Pr(\mathbf{R}) \mid M_\mu \in \mathcal{L}_{\mathfrak{P}} \}$, $\mathcal{L}_{bd} \subseteq \mathcal{L}_{\mathfrak{P}}$ *and* $W_{bd} \subseteq W_{\mathfrak{P}}$

(2) $\quad \mathrm{A}_{\mathfrak{P}}$ *and* $W_{\mathfrak{P}}$ *are complete sets of tests functions for* $\mathfrak{P}$ *and* $\mathrm{A}_{\mathfrak{P}} \subseteq W_{\mathfrak{P}} = \mathfrak{P}^*$

Let $\alpha > 0$ *be any given positive number and let* $\mu \in \Pr(\mathbf{R})$ *be a given probability measure. If* $\mathrm{A} \subseteq \mathrm{B}_+(\mathbf{R})$ *is any complete set of test function for* $\mathfrak{P}$*, then* $\mathrm{A} \subseteq W_{\mathfrak{P}}$ *and we have the following equivalence scheme:*

(3) $$\mu \in \mathfrak{P} \ \Leftrightarrow \ W_{\mathfrak{P}} \cap L^\alpha(\mu) \neq \emptyset \ \Leftrightarrow \ \mathrm{A} \cap L^1(\mu) \neq \emptyset \ \Leftrightarrow \ \rho_\mu(x)^\alpha \in W_{\mathfrak{P}}$$

$$\Leftrightarrow \ \exists \, w \in W_{\mathfrak{P}} \ \text{so that} \ \limsup_{|x| \to \infty} w(x)^\alpha \, R_\mu(|x|) < \infty$$

Proof: By the definition of $\mathcal{L}_{\mathfrak{P}}$ and $W_{\mathfrak{P}}$, we see that $\mathcal{L}_{\mathfrak{P}}$ is lower $(\vdash)$-directed, and that $W_{\mathfrak{P}}$ is upper $(\models)$-directed.

(1): Since $W_{\mathfrak{P}}$ is upper $(\models)$-directed, then by Cor.4.2.1, we have $W_{bd} \subseteq W_{\mathfrak{P}}$ and $\mathcal{L}_{bd} \subseteq \mathcal{L}_{\mathfrak{P}}$ and since $\mathfrak{P}$ is lower $(\preceq)$-directed, we have $\mathfrak{P}_{bd} \subseteq \mathfrak{P}$. Let $\mu \in \Pr(\mathbf{R})$ be a given probability measure satisfying $M_\mu \in \mathcal{L}_{\mathfrak{P}}$. Then there exists $\nu \in \mathfrak{P}$ such that $M_\mu \vdash M_\nu$ or equivalently $\mu \preceq \nu$ and since $\mathfrak{P}$ is lower $(\preceq)$-directed, we have $\mu \in \mathfrak{P}$ and since $M_\mu \in \mathcal{L}_{\mathfrak{P}}$ for all $\mu \in \mathfrak{P}$, we have proved (1).

(2): Let $w \in W_{\mathfrak{P}}$ and $\mu \in \Pr_w$ be given. Since $w \in L^1(\mu)$, then by Lem.2.1.3 and Lem.2.2.3 there exists $C > 0$ such that $M_\mu(q) \leq C \, \widehat{w}(q)$ for all $q > 0$. Hence, we have $M_\mu \vdash \widehat{w}$ and since $\widehat{w} \in \mathcal{L}_{\mathfrak{P}}$ and $\mathcal{L}_{\mathfrak{P}}$ is lower directed, we conclude that $M_\mu \in \mathcal{L}_{\mathfrak{P}}$. So by (1) we have $\mu \in \mathfrak{P}$ for all $\mu \in \Pr_w$; that is, $W_{\mathfrak{P}} \subseteq \mathfrak{P}^*$. Let $w \in \mathfrak{P}^*$ be given. If $w \in W_{bd}$, then by (1) we have $w \in W_{\mathfrak{P}}$. So suppose that $w \notin W_{bd}$. By Thm.3.3.2 and Prop.4.4, there exists a probability measure $\mu \in \Pr_w$ such that $\widehat{w} \vdash M_\mu$. Since $w \in \mathfrak{P}^*$ and $\mu \in \Pr_w$, we have $\mu \in \mathfrak{P}$ and $M_\mu \in \mathcal{L}_{\mathfrak{P}}$ and since $\widehat{w} \vdash M_\mu$ and $\mathcal{L}_{\mathfrak{P}}$ is lower $(\vdash)$-directed, we see that $\widehat{w} \in \mathcal{L}_{\mathfrak{P}}$ or equivalently $w \in W_{\mathfrak{P}}$ for all $w \in \mathfrak{P}^*$. In view of the inclusion proved above, we conclude that $\mathfrak{P}^* = W_{\mathfrak{P}}$.

Let $w \in \mathrm{A}_{\mathfrak{P}}$ be given. Then there exists $\mu \in \mathfrak{P}$ such that $w(x) = \sum_{n=1}^\infty M_\mu(n)^{-1} |x|^n$ for all $x \in \mathbf{R}$. Let $n \geq 1$ be a given integer. Since $w(x) \vee 1 \geq |x|^n \, M_\mu(n)^{-1}$ for all $x \in \mathbf{R}$, we see that $\widehat{w}(n) \leq M_\mu(n)$ for all $n \in \mathbf{N}$. Let $q \geq 1$ be a given number and let $n \geq 1$ denote the unique integer satisfying $n \leq q < n+1$. Since $n+1 \leq 2n \leq 2q$ and M_μ and $\widehat{w}$ belong to $\mathcal{M}$, we have

$$\widehat{w}(q)^{\frac{1}{q}} \leq \widehat{w}(n+1)^{\frac{1}{n+1}} \leq M_\mu(n+1)^{\frac{1}{n+1}} \leq M_\mu(2q)^{\frac{1}{2q}}$$

for all $q \geq 1$. Hence, we have $\widehat{w} \vdash M_\mu \in \mathcal{L}_{\mathfrak{P}}$ and since $\mathcal{L}_{\mathfrak{P}}$ is lower directed, we have $\widehat{w} \in \mathcal{L}_{\mathfrak{P}}$ or equivalently $w \in W_{\mathfrak{P}}$; that is, $\mathrm{A}_{\mathfrak{P}} \subseteq W_{\mathfrak{P}} = \mathfrak{P}^*$. Let $\mu \in \mathfrak{P}$ be a given and let ν denote the image measure of μ under the function $x \curvearrowright 2x$. Since $\mathfrak{P}$ is lower $(\preceq)$-directed, then by Lem.4.3.3 we have $\nu \in \mathfrak{P}$. Hence, if we define $w(x) := \sum_{n=1}^\infty M_\nu(n)^{-1}|x|^n$, then $w \in \mathrm{A}_{\mathfrak{P}}$ and since $M_\nu(q) = 2^q\, M_\mu(q)$, we have

$$\int_{\mathbf{R}} w\, d\mu = \sum_{n=1}^\infty M_\mu(n)\, M_\nu(n)^{-1} \leq \sum_{n=1}^\infty 2^{-n} = 1$$

So we have $w \in \mathrm{A}_{\mathfrak{P}} \cap L^1(\mu)$ and since $\mathrm{A}_{\mathfrak{P}} \subseteq W_{\mathfrak{P}} = \mathfrak{P}^*$, then by (1.4) we conclude that $\mathrm{A}_{\mathfrak{P}}$ and $W_{\mathfrak{P}}$ are complete set of test functions for $\mathfrak{P}$ which completes the proof of (2).

(3): Since $\mathrm{A} \subseteq \mathrm{B}_+(\mathbf{R})$ is a complete set of tests functions for $\mathfrak{P}$, then by (2) we have $\mathrm{A} \subseteq W_{\mathfrak{P}}$ and since $W_{\mathfrak{P}}$ is upper $(\models)$-directed, then by Cor.4.2 we see that $W_{\mathfrak{P}} \cap L^1(\mu) \neq \emptyset$ if and only if $W_{\mathfrak{P}} \cap L^\alpha(\mu) \neq \emptyset$. Hence, we see that the first equivalence in (3) follows from (2), and the second equivalence follows directly from the definition of completeness. Since $W_{\mathfrak{P}}$ is upper $(\models)$-directed and $\mathcal{L}_{\mathfrak{P}}$ is lower $(\vdash)$-directed, then by (1), Cor.4.2 and Lem.4.3.1 we have

(*) $\quad \rho_\mu(x)^\alpha \in W_{\mathfrak{P}} \Leftrightarrow \rho_\mu(x) \in W_{\mathfrak{P}} \Leftrightarrow \widehat{\rho}_\mu \in \mathcal{L}_{\mathfrak{P}} \Leftrightarrow M_\mu \in \mathcal{L}_{\mathfrak{P}} \Leftrightarrow \mu \in \mathfrak{P}$

which proves the third equivalence in (3). If $\rho_\mu(x)^\alpha \in W_{\mathfrak{P}}$, then by Lem.4.1 we have $\rho_\mu(x)^{1/\alpha} \in W_{\mathfrak{P}}$ and so we the last condition in (3) holds. Suppose that the last condition in (3) holds. By Lem.4.1 we have $w(x)^\alpha \models \rho_\mu(x)$ and since $W_{\mathfrak{P}}$ is upper directed, then by (*) and Cor.4.2 we see that $\mu \in \mathfrak{P}$ which completes the proof of (3). $\square$

Theorem 4.6: *Let $\mathcal{L} \subseteq \mathcal{M}$ be a non-empty lower $(\vdash)$-directed set and let us define*

$$\mathfrak{P}_{\mathcal{L}} := \{\, \mu \in \mathrm{Pr}(\mathbf{R}) \mid M_\mu \in \mathcal{L}\,\} \;\;,\;\; W_{\mathcal{L}} := \{\, w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}\}$$

$$\mathrm{A}_{\mathcal{L}} := \Big\{\, w \in \mathrm{B}_+(\mathbf{R}) \;\Big|\; \exists\, \mu \in \mathcal{P}_{\mathcal{L}} \;:\; w(x) = \sum_{n=1}^\infty M_\mu(n)^{-1}\, |x|^n \;\; \forall\, x \in \mathbf{R}\Big\}$$

Then $\mathfrak{P}_{\mathcal{L}}$ is lower $(\preceq)$-directed, $W_{\mathcal{L}}$ is upper $(\models)$-directed and we have

(1) $\quad \mathfrak{P}_{bd} \subseteq \mathfrak{P}_{\mathcal{L}}$, $\mathcal{L}_{bd} \subseteq \mathcal{L} = \{\, L \in \mathcal{M} \mid \exists\, \mu \in \mathfrak{P}_{\mathcal{L}} \;:\; L \vdash M_\mu\}$ *and* $W_{bd} \subseteq W_{\mathcal{L}}$

(2) $\quad \mathrm{A}_{\mathcal{L}}$ *and* $W_{\mathcal{L}}$ *are complete sets of tests functions for* $\mathfrak{P}_{\mathcal{L}}$ *and* $\mathrm{A}_{\mathcal{L}} \subseteq W_{\mathcal{L}} = \mathfrak{P}_{\mathcal{L}}^*$

Let $\alpha > 0$ be any given positive number and let $\mu \in \mathrm{Pr}(\mathbf{R})$ be a given probability measure. If $\mathrm{A} \subseteq \mathrm{B}_+(\mathbf{R})$ is any complete set of test function for $\mathfrak{P}_{\mathcal{L}}$, then $\mathrm{A} \subseteq W_{\mathcal{L}}$ and we have the following equivalence scheme:

(3) $\quad \mu \in \mathfrak{P}_{\mathcal{L}} \Leftrightarrow W_{\mathcal{L}} \cap L^\alpha(\mu) \neq \emptyset \Leftrightarrow \mathrm{A} \cap L^1(\mu) \neq \emptyset \Leftrightarrow \rho_\mu(x)^\alpha \in W_{\mathcal{L}}$

$$\Leftrightarrow \exists\, w \in W_{\mathcal{L}} \text{ so that } \limsup_{|x|\to\infty} w(x)^\alpha\, R_\mu(|x|) < \infty$$

Proof: Since $\mathcal{L}$ is lower $(\vdash)$-directed, we see that $\mathfrak{P}_{\mathcal{L}}$ is non-empty and lower $(\preceq)$-directed. Let $\mu \in \mathfrak{P}_{\mathcal{L}}$ and $L \in \mathcal{M}$ be given such that $L \vdash M_\mu$. Since $M_\mu \in \mathcal{L}$ and $\mathcal{L}$ is lower $(\vdash)$-directed, we have $L \in \mathcal{L}$. Let $L \in \mathcal{L}$ be given and let me show that $L \vdash M_\mu$ for some $\mu \in \mathfrak{P}_{\mathcal{L}}$. If $L(q) = 0$ for some $q > 0$, then by (2.2) we see that this holds trivially. If $L \notin \mathcal{L}_\infty$, then there exists $r > 0$ such that $L(q) = \infty$ for all $q \geq r$ and since $\mathcal{L}$ is lower $(\vdash)$-directed, we have $\mathcal{L} = \mathcal{M}$ and $\mathfrak{P}_{\mathcal{L}} = \mathrm{Pr}(\mathbf{R})$ and so the claim holds trivially (take μ to be any probability measure with $M_\mu \equiv \infty$). So suppose that $0 < L(q) < \infty$ for all $q > 0$ let us define $v(x) := \sup_{q>0} |x|^q \, L(q)^{-1}$ for all $x \in \mathbf{R}$. Since $v(x)^{-1} \leq |x|^{-q} \, L(q)$ for all $x \in \mathbf{R}$, we see that $\widehat{v}(q) \leq L(q)$ for all $q > 0$. Let $q > 0$ be given. Since $f(r) := \log L(r)$ is finite and convex on $(0, \infty)$, we have $f(r) \geq \alpha r + \beta$ for all $r > 0$ and $f(q) = \alpha q + \beta$ where $\alpha := f'(q)$ the right hand derivative of f at q and $\beta = f(q) - \alpha q$. Since $\frac{1}{r} \, f(r)$ is increasing on $(0, \infty)$, we have $f(r) - f(q) \geq \frac{r-q}{q} \, f(q)$ for all $r > q$. Hence, we have $\alpha q \geq f(q)$ or equivalently $\beta \leq 0$ and so we have $e^{-\beta} \geq 1$ and

$$v(e^\alpha) = \sup_{r>0} e^{\alpha r} \, L(r)^{-1} = \sup_{r>0} e^{\alpha r - f(r)} \leq e^{-\beta}$$

$$\widehat{v}(q) \geq e^{\alpha q} \, (v(e^\alpha) \vee 1)^{-1} \geq e^{\alpha q + \beta} = e^{f(q)} = L(q) \geq \widehat{v}(q)$$

Thus, we see that $\widehat{v}(q) = L(q)$ for all $q > 0$. Since $L(q)^{-1} > 0$ for all $q > 0$, we have $\lim_{x \to \infty} v(x) = \infty$ and since v is even lower semicontinuous on $\mathbf{R}$ and increasing on $[0, \infty)$ with $v(0) = 0$, then by (2.4) there exists $\mu \in \mathrm{Pr}(\mathbf{R})$ such that $\rho_\mu(x) = v(x) \vee 1$ for all $x \in \mathbf{R}$. Hence, we have $\widehat{\rho}_\mu(q) = \widehat{v}(q) = L(q)$ for all $q > 0$ and so by Lem.4.3.1 we have $L \vdash M_\mu$ and $M_\mu \vdash L$. Since $\mathcal{L}$ is lower $(\vdash)$-directed, we have $M_\mu \in \mathcal{L}$ or equivalently $\mu \in \mathfrak{P}_{\mathcal{L}}$ which proves the claim. Hence, we see that $\mathfrak{P}_{\mathcal{L}}$ is a non-empty lower $(\preceq)$-directed set of probability measures such that $L \in \mathcal{M}$ if and only if $L \vdash M_\mu$ for some $\mu \in \mathfrak{P}_{\mathcal{L}}$. Hence, we see that the remaining parts of the theorem follows from Thm.4.5 applied the set $\mathfrak{P} := \mathfrak{P}_{\mathcal{L}}$. $\square$

Theorem 4.7: *Let* $W \subseteq \mathrm{B}_+(\mathbf{R})$ *be a non-empty upper* $(\models)$*-directed set and let us define*

$$\mathcal{L}_W := \{ L \in \mathcal{M} \mid \exists \, w \in W \, : \, L \vdash \widehat{w} \} \ , \ \mathfrak{P}_W := \{ \mu \in \mathrm{Pr}(\mathbf{R}) \mid M_\mu \in \mathcal{L}_W \}$$

$$\mathrm{A}_W := \Big\{ w \in \mathrm{B}_+(\mathbf{R}) \ \Big| \ \exists \, \mu \in \mathfrak{P}_W \, : \, w(x) = \sum_{n=1}^{\infty} M_\mu(n)^{-1} \, |x|^n \ \ \forall \, x \in \mathbf{R} \Big\}$$

Then $\mathfrak{P}_W$ *is lower* $(\preceq)$*-directed,* $\mathcal{L}_W$ *is lower* $(\vdash)$*-directed and we have*

(1) $\mathfrak{P}_{bd} \subseteq \mathfrak{P}_W$, $\mathcal{L}_{bd} \subseteq \mathcal{L}_W$ *and* $W_{bd} \subseteq W = \{ w \in \mathrm{B}_+(\mathbf{R}) \mid \widehat{w} \in \mathcal{L}_W \}$

(2) A_W *and* W *are complete sets of tests functions for* $\mathfrak{P}_W$ *and* $\mathrm{A}_W \subseteq W = \mathfrak{P}_W^*$

Let $\alpha > 0$ *be any given positive number and let* $\mu \in \mathrm{Pr}(\mathbf{R})$ *be a given probability measure. If* $\mathrm{A} \subseteq \mathrm{B}_+(\mathbf{R})$ *is any complete set of test function for* $\mathfrak{P}_W$ *, then* $\mathrm{A} \subseteq W$ *and we have the following equivalence scheme:*

$$(3) \qquad \mu \in \mathfrak{P}_W \;\Leftrightarrow\; W \cap L^\alpha(\mu) \neq \emptyset \;\Leftrightarrow\; \mathrm{A} \cap L^1(\mu) \neq \emptyset \;\Leftrightarrow\; \rho_\mu(x)^\alpha \in W$$
$$\Leftrightarrow\; \exists\, w \in W \text{ so that } \limsup_{|x|\to\infty} w(x)^\alpha R_\mu(|x|) < \infty$$

Proof: By the definition of $\mathcal{L}_W$, we see that $\mathcal{L}_W$ is non-empty and lower $(\vdash)$-directed. Let $w \in \mathrm{B}_+(\mathbf{R})$ be a given function such that $\widehat{w} \in \mathcal{L}_W$. Then there exists $v \in W$ such that $\widehat{w} \vdash \widehat{v}$ or equivalently $v \models w$ and since W is upper $(\models)$-directed, we see that $w \in W$. Hence, we see that $\mathcal{L}_W$ is a non-empty lower $(\vdash)$-directed set such that $\widehat{w} \in \mathcal{L}_W$ if and only if $w \in W$ and so we see that the remaining parts of theorem follows from Thm.4.6 with $\mathcal{L} := \mathcal{L}_W$. ☐

Theorem 4.8: *If "x" stands for one of the five symbols "bd" or "∞" or "P" or "R" or "C" and* A_x *is defined as follows:*

$$\mathrm{A}_x := \Big\{ w \in \mathrm{B}_+(\mathbf{R}) \;\Big|\; \exists\, \mu \in \mathcal{P}_x \text{ so that } w(x) = \sum_{n=1}^{\infty} M_\mu(n)^{-1} |x|^n \;\; \forall\, x \in \mathbf{R} \Big\}$$

then $\mathcal{L}_x$ *is lower* $(\vdash)$*-directed,* $\mathfrak{P}_x$ *is lower* $(\preceq)$*-directed,* W_x *is upper* $(\models)$*-directed and we have*

(1) A_x *and* W_x *are complete sets of test functions for* $\mathfrak{P}_x$ *and* $\mathrm{A}_x \subseteq W_x = \mathfrak{P}_x^*$

(2) *The set* $\mathrm{B}_P := \{e^{\alpha|x|} \mid \alpha > 0\}$ *is a complete set of test functions for* $\mathfrak{P}_P$

(3) *The set* B_R *of all functions of the form* $w(x) = \sum_{n \in L} \frac{|\alpha x|^n}{n!}$ *for some* $\alpha > 0$ *and some infinite set* $L \subseteq \mathbf{N}$ *is a complete set of test functions for* $\mathfrak{P}_R$

Let $\alpha > 0$ *be any given positive number and let* $\mu \in \mathrm{Pr}(\mathbf{R})$ *be a given probability measure. If* $\mathrm{A} \subseteq \mathrm{B}_+(\mathbf{R})$ *is any complete set of test function for* $\mathfrak{P}_x$ *, then* $\mathrm{A} \subseteq W_x$ *and we have the following equivalence scheme:*

$$(4) \qquad \mu \in \mathfrak{P}_x \;\Leftrightarrow\; W_x \cap L^\alpha(\mu) \neq \emptyset \;\Leftrightarrow\; \mathrm{A} \cap L^1(\mu) \neq \emptyset \;\Leftrightarrow\; \rho_\mu(x)^\alpha \in W_x$$
$$\Leftrightarrow\; \exists\, w \in W_x \text{ so that } \limsup_{|x|\to\infty} w(x)^\alpha R_\mu(|x|) < \infty$$

Remarks: Note that Thm.4.5–4.8 meets our objective classifying sets of probability measures which admits a complete set of test functions and of finding decent complete sets of test function for the sets $\mathfrak{P}_x$ for $\mathcal{P}_x$ when "x" equals one of the symbols "bd" or "∞" or "P" or "R" or "C". Recall that $\mathfrak{P}_P \subseteq \mathfrak{P}_R \subseteq \mathfrak{P}_C$ and by (1) and by Example 3.5 we see that each of these inclusions are strict.

Proof: Let $M, L \in \mathcal{M}$ be given functions such that $L \vdash M$. Then there exist positive numbers $\alpha > 0$, $C \geq 1$ and $r > \frac{1}{\alpha} + 1$ such that $L(\alpha q) \leq C^{\alpha q}\,(1 + M(q)^{\alpha})$ for all $q \geq r$. Since $\alpha r > 1$, we have $L(\alpha q)^{1/(\alpha q)} \leq C\,(1 + M(q)^{1/q})$ for all $q \geq r$ and since $L(q)^{1/q}$ is increasing on $(0, \infty)$, we see that $M \in \mathcal{L}_x$ implies $L \in \mathcal{L}_x$ whenever "x" equals "bd" or "∞" or "P" or "R". Hence, we see that the sets $\mathcal{L}_{bd}$, $\mathcal{L}_{\infty}$, $\mathcal{L}_P$ and $\mathcal{L}_R$ are lower $(\vdash)$-directed. So suppose that $M \in \mathcal{L}_C$. If $M \in \mathcal{L}_{bd}$, then by Cor.3.2.3 we have $L \in \mathcal{L}_{bd} \subseteq \mathcal{L}_C$. So suppose that $M \notin \mathcal{L}_{bd}$. Since $M(q)^{1/q}$ is increasing and unbounded on $(0, \infty)$ there exists $u \geq 1 + r$ such that $M(q) \geq 1$ for all $q \geq u$. Hence, we have $L(\alpha q)^{1/(\alpha q)} \leq 2C\,M(q)^{1/q}$ for all $q \geq u$ and since $M \in \mathcal{L}_C$, then by Thm.3.1.3 we have

$$\alpha \int_u^{\infty} L(\alpha q)^{-\frac{1}{\alpha q}}\,dq \geq \tfrac{\alpha}{2C} \int_u^{\infty} M(q)^{-\frac{1}{q}}\,dq = \infty$$

Applying Thm.3.1.3 once more, we see that $L \in \mathcal{L}_C$ and so we see that $\mathcal{L}_C$ is lower $(\vdash)$-directed.

Thus, we see that $\mathcal{L}_x$ is lower $(\vdash)$-directed and by Cor.3.1.1 we have that $\mathcal{P}_x = \{\mu \in \Pr(\mathbf{R}) \mid M_\mu \in \mathcal{L}_x\}$. Hence, we see that (1) and (4) follow from Thm.4.6, and that W_x is upper $(\models)$-directed and $\mathfrak{P}_x$ is lower $(\preceq)$-directed.

(2): Follows from (1.3).

(3): By (1) and Example 3.5.3+4, we see that $\mathrm{B}_R \subseteq W_R = \mathfrak{P}_R^*$. Let $\mu \in \mathfrak{P}_R$ be given. Since $M_\mu \in \mathcal{L}_R$, then by Lem.3.1.2 there exist a positive number $\alpha > 0$ and an infinite set $L \subseteq \mathbf{N}_0$ such that $M_\mu(n) \leq \alpha^{-n}\,n!$ for all $n \in L$. Hence, if we define $w(x) := \sum_{n \in L} \frac{|\alpha x|^n}{n!}$, then $w \in \mathrm{B}_R$ and we have

$$\int_{\mathbf{R}} w\,d\mu = \sum_{n \in L} M_\mu(n)\,\tfrac{\alpha^n}{n!} \leq \sum_{n \in L} \tfrac{1}{n!} \leq e < \infty$$

Thus, we see that $w \in \mathrm{B}_R \cap L^1(\mu)$ which completes the proof of (3). $\square$

Theorem 4.9: *Let $\mathfrak{P} \subseteq \Pr(\mathbf{R})$ be a non-empty lower $(\preceq)$-directed set of probability measures and let us define $W := \mathfrak{P}^*$. Let $\mu \in \Pr(\mathbf{R})$ be a probability measure and let ν be a Borel measure on $\mathbf{R}$ such that $d\mu = \phi \cdot d\nu$ for some Borel function $\phi : \mathbf{R} \to [0, \infty]$. If we define $S := \{x \in \mathbf{R} \mid 0 < \phi(x) < \infty\}$, then we have*

(1) $\mu \in \mathfrak{P} \;\Leftrightarrow\; \exists\,\alpha, c, p, q > 0\;\exists\,\beta \in \mathbf{R}\;\exists\,w \in W \;:\; \int_{|x| \geq c} \frac{w(\alpha x + \beta)^q}{1 + |x|^p}\,\mu(dx) < \infty$

(2) *If $\phi(x)^{-\alpha} \in L^1(\mu)$ and $\phi(x)^{-\beta} \in W$ for some $\alpha, \beta > 0$, then $\mu \in \mathfrak{P}$*

(3) *If $1_S(x)\,(1 + |x|)^{-p} \in L^1(\nu)$ and $\phi(x)^{-\beta} \in W$ for some $p, \beta > 0$, then $\mu \in \mathfrak{P}$*

(4) *If $\nu \in \mathfrak{P}$ and $\limsup_{|x| \to \infty} |x|^{-p}\,\phi(x) < \infty$ for some $p > 0$, then $\mu \in \mathfrak{P}$*

Let $(J_x \mid x \in \mathbf{R})$ *be a family of Borel subsets of* $\mathbf{R}$ *and let* $C\,,\,a\,,\,p\,,\,q\,,\,\delta > 0$ *and be positive numbers satisfying*

(5) $\quad J_x \subseteq [\delta x, \infty)\ \ \forall\, x > 0\ \ ,\ \ J_x \subseteq (-\infty, \delta x]\ \ \forall\, x < 0$ *and* $\liminf_{|x|\to\infty} |x|^p\, \nu(J_x) > 0$

(6) $\quad \left(\frac{\phi(x)}{1+\phi(x)}\right)^q \leq C\,\phi(y) \quad \forall\, y \in S \cap J_x\ \forall\, |x| \geq a$

Then we have

(7) $\quad \mu \in \mathfrak{P} \quad \Rightarrow \quad \phi(x)^{-\alpha} \in W \quad \forall\, \alpha > 0$

Remarks: (a): Suppose that ν is the Lebesgue measure on $\mathbf{R}$. Then we have $(1+|x|)^{-p} \in L^1(\nu)$ for all $p > 1$ and if $0 < \delta < 1$ is any given number and $J_x := [\delta x, x]$ if $x \geq 0$ and $J_x := [x, \delta x]$ if $x \leq 0$, then ν satisfies condition (5) for all $p > 0$. Hence, if $\phi(x)^{-\beta} \in W$ for some $\beta > 0$, then by (2) we have $\mu \in \mathfrak{P}$ and if ϕ satisfies condition (6) with this choice of (J_x) (for instance, there exists $b > 0$ such that ϕ is decreasing on $[b, \infty)$ and increasing on $(-\infty, -b]$), then by (7) we have that $\mu \in \mathfrak{P}$ implies $\phi(x)^{-\alpha} \in W$ for all $\alpha > 0$.

(b): Let $D \subseteq \mathbf{R}$ be a countable set and let $\nu := \sum_{x \in L} \delta_x$ denote the counting measure on D . Then we have $\sum_{x\in D} \phi(x) = 1$ and $\mu = \sum_{x \in D} \phi(x)\, \delta_x$. If $\sum_{x\in D} (1+|x|)^{-p} < \infty$ for some $p > 0$ and $\phi(x)^{-\beta} \in W$ for some $\beta > 0$, then by (3) we have $\mu \in \mathfrak{P}$. Suppose that $D = \{a_n \mid n \in \mathbf{Z}\}$ where $(a_n \mid n = 0, \pm 1, \pm 2, \ldots)$ is a strictly increasing sequence satisfying

(7) $\quad \lim_{n\to\infty} a_n = \infty\ ,\ \lim_{n\to-\infty} a_n = -\infty\ ,\ \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty\ ,\ \limsup_{n\to-\infty} \frac{a_n}{a_{n+1}} < \infty$

Then it follows easily that condition (5) holds for some $0 < \delta < 1$ with $p = 1$ and $J_x := [\delta x, x]$ if $x \geq 0$ and $J_x := [x, \delta x]$ if $x \leq 0$. Hence, if the density ϕ satisfies condition (6) with this choice of (J_x) (for instance, there exists $b > 0$ such that ϕ is decreasing on $[b, \infty)$ and increasing on $(-\infty, -b]$), then by (7) we have that $\mu \in \mathfrak{P}$ implies $\phi(x)^{-\alpha} \in W$ for all $\alpha > 0$.

(c): If $\mathfrak{P} := \mathfrak{P}_x$ where "x" denotes on of the symbols "bd" or "P" or "R" or "C" or "∞", then by Thm.4.8 we have $\mathfrak{P}^* = W_x$. In particular, we see that W_x satisfies stability conditions of Cor.4.2 and in view of the remarks above and the and results of Section 3, we see that the theorem provide a powerful method of verifying or falsifying the statement " $\mu \in \mathfrak{P}_x$ ".

Proof: (1): Let $\alpha, c, p, q > 0$, $\beta \in \mathbf{R}$ and $w \in W$ be given and let us define $v(x) := (1+|x|^p)^{-1}\, w(\alpha x + \beta)^q$ for $|x| > c$ and $v(x) = 0$ for $|x| \leq c$. Since $w(\alpha x + \beta) = (1+|x|^p)^{1/q}\, v(x)^{1/q}$ for all $|x| > c$, then by Lem.4.1 we have $w \models v$ and since $\mathfrak{P}$ is lower $(\preceq)$-directed, then by Thm.4.5 we have that W is an upper $(\models)$-directed complete set of test functions for $\mathfrak{P}$. Hence, we see that $v \in W$ and that (1) holds.

(2): Immediate consequence of (1).

(3): Since $1_S(x)\,(1+|x|)^{-p} \in L^1(\nu)$, we have $(1+|x|)^{-p}\,\phi(x)^{-1} \in L^1(\mu)$ and since $\phi(x)^{-\beta} \in W$, we see that (3) follows from (1).

(4): Suppose that $M_\nu(r) = \infty$ for some $r > 0$. Then $M_\mu(q) = \infty$ for all $q \geq r$ and so we see that $L \vdash M_\nu$ for all $L \in \mathcal{M}$. Hence, by Thm.4.5.1 we have $\mathfrak{P} = \Pr(\mathbf{R})$ and so we have $\mu \in \mathfrak{P}$. Suppose that $M_\nu(r) < \infty$ for all $r > 0$. By assumption, there exist $c, p > 0$ such that $\phi(x) \leq c\,|x|^p$ for all $|x| \geq c$. Hence, by Cauchy-Schwartz' inequality, we have

$$
\begin{aligned}
M_\mu(q) &\leq c^q + \int_{|x1\geq c} |x|^q\,\phi(x)\,\nu(dx) \leq c^q + c\int_{|x1\geq c} |x|^{p+q}\,\nu(dx) \\
&\leq c^q + c\,M_\nu(2q)^{1/2}\,M_\nu(p)^{1/2}
\end{aligned}
$$

for all $q > 0$ and since $M_\nu(p) < \infty$, there exists $C > 1$ such that $M_\mu(q) \leq C^q\,(1 + M_\nu(2q)^{1/2})$ for all $q \geq 1$. Hence, we see that $M_\mu \vdash M_\nu$ or equivalently $\mu \preceq \nu$ and since $\nu \in \mathfrak{P}$ and $\mathfrak{P}$ is lower $(\preceq)$-directed, we conclude that $\mu \in \mathfrak{P}$.

(7): Let us define $\phi_0(x) := \frac{\phi(x)}{1+\phi(x)}$ and $\psi(x) := \phi_0(x)^q + \phi_0(-x)^q$ for all $x \in \mathbf{R}$. By (5), there exist $\gamma > 0$ and $b > a$ such that $\nu(J_x) \geq \gamma\,|x|^{-p}$ for all $|x| \geq b$. Let $x \geq b$ be given. By (5) and (6) we have

$$
\begin{aligned}
C\,R_\mu(\delta x) &\geq \int_{J_x} C\phi\,d\nu + \int_{J_{-x}} C\phi\,d\nu \geq \phi_0(x)^q\,\nu(J_x) + \phi_0(-x)^q\,\nu(J_{-x}) \\
&\geq \gamma\,x^{-p}\,(\phi_0(x)^q + \phi_0(-x)^q) = \gamma\,x^{-p}\,\psi(x)
\end{aligned}
$$

and since ψ is an even function such that $\psi \geq \phi_0$, we have

$$
\begin{aligned}
\rho_\mu(\delta x) &= R_\mu(|\delta x|)^{-1} \leq \tfrac{C}{\gamma}\,|x|^p\,\psi(x)^{-1} \leq \tfrac{C}{\gamma}\,|x|^p\,\phi_0(x)^{-q} \\
&= \tfrac{C}{\gamma}\,|x|^p\,\big(1 + \phi(x)^{-1}\big)^q \leq 2^q\,\tfrac{C}{\gamma}\,|x|^p\,\big(1 + \phi(x)^{-q}\big)
\end{aligned}
$$

for all $|x| \geq b$. So by Lem.4.1 we see that $\rho_\mu \models \phi(x)^{-\alpha}$ for all $\alpha > 0$ and since $\mu \in \mathfrak{P}$ and $\mathfrak{P}$ is lower $(\preceq)$-directed, then by Thm.4.5 we have $\rho_\mu \in W$ and that W is upper $(\models)$-directed. Hence, we see that $\phi(x)^{-\alpha} \in W$ for all $\alpha > 0$. $\square$

Corollary 4.10: *Let $\mu \in \Pr(\mathbf{R})$ be a given probability measure and let ν be a Borel measure on $\mathbf{R}$ such that $d\mu = \phi \cdot d\nu$ for some Borel function $\phi : \mathbf{R} \to [0,\infty]$ and $(1+|x|)^{-r} \in L^1(\nu)$ for some $r > 0$. Let $a > 0$ be a given number and let $f : [a,\infty) \to [0,\infty]$ be a log-exp-convex function satisfying*

(1) $\quad f(a) < \infty$ *and* $\displaystyle\int_a^\infty \frac{\log^+ f(x)}{1+x^2}\,dx = \infty$

Let $c\,,\,p\,,\,q\,,\,\alpha>0$ *and* $\beta\in\mathbf{R}$ *be given numbers. Then we have*

(2) *If* $\int_{|x|>c}\frac{w(\alpha x+\beta)^q}{1+|x|^p}\,\mu(dx)<\infty$ *for some* $w\in W_C$ *, then* $\mu\in\mathfrak{P}_D$

(3) *If* $\phi(x)^{-p}\in W_C$ *, then* $\mu\in\mathfrak{P}_C\subseteq\mathfrak{P}_D$

(4) *If* $\phi(x)\wedge 1\leq c\,(1+|x|^q)\,f(|x|)^{-p}$ *for all* $|x|>a$ *, then* $\mu\in\mathfrak{P}_C\subseteq\mathfrak{P}_D$

Suppose that $\mu([0,\infty))=1$ *and let* $h:[a,\infty)\to[0,\infty)$ *be a log-exp-convex function satisfying*

(6) $h(a)<\infty$ *and* $\int_a^\infty\frac{\log^+h(x)}{1+x^{3/2}}\,dx=\infty$

Then we have

(7) *If* $\sum_{n=1}^{\infty}M_\mu(n)^{-\frac{1}{2n}}=\infty$ *, then* $\mu\in\mathfrak{P}_D$

(8) *If* $\int_c^\infty\frac{w\left(\sqrt{x}\right)^q}{1+|x|^p}\,\mu(dx)<\infty$ *for some* $w\in W_C$ *, then* $\mu\in\mathfrak{P}_D$

(9) *If* $\phi(x^2)^{-p}\in W_C$ *, then* $\mu\in\mathfrak{P}_D$

(10) *If* $\phi(x^2)\wedge 1\leq c\,(1+|x|^q)\,f(x)^{-p}$ *for all* $x>a$ *, then* $\mu\in\mathfrak{P}_D$

(11) *If* $\phi(x)\wedge 1\leq c\,(1+|x|^q)\,h(x)^{-p}$ *for all* $x>a$ *, then* $\mu\in\mathfrak{P}_D$

Remark: Let $a>0$ be given number and let $f:(a,\infty)\to(0,\infty)$ be given function. Since f is log-exp-convex on (a,∞) if and only if $\log f(e^x)$ is convex on $(\log a,\infty)$, it follows easily that f is log-exp-convex on (a,∞) if and only if f is absolutely continuous with an a.e. derivative f' satisfying $x\curvearrowright\frac{x\,f'(x)}{f(x)}$ is increasing on (a,∞). In particular, we see the so-called *Lin condition* in [18] is equivalent to log-exp-convexity of $\phi(x)^{-1}$ on (a,∞) for some $a>0$. Since (4) and (10) hold with $f(x)=h(x):=\phi(x)^{-1}$ and the Lebesgue measure λ satisfies the integrability condition $(1+|x|)^{-r}\in L^1(\lambda)$ for all $r>1$, we see that the corollary extends the results in [18]. Moreover, we see that the results in [18] holds whenever the density is taken with respect to a Borel measure ν satisfying $(1+|x|)^{-r}\in L^1(\nu)$ for some $r>0$ and the density $\phi(x)$ satisfies one of the hypotheses of (2)–(4) or (7)–(10). In particular, we see that the results of the corollary (and of [18]) applies to discrete probability measures supported by a countable set D satisfying $\sum_{x\in D}(1+|x|)^{-r}<\infty$ for some $r>0$ (see [12] and [13] for more information about determinacy of discrete measures).

Proof: (1) and (2) are immediate consequences of Thm.4.8 and Thm.4.9.1+2. Let us define $v(x):=f(|x|)$ if $|x|>a$ and $v(x):=0$ if $|x|\leq a$. By (1) and (2.2), we see that $0<f(x)\leq\infty$ for all $x\geq a$ and that $J:=\{x\geq a\mid f(x)<\infty\}$ is an interval containing a . Suppose that J is unbounded and let $b > a$ be given. If $a < x < b$, then we have $x = a^{1-\lambda} b^{\lambda}$ for some $0 < \lambda < 1$ and since $J = [a, \infty)$ and f is log-exp-convex, we have

$$0 < f(x) \le f(a)^{1-\lambda} f(b)^{\lambda} \le (1 + f(a))\,(1 + f(b)) < \infty \quad \forall\, a < x < b$$

Hence, if J is unbounded, then by (1) we have $\int_\beta^\infty \frac{\log^+ f(x)}{1+x^2}\, dx = \infty$ for all $\beta \ge a$ and if J is bounded this holds trivially. So, by Lem.2.5.7 and Thm.3.3.7 we see that $v \in W_C$ and if $\phi(x) \wedge 1 \le c\,(1 + |x|^q)\, f(|x|)^{-p}$ for all $|x| > a$, then we have

$$v(x)^p = f(|x|)^p \le c\,(1 + |x|^q)\,(u(x) \vee 1) \le c\,(1 + |x|^q)\,(1 + u(x)) \;\; \forall\, |x| > a$$

where $u(x) := \phi(x)^{-1}$. Since W_C is ($\models$)-directed (see Thm.4.8), then by Lem.4.1 we have $u \in W_C$ and so we see that (3) follows from (2).

(7): Suppose that $\mu([0, \infty)) = 1$ and that $\sum_{n=1}^\infty M_\mu(n)^{-\frac{1}{2n}} = \infty$. Let $\mu_1(B) = \mu(B - 1)$ denote image measure of μ under the map $x \curvearrowright x + 1$. If $\mu \in \mathfrak{P}_{bd}$, then (7) holds trivially. So suppose that $\mu \notin \mathfrak{P}_{bd}$. Then we have $\lim_{q\to\infty} M_\mu(q) = \infty$ and so there exists $k \ge 1$ such that $M_\mu(q) \ge 1$ for all $q \ge 1$. Let $n \ge k$ be a given integer. Since $x \curvearrowright (1 + x)^n$ is convex on $[0, \infty)$, we have $(1 + x)^n \le 2^{n-1}(1 + x^n)$ for all $x \ge 0$ and since $M_\mu(n) \ge 1$ and $\mu([0, \infty)) = 1$, we have

$$M_{\mu_1}(n) = \int_{\mathbf{R}} (1 + x)^n\, \mu(dx) = \int_0^\infty (1 + x)^n\, \mu(dx) \le 2^{n-1}(1 + M_\mu(n)) \le 2^n\, M_\mu(n)$$

Hence, we have

$$\sum_{n=1}^\infty M_{\mu_1}(n)^{-\frac{1}{2n}} \ge \sum_{n=k}^\infty M_{\mu_1}(n)^{-\frac{1}{2n}} \ge \frac{1}{\sqrt{2}} \sum_{n=k}^\infty M_\mu(n)^{-\frac{1}{2n}} = \infty$$

So by Thm.1.11 in [16; p.20] we have that μ_1 is Stieltjes determined and since $\mu_1([1, \infty)) = 1$, then by [6; Corollary p.481] we have that $\mu_1 \in \mathfrak{P}_D$ but then it follows easily that $\mu \in \mathfrak{P}_D$.

(8)–(11): Suppose that $\mu([0, \infty)) = 1$. Then we have $d\mu = \phi \cdot d\nu_0$ where $\nu_0(B) := \nu(B \cap [0, \infty))$ for all $B \in \mathcal{B}(\mathbf{R})$. Let κ and ξ denote the image measures of μ and ν_0 under the map $x \curvearrowright \sqrt{|x|}$. Since $d\mu = \phi \cdot d\nu_0$, we have $\kappa(dx) = \phi(x^2)\,\xi(dx)$ and since $(1 + |x|)^{-p} \in L^1(\nu)$, it follows easily that we have $(1 + |x|)^{-2p} \in L^1(\xi)$. Since

$$\int_{\mathbf{R}} u(x)\,\kappa(dx) = \int_0^\infty u(x)\,\kappa(dx) = \int_0^\infty u\big(\sqrt{x}\,\big)\,\mu(dx) \quad \forall\, u \in \mathrm{B}_+(\mathbf{R})$$

then by (1)–(3) we see that $\kappa \in \mathfrak{P}_C$ under each of the hypotheses in (8)–(10). Since h is log-exp-convex on $[a, \infty)$, then so is $g(x) := h(x^2)$ on $[\sqrt{a}, \infty)$ and a simple substitution shows that g satisfies (1) with a replaced by $\sqrt{a}$. Hence, by (3) we see that $\kappa \in \mathfrak{P}_C$ under each of the hypotheses in (8)–(11) and since $M_\mu(q) = M_\kappa(2q)$ for all $q > 0$, then by Thm.3.1.3 we have

$$\sum_{n=1}^\infty M_\mu(n)^{-\frac{1}{2n}} = \sum_{n=1}^\infty M_\kappa(2n)^{-\frac{1}{2n}} = \infty$$

So by (7) we conclude that $\mu \in \mathfrak{P}_D$ under each of the hypotheses in (8)–(11). $\square$

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