Maximum Process Problems in Optimal Control Theory

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Abstract

Given a standard Brownian motion $(B_t)_{t \geq 0}$ and the equation of motion:

$$dX_t = v_t \, dt + \sqrt{2} \, dB_t$$

we set $S_t = \max_{0 \leq s \leq t} X_s$ and consider the optimal control problem:

$$\sup_v E\big(S_\tau - c\tau\big)$$

where $c > 0$ and the supremum is taken over all admissible controls v satisfying $v_t \in [\mu_0, \mu_1]$ for all t up to $\tau = \inf\{\, t > 0 \mid X_t \notin (\ell_0, \ell_1)\,\}$ with $\mu_0 < 0 < \mu_1$ and $\ell_0 < 0 < \ell_1$ given and fixed. The following control v^* is proved optimal: 'Pull as hard as possible' that is $v_t^* = \mu_0$ if $X_t < g_*(S_t)$, and 'push as hard as possible' that is $v_t^* = \mu_1$ if $X_t > g_*(S_t)$, where $s \mapsto g_*(s)$ is a switching curve that is determined explicitly (as a unique solution to a nonlinear equation). The solution found demonstrates that the problem formulations based on a maximum functional can be successfully included into optimal control theory (calculus of variations) in addition to the classic problem formulations due to Lagrange, Mayer and Bolza.

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