Lévy Laws and Processes in Free Probability

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Introduction.

In this note we outline certain aspects of the theory of infinite divisibility and Lévy processes in the framework of Voiculescu's free probability. Our studies are concentrated around the bijection, introduced by Bercovici and Pata in [7], between the class of classically infinitely divisible probability measures and the class of freely infinitely divisible probability measures. We derive, in Section 5, certain algebraic and topological properties of this bijection, in the present paper denoted Λ, and explain how these properties imply that Λ maps certain canonical subclasses of classically infinitely divisible probability measures onto their natural free counterparts. We show also, in Section 6, how Λ, by virtue of the afore-mentioned properties, gives rise to a one-to-one (in law) correspondence between classical and free Lévy processes. In Section 7 we use the properties of Λ to construct certain stochastic integrals w.r.t. free Lévy processes, and we derive the free counterpart of the well-known integral representation of classically selfdecomposable random variables. Finally, in Section 8, we describe a free version of the Lévy-Itô decomposition of a classical Lévy process into the sum of two independent processes: a Brownian motion (with drift) and a pure jump process. Sections 1-2 provide background material on non-commutative probability in general and free probability in particular. Sections 3-4 review, briefly, the theory of convolution and infinite divisibility in free (and classical) probability. The theory, outlined in sections 5-8, is developed, in detail, in the forthcoming papers [5] and [6].

1 Non-commutative Probability.

In classical probability, the basic objects of study are random variables, i.e. measurable functions from a probability space $(\Omega, \mathcal{F}, P)$ into the real numbers $\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}$. To any such random variable $X : \Omega \to \mathbb{R}$ there is associated a probability measure μ_X on $(\mathbb{R}, \mathcal{B})$, defined by $\mu_X(B) = P(X \in B) = P(X^{-1}(B))$, for any Borel set B. The measure μ_X is called the distribution of X (w.r.t. P), and it satisfies the property that:

$$\int_{\mathbb{R}} f(t)\ \mu_X(dt) = \mathbb{E}(f(X)),$$

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for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$, and where $\mathbb{E}$ denotes expectation (or integration) w.r.t. P. We shall also use the notation $L\{X\}$ for μ_X (where L stands for "law").

In *non-commutative* probability, one replaces the random variables by (selfadjoint) operators on a Hilbert space $\mathcal{H}$. These operators are then referred to as "non–commutative random variables". The term non-commutative refers to the fact that, in this setting, the multiplication of "random variables" (i.e. composition of operators) is no longer commutative, as opposed to the usual multiplication of classical random variables. The non-commutative situation is often remarkably different from the classical one, and most often more complicated. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of all bounded operators on $\mathcal{H}$. Recall that $\mathcal{B}(\mathcal{H})$ is equipped with an involution (the adjoint operation) $a \mapsto a^*\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, which is given by:

$$\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle, \quad (a \in \mathcal{B}(\mathcal{H}),\ \xi, \eta \in \mathcal{H}).$$

Instead of working with the whole algebra $\mathcal{B}(\mathcal{H})$ as the set of "random variables" under consideration, it is, for most purposes, natural to restrict attention to certain subalgebras of $\mathcal{B}(\mathcal{H})$. In this note, we shall only consider the nicest cases of such subalgebras, the *von Neumann algebras*, although much of what follows is also valid for more general classes of "non-commutative probability spaces". A von Neumann algebra, acting on a Hilbert space $\mathcal{H}$, is a subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the multiplicative unit $\mathbf{1}$ of $\mathcal{B}(\mathcal{H})$ (i.e. $\mathbf{1}$ is the identity mapping on $\mathcal{H}$), and which is closed under the adjoint operation and closed in the weak operator topology on $\mathcal{B}(\mathcal{H})$ (i.e. the weak topology on $\mathcal{B}(\mathcal{H})$ induced by the linear functionals: $a \mapsto \langle a\xi, \eta\rangle,\ \xi, \eta \in \mathcal{H}$). A tracial state on a von Neumann algebra $\mathcal{A}$ is a positive linear functional $\tau\colon \mathcal{A} \to \mathbb{C}$, taking the value 1 at the identity operator $\mathbf{1}$ on $\mathcal{H}$, and satisfying the trace property[1]:

$$\tau(ab) = \tau(ba), \quad (a, b \in \mathcal{A}).$$

1.1 Definition. A W^*-probability space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$ and τ is a faithful tracial state on $\mathcal{A}$.

The assumed faithfulness of τ in Definition 1.1 means that τ does not annihilate any non-zero positive operator. It implies that $\mathcal{A}$ is finite in the sense of F. Murray and J. von Neumann.

Suppose now that $(\mathcal{A}, \tau)$ is a W^*-probability space and that a is a selfadjoint operator (i.e. $a^* = a$) in $\mathcal{A}$. Then, as in the classical case, we can associate a (spectral) distribution to a in a natural way: Indeed, by the Riesz representation theorem, there exists a unique probability measure μ_a on $(\mathbb{R}, \mathcal{B})$, satisfying that

$$\int_{\mathbb{R}} f(t)\ \mu_a(dt) = \tau(f(a)), \tag{1.1}$$

for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$. In formula (1.1), $f(a)$ has the obvious meaning if f is a polynomial. For general Borel functions f, $f(a)$ is defined in terms of spectral theory (see e.g. [22]).

The (spectral) distribution μ_a of a selfadjoint operator a in $\mathcal{A}$ is automatically concentrated on the spectrum $\mathrm{sp}(a)$, and is thus, in particular, compactly supported. If one wants to be able to consider any probability measure μ on $\mathbb{R}$ as the spectral distribution of some selfadjoint

[1] In quantum physics, τ is of the form $\tau(a) = \mathrm{tr}(\rho a)$, where ρ is a trace class selfadjoint operator on $\mathcal{H}$ with trace 1, that expresses the state of a quantum system, and a would be an observable, i.e. a selfadjoint operator on $\mathcal{H}$, the mean value of the outcome of observing a being $\tau(a) = \mathrm{tr}\{\rho a\}$.

operator, then it is necessary to take unbounded (i.e. non-continuous) operators into account. Such an operator a is, generally, not defined on all of $\mathcal{H}$, but only on a subspace $\mathcal{D}(a)$ of $\mathcal{H}$, called the domain of a. We say then that a is an operator *in* $\mathcal{H}$ rather than *on* $\mathcal{H}$. For most of the interesting examples, $\mathcal{D}(a)$ is a dense subspace of $\mathcal{H}$, in which case a is said to be densely defined.

If $(\mathcal{A}, \tau)$ is a W^*-probability space acting on $\mathcal{H}$ and a is an unbounded operator in $\mathcal{H}$, a cannot be an element of $\mathcal{A}$. The closest a can get to $\mathcal{A}$ is to be *affiliated* with $\mathcal{A}$, which means that a commutes with any unitary operator u, that commutes with all elements of $\mathcal{A}$. If a is selfadjoint, a is affiliated with $\mathcal{A}$ if and only if $f(a) \in \mathcal{A}$ for any bounded Borel function $f: \mathbb{R} \to \mathbb{R}$. In this case, (1.1) determines, again, a unique probability measure μ_a on $\mathbb{R}$, which we also refer to as the (spectral) distribution of a, and which generally has unbounded support. Furthermore, any probability measure on $\mathbb{R}$ can be realized as the (spectral) distribution of some selfadjoint operator affiliated with some W^*-probability space. In the following we shall also use the notation $L\{a\}$ for the distribution of a (possibly unbounded) operator a affiliated with $(\mathcal{A}, \tau)$.

Although there is no problem in defining the distribution of a single selfadjoint operator affiliated with $(\mathcal{A}, \tau)$, one major difficulty in non-commutative probability is the lack of a notion of the joint distribution of two non-commuting operators. This fact means that many arguments from classical probability cannot, directly, be carried over to the non-commutative case.

2 Free independence.

The key concept on relations between classical random variables X and Y is *independence*. One way of defining that X and Y are independent is to ask that all compositions of X and Y with bounded Borel functions be uncorrelated:

$$\mathbb{E}\big\{[f(X) - \mathbb{E}\{f(X)\}] \cdot [g(Y) - \mathbb{E}\{g(Y)\}]\big\} = 0,$$

for any bounded Borel functions $f, g: \mathbb{R} \to \mathbb{R}$.

In the early 1980's, D.V. Voiculescu introduced the notion of *free independence* among non-commutative random variables:

2.1 Definition. Let $a_1, a_2, \dots, a_r$ be selfadjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. We say then that $a_1, a_2, \dots, a_r$ are *freely independent* w.r.t. τ, if

$$\tau\big\{[f_1(a_{i_1}) - \tau(f_1(a_{i_1}))][f_2(a_{i_2}) - \tau(f_2(a_{i_2}))] \cdots [f_p(a_{i_p}) - \tau(f_p(a_{i_p}))]\big\} = 0,$$

for any p in $\mathbb{N}$, any bounded Borel functions $f_1, f_2, \dots, f_p: \mathbb{R} \to \mathbb{R}$ and any indices $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$ satisfying that $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{p-1} \neq i_p$.

At a first glance, the definition of free independence looks, perhaps, quite similar to the definition of classical independence given above, and indeed, in many respects free independence is conceptually similar to classical independence. For example, if $a_1, a_2, \dots, a_r$ are freely independent selfadjoint operators in $(\mathcal{A}, \tau)$, then all numbers of the form $\tau\{f_1(a_{i_1}) f_2(a_{i_2}) \cdots f_p(a_{i_p})\}$ (where $i_1, i_2, \dots, i_p \in \{1, 2, \dots, r\}$ and $f_1, f_2, \dots, f_p: \mathbb{R} \to \mathbb{R}$ are bounded Borel functions), are uniquely determined by the distributions $L\{a_i\}$, $i = 1, 2, \dots, r$. On the other hand, free independence is a truly non-commutative notion, which can be seen, for instance, from the fact that two classical random variables are never freely independent, unless one of them is trivial, i.e. constant with probability one. Indeed, suppose X and Y are classical random variables, and assume, for simplicity, that X and Y are both bounded with mean 0. If X and Y are freely independent w.r.t. expectation $\mathbb{E}$, then, since X and Y commute, we have

$$0 = \mathbb{E}\{XYXY\} = \mathbb{E}\{X^2Y^2\} = \mathbb{E}\{X^2\}\mathbb{E}\{Y^2\},$$

where the last equality is due to the fact that X^2 and Y^2 are necessarily freely independent too. The calculation above implies that either $\mathbb{E}\{X^2\} = 0$ or $\mathbb{E}\{Y^2\} = 0$, as asserted.

Voiculescu originally introduced free independence in connection with his deep studies of the von Neumann algebras associated to the free group factors (see [25], [27], [28]). We prefer in this note, however, to indicate the significance of free independence by explaining its connection with random matrices. In the 1950's, the phycisist E.P. Wigner showed that the spectral distribution of large selfadjoint random matrices with independent complex Gaussian entries is, approximately, the semi-circle distribution, i.e. the distribution on $\mathbb{R}$ with density $s \mapsto \sqrt{4-s^2} \cdot 1_{[-2,2]}(s)$ w.r.t. Lebesgue measure. More precisely, for each n in $\mathbb{N}$, let $X^{(n)}$ be a selfadjoint complex Gaussian random matrix of the kind considered by Wigner (and suitably normalized), and let tr_n denote the (usual) tracial state on the $n \times n$ matrices $M_n(\mathbb{C})$. Then for any positive integer p, Wigner showed that

$$\mathbb{E}\big\{\mathrm{tr}_n\big[(X^{(n)})^p\big]\big\} \underset{n\to\infty}{\longrightarrow} \int_{-2}^{2} s^p\sqrt{4-s^2}\, ds.$$

In the late 1980's, Voiculescu generalized Wigner's result to families of independent selfadjoint Gaussian random matrices (cf. [27]): For each n in $\mathbb{N}$, let $X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)}$ be independent[2] random matrices of the kind considered by Wigner. Then for any indices $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$,

$$\mathbb{E}\big\{\mathrm{tr}_n\big[X_{i_1}^{(n)}X_{i_2}^{(n)}\cdots X_{i_p}^{(n)}\big]\big\} \underset{n\to\infty}{\longrightarrow} \tau\{x_{i_1}x_{i_2}\cdots x_{i_p}\},$$

where $x_1, x_2, \dots, x_r$ are freely independent selfadjoint operators in a W^*-probability space $(\mathcal{A}, \tau)$, and such that $L\{x_i\}$ is the semi-circle distribution for each i.

By Voiculescu's result, free independence describes what the assumed classical independence between the random matrices is turned into, as $n \to \infty$. Also, from a classical probabilistic point of view, free probability theory may be considered as (an aspect of) the probability theory of large random matrices.

Voiculescu's result reveals another general fact in free probability, namely that the role of the Gaussian distribution in classical probability is taken over by the semi-circle distribution in free probability. In particular, as also proved by Voiculescu, the limit distribution appearing in the free version of the central limit theorem is the semi-circle distribution (see e.g. [24]).

3 Classical and Free Convolution.

In classical probability, the convolution $\mu_1 * \mu_2$ of two probability measures μ_1 and μ_2 on $\mathbb{R}$ is defined as the distribution of the sum $X_1 + X_2$ of two independent random variables X_1 and X_2 with distributions μ_1 respectively μ_2. The existence of two independent random variables

[2]in the classical sense; at the level of the entries.

X_1 and X_2 defined on the same probability space and with prescribed distributions μ_1 and μ_2 follows from a tensor-product construction. In free probability, the corresponding existence result follows from a similar construction, where the tensor-product is replaced by the so-called free product (we refer to [24] for details). Furthermore, as previously indicated, if x_1 and x_2 are freely independent selfadjoint operators with spectral distributions μ_1 and μ_2, the distribution $L\{x_1 + x_2\}$ depends only on μ_1 and μ_2. Hence, it makes sense to define the free convolution $\mu_1 \boxplus \mu_2$ of μ_1 and μ_2 by setting $\mu_1 \boxplus \mu_2 = L\{x_1 + x_2\}$. Once the free convolution $\boxplus$ has, thus, been defined, one could, from a probabilistic point of view, forget about the underlying operator construction, and merely consider $\boxplus$ as a new type of convolution on the set of probability measures on $\mathbb{R}$. To a large extent, this approach can, in fact, be followed through by virtue of the analytical function tools that we describe next.

The main tool for dealing with classical convolution is the Fourier transform. The Fourier transform (or characteristic function) of a probability measure μ on $\mathbb{R}$ is the function $f_\mu \colon \mathbb{R} \to \mathbb{C}$ given by:

$$f_\mu(u) = \int_{\mathbb{R}} e^{isu}\ \mu(ds), \quad (u \in \mathbb{R}).$$

The key property of the Fourier transform, in this connection, is that

$$f_{\mu_1 * \mu_2}(u) = f_{\mu_1}(u) \cdot f_{\mu_2}(u), \quad (u \in \mathbb{R}),$$

for any probability measures μ_1, μ_2 on $\mathbb{R}$. Thus, the logarithm of the Fourier transform (the so-called cumulant transform) linearizes classical convolution. In the paper [26], Voiculescu found a transformation which linearizes free convolution; the so-called R-transform. Since then, several modifications of Voiculescu's R-transform have appeared in the literature. We prefer, here, to work with what we shall refer to as the free cumulant transform, which we introduce next.

By $\mathbb{C}^+$ (respectively $\mathbb{C}^-$) we denote the strictly upper (respectively strictly lower) complex half-plane. For a probability measure μ on $\mathbb{R}$, the *Cauchy transform* $G_\mu \colon \mathbb{C}^+ \to \mathbb{C}^-$ is defined by:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\ \mu(dt), \quad (z \in \mathbb{C}^+).$$

It turns out that the mapping $F_\mu := \frac{1}{G_\mu} \colon \mathbb{C}^+ \to \mathbb{C}^+$ always has a right inverse, F_μ^{-1}, defined on a region of the form: $\Gamma(\eta, M) = \{x + iy \in \mathbb{C}^+ \mid x^2 + y^2 > M^2,\ |x| < \eta y\}$, where η and M are positive numbers (see [9]). The free cumulant transform $\mathcal{C}_\mu$ is then defined by

$$\mathcal{C}_\mu(z) = zF_\mu^{-1}(\tfrac{1}{z}) - 1,$$

for $\frac{1}{z}$ in $\Gamma(\eta, M)$, i.e. for z in the region $\{x - iy \in \mathbb{C}^- \mid x^2 + y^2 < M^{-2},\ |x| < \eta y\}$. As indicated above, the key property of the free cumulant transform is that

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}_{\mu_1}(z) + \mathcal{C}_{\mu_2}(z),$$

for any probability measures μ_1, μ_2 on $\mathbb{R}$.

The reason we have chosen to work with the free cumulant transform, rather than the R-transform or other modifications of it, is that this particular modification is especially close in nature to the classical cumulant transform. In particular, it behaves exactly like the classical cumulant transform w.r.t. scalar-multiplication, which is important for the discussion of *free selfdecomposability*, introduced in [5] (see Section 4 below). Indeed, if μ is the distribution of a random variable X and $c \geq 0$, then, denoting by $D_c\mu$ the distribution of cX, we have the relation:

$$\mathcal{C}_{D_c\mu}(z) = \mathcal{C}_\mu(cz). \tag{3.1}$$

Furthermore, in terms of $\mathcal{C}_\mu$ the free Lévy-Khintchine representation of freely infinitely divisible probability measures resembles more closely the classical Lévy-Khintchine representation, as we shall see in Section 5 below.

4 Infinite Divisibility, Selfdecomposability and Stability.

In classical probability theory one has the following hierarchy of classes of probability measures on $\mathbb{R}$:

$$\mathcal{G}(*) \subset \mathcal{S}(*) \subset \mathcal{L}(*) \subset \mathcal{ID}(*) \subset \mathcal{P}$$

where

(i) $\mathcal{P}$ is the class of all probability measures on $\mathbb{R}$.

(ii) $\mathcal{ID}(*)$ is the class of infinitely divisible probability measures on $\mathbb{R}$, i.e.

$$\mu \in \mathcal{ID}(*) \iff \forall n \in \mathbb{N}\ \exists \mu_n \in \mathcal{P}\colon \mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ terms}}.$$

(iii) $\mathcal{L}(*)$ is the class of selfdecomposable probability measures on $\mathbb{R}$, i.e.

$$\mu \in \mathcal{L}(*) \iff \forall c \in]0,1[\ \exists \mu_c \in \mathcal{P}\colon \mu = D_c\mu * \mu_c.$$

(iv) $\mathcal{S}(*)$ is the class of stable probability measures on $\mathbb{R}$, i.e.

$$\mu \in \mathcal{S}(*) \iff \begin{array}{l}\{\psi(\mu) \mid \psi\colon \mathbb{R} \to \mathbb{R}, \text{ increasing affine transformation}\} \\ \text{is closed under convolution } * .\end{array}$$

(v) $\mathcal{G}(*)$ is the class of Gaussian (or normal) distributions on $\mathbb{R}$.

The classes of probability measures defined above, are all of great importance in classical probability. This is, partly, explained by their characterizations as limit distributions of certain types of sums of independent random variables (see e.g. [16] or [14]).

In free probability, we denote by $\mathcal{ID}(\boxplus)$, $\mathcal{L}(\boxplus)$ and $\mathcal{S}(\boxplus)$ the classes of, respectively, freely infinitely divisible, freely selfdecomposable and freely stable probability measures on $\mathbb{R}$. These classes are defined exactly as the corresponding classical classes, except that one replaces classical convolution $*$ by free convolution $\boxplus$ throughout in (ii)-(iv) above. Furthermore, we shall denote by $\mathcal{G}(\boxplus)$ the class of free Gaussian distributions, i.e. that of semi-circle distributions. It turns out, then, that in free probability, we also have the hierarchy:

$$\mathcal{G}(\boxplus) \subset \mathcal{S}(\boxplus) \subset \mathcal{L}(\boxplus) \subset \mathcal{ID}(\boxplus) \subset \mathcal{P}.$$

The first inclusion is well-known and easily verified, and the second one is not hard to prove by application of the free cumulant transform. The third inclusion is of deeper nature. As in the classical case, it is a consequence of the fact that the infinitely divisible distributions may be characterized as the possible limit distributions, as $n \to \infty$, of sums $S_n = X_{n,1} + \cdots + X_{n,k_n}$ of (freely) independent random variables, such that the terms $X_{n,1}, \ldots, X_{n,k_n}$ are uniformly negligible (in probability) as $n \to \infty$. The latter result was proved in 1937 by Khintchine (cf. [19]) in the classical case, and recently by Bercovici and Pata in the free case (cf. [8]). Based on this, it remains to remark that by successive applications of (iii) above, a measure μ in $\mathcal{L}(\boxplus)$ can be considered as the (fixed) distribution of sums S_n of the kind described above (see [5] for details).

We shall focus, here, mostly on the class $\mathcal{L}(*)$ of selfdecomposable measures (and its free counterpart), which, until fairly recently, seemed to be more or less forgotten, except by a few experts, even though it did receive considerable attention in the early studies of infinite divisibility. It was first introduced as a class of limit distributions by P. Lévy and is now playing a substantial role in mathematical finance (see the contribution by Barndorff-Nielsen and Shephard in [3]).

A random variable Y has distribution in $\mathcal{L}(*)$ if and only if Y has a representation in law of the form[3]

$$Y \stackrel{\mathrm{d}}{=} cY + Y_c,$$

for some random variable Y_c which is independent of Y. This latter formulation makes the idea of selfdecomposability of immediate appeal from the viewpoint of mathematical modeling. Yet another key characterization is given by the following result which was first proved by Wolfe in [30] and later generalized and strengthened by Jurek and Verwaat in [18]: A random variable Y has law in $\mathcal{L}(*)$ if and only if Y has a representation in the form

$$Y \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-t}\, dX_t, \tag{4.1}$$

where X_t is a Lévy process (see Section 6 below) satisfying $\mathbb{E}\{\log(1 + |X_1|)\} < \infty$. The process $X = (X_t)_{t \geq 0}$ is termed the *background driving Lévy process* or the BDLP corresponding to Y; this is due to its role for processes of Ornstein-Uhlenbeck type (see [2]). One of the main results, to be outlined in the present paper, is a representation in the form (4.1), for any selfadjoint operator y with (spectral) distribution in $\mathcal{L}(\boxplus)$ (see Section 7).

5 The Bercovici-Pata bijection.

We present next a bijection between the classes $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$, which was introduced by Bercovici and Pata in [7]. The bijection is defined in terms of the Lévy-Khintchine representations of classical and free infinitely divisible probability measures.

In the classical case, a famous result, due to Lévy and Khintchine (who build on initial work by Kolmogorov) states that a probability measure μ on $\mathbb{R}$ is in $\mathcal{ID}(*)$ if and only if its characteristic function f_μ has a representation in the form:

$$\log f_\mu(u) = i\gamma u - \tfrac{1}{2}au^2 + \int_{\mathbb{R}} \left(e^{iut} - 1 - iut1_{[-1,1]}(t)\right)\, \rho(dt), \quad (u \in \mathbb{R}), \tag{5.1}$$

[3]The symbol "$\stackrel{\mathrm{d}}{=}$" means "has the same distribution as".

where γ is a real constant, a is a non-negative constant and ρ is a measure on $\mathbb{R}$ satisfying the conditions:

$$\rho(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, t^2\}\ \rho(dt) < \infty,$$

i.e. ρ is a *Lévy measure*. The triplet (a, ρ, γ) is uniquely determined and is called the generating triplet for μ.

The free version of the Lévy-Khintchine representation was proved, in the general case, by Bercovici and Voiculescu in [9]. Reformulated in terms of the free cumulant transform (rather than the Voiculescu transform), it asserts that a probability measure μ on $\mathbb{R}$ is in $\mathcal{ID}(\boxplus)$ if and only if its free cumulant transform $\mathcal{C}_\mu$ has a representation in the form:

$$\mathcal{C}_\nu(z) = \gamma z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\ \rho(dt), \tag{5.2}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure. Again, the triplet (a, ρ, γ) is uniquely determined, and we call it the free generating triplet of μ.

There are several alternative ways of writing the Lévy-Khintchine representations. We have chosen, in this note, to use the representations (5.1) and (5.2), since they are very similar, and since (5.1) seems to be the preferred representation in recent literature (see e.g. [23]). In addition, both (5.1) and (5.2) clearly exhibit how μ is always the convolution of a Gaussian distribution (respectively a semi-circle distribution) and a distribution of generalized Poisson type (cf. also the Lévy-Itô decomposition in Section 8). In particular, the cumulant transform for the Gaussian distribution with mean γ and variance a is: $u \mapsto i\gamma u - \frac{1}{2}au^2$, and the free cumulant transform for the semi-circle distribution with mean γ and variance a is $z \mapsto \gamma z + az^2$.

5.1 Definition. The Bercovici-Pata bijection is the mapping $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ defined in the following way: Suppose μ is in $\mathcal{ID}(*)$ and has generating triplet (a, ρ, γ). Then $\Lambda(\mu)$ is the measure in $\mathcal{ID}(\boxplus)$ with free generating triplet (a, ρ, γ).

From the characterizations of $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ in terms of the Lévy-Khintchine representations, it is immediate that Λ is, in fact, a bijection. However formal Λ may seem at a first glance, it is clear from the definition that Λ maps the Gaussian distributions onto the semi-circle distributions. Furthermore, it was proved by Bercovici and Pata in [7] that Λ actually preserves stability[4], i.e. $\Lambda(\mathcal{S}(*)) = \mathcal{S}(\boxplus)$. When investigating the corresponding question for selfdecomposability we realized that, in fact, Λ has the following algebraic properties:

5.2 Theorem. *The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$, satisfies:*

(i) *If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.*

(ii) *If $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$.*

The proof of (i) is actually a straightforward consequence of the fact that the classical (respectively free) cumulant transform linearizes classical (respectively free) convolution. To prove (ii), one has to verify that the operation D_c has the exact same effect on classical and free generating

[4]Bercovici and Pata actually proved an even stronger result, namely that Λ preserves the so-called partial domain of attraction.

triplets. By virtue of (3.1) (and the corresponding classical result) and the striking similarity between (5.1) and (5.2), this ends up being a straightforward observation too.

Together with the, easily checked, property that all Dirac measures are fixed points of Λ, Theorem 5.2 shows that Λ preserves the affine structure on $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$. This provides another explanation of the fact that Λ preserves stability, and it also shows that the same holds for selfdecomposability, i.e. that $\Lambda(\mathcal{L}(*)) = \mathcal{L}(\boxplus)$. Indeed, suppose that $\mu \in \mathcal{L}(*)$ and that $c \in]0,1[$. Then $\mu = D_c\mu * \mu_c$ for some probability measure μ_c. It is a well-known fact that μ_c is automatically in $\mathcal{ID}(*)$ (see [16]) and hence, by Theorem 5.2,

$$\Lambda(\mu) = \Lambda(D_c\mu * \mu_c) = D_c\Lambda(\mu) \boxplus \Lambda(\mu_c),$$

which shows that $\Lambda(\mu) \in \mathcal{L}(\boxplus)$. The same argumentation applies to the converse inclusion.

In the paper [5] we also studied the topological properties of Λ. Recall that a sequence (σ_n) of finite measures on $\mathbb{R}$ is said to converge weakly to a finite measure σ on $\mathbb{R}$ if $\int_{\mathbb{R}} f(s)\ \sigma_n(ds) \to \int_{\mathbb{R}} f(s)\ \sigma(ds)$ for any continuous bounded function $f\colon \mathbb{R} \to \mathbb{R}$. In that case we write $\sigma_n \xrightarrow{\text{w}} \sigma$.

5.3 Theorem. *The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is a homeomorphism w.r.t. weak convergence. In other words, if (μ_n) is a sequence of measures in $\mathcal{ID}(*)$ and μ is another measure in $\mathcal{ID}(*)$, then $\mu_n \xrightarrow{\text{w}} \mu$ if and only if $\Lambda(\mu_n) \xrightarrow{\text{w}} \Lambda(\mu)$.*

The proof of Theorem 5.3 is based on a result by B.V. Gnedenko (see [16]), which gives a necessary and sufficient condition for weak convergence in $\mathcal{ID}(*)$ in terms of the generating triplets. Suppose (μ_n) is a sequence in $\mathcal{ID}(*)$ with corresponding generating triplets (a_n, ρ_n, γ_n), and suppose that μ is another measure in $\mathcal{ID}(*)$ with generating triplet (a, ρ, γ). Consider the finite measures σ_n and σ on $\mathbb{R}$ defined by:

$$\sigma(dt) = a\delta_0(dt) + \tfrac{t^2}{1+t^2}\rho(dt) \quad \text{and} \quad \sigma_n(dt) = a_n\delta_0(dt) + \tfrac{t^2}{1+t^2}\rho_n(dt), \quad (n \in \mathbb{N}),$$

where δ_0 is the Dirac measure at 0. If we assume, for simplicity, that the Lévy measure ρ for μ has no atoms at -1 or 1 (the discontinuity points of the integrands in (5.1) and (5.2)), then Gnedenko's result asserts that $\mu_n \xrightarrow{\text{w}} \mu$, if and only if $\sigma_n \xrightarrow{\text{w}} \sigma$ and $\gamma_n \to \gamma$. The remaining task, then, is to prove a free version of Gnedenko's result. This can be done by fairly standard measure theoretic techniques, based on the description of weak convergence in terms of the free cumulant transform, which was established by Bercovici and Voiculescu in [9]. We refer to [5] for details.

6 Lévy processes in Free Probability.

In classical probability, Lévy processes form a very important area of research, both from theoretical and applied points of view (see [23],[10],[11],[20],[3]). In free probability, such processes have already received quite a lot of attention (see e.g. [1], [12] and [13]).

6.1 Definition. A *free Lévy process (in law)*, affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, is a process $(Z_t)_{t\geq 0}$ of selfadjoint operators affiliated with $\mathcal{A}$, which satisfies the following conditions:

(i) whenever $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the increments

$$Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \dots, Z_{t_n} - Z_{t_{n-1}},$$

are freely independent random variables.

(ii) $Z_0 = 0$.

(iii) for any s, t in $[0, \infty[$, the (spectral) distribution of $Z_{s+t} - Z_s$ does not depend on s.

(iv) for any s in $[0, \infty[$, $Z_{s+t} \to Z_s$ *in probability*, as $t \to 0$, i.e. the (spectral) distributions $L\{Z_{s+t} - Z_s\}$ converge weakly to δ_0, as $t \to 0$.

A classical Lévy process in law is a family $(X_t)_{t\geq 0}$ of random variables on a probability space $(\Omega, \mathcal{F}, P)$, which satisfies the conditions (i)-(iv) above, except that free independence has to be replaced by classical independence in (i). Such a process (X_t) is called a (genuine) Lévy process, if it satisfies, in addition, the requirement that for almost all ω in Ω, the sample path $t \mapsto X_t(\omega)$ is right continuous with left limits.

Let (Z_t) be a free Lévy process and let (μ_t) be the family of marginal distributions, i.e. $\mu_t = L\{Z_t\}$ for all t. As in the classical case, it is an immediate consequence of conditions (i) and (iii) that μ_t is $\boxplus$-infinitely divisible for all t. Note also that the following conditions are satisfied:

$$\mu_s \boxplus \mu_{t-s} = \mu_t, \quad (0 \leq s < t), \tag{6.1}$$

and

$$\mu_t \xrightarrow{\text{w}} \delta_0, \quad \text{as } t \searrow 0. \tag{6.2}$$

Conversely, given any family (μ_t) of probability measures on $\mathbb{R}$, which satisfies (6.1) and (6.2), there exists a W^*-probability space $(\mathcal{A}, \tau)$ and a free Lévy process in law $(Z_t)_{t\geq 0}$ affiliated with $(\mathcal{A}, \tau)$, such that $L\{Z_t\} = \mu_t$ for all t. As noted in [12] and [29], $(\mathcal{A}, \tau)$ can be constructed, loosely speaking, as the inductive limit of a directed system of free product von Neumann algebras. In classical probability, the corresponding existence result for classical Lévy processes in law follows by an application of Kolmogorov's consistency theorem. Since the Bercovici-Pata bijection preserves both conditions (6.1) and (6.2), it follows, then, that we have the following correspondence between classical and free Lévy processes in law:

6.2 Proposition. *Let $(Z_t)_{t\geq 0}$ be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and with marginal distributions (μ_t). Then there exists a (classical) Lévy process (in law) $(X_t)_{t\geq 0}$, with marginal distributions $(\Lambda^{-1}(\mu_t))$.*

Conversely, for any (classical) Lévy process (in law) (X_t) with marginal distributions (μ_t), there exists a free Lévy process (in law) (Z_t) with marginal distributions $(\Lambda(\mu_t))$.

7 Selfdecomposability and Free Stochastic Integration.

Let Y be a classical random variable on $(\Omega, \mathcal{F}, P)$. As mentioned previously, $L\{Y\}$ is in $\mathcal{L}(*)$ if and only if there exists a (classical) Lévy process (X_t), defined on some probability space $(\Omega', \mathcal{F}', P')$, and such that

$$\mathbb{E}\{\log(1 + |X_1|)\} < \infty, \tag{7.1}$$

and

$$Y \stackrel{\text{d}}{=} \int_0^\infty e^{-t}\, dX_t. \tag{7.2}$$

Condition (7.1) is equivalent to asking that $\int_{\mathbb{R}\setminus[-1,1]} \log(1+|s|)\ \rho_1(ds) < \infty$, where ρ_1 is the Lévy measure appearing in the generating triplet for $L\{X_1\}$. Moreover, this condition is necessary and sufficient for the integrals $\int_0^R e^{-t}\ dX_t$ to converge, in probability, as $R \to \infty$; the limit being, by definition, the right hand side of (7.2) (see [18]). The integrals $\int_0^R e^{-t}\ dX_t$, in turn, are defined as the limit, in probability, of Riemann sums

$$S_n = \sum_{j=1}^{n} e^{-t_{n,j}^{\#}} \cdot (X_{t_{n,j}} - X_{t_{n,j-1}}),$$

corresponding to subdivisions:

$$0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,n} = R, \quad t_{n,j}^{\#} \in [t_{n,j-1}, t_{n,j}],\ (j = 1, 2, \dots, n), \tag{7.3}$$

subject to the condition that $\max\{t_{n,j} - t_{n,j-1} \mid j = 1, 2, \dots, n\} \to 0$, as $n \to \infty$. Using, once again, the algebraic and topological properties of Λ, we derived, in [5], the following free analog of the classical result described above:

7.1 Theorem. *Let y be a selfadjoint operator affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. Then the distribution of y is $\boxplus$-selfdecomposable if and only if y has a representation in law of the form:*

$$y \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-t}\ dZ_t, \tag{7.4}$$

for some free Lévy process (in law) (Z_t) affiliated with a W^*-probability space $(\mathcal{A}', \tau')$, and satisfying that $\int_{\mathbb{R}\setminus[-1,1]} \log(1 + |s|)\ \rho_1(ds) < \infty$, where ρ_1 is the Lévy measure appearing in the free generating triplet for Z_1.

The key ingredient in the proof of Theorem 7.1 is the following observation: Let (X_t) and (Z_t) be, respectively, classical and free Lévy processes corresponding to each other as in Proposition 6.2. Then, if we form Riemann sums S_n and T_n w.r.t. (X_t) and (Z_t), corresponding to subdivisions as in (7.3), it follows from the algebraic properties of Λ, that $\Lambda(L\{S_n\}) = L\{T_n\}$, and similarly $\Lambda(L\{S_n - S_m\}) = L\{T_n - T_m\}$. Using then, in addition, the continuity of Λ, it follows that (T_n) is a Cauchy sequence w.r.t. convergence in probability. Since the set of selfadjoint operators affiliated with a W^*-probability space is complete w.r.t. convergence in probability (cf. [21]), we conclude that (T_n) converges in probability to a selfadjoint operator, which we may then denote by $\int_0^R e^{-t}\ dZ_t$.

Having established integration w.r.t. (Z_t) by taking limits of Riemann sums, Theorem 7.1 is now easily derived by using the corresponding classical result, as well as the fact that Λ preserves selfdecomposability and distributions of Riemann sums w.r.t. corresponding Lévy processes as in Proposition 6.2.

8 The Lévy-Itô Decomposition.

Historically, P. Lévy derived the Lévy-Khintchine representation of a measure μ in $\mathcal{ID}(*)$ by establishing, first, a decomposition of any (classical) Lévy process into two independent parts: a continuous part and a part which is, loosely speaking, the sum of the jumps of the process. This decomposition, now known as the Lévy-Itô decomposition, was later proved rigorously by K. Itô, and is, from the probabilistic viewpoint, more basic than the Lévy-Khintchine representation. In order to describe, precisely, the sum of jumps of a Lévy process, one needs to introduce the concept of Poisson random measures. Before doing so, we recall that for any positive number λ, the Poisson distribution P_λ is the measure on the non-negative integers, given by:

$$P_\lambda(\{n\}) = \frac{\lambda^n}{n!}e^{-\lambda}, \quad (n \in \mathbb{N}_0).$$

8.1 Definition. Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space. A Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ is a collection $\{N(E) \mid E \in \mathcal{E}\}$ of random variables (defined on some probability space $(\Omega, \mathcal{F}, P)$), satisfying the following conditions:

(i) For each E in $\mathcal{E}$, $L\{N(E)\} = P_{\nu(E)}$.

(ii) If $E_1, \dots, E_n$ are disjoint sets from $\mathcal{E}$, then $N(E_1), \dots, N(E_n)$ are independent random variables.

(iii) For each fixed ω in Ω, the mapping $E \mapsto N(E, \omega)$ is a measure on $\mathcal{E}$.

In case $\nu(E) = \infty$, condition (i) in the definition above means, by convention, that $N(E, \omega) = \infty$ for all ω in Ω. Recall next that a (standard) Brownian motion is a classical Lévy process (B_t) for which $L\{B_t\}$ is the Gaussian distribution with mean 0 and variance t. We are then ready to state the Lévy-Itô result mentioned above:

8.2 Theorem. (Lévy-Itô) *Let (X_t) be a classical genuine Lévy process and let ρ be the Lévy measure appearing in the generating triplet for $L\{X_1\}$. Assume, for simplicity, that $\int_{-1}^{1} |x| \; \rho(dx) < \infty$. Then (X_t) has a representation in the form:*

$$X_t \overset{\text{a.s.}}{=} \gamma t + \sqrt{a} B_t + \int_{]0,t] \times \mathbb{R}} x \; N(ds, dx), \tag{8.1}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, (B_t) is a Brownian motion and N is a Poisson random measure on $(]0, \infty[\times\mathbb{R}, \mathcal{B}(]0, \infty[\times\mathbb{R}), \mathrm{Leb} \otimes \rho)$ (here $\mathcal{B}$ denotes Borel σ-algebra and Leb *denotes Lebesgue measure). Furthermore, the last two terms on the right hand side of* (8.1) *are independent processes.*

The symbol $\overset{\text{a.s.}}{=}$ in (8.1) means that the two random variables are equal with probability 1 (a.s. stands for "almost surely"). The Poisson random measure N appearing in the right hand side of (8.1) is, specifically, given by

$$N(E, \omega) = \#\big\{s \in]0, \infty[\,\big|\, (s, \Delta X_s(\omega)) \in E\big\},$$

for any Borel subset E of $]0, \infty[\times\mathbb{R}$, and where $\Delta X_s = X_s - \lim_{u \nearrow s} X_u$. Consequently, the integral in the right hand side of (8.1) is, indeed, the sum of the jumps of X_t until time t: $\int_{]0,t]\times\mathbb{R}} x \; N(ds, dx) = \sum_{s \leq t} \Delta X_s$. The condition $\int_{-1}^{1} |x| \; \rho(dx) < \infty$ ensures that this sum converges. Without that assumption, one still has a Lévy-Itô decomposition, but it is slightly more complicated than (8.1). In particular, the sum of jumps interpretation does not make sense, directly, in a rigorous fashion. We emphasize, though, that for applied purposes, the most interesting examples actually appear when the afore-mentioned condition is *not* satisfied.

In the forthcoming paper [6], we prove the following free analog of the Lévy-Itô decomposition. For simplicity, we restrict attention, here, to the case where $\int_{-1}^{1} |x| \; \rho(dx) < \infty$.

8.3 Theorem. *Let (Z_t) be a free Lévy process affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and let ρ be the Lévy measure appearing in the free generating triplet for $L\{Z_1\}$. Assume, for simplicity, that $\int_{-1}^{1} |x| \; \rho(dx) < \infty$. Then (Z_t) has a representation in the form:*

$$X_t \overset{\mathrm{d}}{=} \gamma t + \sqrt{a} W_t + \int_{]0,t] \times \mathbb{R}} x \; M(ds, dx), \tag{8.2}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, (W_t) is a free Brownian motion and M is a free Poisson random measure on $(]0,\infty[\times\mathbb{R}, \mathcal{B}(]0,\infty[\times\mathbb{R}), \mathrm{Leb} \otimes \rho)$. Furthermore, the last two terms on the right hand side of (8.2) are freely independent processes, and the right hand side of (8.2) is a free Lévy process.

Some explanatory comments are in order: Recall that the symbol $\overset{\mathrm{d}}{=}$ in (8.2) only means that the two operators have the same (spectral) distribution. Therefore, it is necessary to specify that the right hand side of (8.2) is, indeed, a free Lévy process. A free Brownian motion is a free Lévy process with semi-circular distributed increments. It corresponds, thus, to a classical Brownian motion via the correspondence described in Proposition 6.2. Finally, a free Poisson random measure is defined as follows:

8.4 Definition. Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space, and put

$$\mathcal{E}_f = \{E \in \mathcal{E} \mid \nu(E) < \infty\}.$$

A Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ is a collection $\{M(E) \mid E \in \mathcal{E}_f\}$ of selfadjoint operators (affiliated with some W^*-probability space $(\mathcal{A}, \tau)$), satisfying the following conditions:

(i) For each E in $\mathcal{E}_f$, $L\{M(E)\} = \Lambda(P_{\nu(E)})$, where Λ is the Bercovici-Pata bijection.

(ii) If $E_1, \dots, E_n$ are disjoint sets from $\mathcal{E}_f$, then $M(E_1), \dots, M(E_n)$ are freely independent.

(iii) If $E_1, \dots, E_n$ are disjoint sets from $\mathcal{E}_f$, then $M(\cup_{j=1}^{n} E_j) = \sum_{j=1}^{n} M(E_j)$.

The above definition of a free Poisson random measure may seem a little "poor", compared to that of a classical Poisson random measure. This definition is, however, sufficient to develop the integration theory needed to establish (8.2).

9 Concluding remarks

Processes with 'independent' increments, where 'independent' can have a variety of meanings, are objects of wide current interest in stochastics (i.e. probability and statistics together) and in mathematical physics. Not only are processes of this type of great interest in themselves but they occur as important building blocks in other more structured processes. The reference [3] contains state of the art papers discussing a variety of aspects of this. But a number of the topics in question are, however, not treated there. Some of the further recent developments are presented in [17, Chapters 4 and 5] and [15].

In the present paper we have indicated several results on processes with freely independent (and stationary) increments that are closely parallel to key results from the theory of Lévy processes in classical probability theory. In the light of these, and other related findings, it seems certain that much more of interest can be done in studying the similarities, as well as the intriguing differences, between processes based on classical stochastic independence and free independence, respectively. It is also feasible to link the two types of processes more directly, for instance by using classical subordinators as time changes of free Lévy processes, a topic that we are currently considering. (For information on subordination in classical probability, see [10],[11] and [4]).

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