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The Existence and Stability of Noncommutative Scalar Solitons

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Abstract. We establish existence and stabilty results for solitons in noncommutative scalar field theories in even space dimension $2d$. In particular, for any finite rank spectral projection P of the number operator $\mathcal{N}$ of the d-dimensional harmonic oscillator and sufficiently large noncommutativity parameter θ we prove the existence of a rotationally invariant soliton which depends smoothly on θ and converges to a multiple of P as $\theta \to \infty$.

In the two-dimensional case we prove that these solitons are stable at large θ, if $P = P_N$, where P_N projects onto the space spanned by the $N+1$ lowest eigenstates of $\mathcal{N}$, and otherwise they are unstable. We also discuss the generalisation of the stability results to higher dimensions. In particular, we prove stability of the soliton corresponding to $P = P_0$ for all θ in its domain of existence.

Finally, for arbitrary d and small values of θ, we prove without assuming rotational invariance that there do not exist any solitons depending smoothly on θ.

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1 Introduction

Recent progress in string theory has stimulated interest in solitons in noncommutative field theories [1, 2, 3]. Several authors have found explicit solitons in gauge theories with and without matter fields [4, 5, 6, 7, 8]. In [9] solitons in scalar field theories were studied and it was shown that in the case of an infinite noncommutativity parameter θ, where the kinetic term in the action can be neglected, large families of solitons exist. This is in a stark contrast to the commutative case where there are no solitons [10]. Various aspects of solitons in noncommutative scalar field theories are discussed in [11, 12, 13, 14, 15, 16, 17, 18, 19]. For background and a recent review of some of these results, see [20].

The problem we discuss can be formulated either in terms of functions on $\mathbf{R}^{2d}$, or, by applying a quantization map, in terms of operators on $L^2(\mathbf{R}^d)$, as explained e.g. in [9, 20]. In this paper we do not make use of the former formulation, except for some technical purposes in the final section. Thus we define *solitons* as critical points of the energy functional

$$S(\varphi) = \mathrm{Tr}\left(\sum_{k=1}^{d}[\varphi, a_k^*][a_k, \varphi] + \theta V(\varphi)\right) ,$$

where a_k and a_k^* are the standard annihilation and creation operators of the d-dimensional harmonic oscillator, V is a potential, θ a positive parameter (called the noncommutativity parameter), and φ is a self-adjoint operator on $L^2(\mathbf{R}^d)$.

In [21] we established the existence of spherically symmetric solitons in even dimensional scalar field theories under fairly general conditions on the potential, provided θ is sufficiently large and we proved that no spherically symmetric solutions can exist for small θ.

Throughout the present paper we assume that V *is twice continuously differentiable and positive, except for a second order zero at $x = 0$. Furthermore, we assume that $V'(x)$ is strictly negative for $x < 0$ and has exactly two zeroes at positive values t and s corresponding to a local maximum and a local minimum of V*, see Fig. 1. The techniques developed here can be adapted to potentials with more local maxima and minima. For the proof of Theorem 5 and for the discussion of stability in higher dimensions, we shall assume that V is analytic, although this assumption can presumably be relaxed.

Our results can be divided into two classes, one concerning general solitons and another concerning solitons that are diagonal in the harmonic oscillator basis consisting of the joint eigenfunctions of $a_k^* a_k$. In the $d = 1$ case the latter solitons correspond to rotationally invariant functions under the quantization map but in higher dimensions these solitons correspond to functions that are invariant under rotations in each of the d quantization planes. For $d > 1$ the rotationally invariant solitons are those which are functions of the number operator $\mathcal{N}$.

In the first category we have the following results for any nonzero critical point φ of S:

- φ is a positive operator, whose operator norm satisfies

$$\|\varphi\| \leq s$$

  independently of the value of θ.

- φ is of trace class and $\operatorname{Tr} V'(\varphi) = 0$.

- There exists a nonzero constant c depending only on the potential V such that the Hilbert-Schmidt norm of φ, denoted $\|\varphi\|_2$, satisfies

$$\|\varphi\|_2 \geq c\theta^{-\frac{d}{2}} .$$

  As a corollary we find that any family φ_θ of solitons depending smoothly on the noncommutativity parameter θ (in a sense made precise in Section 3) has a diverging energy at some strictly positive value of θ. Hence, such families cannot exist for arbitrarily small values of θ. This result can be viewed as a noncommutative version of Derrick's theorem [10].

Of results in the second category we mention, in particular, the following.

- For any finite rank spectral projection P of the number operator $\mathcal{N} = \sum_{k=1}^d a_k^* a_k$ there exists a maximal smooth family

$$(\theta_P, \infty) \ni \theta \mapsto \varphi_\theta$$

  of solitons such that $V''(\varphi_\theta) > 0$ and

$$\varphi_\theta \to sP \quad \text{as} \quad \theta \to \infty .$$

- If $d = 1$ and P equals the projection P_N onto the space spanned by the $N + 1$ lowest eigenstates of $\mathcal{N}$, the solitons φ_θ are stable for θ sufficiently large. For all other P the corresponding solitons are unstable in their full range of existence.

- For $P = P_0$ the corresponding solitons are stable for all $d \geq 1$ in their full range of existence.

This paper is organized as follows. In a preliminary section we describe the mathematical setting of the problem, recall results from [21] and prove some technical results on general properties of solitons.

In Section 3 we establish the main existence theorem for solitons. We actually give two proofs, one elementary, generalizing [21], based on an analysis of the difference equation for the eigenvalues of φ obtained from the Euler-Lagrange equation for the variational problem for S, and another proof based on an application of a fixed point theorem. While less elementary, the latter approach has the advantage of giving smoothness of the solitons as a function of θ. A related existence proof has been obtained independently in [24].

The results on stability are proven in Section 4, which also contains a discussion of the extension of our approach to higher dimensions without giving full details, except in the case $P = P_0$.

Finally, in Section 5 we prove non-existence of smooth families of solitons for small values of θ. It should be stressed that this result only rules out the existence of smooth families contrary to the nonexistence theorem in [21] for rotationally invariant solitons which rules out the existence of any rotationally invariant solitons for θ smaller than some positive θ_0 depending only on V and d. It is an interesting unsolved question whether this stronger result also holds without the assumption of rotational invariance .

Another interesting unsolved problem concerns existence of general non-rotationally invariant solutions, in particular the so called multi-soliton solutions described in [9]. The solitons discussed in this paper are special cases corresponding to overlapping solitons sitting at the origin. In [17] and [23] properties of moduli spaces of multi-solitons are discussed perturbatively in θ^{-1}. The latter paper contains a discussion of stability perturbatively to first order in θ^{-1}. Stability of scalar solitons under radial fluctuations is also discussed in [22].

2 General properties of solitons

Solitons in a noncommutative $2d$-dimensional scalar field theory with a potential V are finite energy solutions to the variational equation of the energy functional

$$S(\varphi) = \mathrm{Tr}\left(\sum_{k=1}^{d}[\varphi, a_k^*][a_k, \varphi] + \theta V(\varphi)\right), \tag{1}$$

where a_k^* and a_k are the usual raising and lowering operators of the d-dimensional simple harmonic oscillator and φ is a self-adjoint operator on $L^2(\mathbf{R}^d)$. We assume that the potential V is at least twice continuously differentiable with a second order zero at $x = 0$ and that $V(x) > 0$ if $x \neq 0$. Hence, finiteness of the potential energy $\theta \mathrm{Tr} V(\varphi)$ requires φ to belong to the space $\mathcal{H}_2$ of Hilbert-Schmidt operators. Consequently, S is defined and finite on the space $\mathcal{H}_{2,2}$ of self-adjoint Hilbert-Schmidt operators φ for which $[a_k, \varphi]$ is also Hilbert-Schmidt. We note that $\mathcal{H}_{2,2}$ is a Hilbert space with norm $\|\cdot\|_{2,2}$ given by

$$\|\varphi\|_{2,2}^2 = \sum_k \mathrm{Tr}\,([\varphi, a_k^*][a_k, \varphi]) + \mathrm{Tr}\,\varphi^2 = \sum_k \|[a_k, \varphi]\|_2^2 + \|\varphi\|_2^2\,, \tag{2}$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. It is easy to see that the space $\mathcal{H}_0$ consisting of operators that are represented by finite matrices (i.e. matrices with only finitely many non-zero entries) in the standard harmonic oscillator basis form a dense subspace of $\mathcal{H}_{2,2}$.

The variational equation of the functional (1) is

$$2\sum_{k=1}^{d}[a_k^*, [a_k, \varphi]] = -\theta V'(\varphi). \tag{3}$$

We regard this equation as an equality between two Hilbert-Schmidt operators on $L^2(\mathbf{R}^d)$. Thus, a solution φ to Eq. (3) belongs to $\mathcal{H}_{2,2}$ and has the property that the left hand side of Eq. (3), interpreted as a quadratic form on the domain of $\mathcal{N}^{\frac{1}{2}}$, where $\mathcal{N}$ denotes the number operator

$$\mathcal{N} = \sum_{k=1}^{d} a_k^* a_k\,,$$

is Hilbert-Schmidt. We denote the space of such operators by $\mathcal{D}$. Alternatively, $\mathcal{D}$ is the space of operators φ in $\mathcal{H}_{2,2}$ such that the linear form

$$\mathcal{H}_{2,2} \ni \psi \mapsto \sum_k \mathrm{Tr}\,([a_k^*, \psi][a_k, \varphi]) \tag{4}$$

is continuous in the Hilbert Schmidt norm $\|\cdot\|_2$.

This operator theoretic formulation of the problem is the most convenient one for our discussion of the existence and stability results in Sections 3 and 4. For the non-existence results in Section 5 we shall also make use of the alternative formulation in terms of ordinary functions and a quantization map (see e.g. [20]). Choosing the harmonic oscillator eigenstates $|n_1,\ldots,n_d\rangle$, $n_i = 0,1,\ldots$, $a_k^* a_k|n_1,\ldots,n_d\rangle = n_k|n_1,\ldots,n_d\rangle$, as the basis for the Hilbert space $L^2(\mathbf{R}^d)$, rotationally symmetric functions correspond, under the standard Weyl quantization, to diagonal operators whose eigenvalues only depend on $n_1+\cdots+n_d$. If we consider a diagonal operator with eigenvalues λ_n, $n=0,1,2,\ldots$, Eq. (3) reduces, for $d=1$, to [9, 11]

$$(n+1)\lambda_{n+1}-(2n+1)\lambda_n+n\lambda_{n-1} = \frac{\theta}{2}V'(\lambda_n), \quad n\geq 1 \tag{5}$$

$$\lambda_1-\lambda_0 = \frac{\theta}{2}V'(\lambda_0). \tag{6}$$

Summing the second order finite difference equation for λ_n from $n=0$ to $n=m$ yields the first order equation

$$\lambda_{m+1}-\lambda_m = \frac{\theta}{2(m+1)}\sum_{n=0}^{m}V'(\lambda_n), \quad m\geq 0. \tag{7}$$

A necessary condition for the energy to be finite is clearly that

$$\lambda_m \to 0 \quad \text{as} \quad m\to\infty. \tag{8}$$

Actually, this condition implies $\varphi\in\mathcal{H}_{2,2}$ by Lemma 1 below. In [21] we proved the existence of solutions to Eq. (7) satisfying the boundary condition (8) under fairly general conditions on the potential V. In the next section we generalize that result.

In addition to the conditions on V which have been imposed above we assume that V has only one local minimum in addition to $x=0$. Let the other local minimum be at $s>0$. Let $r\in(0,s)$ be a point where V has a local maximum and for technical convenience assume that V' does not vanish except at $0,r$ and s. Then $V'(x)<0$ for $x<0$ or $x\in(r,s)$ and $V'(x)>0$ for $x>s$ or $x\in(0,r)$ (see Fig. 1).

The following result which will be needed in the next section was proven in [21]. We state the result for $d=1$, but its generalisation to arbitrary $d\geq 1$ is straightforward as explained in [21].

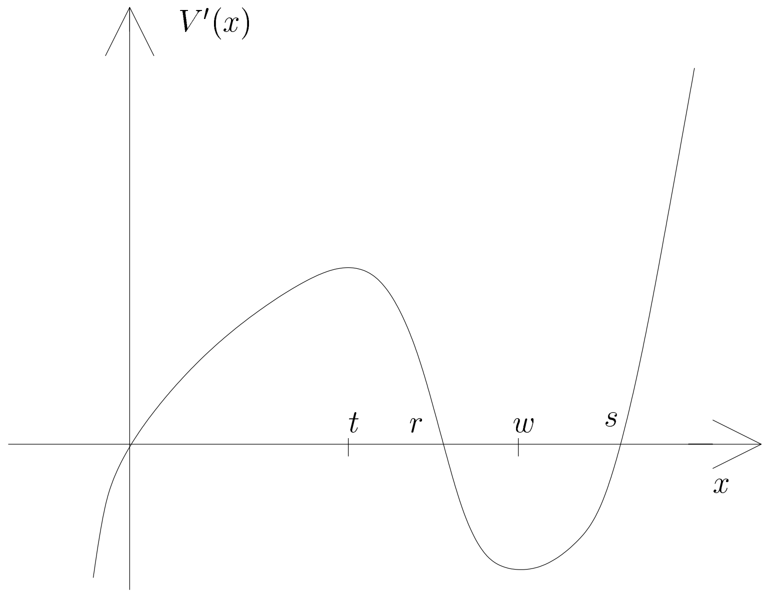

Figure 1: A graph of the derivative of a generic potential V which satisfies our assumptions.

Lemma 1. *Let $\{\lambda_m\}$ be a sequence of real numbers which satisfy Eq. (7). If $\lambda_n > s$ for some n then $\{\lambda_m\}$ is increasing for $m \geq n$ and $\lambda_m \to \infty$ as $m \to \infty$. If $\lambda_n \leq 0$ for some n then $\{\lambda_m\}$ is decreasing for $m \geq n$ and $\lambda_m \to -\infty$ as $m \to \infty$.*

If the sequence $\{\lambda_m\}$ also satisfies the boundary condition (8) and the λ_m's are not all zero then

(i) $0 < \lambda_m < s$, for all m.

(ii) λ_m tends monotonically to 0 for m large enough.

(iii) $\sum_m V'(\lambda_m) = 0$ and $\sum_m \lambda_m < \infty$.

Dropping the assumption of rotational symmetry we have the following generalization of (i) and (iii), which, apart from being of some independent interest, we will use in Section 5. The remainder of the present section is not needed for the existence and stability results in the following two sections.

Lemma 2. *Let φ be a nonzero solution to Eq. (3). Then*

(i) the operator φ is positive and its norm satisfies the inequality

$$\|\varphi\| \leq s. \tag{9}$$

(ii) φ is of trace class and $\mathrm{Tr}\,(V'(\varphi)) = 0$.

Before proving the above lemma we need the following result, where $\varphi_\pm$ denote the positive and negative parts of a bounded selfadjoint operator φ, defined by

$$\varphi = \varphi_+ - \varphi_- \ , \quad \varphi_+\varphi_- = 0 \ , \quad \varphi_\pm \geq 0 \ . \tag{10}$$

Lemma 3. *The maps*

$$\varphi \mapsto \varphi_\pm$$

are well defined and continuous from $\mathcal{H}_{2,2}$ to itself.

Proof. Since

$$\|\varphi_\pm\|_2 \leq \|\varphi\|_2, \tag{11}$$

it suffices to show that, for all k,

$$\|[a_k, \varphi_\pm]\|_2 \leq \ \text{const} \ \ \|[a_k, \varphi]\|_2. \tag{12}$$

We will prove below that this holds with the constant equal to $\sqrt{3}$. Since $\mathcal{H}_0$ is dense in $\mathcal{H}_{2,2}$ we can assume $\varphi \in \mathcal{H}_0$. It is clear that the spectral projections of finite rank operators corresponding to non-zero eigenvalues belong to $\mathcal{H}_0$ and the same applies to the spectral projections of $\varphi_\pm$. In order to estimate the norms of $\varphi_\pm$ it is convenient to write

$$\varphi_+ = \frac{1}{2\pi i}\int_\gamma \frac{z}{z-\varphi}\, dz \ , \tag{13}$$

where γ is a simple closed positively oriented contour in the complex plane enclosing the positive eigenvalues $\{\lambda_i\}$ of φ but not the non-positive eigenvalues $\{\mu_j\}$. Then

$$[a_k, \varphi_+] = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-\varphi}[a_k, \varphi]\frac{1}{z-\varphi} z dz \ . \tag{14}$$

Denoting the spectral projection corresponding to λ_i by e_i and the one of μ_j by f_j, we have

$$\frac{1}{z-\varphi} = \sum_i \frac{1}{z-\lambda_i}e_i + \sum_j \frac{1}{z-\mu_j}f_j \ . \tag{15}$$

Inserting the above identity into Eq. (14) and computing residues one obtains

$$[a_k, \varphi_+] = e_+[a_k, \varphi]e_+ + \sum_{i,j}\frac{\lambda_i}{\lambda_i - \mu_j}(e_i[a_k, \varphi]f_j + f_j[a_k, \varphi]e_i) \ , \tag{16}$$

where $e_+ = \sum_i e_i$ is the support projection of φ_+. Hence,

$$\begin{aligned}
\mathrm{Tr}\,([a_k,\varphi_+]^*[a_k,\varphi_+]) &= \mathrm{Tr}\,(e_+[a_k,\varphi]^*e_+[a_k,\varphi]e_+) \\
&\quad + \sum_{i,j}\left(\frac{\lambda_i}{\lambda_i-\mu_j}\right)^2 \mathrm{Tr}\,(e_i[a_k,\varphi]^*f_j[a_k,\varphi]e_i + f_j[a_k,\varphi]^*e_i[a_k,\varphi]f_j) \\
&\leq \mathrm{Tr}\,(e_+[a_k,\varphi]^*[a_k,\varphi]e_+) \\
&\quad + \sum_{i,j}\mathrm{Tr}\,(e_i[a_k,\varphi]^*f_j[a_k,\varphi]e_i + f_i[a_k,\varphi]^*e_j[a_k,\varphi]f_i) \\
&\leq 3\,\mathrm{Tr}\,([a_k,\varphi]^*[a_k,\varphi]), \qquad (17)
\end{aligned}$$

where we used the fact that

$$0 \leq \frac{\lambda_i}{\lambda_i-\mu_j} \leq 1. \qquad (18)$$

Clearly, the same estimate applies to $\mathrm{Tr}\,([a_k,\varphi_-]^*[a_k,\varphi_-])$ and the claimed result follows.

Proof of Lemma 2. (i) We first show that $\varphi \geq 0$. Suppose on the contrary that $\varphi_- \neq 0$. Then, since $V'(-\varphi_-) < 0$, we have for any integer $n > 2$ that

$$2\sum_{k=1}^{d}\mathrm{Tr}\,(\varphi_-^n[a_k^*,[\varphi,a_k]]) = \theta\mathrm{Tr}\,(\varphi_-^nV'(\varphi)) < 0\,. \qquad (19)$$

But, using the cyclicity of the trace,

$$\begin{aligned}
\mathrm{Tr}\,(\varphi_-^n[a_k^*,[\varphi,a_k]]) &= \mathrm{Tr}\,(\varphi_-^n[a_k^*,[\varphi_+,a_k]]) - \mathrm{Tr}\,(\varphi_-^n[a_k^*,[\varphi_-,a_k]]) \\
&= \mathrm{Tr}\,([a_k^*,\varphi_-^n][\varphi_-,a_k]) - \mathrm{Tr}\,([a_k^*,\varphi_-^n][\varphi_+,a_k]) \\
&= \sum_{p+q=n-1}\mathrm{Tr}\,(\varphi_-^p[a_k^*,\varphi_-]\varphi_-^q[\varphi_-,a_k]) \\
&\quad - \sum_{p+q=n-1}\mathrm{Tr}\,(\varphi_-^p[a_k^*,\varphi_-]\varphi_-^q[\varphi_+,a_k]) \\
&= \sum_{p+q=n-1}\mathrm{Tr}\,(\varphi_-^{\frac{p}{2}}[a_k^*,\varphi_-]\varphi_-^q[\varphi_-,a_k]\varphi_-^{\frac{p}{2}}) \\
&\quad +\mathrm{Tr}\,(\varphi_+^{\frac{1}{2}}[\varphi_-,a_k]\varphi_-^{n-2}[a_k^*,\varphi_-]\varphi_+^{\frac{1}{2}}) + \mathrm{Tr}\,(\varphi_+^{\frac{1}{2}}[a_k^*,\varphi_-]\varphi_-^{n-2}[\varphi_-,a_k]\varphi_+^{\frac{1}{2}}) \\
&\geq 0\,, \qquad (20)
\end{aligned}$$

which contradicts the inequality (19).

To prove the inequality in (i) we note that the equation of motion (3) implies that

$$\|\varphi\|^{-n}\theta\mathrm{Tr}\,(\varphi^nV'(\varphi)) = 2\sum_k\mathrm{Tr}\,(\|\varphi\|^{-n}\varphi^n[a_k^*,[\varphi,a_k]])$$

$$= \quad -2\sum_k \|\varphi\|^{-n}\mathrm{Tr}\,[a_k^*,\varphi_n][\varphi,a_k] < 0. \tag{21}$$

We also have

$$\lim_{n\to\infty} \|\varphi\|^{-n}\theta\mathrm{Tr}\,(\varphi^n V'(\varphi)) = \theta V'(\|\varphi\|)\mathrm{Tr}\,e\ , \tag{22}$$

where e is the spectral projection of the operator φ corresponding to the eigenvalue $\|\varphi\|$. In particular,

$$\theta V'(\|\varphi\|)\mathrm{Tr}\,e \leq 0\ , \tag{23}$$

which implies the desired inequality by the assumed form of the potential V.

(ii) Let P_m, $m=0,1,2,\ldots$, denote the orthogonal projection onto the eigenspace of the number operator $\mathcal{N}$ corresponding to eigenvalue m, and set

$$\lambda_m = \mathrm{Tr}\,(P_m\varphi)\ . \tag{24}$$

Then the equation of motion (3) gives

$$\frac{1}{2}\theta\,\mathrm{Tr}\,(P_m V'(\varphi)) = (m+1)\lambda_{m+1} - (2m+d)\lambda_m + (m+d-1)\lambda_{m-1}. \tag{25}$$

Summing this identity over $m \leq n$ we get (as in the spherically symmetric case)

$$(n+1)\lambda_{n+1} - (n+d)\lambda_n = \theta\sum_{i\leq n}\mathrm{Tr}\,(P_i V'(\varphi)), \tag{26}$$

and, finally, summing over $n \leq p$,

$$\lambda_{p+1} - \lambda_0 = \theta\sum_{n\leq p}\frac{1}{(n+1)}\left((d-1)\lambda_n + \sum_{i\leq n}\mathrm{Tr}\,(P_i V'(\varphi))\right). \tag{27}$$

Besides this equation we shall also make use of the fact that

$$V'(\varphi) = a\varphi + O(\varphi^2) \tag{28}$$

for some positive constant a as a consequence of the assumptions made on V. Since φ is Hilbert-Schmidt it follows from this that $V'(\varphi)$ is of trace class if and only if φ is of trace class. We first prove that this is the case if (and only if) $\lim_{m\to\infty}\lambda_m = 0$ and in this case, Tr $V'(\varphi) = 0$. In fact, by (28),

$$\sum_{i\leq n}\mathrm{Tr}\ P_i(V'(\varphi)) = \sum_{i\leq n}(a\lambda_i + c_i)\ , \tag{29}$$

where $\sum_i c_i$ is absolutely convergent while all the terms in $\sum_{i\leq n}\lambda_i$ are positive, since φ is a positive operator by (i). It follows that the sum $\sum_{i\leq n}$ Tr $P_i V'(\varphi)$ has a limit

L, finite or $+\infty$, as $n \to \infty$. On the other hand, it follows from our assumptions that the right hand side of Eq. (27) converges as $p \to \infty$ and consequently, since the λ_m's are nonnegative, L must be zero. Hence, Eq. (29) implies that $\sum_i \lambda_i$ converges, i.e., φ is of trace class, and the trace L of $V'(\varphi)$ is zero as claimed.

It remains to show that $\lambda_m \to 0$ as $m \to \infty$. Assume this is not the case. Then $\sum_i \lambda_i = +\infty$ and therefore, by Eq. (29), we have

$$\sum_{i \le m} \mathrm{Tr}\ (P_i V'(\varphi)) > 1 \tag{30}$$

for m large enough. Thus, by Eq. (27),

$$\lambda_p \ge \theta \sum_{n \le p-1} \frac{1}{n+1} \sum_{i \le n} \mathrm{Tr}\ (P_i V'(\varphi)) \ge \text{const}\ \ \ln p\ , \tag{31}$$

for p large enough. Repeating the argument with the inductive assumption $\lambda_p \ge$ const p^l, for sufficienly large p, where l is a nonegative integer, leads to $\lambda_p \ge$ const p^{l+1} for p sufficeiently large. Hence, λ_m increases faster than any power of m, if it does not tend to zero. But this is not possible since, by the Cauchy-Schwarz inequality,

$$\lambda_m^2 = (\mathrm{Tr}\ P_m\varphi)^2 \le \mathrm{Tr}\ (P_m\varphi^2)\mathrm{Tr}\ P_m \le \text{const}\ m^{d-1}\mathrm{Tr}\ (P_m\varphi^2) \tag{32}$$

and hence,

$$\sum_m \frac{\lambda_m^2}{m^{d-1}} \le \sum_m \mathrm{Tr}\ (P_m\varphi^2) = \|\varphi\|_2^2 < \infty\ . \tag{33}$$

This finishes the proof of Lemma 2.

3 Existence

We now proceed to discuss the existence of rotationally invariant solutions to Eq. (3). Let t be the location of the maximum of V' in the interval $[0, s]$ and let w be the location of the minimum of V' in the same interval (see Fig. 1). As above we denote by $P_0, P_1, \ldots$ the orthogonal projections onto the eigenspaces of the number operator of the d-dimensional harmonic oscillator. The purpose of this section is to prove the following theorem.

Theorem 1. *For any projection P on $L^2(\mathbf{R}^d)$, which is the sum of a finite number of the projections P_n, there is a unique maximal family φ_θ, $\theta > \theta_P$, of rotationally* *invariant solutions of Eq. (3), which depends smoothly on θ, i.e., is continuously differentiable with respect to the norm $\|\cdot\|_{2,2}$, and fulfills*

$$V''(\varphi_\theta) > 0 \,, \tag{34}$$

as well as

$$\varphi_\theta \to s\,P \tag{35}$$

in Hilbert-Schmidt norm as $\theta \to \infty$.

Proof. We shall give two proofs of existence of solutions for sufficiently large θ. The first proof is an extension of the proof given in [21] for $P = P_0$. For simplicity we restrict to $d = 1$ and to $P = P_0 + \cdots P_N$, the adaptation of the arguments to arbitrary $d \geq 1$ being explained in [21].

First, assume θ is so large that

$$\frac{\theta}{2(N+1)}|V'(w)| \geq w. \tag{36}$$

In this case we claim there is a unique $\underline{\lambda} \in [w, s)$ such that if we set $\lambda_0 = \underline{\lambda}$ and define λ_i for $i > 0$ by the recursion (7) then

$$\lambda_0 > \lambda_1 > \ldots \lambda_N \geq w \tag{37}$$

and $\lambda_{N+1} = 0$. In order to prove the claim we begin by choosing λ_0 close to but smaller than s so that (37) holds, which clearly is possible. Then $\lambda_N > \lambda_{N+1}$ by (7), and if $\lambda_{N+1} = 0$ we are done. Note that all the λ_i's depend continuously on λ_0 and $\lambda_{N+1} \to s$ as $\lambda_0 \to s$. If $\lambda_{N+1} < 0$ we increase λ_0 until $\lambda_{N+1} = 0$ and the inequalities (37) still hold because $\lambda_1, \ldots \lambda_N$ all increase with λ_0. If $\lambda_{N+1} > 0$ we decrease λ_0 until $\lambda_{N+1} = 0$ and (37) still holds due to the inequality (36). This proves the existence of $\underline{\lambda}$.

Next take θ still larger, if necessary, so that

$$V'(t) \geq (N+1)|V'(\underline{\lambda})|. \tag{38}$$

This is clearly possible because $\underline{\lambda} \to s$ as $\theta \to \infty$. We now claim there exists $\bar{\lambda} \in (\underline{\lambda}, s)$ such that if we take $\lambda_0 = \bar{\lambda}$ then (37) holds and $\lambda_{N+1} = \lambda_{N+2}$, i.e.

$$0 = \frac{\theta}{2(N+2)} \sum_{i=0}^{N+1} V'(\lambda_i). \tag{39}$$

In order to verify the existence of $\bar{\lambda}$ we note that, as a consequence of (7), for λ_0 greater than but close to $\underline{\lambda}$ we have λ_{N+1} is greater than but close to 0, and λ_{N+1} increases with λ_0. Hence, in view of (38) and the fact that $\lambda_1, \ldots, \lambda_N$ are also increasing functions of λ_0, there is a $\lambda_0 \equiv \overline{\lambda} \in (\underline{\lambda}, s)$ such that

$$V'(\lambda_{N+1}) = -\sum_{i=0}^{N} V'(\lambda_i) \tag{40}$$

which establishes the claim. We note that for $\lambda_0 = \overline{\lambda}$ we have $\lambda_{N+1} \in (0, t)$.

If a sequence $\{\lambda_i\}$ obeys the recursion (7) and has the property $\lambda_0 > \lambda_1 > \ldots > \lambda_p$, but $\lambda_{p+1} \geq \lambda_p$, we say that the sequence *turns at* p. We note that in this case $\lambda_p > 0$ by Lemma 1 and if $\lambda_{p+1} = \lambda_p$ then $\lambda_{p+2} > \lambda_{p+1}$ by (7).

Define the set

$$A = \{\lambda_0 \in [\underline{\lambda}, \bar{\lambda}] : \{\lambda_i\} \text{ turns at some } p\}. \tag{41}$$

By construction $\underline{\lambda} \notin A$ and $\bar{\lambda} \in A$. Put $\Lambda_0 = \inf A$. Since each λ_i depends continuously on the initial value λ_0 it follows that $\Lambda_0 \notin A$.

Now consider the sequence defined by $\lambda_0 = \Lambda_0$ and Eq. (7). Since this sequence does not turn it is monotonically decreasing. In order to show that this sequence provides a solution to our problem it therefore suffices to show that $\lambda_i \to 0$ as $i \to \infty$. Suppose λ_i becomes negative for some i. Then Lemma 1 implies that $\lambda_i \to -\infty$. By the continuity of λ_i as a function of λ_0 it follows that for λ_0 sufficiently close to Λ_0 the sequence λ_i tends monotonically to $-\infty$ but this contradicts the definition of Λ_0. We conclude that the limit $\lim_{i\to\infty} \lambda_i = a \geq 0$ exists and by (7) we have

$$V'(a) = \frac{2}{\theta} \lim_{i\to\infty} (\lambda_{i+1} - \lambda_i) = 0. \tag{42}$$

Hence, $a = 0$ since $\lambda_i < r$ for $i > N$. This completes the proof of the existence of rotationally invariant solutions φ_θ for large enough θ and it follows easily from the construction that $\varphi_\theta \to sP$ in operator norm as $\theta \to \infty$.

It is worth while noting that the proof given here shows that the sequence of eigenvalues $\{\lambda_i\}$ of φ_θ is strictly decreasing for θ large enough. This is special for the choice of projection P made above. The same technique can be applied to demonstrate existence of solutions converging to any projection of the type stated in the theorem, but since this result as well as the claim of differentiability are obtained in a more uniform manner by the second method of proof, we shall not discuss that approach in more detail here. Also, the above proof can easily be generalized to establish the existence of solutions which converge to finite rank operators of the form $tP + sP'$, $PP' = 0$, as $\theta \to \infty$.

The second proof of existence is by use of a fixed point theorem. Let us first note that the operator Δ, defined by

$$\Delta\varphi = \sum_{k=1}^{d} [a_k^*, [a_k, \varphi]] \ , \tag{43}$$

is self-adjoint and positive on $\mathcal{H}_2$ with domain $\mathcal{D}$. Indeed, as explained in Section 5, it is unitarily equivalent to the standard Laplace operator on $L^2(\mathbf{R}^{2d})$ via a quantization map $\pi_W : L^2(\mathbf{R}^{2d}) \to \mathcal{H}_2$, which justifies the notation Δ for this operator in the remainder of this proof. Given a bounded self-adjoint operator B on $L^2(\mathbf{R}^d)$, it defines by left multiplication a bounded self-adjoint operator on $\mathcal{H}_2$, which we shall also denote by B. By the Kato-Rellich theorem $\Delta + B$ is self-adjoint with domain $\mathcal{D}$. Assuming $B \geq c > 0$ we have $\Delta + B \geq c$ and hence $\Delta + B$ maps $\mathcal{D}$ bijectively onto $\mathcal{H}_2$ with bounded inverse

$$(\Delta + B)^{-1} \leq c^{-1} \ . \tag{44}$$

The same statement holds if B is of the form

$$B = \sum_{n=0}^{\infty} b_n P_n \tag{45}$$

and we restrict $\Delta + B$ to $\mathcal{D}' = \mathcal{D} \cap \mathcal{H}_2'$, where $\mathcal{H}_2'$ is the Hilbert subspace of $\mathcal{H}_2$ consisting of diagonal operators of the form (45). This follows by using that $\mathcal{H}_2'$ corresponds under the quantization map π_W to rotation invariant functions in $L^2(\mathbf{R}^{2d})$ on which the Laplace operator is known to be self-adjoint. Alternatively, one can use the explicit form

$$\Delta\varphi = -\sum_{n=0}^{\infty} \{(n+d)\lambda_{n+1} - (2n+d)\lambda_n + n\lambda_{n-1}\} P_n \ , \tag{46}$$

where $\varphi = \sum_{n=0}^{\infty} \lambda_n P_n$, and the domain $\mathcal{D}'$ consists of those φ which fulfill

$$\sum_{n=0}^{\infty} |(n+d)\lambda_{n+1} - (2n+d)\lambda_n + n\lambda_{n-1}|^2 < \infty \, . \tag{47}$$

Since $\Delta + B$ is a closed symmetric operator it suffices to verify that the orthogonal complement to its range is $\{0\}$. But it is easily seen that φ belongs to this orthogonal complement if and only if

$$(n+d)\lambda_{n+1} - (2n+d)\lambda_n + n\lambda_{n-1} = b_n\lambda_n, \tag{48}$$

for $n \geq 0$. The proof of Lemma 1 shows that any non-trivial solution $\{\lambda_n\}$ of this recursion relation diverges to $\pm\infty$, since $b_n \geq c > 0$. Hence $\varphi = 0$ if $\varphi \in \mathcal{H}_2'$, as desired.

As a consequence, we note that for $\rho \geq 0$ and B and c as above, the operator $\rho\Delta + B$ has a bounded inverse on $\mathcal{H}_2'$ fulfilling

$$(\rho\Delta + B)^{-1} \leq c^{-1} \ , \tag{49}$$

the case $\rho = 0$ being obvious.

In view of these preparatory remarks, we rewrite Eq. (3) as

$$\rho\Delta\varphi + V'(\varphi) = 0 \ , \tag{50}$$

where $\rho = 2\theta^{-1}$. Then $\psi_0 = sP$ is a solution for $\rho = 0$. Since $\psi_0 \in \mathcal{H}_2'$ and

$$V''(\psi_0) \geq \min\{V''(0), V''(s)\} \equiv c_0 > 0 \ , \tag{51}$$

by assumption, we can, for $\rho \geq 0$, further rewrite the equation in the form

$$\varphi = (\rho\Delta + V''(\psi_0))^{-1}\{V''(\psi_0)\psi_0 + V'(\psi_0) - V'(\varphi) - V''(\psi_0)(\psi_0 - \varphi)\} \equiv T_\rho(\varphi) \ . \tag{52}$$

Since V is C^2 by assumption we have

$$\|V'(\varphi) - V'(\psi_0) - V''(\psi_0)(\varphi - \psi_0)\|_2 = o(\|\varphi - \psi_0\|_2) \,, \tag{53}$$

and also

$$\|(\rho\Delta + V''(\psi_0))^{-1}V''(\psi_0)\psi_0 - \psi_0\|_2 = \rho\|(\rho\Delta + V''(\psi_0))^{-1}\Delta\psi_0\|_2 \leq c_1\rho \ , \tag{54}$$

where $c_1 = c_0^{-1}\|\Delta\psi_0\|_2$.

For φ in the ball

$$B_\varepsilon(\psi_0) = \{\varphi \in \mathcal{H}_2' : \|\varphi - \psi_0\|_2 \leq \varepsilon\} \ , \tag{55}$$

we then have

$$\|T_\rho(\varphi) - \psi_0\|_2 \leq c_1\rho + o(1)\|\varphi - \psi_0\|_2 \ , \tag{56}$$

and hence, $T_\rho(\varphi) \in B_\varepsilon(\psi_0)$ if ρ and ε are sufficiently small. Similarly, one sees that

$$\|T_\rho(\varphi) - T_\rho(\psi)\|_2 \leq o(1)\,\|\varphi - \psi_0\|_2 \ , \tag{57}$$

so T_ρ is a contraction on $B_\varepsilon(\psi_0)$, if ρ and ε are sufficiently small. Fixing ε accordingly, Banach's fixed point theorem implies the existence of a unique solution ψ_ρ of Eq. (50) in $B_\varepsilon(\psi_0)$ for $0 \leq \rho \leq \delta$ and δ small enough.

For $0 \leq \rho,\ \rho_0 \leq \delta$, we have

$$\psi_\rho - \psi_{\rho_0} = (\rho\Delta + V''(\psi_0))^{-1}\{(\rho_0 - \rho)\Delta\psi_{\rho_0} + V'(\psi_{\rho_0}) - V'(\psi_\rho) - V''(\psi_0)(\psi_\rho - \psi_{\rho_0})\} \tag{58}$$

from which we get

$$\|\psi_\rho - \psi_{\rho_0}\|_2 \leq c_2|\rho - \rho_0| + o(\|\psi_\rho - \psi_{\rho_0}\|_2) \ , \tag{59}$$

where the constant c_2 depends only on ρ_0, and we have assumed ε is small enough such that $V''(\psi_\rho) > 0$. This inequality implies that ψ_ρ is a Lipschitz continuous function of ρ if ε is small enough. In turn, Eq. (58) implies that ψ_ρ is differentiable in the $\|\cdot\|_2$-norm with

$$\frac{d\psi_\rho}{d\rho} = (\rho\Delta + V''(\psi_0))^{-1}\Delta\psi_\rho \ . \tag{60}$$

By standard arguments, the family $\psi_\rho,\ 0 \leq \rho < \delta$ extends to a maximal family, differentiable in the $\|\cdot\|_2$-norm, and such that $V''(\psi_\rho) > 0$.

It remains to establish the stronger claim of smoothness in the norm $\|\cdot\|_{2,2}$ for $\rho > 0$. In order to obtain this, it is sufficient to verify that the bijective operator $(\rho\Delta + V''(\varphi))^{-1}$ from $\mathcal{H}_2$ onto $\mathcal{D}'$ is bounded, when is $\mathcal{D}'$ equipped with the $\|\cdot\|_{2,2}$-norm, for $\rho > 0$ and $V''(\varphi) > 0$. It is straightforward to verify that under these conditions $(\rho\Delta + V''(\varphi))^{-1}$ is bounded (and $\rho\Delta + V''(\varphi)$ as well, in fact), when $\mathcal{D}'$ is equipped with the norm

$$\|\varphi\|_{4,2} = \left(\|\Delta\varphi\|_2^2 + \|\varphi\|_2^2\right)^{\frac{1}{2}} \ , \tag{61}$$

which is easily seen to be stronger than $\|\cdot\|_{2,2}$. In addition, simple estimates show that the derivative given by Eq. (60) is continuous in this norm.

This completes the proof of the theorem with $\varphi_\theta = \psi_\rho$ for $\rho = 2\theta^{-1}$.

We remark that the above argument can easily be generalized to prove the existence of solutions to Eq. (3) which converge to sP, where P is a projector onto space spanned by a finite number of the joint eigenfunctions of the number operators $a_k^* a_k$. As remarked above, these solutions are not rotationally invariant but only invariant under rotations in the d two-dimensional quantization planes.

4 Stability

In this section we study the stability of solutions to Eq. (3) in the case $d = 1$. Extension to $d > 1$ is briefly discussed at the end of the section.

A solution φ is defined to be stable if the second functional derivative of the action S at φ is a positive semidefinite quadratic form at φ, i.e.,

$$\Sigma(\omega) \equiv \frac{1}{2} \frac{d^2}{d\epsilon^2} S(\varphi + \epsilon\omega)\bigg|_{\epsilon=0} \geq 0 \ . \tag{62}$$

The natural domain of definition of the quadratic form Σ depends generally both on the potential V and on φ. Under the previously stated assumptions on V the domain contains at least the space $\mathcal{H}_0$ for the rotationally symmetric solutions that we consider here. If Σ is continuous with respect to the norm $\|\cdot\|_{2,2}$ it is sufficient to show stability for perturbations ω in $\mathcal{H}^0$. Since the kinetic term in $S(\varphi)$ is quadratic, continuity of Σ means that the second functional derivative of V is a continuous quadratic form with respect to the Hilbert-Schmidt norm. This continuity is easy to check, using the analytic functional calculus, if V is analytic in a neighborhood of the interval $[0, s]$ which we will assume to be the case from now on. For this reason we restrict attention below to $\omega \in \mathcal{H}^0$. Our results about stability can be summarized in the following three theorems.

Theorem 2. *Let φ be a rotationally invariant, finite energy solution to (3) with a nondegenerate spectrum and let $\lambda_0, \lambda_1, \ldots$ denote the eigenvalues of φ in the harmonic oscillator basis. Then φ is unstable unless $\{\lambda_n\}$ is a decreasing sequence.*

This theorem implies that only the solutions corresponding to $P = P_0 + \ldots + P_N$ in Theorem 1 can possibly be stable. By abuse of notation we denote this solution by φ_N, for a fixed value on θ, in the remainder of this section.

Theorem 3. *The solution φ_0 of Eq. (3) constructed in the previous section is stable for all values of θ in the maximal range.*

Theorem 4. *For any $N \geq 0$ the solution φ_N constructed in the previous section is stable for θ sufficiently large.*

We note that Theorem 3 implies Theorem 4 in the case $N = 0$. We choose to state and prove Theorem 3 separately because it is stronger than Theorem 4 for $N = 0$ and the proof is simpler. In the proof of Theorem 4 we have to rely on asymptotic expansions of the eigenvalues for large θ which are not needed in the proof of Theorem 3. We remark further that solutions with eigenvalues λ_n some of which lie in the region where $V'' < 0$ are in general unstable but one can construct examples of stable solutions with eigenvalues in the region where $V'' < 0$.

Before proving the theorems we do some groundwork and establish notation. Let

$$\begin{aligned} K(\varphi) &= \mathrm{Tr}\,[\varphi, a^*][a, \varphi] \\ &= \sum_{n,m=0}^{\infty} |\langle n|[a,\varphi]|m\rangle|^2 \end{aligned} \tag{63}$$

denote the kinetic energy functional. Let φ be a rotationally invariant solution of Eq. (3) with a nondegenerate spectrum. Then we can write

$$\varphi + \epsilon\omega = U_\epsilon^* \varphi_\epsilon U_\epsilon \tag{64}$$

where U_ϵ is unitary and φ_ϵ is diagonal in the harmonic oscillator basis. It follows that

$$\frac{d^2}{d\epsilon^2} S(\varphi + \epsilon\omega)\bigg|_{\epsilon=0} = 2K(\omega) + \theta\,\frac{d^2}{d\epsilon^2}\mathrm{Tr}\;V(\varphi_\epsilon)\bigg|_{\epsilon=0}. \tag{65}$$

Notice that the assumption $\omega \in \mathcal{H}_0$ implies that only finitely many of the eigenvalues and eigenvectors of φ are perturbed, and we can apply standard non-degenerate perturbation theory. Let $\lambda_n(\epsilon)$ denote the eigenvalue of φ_ϵ which converges to λ_n as $\epsilon \to 0$. Then $\lambda_n(\epsilon)$ is real analytic in ϵ, and

$$\frac{d^2}{d\epsilon^2}\mathrm{Tr}\;V(\varphi_\epsilon)\bigg|_{\epsilon=0} = \sum_{n=0}^{\infty} \Big(\lambda_n''(0)V'(\lambda_n) + (\lambda_n'(0))^2 V''(\lambda_n)\Big)\,. \tag{66}$$

From standard perturbation theory we know that

$$\lambda_n'(0) = \langle n|\omega|n\rangle \tag{67}$$

and

$$\lambda_n''(0) = 2 \sum_{m \neq n} \frac{|\langle n|\omega|m\rangle|^2}{\lambda_n - \lambda_m}. \tag{68}$$

The condition for stability can therefore be written as

$$\begin{aligned} \Sigma(\omega) &= K(\omega) + \theta \sum_{m \neq n} \frac{|\langle n|\omega|m\rangle|^2}{\lambda_n - \lambda_m} V'(\lambda_n) + \frac{\theta}{2} \sum_{n=0}^{\infty} |\langle n|\omega|n\rangle|^2 V''(\lambda_n) \\ &= K(\omega) + \theta \sum_{m<n} |\langle n|\omega|m\rangle|^2 \frac{V'(\lambda_n) - V'(\lambda_m)}{\lambda_n - \lambda_m} + \frac{\theta}{2} \sum_{n=0}^{\infty} |\langle n|\omega|n\rangle|^2 V''(\lambda_n) \\ &\geq 0. \end{aligned} \tag{69}$$

We remark that the last term in Σ is nonnegative if $V''(\lambda_n) \geq 0$ for all n. The kinetic energy term can be written

$$\sum_{n,m=0}^{\infty} |\sqrt{n+1}\,\langle n+1|\omega|m\rangle - \sqrt{m}\,\langle n|\omega|m-1\rangle|^2 \,, \tag{70}$$

where $\sqrt{m}\,\langle n|\omega|m-1\rangle$ is set to zero for $m = 0$, and we see that the kinetic energy couples the matrix elements of ω to their nearest neighbours along diagonals with $n - m$ fixed. On the other hand, the potential part of Σ does not couple different matrix elements of ω. Note that $\langle n|\omega|m\rangle = \langle m|\omega|n\rangle^*$ since ω is self-adjoint but otherwise the matrix elements of ω can be chosen arbitrarily.

Proof of Theorem 2. We will show that there exists a perturbation ω such that $\Sigma(\omega) < 0$ unless the λ_n's are decreasing. We take ω such that $\langle n|\omega|m\rangle = 0$ for $|n - m| \neq 1$. Then we can write

$$\begin{aligned} \Sigma(\omega) = & \sum_{n=0}^{\infty} \Big(|\sqrt{n+1}\,\langle n+1|\omega|n\rangle - \sqrt{n}\,\langle n|\omega|n-1\rangle|^2 \\ & + |\sqrt{n+1}\,\langle n+1|\omega|n+2\rangle - \sqrt{n+2}\,\langle n|\omega|n+1\rangle|^2 \Big) \\ & + \theta \sum_{n=0}^{\infty} \frac{|\langle n|\omega|n+1\rangle|^2}{\lambda_n - \lambda_{n+1}} \left(V'(\lambda_n) - V'(\lambda_{n+1}) \right). \end{aligned} \tag{71}$$

The above expression is quadratic in the variables

$$\alpha_n = \langle n+1|\omega|n\rangle, \tag{72}$$

$n = 0, 1, 2, \ldots$. Assuming without loss of generality that the α_n's are real we have

$$\Sigma(\omega) = 2 \sum_{n,m} q_{nm} \alpha_n \alpha_m \,, \tag{73}$$

where the symmetric matrix q_{nm} has only nonvanishing matrix elements on the diagonal and next to the diagonal which are given by

$$\begin{aligned} q_{nn} &= 2(n+1) + \gamma_n && (74) \\ q_{nn+1} &= -\sqrt{n+1}\,\sqrt{n+2} && (75) \\ q_{nn-1} &= -\sqrt{n}\,\sqrt{n+1}, && (76) \end{aligned}$$

where

$$\gamma_n = \frac{\theta}{2}\frac{V'(\lambda_{n+1}) - V'(\lambda_n)}{\lambda_{n+1} - \lambda_n}. \tag{77}$$

We need to show that q_{nm} is a positive semidefinite matrix. This is most easily done by diagonalising q_{nm}, using elementary row and column operations, and verifying that the diagonal entries $C_0, C_1, \ldots$ in the resulting diagonal matrix C are non-negative. In the first step we divide the first row by q_{00}, multiply it by $-q_{10}$ and add the resulting row to the second row. Then we see that the first two diagonal entries of C are

$$\begin{aligned} C_0 &= q_{00} && (78) \\ C_1 &= q_{11} - \frac{q_{10}^2}{q_{00}}. && (79) \end{aligned}$$

Inductively we find

$$C_k = q_{kk} - \frac{q_{kk-1}^2}{C_{k-1}}. \tag{80}$$

We can evaluate C_0 and C_1 directly using the equation of motion (5) and find

$$\begin{aligned} C_0 &= 2\,\frac{\lambda_2 - \lambda_1}{\lambda_1 - \lambda_0}, && (81) \\ C_1 &= 3\,\frac{\lambda_3 - \lambda_2}{\lambda_2 - \lambda_1}. && (82) \end{aligned}$$

Now it is straightforward to prove from Eq. (80) by induction that

$$C_k = (k+2)\,\frac{\lambda_{k+2} - \lambda_{k+1}}{\lambda_{k+1} - \lambda_k} \tag{83}$$

and we conclude that $C_k > 0$ for all k if and only if the sequence $\{\lambda_n\}$ is monotone. Obviously, the sequence cannot be increasing since $\lambda_n > 0$ for all n and $\lambda_n \to 0$ as $n \to \infty$.

Proof of Theorem 3. Let λ_n be the eigenvalue of φ_0 corresponding to the eigenvector $|n\rangle$, $n = 0, 1, 2, \ldots$. Since $V''(\lambda_n) \geq 0$ for all n, by hypothesis, and the kinetic energy only couples the matrix elements of ω along diagonals it is sufficient and also necessary, in view of Eq. (69), to prove that

$$\begin{aligned} \Sigma_k(\omega) &\equiv \sum_{n-m=k} \left(|\langle n|[a,\omega]|m\rangle|^2 + |\langle m|[a,\omega]|n\rangle|^2 + \theta|\langle n|\omega|m\rangle|^2 \frac{V'(\lambda_n) - V'(\lambda_m)}{\lambda_n - \lambda_m} \right) \\ &\geq 0 \end{aligned} \tag{84}$$

for $k \geq 1$. For each fixed k the argument is quite similar to the proof of the previous theorem. We put $\alpha_n = \langle n+k|\omega|n\rangle$ which can be assumed to be real for the purpose of proving positivity. We see that $\Sigma_k(\omega)$ is a quadratic form $2Q_k$ in the variables α_n. As in the previous proof the matrix representing Q_k has only nonvanishing matrix elements on the diagonal and next to it, and they are given by

$$q_{nn} = 2n + 1 + k + \gamma_n \tag{85}$$

$$q_{nn-1} = -\sqrt{n(n+k)} \tag{86}$$

$$q_{nn+1} = -\sqrt{(n+1)(n+1+k)} \tag{87}$$

and

$$\gamma_n = \frac{\theta}{2} \frac{V'(\lambda_n) - V'(\lambda_{n+k})}{\lambda_n - \lambda_{n+k}}. \tag{88}$$

The positivity of this form is equivalent to the positivity of the numbers C_n defined inductively by

$$C_0 = 1 + k + \gamma_0 \tag{89}$$

and

$$C_n = 2n + 1 + k + \gamma_n - \frac{n(n+k)}{C_{n-1}}, \quad n = 1, 2, \ldots \tag{90}$$

by the same row and column argument as in the proof of Theorem 1. The case $k = 1$ is taken care of by the argument in Theorem 1 since the eigenvalues λ_n form a decreasing sequence. In order to prove the positivity of C_n for general values of k we observe, using Eq.(5), that

$$q_{nn} = (2n+k+1)\frac{\lambda_{n+1} - \lambda_{n+k+1}}{\lambda_n - \lambda_{n+k}} + n\frac{\Delta\lambda_n - \Delta\lambda_{n+k+1}}{\lambda_n - \lambda_{n+k}} + (n+k)\frac{\Delta\lambda_{n+1} - \Delta\lambda_{n+k}}{\lambda_n - \lambda_{n+k}}\ , \tag{91}$$

where $\Delta\lambda_n = \lambda_{n-1} - \lambda_n\ , \quad n \geq 1$. Furthermore,

$$\Delta\lambda_n > \Delta\lambda_{n+1} \tag{92}$$

for $n \geq 1$, since $V'(\lambda_n) > 0$ for $n \geq 1$ in the case at hand, $N = 0$. We have

$$C_0 = (k+1)\frac{\lambda_1 - \lambda_{k+1}}{\lambda_0 - \lambda_k} + k\frac{\Delta\lambda_1 - \Delta\lambda_k}{\lambda_0 - \lambda_k} \tag{93}$$

and therefore, since $\Delta\lambda_1 \geq \Delta\lambda_k$,

$$C_0 \geq (k+1)\frac{\lambda_1 - \lambda_{k+1}}{\lambda_0 - \lambda_k}. \tag{94}$$

Finally, using Eqs. (91) and (92), it follows by induction that

$$C_n \geq (n+1+k)\frac{\lambda_{n+1} - \lambda_{n+k+1}}{\lambda_n - \lambda_{n+k}} \tag{95}$$

and the proof is complete.

Proof of Theorem 4. We only need to consider $N \geq 1$. As explained in the proof of Theorem 3 it suffices to show that there exists a number θ_c such that the C_i's, defined inductively by Eqs. (89) and (90), are positive for each value of $k = 0, 1, 2, \ldots$, provided $\theta \geq \theta_c$. Note that for $k = 0$ we simply have $\gamma_i = \frac{1}{2}\theta V''(\lambda_i)$.

We begin by discussing the case $k = 0$ and choose θ_c such that

$$V''(\lambda_m) \geq 0 \tag{96}$$

for $\theta \geq \theta_c$ and $m = 0, 1, \ldots$. Then $C_0 \geq 1$ and it follows easily by induction that $C_m \geq m + 1$ for $m > 0$.

The case $k = 1$ follows from the proof of Theorem 2 since $\{\lambda_n\}$ is by construction monotonically decreasing.

In general $\{V'(\lambda_n)\}$ is not a positive decreasing sequence for $n \geq 1$ so the argument used in the proof of Theorem 3 does not generalize and we will need to use information about the asymptotic behaviour of the eigenvalues of φ_N as $\theta \to \infty$.

We begin by analysing the asymptotic beahaviour of the eigenvalues of φ_N regarded as functions of θ. By Theorem 1 we can write the eigenvalues as

$$\lambda_i(\theta) = s - r_i(\theta), \quad i = 0, 1, \ldots, N \tag{97}$$

$$\lambda_i(\theta) = r_i(\theta), \quad i = N+1, N+2, \ldots, \tag{98}$$

where $r_i(\theta) \to 0$ as $\theta \to \infty$ for all i. The potential function V is assumed to be C^2 and $V''(0) > 0$, $V''(s) > 0$ so the equation of motion (7) used for $m = 0$ implies that

$$r_0(\theta) - r_1(\theta) = -\frac{\theta}{2}\left[V''(s)r_0(\theta) + o(r_0(\theta))\right] \tag{99}$$

which shows that $\theta r_0(\theta) \to 0$ as $\theta \to \infty$. Repeating this argument for the next values of m we find that

$$\theta r_i(\theta) \to 0, \quad i = 0, 1, \ldots N-1. \tag{100}$$

Using (100) in the equation of motion for $m = 0, 1, \ldots, N-1$ we find by an analogous argument that

$$\theta^2 r_i(\theta) \to 0, \quad i = 0, 1, \ldots N-2. \tag{101}$$

Continuing in the same vein we obtain

$$\theta^{N-i} r_i(\theta) \to 0 \ \text{ as } \ i = 0, 1, \ldots N-1. \tag{102}$$

Using (102) in Eq. (7) with $m = N$ gives

$$\theta V'(\lambda_N(\theta)) \to -2(N+1)s, \tag{103}$$

which implies

$$r_N(\theta) = \frac{2(N+1)s}{V''(s)\theta} + o(\theta^{-1}). \tag{104}$$

Continuing this argument we find

$$r_N(\theta) \sim \frac{d_N}{\theta}, \ r_{N-1}(\theta) \sim \frac{d_{N-1}}{\theta^2}, \ldots, r_0(\theta) \sim \frac{d_0}{\theta^{N+1}}, \tag{105}$$

where

$$d_N = \frac{2(N+1)s}{V''(s)}. \tag{106}$$

We do not need the explicit values of d_i for $i = 0, \ldots N-1$. Using (105) in Eq. (7) with $m = N+1$ yields

$$r_{N+1}(\theta) \sim \frac{d_{N+1}}{\theta}, \tag{107}$$

where

$$V''(0) d_{N+1} = V''(s) d_N = 2(N+1)s\ . \tag{108}$$

Taking now $m > N+1$ in Eq. (7) we find

$$\theta r_i(\theta) \to 0 \ \text{ as } \ \theta \to \infty \tag{109}$$

for $i \geq N+2$. Continuing the analysis in the same fashion as for $i \leq N$ we obtain the bound

$$r_i(\theta) = O(\theta^{N-i}) \tag{110}$$

for $i \geq N+2$. This completes our discussion of the behaviour of the eigenvalues of φ_N for large θ.

We now use the asymptotic behaviour of the λ_i's to find the asymptotic behaviour of the γ_i's. This is a straightforward calculation using Eq. (5) and Eqs. (105)-(110). The results can be summarized as follows:

k = 2

$$m \leq N-2 \quad : \quad \gamma_m = \frac{\theta}{2} V''(s) + O(1) \tag{111}$$

$$m \geq N+1 \quad : \quad \gamma_m = \frac{\theta}{2} V''(0) + O(1) \tag{112}$$

$$m = N-1 \quad : \quad \gamma_m = -(N+1) + \frac{(N+2)d_{N+1} + d_N}{s\theta} + O(\theta^{-2}) \tag{113}$$

$$m = N \quad : \quad \gamma_m = -(N+1) + \frac{N d_N - d_{N+1}}{s\theta} + O(\theta^{-2}) \tag{114}$$

k ≥ 3

$$m+k \leq N \quad : \quad \gamma_m = \frac{\theta}{2} V''(s) + O(1) \tag{115}$$

$$m \geq N+1 \quad : \quad \gamma_m = \frac{\theta}{2} V''(0) + O(1) \tag{116}$$

$$m+k = N+1 \quad : \quad \gamma_m = -(N+1) + \frac{(N+1)d_N + (N+2)d_{N+1}}{s\theta} + O(\theta^{-2}) \tag{117}$$

$$m = N \quad : \quad \gamma_m = -(N+1) + \frac{(N+1)d_{N+1} + N d_N}{s\theta} + O(\theta^{-2}) \tag{118}$$

$$m+k = N+2 \quad : \quad \gamma_m = -\frac{N d_N \delta_{k3} + (N+2)d_{N+1}}{s\theta} + O(\theta^{-2}) \tag{119}$$

$$m = N-1 \quad : \quad \gamma_m = -\frac{(N+2)d_{N+1}\delta_{k3} + N d_N}{s\theta} + O(\theta^{-2}) \tag{120}$$

$$\text{All other cases} \quad : \quad \gamma_m = O(\theta^{-2}). \tag{121}$$

All the correction terms to the above asymptotic expressions are uniform in k and m for $\theta \geq \theta_c$ and θ_c sufficiently large.

We are now ready to show that $C_m > 0$ for all $k \geq 2$ provided θ is sufficiently large. First, we note that it is an immediate consequence of the preceding asymptotic formulae and the recursion relations (89) and (90) that $C_m > 0$ for $n \leq N-k$ and $\theta \geq \theta_c$, if θ_c is large enough. It is convenient to separate the discussion of the remaining values of m into two cases depending on whether $N-k \geq 0$ or not.

Case I. $N - k \geq 0$. By Eqs. (111) and (115),

$$C_0 = k + 1 + \gamma_0 \geq k + 1 + \frac{\theta}{2} V''(s) + O(1). \tag{122}$$

Choosing θ_c sufficiently large we also have

$$\gamma_0, \ldots, \gamma_{N-k} > 0 \tag{123}$$

and by induction

$$C_m \geq m + k + 1 + \gamma_m \geq \frac{\theta}{2} V''(s) + O(1) \tag{124}$$

for $m = 0, 1, \ldots, N - k$.

I.a. Assume first that $k = 2$. Then we find, using the asymptotic formulae above,

$$C_{N-1} = N + \frac{(N+2)d_{N+1} - (N-2)d_N}{s\theta} + O(\theta^{-2}) \tag{125}$$

and

$$C_N = \frac{4d_N + (N^2 + 3N + 4)d_{N+1}}{Ns\theta} + O(\theta^{-2}). \tag{126}$$

Choosing θ_c large enough C_{N-1} and C_N are positive and

$$\begin{aligned} C_{N+1} &= 2(N+1) + 3 + \gamma_{N+1} - \frac{(N+1)(N+3)}{C_N} \\ &\geq \frac{2\theta V''(0)}{N^2 + 3N + 4} + O(1). \end{aligned} \tag{127}$$

For θ sufficiently large $C_{N+1} \geq N + 2$ and it follows by induction that $C_m \geq m + 1$ for $m \geq N + 2$ if θ_c is so large that $\gamma_m \geq 0$ for $m \geq N + 2$.

I.b. Assume next that $k = 3$. Then we find by a calculation similar to the one in I.a:

$$C_{N-2} = N - 1 + \frac{3d_N + (N+2)d_{N+1}}{s\theta} + O(\theta^{-2}) \tag{128}$$

$$C_{N-1} = N + \frac{3(N+2)(d_N + d_{N+1})}{(N-1)s\theta} - \frac{Nd_N}{s\theta} + O(\theta^{-2}) \tag{129}$$

$$C_N = \frac{(N+1)d_{N+1} - 3d_N}{s\theta} + 3\,\frac{(N+2)(N+3)(d_{N+1} + d_N)}{N(N-1)s\theta} + O(\theta^{-2}) \tag{130}$$

$$C_{N+1} = \frac{\theta}{2}\frac{18(N+1)V''(0)}{(N+1)^3 + 11N + 17} + O(1). \tag{131}$$

Choosing θ_c sufficiently large the above coefficients are all positive and taking θ_c so large that $C_{N+1} \geq N + 2$ and $\gamma_m \geq 0$ for $m \geq N + 2$ we conclude by induction that all the C_m's are positive.

I.c. Now we consider the case $k \geq 4$. The calculation is analogous to the one given above for $k = 2$ and $k = 3$. We evaluate $C_{N+1-k}, C_{N+2-k}, \ldots C_N$ to order θ^{-1} and find that $C_{N+1-i} = N + 2 - i + O(\theta^{-1})$ for $i = 2, \ldots, k$ and then

$$C_N \geq \left(N + 1 + k\frac{(N+k)\cdots(N+2)}{N\cdots(N+2-k)}\right)\frac{d_{N+1}}{s\theta} + O(\theta^{-2}) \tag{132}$$

$$C_{N+1} \geq \frac{\theta}{2}V''(0)\left(1 - (N+1+k)\left(N + 1 + k\frac{(N+k)\cdots(N+2)}{N\cdots(N+2-k)}\right)^{-1}\right) + O(1)\,.$$

Noting that the coefficient of θ in the last expression is positive we proceed to show by induction as before that $C_m > 0$ for all m provided θ_c is chosen large enough.

Case II. $k \geq N + 1$. Again it is convenient to split the argument into different subcases.

II.a. If $N + 1 = k = 2$ then from the asymptotic formulae we find

$$C_0 = 1 + \frac{3d_2 + d_1}{s\theta} + O(\theta^{-2}) \tag{133}$$

$$C_1 = \frac{4d_1 + 8d_2}{s\theta} + O(\theta^{-2}) \tag{134}$$

$$C_2 \geq \frac{\theta}{4}V''(0) + O(1) \tag{135}$$

and the argument can be completed by induction as before, provided θ_c is taken large enough.

II.b. In the case $N = 1$ and $k \geq 3$ we find

$$C_0 = k + 1 - \frac{3d_2\delta_{k3} + d_1}{s\theta} + O(\theta^{-2}) \tag{136}$$

$$C_1 = k + \left(2 - \frac{3\delta_{k3}}{1+k}\right)\frac{d_2}{s\theta} + \frac{kd_1}{(k+1)s\theta} + O(\theta^{-2}). \tag{137}$$

Choosing θ_c sufficiently large we find that $C_0 > 0$, $C_1 \geq 2$ and $\gamma_m > 0$ for $m \geq 2$. It follows as before that $C_m \geq m + 1$ for $m \geq 2$.

II.c. Consider $N + 1 = k = 3$. The crucial coefficients in this case are C_2 which is of order θ^{-1} and C_3 which diverges at large θ. We find

$$C_2 = \frac{33d_3 + 27d_2}{s\theta} + O(\theta^{-2}) \tag{138}$$

$$\geq \frac{198}{V''(0)\theta} + O(\theta^{-2}) \tag{139}$$

and consequently

$$C_3 = \frac{9V''(0)\theta}{22} + O(1). \tag{140}$$

Taking θ_c large we can now complete the argument by induction as before.

II.d. The case $N = 2$ and $k \geq 4$ is quite similar to II.b. We omit the details which are straightforward.

II.e. Consider the case $N+1 = k \geq 4$. We calculate the C_m inductively, starting with C_0 and keeping terms to order θ^{-1}. We find eventually

$$C_{N-1} = N + \frac{(N+1)(N+2)\cdots(2N)}{2\cdot 3\cdots(N-1)}\left(\frac{d_N + d_{N+1}}{s\theta}\right) - \frac{N d_N}{s\theta} + O(\theta^{-2}) \qquad (141)$$

and after a short calculation

$$C_N \geq \frac{(N+1)d_{N+1}}{s\theta}\left(1 + \frac{(N+2)(N+3)\cdots(2N+1)}{2\cdot 3\cdots N}\right) + O(\theta^{-2}) \qquad (142)$$

which implies

$$C_{N+1} \geq \frac{\theta}{2}V''(0)\left(1 - 2\left(1 + \frac{(2N+1)!}{N!(N+1)!}\right)^{-1}\right) + O(1) \qquad (143)$$

and allows us to complete the argument by induction provided θ_c is large enough.

II.f. The remaining cases $N \geq 3$ and $k \geq N+2$ are simpler than those discussed above. One finds that none of the C_m's approaches zero for large θ. We omit the details.

This completes the proof of Theorem 4.

We end this section by commenting briefly on how to extend the stability results to dimensions $d > 1$. Even though the eigenvalues of the rotationally invariant operators are degenerate in this case the extension of the formula (69) for the stability functional Σ is straightforward to derive if the potential V is analytic in a neighborhood of the interval $[0, s]$, as we are assuming.

If we have a solution $\varphi = \sum \lambda_n P_n$ to Eq. (3), we find by the analytic functional calculus that

$$\begin{aligned} \Sigma(\omega) \quad = \quad & \sum_{n=0}^{\infty}\left(2n + d + \frac{\theta}{2}V''(\lambda_n)\right)\|P_n\omega P_n\|_2^2 \\ + \quad & 2\sum_{m<n}\left(n + m + d + \frac{\theta}{2}\frac{V'(\lambda_n) - V'(\lambda_m)}{\lambda_n - \lambda_m}\right)\|P_n\omega P_m\|_2^2 \\ - \quad & 2\sum_{k=1}^{d}\sum_{\underline{n},\underline{m}}\sqrt{(n_k+1)(m_k+1)}\langle\underline{n}+\delta_k|\omega|\underline{m}+\delta_k\rangle\langle\underline{n}|\omega|\underline{m}\rangle\ , \end{aligned} \qquad (144)$$

where, as usual, the P_n are the spectral projections of the number operator, and the standard harmonic oscillator basis vectors are $|\underline{n}\rangle$, where $\underline{n} = (n_1, \ldots, n_d)$ is a multi-index of non-negative integers. Furthermore, $\delta_1, \ldots, \delta_d$ denotes the standard orthonormal basis for $\mathbf{R}^d$.

We see that Σ only couples the matrix elements of ω diagonally, i.e., it suffices to show that $\Sigma(\omega) \geq 0$ for

$$\langle \underline{n}|\omega|\underline{m}\rangle = 0 \quad \text{unless} \;\; \underline{n} - \underline{m} = \pm\underline{\ell} \,, \tag{145}$$

where $\underline{\ell}$ is an arbitrary integer multi-index, with $|\underline{\ell}| \equiv \ell_1 + \cdots + \ell_d \geq 0$.

Consider first the case $|\underline{\ell}| = 0$, in which the second sum on the right hand side of Eq. (144) does not contribute. If $V''(\lambda_n) \geq 0$ for all n, we have

$$\Sigma(\omega) \geq \Sigma_1^0(\omega) + \cdots + \Sigma_d^0(\omega) \,, \tag{146}$$

where

$$\begin{aligned} \Sigma_k^0(\omega) \;&=\; \sum_{|\underline{n}|=|\underline{m}|} (n_k + m_k + 1)|\langle \underline{n}|\omega|\underline{m}\rangle|^2 \\ &-\;\; 2 \sum_{|\underline{n}|=|\underline{m}|} \sqrt{(n_k+1)(m_k+1)}\langle \underline{n}+\delta_k|\omega|\underline{m}+\delta_k\rangle\langle \underline{n}|\omega|\underline{m}\rangle \,. \end{aligned} \tag{147}$$

The contribution to this expression from any fixed values of n_i and m_i, for $i \neq k$, is a quadratic form in the the matrix elements

$$\langle \underline{n}|\omega|\underline{m}\rangle = \langle n_1, \ldots, m_k + \ell_k, \ldots, n_d|\omega|m_1, \ldots, m_k, \ldots, m_d\rangle \,, \tag{148}$$

that may be assumed to be real. It is a simple matter to verify that this quadratic form is positive definite. Therefore, so is $\Sigma(\omega)$ for $|\underline{\ell}| = 0$, provided the condition $V''(\lambda_n) \geq 0$ holds.

For $|\underline{\ell}| \neq 0$ the first sum on the right hand side of Eq. (144) does not contribute. For the coefficient of $\|P_n\omega P_m\|_2^2$ in the second sum one obtains the value

$$(n+m+d)\frac{\lambda_{n+1} - \lambda_{m+1}}{\lambda_n - \lambda_m} + n\frac{\Delta\lambda_n - \Delta\lambda_{m+1}}{\lambda_n - \lambda_m} + m\frac{\Delta\lambda_{n+1} - \Delta\lambda_m}{\lambda_n - \lambda_m} \tag{149}$$

by using Eq. (3) in the form

$$(n+d)\Delta\lambda_n - n\Delta\lambda_{n-1} = \frac{\theta}{2}V'(\lambda_n) \,, \tag{150}$$

where $\Delta\lambda = \lambda_{n+1} - \lambda_n$, $n \geq 1$. This allows us to write

$$\frac{1}{2}\Sigma(\omega) = \Sigma_1(\omega) + \cdots + \Sigma_d(\omega) \,, \tag{151}$$

where

$$\begin{aligned}\Sigma_k(\omega) &= \sum_{\underline{n}=\underline{m}+\underline{\ell}} \{(n_k+m_k+1)\frac{\lambda_{n+1}-\lambda_{m+1}}{\lambda_n-\lambda_m} \\ &+ n_k\frac{\Delta\lambda_n-\Delta\lambda_{m+1}}{\lambda_n-\lambda_m}+m_k\frac{\Delta\lambda_{n+1}-\Delta\lambda_m}{\lambda_n-\lambda_m}\}|\langle\underline{n}|\omega|\underline{m}\rangle|^2 \\ &- 2\sum_{\underline{n}=\underline{m}+\underline{\ell}}\sqrt{(n_k+1)(m_k+1)}\langle\underline{n}+\delta_k|\omega|\underline{m}+\delta_k\rangle\langle\underline{n}|\omega|\underline{m}\rangle\ . \end{aligned} \tag{152}$$

Considering terms with fixed values of $n_i, m_i,\ i \neq k$, in this expression one obtains a quadratic form in the matrix elements that can be handled by an analysis similar to the one that was carried out for the case $d = 1$. We do not elaborate further on the general case here but note that the analysis of the one-soliton case, $N = 0$, of Theorem 3, generalises immediately to Σ_k. This result is obtained by observing that the sequence $\{\Delta\lambda_n\}$ is again decreasing in this case as a consequence of Eq. (150) since $V'(\lambda_n) > 0$ for $n \geq 1$. Thus, Theorem 3 also holds for $d > 1$.

5 Nonexistence of smooth families

In [21] we proved that rotationally symmetric solutions to Eq. (3) do not exist for sufficiently small values of θ. The purpose of this section is to prove non-existence of smooth families of solutions for small θ without assuming rotational symmetry. By a smooth family of solutions we mean a mapping from an interval $I \subset \mathbf{R}$ to $\mathcal{H}_{2,2}$,

$$I \ni \theta \mapsto \varphi_\theta \in \mathcal{H}_{2,2}, \tag{153}$$

which is continuously differentiable in the norm topology of $\mathcal{H}_{2,2}$.

The proof is based on three lemmas below which are most conveniently established by representing operators by functions via a quantization map. The Weyl or Weyl-Wigner quantization is perhaps the best known quantization map. It can be defined as the mapping π_W which to a function $f(x,p)$ of $2d$ variables, $x, p \in \mathbf{R}^d$, associates an operator $\pi_W(f)$ on $L^2(\mathbf{R}^d)$ whose kernel $K_W(f)$ is given by

$$K_W(f)(x,y) = (2\pi)^{-d}\int_{\mathbf{R}^d} f\left(\frac{x+y}{2},\ p\right)\ e^{i(x-y)\cdot p}dp\ . \tag{154}$$

It is obvious that π_W maps Schwartz functions on $\mathbf{R}^{2d}$ bijectively onto operators whose kernels are Schwartz functions and also maps tempered distributions onto operators whose kernels are tempered distributions. More important for the following is the easily verifiable fact that π_W maps $L^2(\mathbf{R}^{2d})$ isometricaly (up to a factor $(2\pi)^{d/2}$) onto the space of Hilbert-Schmidt operators on $L^2(\mathbf{R}^d)$,

$$\|\pi_W(f)\|_2^2 = \int_{\mathbf{R}^{2d}} |K_W(f)(x,y)|^2 \, dxdy = (2\pi)^{-d} \int_{\mathbf{R}^{2d}} |f(x,p)|^2 \, dxdp. \tag{155}$$

We shall find it more convenient to use the so called Kohn-Nirenberg quantization π for which the kernel $K(f)$ of $\pi(f)$ is given by

$$K(f)(x,y) = (2\pi)^{-d} \int_{\mathbf{R}^d} f(x,p) \, e^{i(x-y)\cdot p} dp. \tag{156}$$

The quantization map π clearly has the same properties as the ones we described for π_W above. Likewise, the following properties of π are shared by π_W except for the last one:

(a) If $\pi(f)$ is of trace class then

$$\mathrm{Tr}\ \pi(f) = \int_{\mathbf{R}^d} K(f)(x,x) dx = (2\pi)^{-d} \int_{\mathbf{R}^{2d}} f(x,p) dxdp\ . \tag{157}$$

(b) If g depends only on x we have

$$\pi(g(x)) = g(x)\ , \tag{158}$$

where the right hand side is to be interpreted as a multiplication operator.

(c) If h depends only on p we have

$$\pi(h(p)) = h(\frac{1}{i}\nabla_x)\ . \tag{159}$$

(d) If g and h are as above, then

$$\pi(g(x)f(x,p)h(p)) = g(x)\pi(f)h(\frac{1}{i}\nabla_x)\ . \tag{160}$$

From (b) and (c) it follows that

$$a_k = \frac{1}{\sqrt{2}}(x_k + \partial_{x_k}) = \frac{1}{\sqrt{2}}\pi(x_k + ip_k) \tag{161}$$

and

$$a_k^* = \frac{1}{\sqrt{2}}(x_k - \partial_{x_k}) = \frac{1}{\sqrt{2}}\pi(x_k - ip_k)\ . \tag{162}$$

From the definition of π one then obtains

$$[a_k, \pi(f)] = \frac{1}{\sqrt{2}}\pi(\partial_{x_k} f + i\partial_{p_k} f) \tag{163}$$

and

$$[a_k^*, \pi(f)] = \frac{-1}{\sqrt{2}}\pi(\partial_{x_k} f - i\partial_{p_k} f) \ . \tag{164}$$

Consequently,

$$2\sum_k [a_k^*, [a_k, \pi(f)]] = \pi(\Delta f) \ , \tag{165}$$

where Δ is the Laplace operator on $\mathbf{R}^{2d}$, and the (complexification of) the space $\mathcal{D}$ introduced in Section 2 is just the image under π of the domain of definition of the self-adjoint operator Δ. Notice, however, that contrary to π_W the quantization map π does not generally map real-valued functions to self-adjoint operators.

There is to our knowledge no known simple characterisation of the subspace of $L^2(\mathbf{R}^{2d})$ consisting of functions f such that $\pi(f)$ is of trace class. We shall need the following result, depending crucially on property (d) above, concerning such functions. Here $\|\cdot\|_1$ denotes the standard trace norm.

Lemma 4. *Suppose f is a square integrable function such that $\pi(f)$ is of trace class. Then its Fourier transform $\mathcal{F}(f)$ is bounded and its uniform norm $\|\mathcal{F}(f)\|_\infty$ satisfies the inequality*

$$\|\mathcal{F}(f)\|_\infty \leq \|\pi(f)\|_1. \tag{166}$$

Proof. First, note that $\pi(e^{-i\xi\cdot x}) = e^{-i\xi\cdot x}$ and $\pi(e^{-ip\cdot\eta}) = e^{-\eta\cdot\nabla_x}$ are unitary operators. Hence,

$$\pi(e^{-i\xi\cdot x}f(x,p)e^{-ip\cdot\eta}) = e^{-i\xi\cdot x}\pi(f)e^{-\eta\cdot\nabla_x} \tag{167}$$

is of trace class and using properties (a) and (d) above we have

$$\begin{aligned}\mathcal{F}(f)(\xi,\eta) &= \int_{\mathbf{R}^{2d}} e^{-i\xi\cdot x}f(x,p)e^{-ip\cdot\eta}dxdp \\ &= \mathrm{Tr}\ \{\pi(e^{-i\xi\cdot x}f(x,p)e^{-ip\cdot\eta})\} = \mathrm{Tr}\ \{e^{-i\xi\cdot x}\pi(f)e^{-\eta\cdot\nabla_x}\}\ ,\end{aligned} \tag{168}$$

and hence

$$|\mathcal{F}(f)(\xi,\eta)| \leq \mathrm{Tr}\ (|\pi(f)|) = \|\pi(f)\|_1\ , \tag{169}$$

which proves the assertion.

Using the above result we get the following a priori estimate relating the Hilbert-Schmidt and trace norms of any solution of Eq. (3).

Lemma 5. *There exists a constant C, depending only on V, such that any solution φ of Eq. (3) fulfills*

$$\|\varphi\|_2 \leq C\theta^{\frac{d}{2}}\|\varphi\|_1. \tag{170}$$

Proof. Since both φ and $V'(\varphi)$ are Hilbert-Schmidt there exist square integrable functions f and F such that $\varphi = \pi(f)$ and $V'(\varphi) = \pi(F)$. By Eq. (165) the equation of motion (3) may be written as

$$\Delta f + \theta F = 0 \tag{171}$$

or, equivalently,

$$\mathcal{F}(f)(\xi,\eta) = \frac{-\theta}{|\xi|^2+|\eta|^2}\mathcal{F}(F)(\xi,\eta). \tag{172}$$

Using Lemma 4 and the fact that for an appropriate constant c,

$$\|\mathcal{F}(F)\|_{L^2} = (2\pi)^{2d}\|V'(\varphi)\|_2 \leq c\|\varphi\|_2, \tag{173}$$

we get

$$\begin{aligned}
(2\pi)^d\|\varphi\|_2^2 &= \|\mathcal{F}(f)\|_{L^2}^2 \\
&= \int_{|\xi|^2+|\eta|^2\leq\delta^2} |\mathcal{F}(f)|^2\, d\xi d\eta + \int_{|\xi|^2+|\eta|^2>\delta^2} |\mathcal{F}(f)|^2\, d\xi d\eta \\
&= \int_{|\xi|^2+|\eta|^2\leq\delta^2} |\mathcal{F}(f)|^2\, d\xi d\eta + \theta^2\int_{|\xi|^2+|\eta|^2>\delta^2} \frac{|\mathcal{F}(F)|^2}{(|\xi|^2+|\eta|^2)^2}\, d\xi d\eta \\
&\leq \text{const }\delta^{2d}\|\mathcal{F}(f)\|_\infty^2 + \frac{\theta^2}{\delta^4}\|\mathcal{F}(F)\|_{L^2}^2 \\
&\leq \text{const }\delta^{2d}\|\varphi\|_1^2 + c\frac{\theta^2}{\delta^4}\|\varphi\|_2^2
\end{aligned} \tag{174}$$

for some constant c. If we now let $\delta^4 = c\theta^2$, the result follows.

Our next goal is to obtain a lower bound on the Hilbert-Schmidt norm of solutions to Eq. (3) .

Lemma 6. *There exists a constant C', depending only on the potential V, such that any non-zero solution φ of Eq. (3) satisfies the inequality*

$$C'\theta^{-\frac{d}{2}} \leq \|\varphi\|_2. \tag{175}$$

Proof. Let $\varphi = \sum_n \lambda_n P_n$ be the spectral decomposition of φ, and set, for $a > 0$,

$$\varphi_{<a} = \sum_{\lambda_n < a} \lambda_n P_n \quad \text{and} \quad \varphi_{\geq a} = \sum_{\lambda_n \geq a} \lambda_n P_n . \tag{176}$$

By our assumptions about V we can fix $a > 0$ and a constant c_1 such that $V'(\varphi_{<a})$ is positive and

$$\|\varphi_{<a}\|_1 \leq c_1 \|V'(\varphi_{<a})\|_1 \ . \tag{177}$$

Now, using that $\|\varphi\| \leq s$ and $\mathrm{Tr}\ (V'(\varphi)) = 0$ by Lemma 2, we can estimate $\|V'(\varphi_{<a})\|_1$ as follows:

$$\|V'(\varphi_{<a})\|_1 = -\mathrm{Tr}\ (V'(\varphi_{\geq a})) \leq \|V'(\varphi_{\geq a})\|_1 \leq c_2 \|\varphi_{\geq a}\|_1 \tag{178}$$

for an appropriate constant c_2. Thus,

$$\|\varphi_{<a}\|_1 \leq c_3 \|\varphi_{\geq a}\|_1 \ , \tag{179}$$

where $c_3 = c_1 c_2$. From this we deduce

$$\begin{aligned} \|\varphi\|_1 &= \|\varphi_{<a}\|_1 + \|\varphi_{\geq a}\|_1 \\ &\leq (1 + c_3) \|\varphi_{\geq a}\|_1 \\ &\leq c_4 \|\varphi\|_2^2 \ , \end{aligned} \tag{180}$$

where $c_4 = (1 + c_3)/a$. Finally, from (180) and the a priori estimate of Lemma 5, we get

$$\|\varphi\|_1 \leq C c_4 \theta^{\frac{d}{2}} \|\varphi\|_2 \|\varphi\|_1 \tag{181}$$

from which the claimed inequality follows.

We are now in a position to prove the announced non-existence result.

Theorem 5. *Let V be analytic on a neighbourhood of the interval $[0, s]$. Suppose*

$$(a, b] \ni \theta \mapsto \varphi_\theta \in \mathcal{H}_{2,2} \ , \tag{182}$$

where $0 \leq a < b$, is a smooth map such that φ_θ is a nonzero solution of the equation of motion (3) for each $\theta \in (a, b)$. Then $a > 0$.

Proof. Since φ_θ is a solution to Eq. (3) the derivative of the energy $S(\varphi_\theta)$ with respect to θ is given by

$$\frac{d}{d\theta} S(\varphi_\theta) = \mathrm{Tr}\ V(\varphi_\theta). \tag{183}$$

This is easy to prove using the analytic functional calculus. Since V is positive definite, it satisfies an estimate of the form

$$V(\varphi) \geq \text{ const } \varphi^2 \tag{184}$$

and hence, by Lemma 6,

$$\frac{d}{d\theta} S(\varphi_\theta) \geq C_V \theta^{-d}, \tag{185}$$

where the constant C_V depends only on V (but not on the given family of solutions). Hence, for $d > 1$, the function

$$\theta \mapsto S(\varphi_\theta) + \frac{C_V}{d-1} \theta^{-d+1} \tag{186}$$

is increasing. Now suppose that $a = 0$. Then

$$S(\varphi_\theta) \leq S(\varphi_b) + \frac{C_V}{d-1}(b^{-d+1} - \theta^{-d+1}) \tag{187}$$

which contradicts positivity of $S(\varphi_\theta)$ for small θ.

For $d = 1$ the expresion $\frac{C_V}{d-1}\theta^{-d+1}$ in (186) should be replaced by $-C_V \ln \theta$ and the same conclusion holds. This proves the theorem.

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