ON A SEMILINEAR BLACK AND SCHOLES PARTIAL DIFFERENTIAL EQUATION FOR VALUING AMERICAN OPTIONS. PART I: VISCOSITY SOLUTIONS AND WELL-POSEDNESS

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ABSTRACT. Using the dynamic programming principle in optimal stopping theory, we derive a semilinear Black and Scholes type partial differential equation set in a fixed domain for the value of an American (call/put) option. The nonlinearity in the semilinear Black and Scholes equation depends discontinuously on the American option value, so that standard theory for partial differential equation does not apply. In fact, it is not clear what one should mean by a solution to the semilinear Black and Scholes equation. Guided by the dynamic programming principle, we suggest an appropriate definition of a viscosity solution. Our main results imply that there exists exactly one such viscosity solution of the semilinear Black and Scholes equation, namely the American option value. In other words, we provide herein a new formulation of the American option valuation problem. Our formulation constitutes a starting point for designing and analyzing "easy to implement" numerical algorithms for computing the value of an American option. The numerical aspects of the semilinear Black and Scholes equation are addressed in [7].

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1. Introduction

From the works by Bensoussan [4] and Karatzas [20], it is well known that the *arbitrage-free* price of an American option is the solution of an *optimal stopping problem.* Roughly speaking, the solution of the optimal stopping problem can be determined via two major methodologies: One is based on the *quasi-variational inequality* formulation in the sense of Bensoussan and Lions [5, 6] (see also [18]), while the other is based on *free boundary problem* formulation due to McKean [25] and van Moerbeke [32]. It is well known that there is no (known) explicit solution formula for the value of an American option, as opposed to European options for which an analytical formula exists. Consequently, with both methodologies one must use numerical algorithms to determine the price of an American option. However, the two methodologies lend themselves to different numerical algorithms, each with its own advantages and disadvantages (see, e.g., the review paper [27]).

In this paper we present and analyze a different formulation of the valuation problem for American options, which to our knowledge has not appeared in the literature before. We shall focus on American call and put options for which the payoff at exercise is given by $g(x) = (x - K)^+$ and $g(x) = (K - x)^+$ respectively. K is the contracted strike price. In our formulation, there are no "side constraints" that need to be fulfilled (as in the quasi-variational inequality formulation) nor is there a free boundary that need to be determined (as in the free boundary problem formulation). Hence the proposed formulation constitutes a starting point for designing "easy to implement" numerical algorithms for computing the value of an American option. Roughly speaking, in the new formulation we seek a function $v = v(t, x)$ (its regularity requirements will be discussed later) that satisfies $v(T, x) = g(x)$ and the following semilinear partial differential equation of the Black and Scholes type:

$$\partial_t v + (r - d)x\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv = -q(x, v), \tag{1.1}$$

where $x \geq 0$, $t \in [0, T)$; r, d, σ are given constants; and the *nonlinear reaction term* q takes the form

$$q(x, v) = \begin{cases} 0, & g(x) - v < 0, \\ c(x), & g(x) - v \geq 0, \end{cases} \tag{1.2}$$

for a "cash flow" function $c = c(x)$ defined as

$$c(x) = \begin{cases} (dx - rK)^+, & \text{call option}, \\ (rK - dx)^+, & \text{put option}. \end{cases} \tag{1.3}$$

Note that (1.1) is set in a fixed domain. However, the nonlinearity $v \mapsto q(x, v)$ in (1.1) is *discontinuous*, and this is the "price" we have to pay for *not* having a free boundary explicitly present in our formulation. The fact that $v \mapsto q(x, v)$ is discontinuous also implies that it is not clear how one should interpret the semilinear Black and Scholes equation (1.1) (luckily the dynamic programming principle will provide some guidance here). In fact, the semilinear partial differential equation as it stands in (1.1) does not uniquely identify the

American option value V as its solution unless it is appropriately interpreted (we will come back to this later).

Before explaining the connection between the American option valuation problem and the semilinear Black and Scholes equation (1.1), we would like to stress that our interest in (1.1) is ultimately linked to a desire to design "easy to implement" numerical algorithms. Indeed, in our companion paper [7] we demonstrate that the semilinear Black and Scholes equation can be used to construct very simple numerical algorithms for valuing American options. Using the mathematical framework developed herein, we also prove in [7] that the approximate solutions generated by the algorithms converge to the American option value as the discretization parameters tend to zero.

Referring to Section 3 for details, we will now briefly try to explain the origin of the semilinear Black and Scholes equation (1.1) and, in particular, how it should be interpreted. Since we want to avoid going into too much details in the introduction, let us here only say that the following formulation of the American option valuation problem comes from the dynamic programming principle in optimal stopping theory: Find a function $v = v(t,x)$ that satisfies $v(T,x) = g(x)$ and

$$\begin{cases} \partial_t v + (r-d)x\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv = 0, & \text{when } v > g, \\ -c(x) \le \partial_t v + (r-d)x\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv \le 0, & \text{when } v = g, \\ \partial_t v + (r-d)x\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv = -c(x), & \text{when } v < g, \end{cases} \tag{1.4}$$

where $x \ge 0$, $t \in [0,T)$, and $c(x)$ is defined in (1.3). In (1.4), the region defined by $v > g$ is known as the *hold* (or *continuation*) region and the region defined by $v = g$ is known as the *exercise* (or *optimal stopping*) region. The natural candidate solution for (1.4) is of course the American option value V. It is well known that $V \ge g$, so that a posteriori we see that the region defined by $v < g$ in (1.4) (and the equation that is defined on it) is irrelevant. However, it will become apparent later that it is technically convenient to have the region defined by $v < g$ explicitly present in the formulation (1.4).

To connect (1.4) to the semilinear Black and Scholes equation (1.1), we recall that for a locally bounded function $f : \mathbb{R}^N \to \mathbb{R}$ of $N (\ge 1)$ variables, its *upper* and *lower semicontinuous envelopes*, denoted by f_* and f^* respectively, are defined as

$$f^*(x) = \limsup_{y \to x} f(y), \qquad f_*(x) = \liminf_{y \to x} f(y). \tag{1.5}$$

Observe now that

$$q^*(x,v) = \begin{cases} 0, & g(x) - v < 0, \\ c(x), & g(x) - v \ge 0, \end{cases} \qquad q_*(x,v) = \begin{cases} 0, & g(x) - v \le 0, \\ c(x), & g(x) - v > 0. \end{cases} \tag{1.6}$$

Using (1.6), the dynamic programming inequalities in (1.4) can be stated compactly as

$$\begin{cases} \partial_t v + (r-d)x\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv \ge -q^*(x,v), \\ \partial_t v + (r-d)x\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv \le -q_*(x,v). \end{cases} \tag{1.7}$$

A function v that satisfies the first inequality in (1.7) is called a subsolution. A function v that satisfies the second inequality in (1.7) is called a supersolution. A solution of (1.1) is a function v that is simultaneously a sub- and supersolution. In other words, we shall take the dynamic programming inequalities in (1.7) as the very definition of a solution to (1.1) (we refer to Section 4 for precise statements). At this stage, we would like to remind the reader of the work by Ishii [16] on a class of first order Hamilton-Jacobi equations with discontinuous Hamiltonians. It is a truly remarkable fact that the dynamic programming interpretation (1.7) of the semilinear Black and Scholes equation (1.1) is nothing but an adaption of the solution concept used by Ishii [16].

Summing up, our starting point is the dynamic programming inequalities in (1.4) (they are derived in Section 3). Being familiar with the work by Ishii [16], we observe then that (1.4) can be taken as the definition of a solution to a certain partial differential equation, namely the semilinear Black and Scholes equation (1.1). Of course, it is not clear a priori that (1.1) (when understood in sense of (1.4)) uniquely identifies the American option value as its solution. One can say that it is a purpose of this paper to prove that is so.

Heuristically, it is not difficult to see why the American option value V ought to satisfy the semilinear Black and Scholes equation. But to easily see this, we need to use the free boundary problem formulation [25, 32] and some properties of the free boundary (at a heuristic level we may allow ourselves to do so). Letting $x(t)$ denote the free boundary, it is known that $x(t) > K$ ($x(t) < K$) for a call option (put option). Furthermore, $V = x - K$ ($V = K - x$) in the exercise region, which coincides with the region $x \geq x(t)$ ($x \leq x(t)$). We see from these properties that the reaction term $q(x, v)$ in the semilinear Black and Scholes equation (1.1) vanishes in the hold region $x < x(t)$ ($x > x(t)$), while it is strictly positive in the exercise region $x \geq x(t) > K$ ($x \leq x(t) < K$). Using this, it is a simple exercise (plug in and equate) to check that V satisfies (1.1).

Motivated by Jamshidian [19] and Barone-Adesi and Elliott [3], Kholodnyi [22] has on a heuristic level already observed that the American option value V should satisfy (1.1). Kholodnyi used (as above) the free boundary formulation and its properties [25, 32] to argue in favor of this. In fact, the work in [22] was the initial motivation for the present study. But we stress that the free boundary problem formulation does not lead to the correct interpretation of (1.1); it can can be used only as a heuristic motivation for setting up (1.1). The rigorous correct way to derive (1.1) goes via the dynamic programming principle in optimal stopping theory and it results in (1.4). As an attempt to understand (1.1) from a rigorous mathematical point of view, in [22] the theory of semigroups generated by multivalued operators in weighted Sobolev spaces is applied to a version of (1.1) where the discontinuous reaction term q defined in (1.2) has been replaced by a continuous function. However, this analysis does not apply to (1.1).

In Section 6 we give yet another heuristic motivation for the semilinear Black and Scholes equation (1.1). There we claim that (1.1) can be viewed as an infinitesimal version of the well known *early exercise premium representation of the American option* [9, 17, 23]. This claim comes from setting up an integral version of (1.1) in terms of the heat kernel.

As already mentioned several times, the value of an American option can be found as the solution of a free boundary problem. Free boundary problems occur in a variety of areas in applied science and the philosophy of embedding the solution of such a problem in a larger (fixed) domain is surely not a new one. Many methods for doing so have been developed over the years (the quasi-variational inequality formulation provides an example). We refer to the books by Crank [12] and Elliott and Ockendon [14] for an overview of some of these methods. We would like to mention the papers by Rogers [30] and Berger, Ciment, and Rogers [8], which deal with a free boundary problem arising in the modeling of absorption of oxygen in tissue. These authors rewrite their free boundary problem in terms of a heat equation with a nonlinear reaction term. The authors then use the semilinear heat equation as a motivation for setting up a certain numerical algorithm for their free boundary problem. Also, let us mention the recent papers by Badea and Wang [1, 2] which formulate the American option valuation problem in terms of a partial differential equation that does not "see" the free boundary. These authors derive and analyze a weak variational inequality for the time value $u := v - g$ of an American call option. Although there are some similarities, the formulations and the mathematical tools used in [8, 30] and [1, 2] are different from ours.

A technical aspect that was set aside in the discussion above was the regularity of the solution of (1.1) (in the sense of (1.4)). Unfortunately, the natural solution candidate for (1.1) might not possess all the continuous derivatives up to first order in t and second order in x, and as such does not satisfy (1.1) pointwise everywhere. In other words, (1.1) might not admit a classical ($C^{1,2}$) solution. To have a flexible mathematical framework in which one can easily prove existence, uniqueness, and convergence of approximate solutions, one needs to relax the notion of classical solution so as to allow functions that are not necessarily $C^{1,2}$ as (generalized) solutions. This can be achieved successfully by introducing the notion of *viscosity solutions*, which allows merely continuous functions to be solutions of fully nonlinear first and second order partial differential equations. We refer to Crandall, Ishii, and Lions [11] and Fleming and Soner [15] for a general overview and introduction to the theory of viscosity solutions.

In this paper, the semilinear Black and Scholes equation (1.1) is interpreted in the sense of viscosity solutions. Roughly speaking, a viscosity solution of (1.1) is a function that satisfies (1.4) in the viscosity solution sense. Remarkably, this definition can be viewed as an adaption to (1.1) of Ishii's definition [16] of a viscosity solution for a class of first order Hamilton-Jacobi equations with discontinuous Hamiltonians. The main theoretical results in this paper implies that there exists exactly one viscosity solution of the semilinear Black and Scholes equation (1.1), and this unique viscosity solution is the American option value V. In other words, our formulation of the American option valuation problem (1.1) makes sense when interpreted in terms of (1.4).

The rest of this paper is organized as follows: In Section 2, we recall some basic parts of the arbitrage-free option valuation theory as well as the quasi-variational inequality and free boundary problem formulations. In Section 3, we motivate and derive the semilinear Black and Scholes equation. In Section 4, we define what is meant by a viscosity solution of the semilinear Black and Scholes equation. The well-posedness (existence and uniqueness) of the viscosity solution is proved in Section 5. Finally, Section 6 is devoted to a representation formula for the viscosity solution, which we show is the well known decomposition of an American option value as the sum of the corresponding European option value and an early exercise premium.

2. American option valuation theory

In this section, we review some results concerning the valuation of American (call and put) options written on a dividend paying stock. We refer to the text books by Duffie [13], Karatzas and Shreve [21], and Musiela and Rutkowksi [26] for further references and historical accounts on the problem of pricing American options. Another excellent reference is the review paper by Myneni [27], which deals exclusively with American options.

Suppose that the price dynamics of a dividend paying stock $X(s) = X^{t,x}(s)$ is governed by a geometric Brownian motion (under the equivalent martingale measure Q), i.e., it evolves according to the stochastic differential equation

$$dX(s) = (r-d)X(s)\,ds + \sigma X(s)\,dW(s), \qquad s \in (t,T], \tag{2.1}$$

where $d \geq 0$ is the constant dividend yield for the stock, $r \geq 0$ is the risk-free interest rate, $\sigma > 0$ is the volatility, $\{W(s) \,|\, s \in [0,T]\}$ is a standard Brownian motion, and T is the expiration time of the option contract. Starting at time t with initial condition $X(t) = x$, it is well known that the arbitrage-free value of an American option is given by

$$V(t,x) = \sup_{t \leq \tau \leq T} \mathbb{E}^{t,x}\left[e^{-r(\tau-t)} g(X(\tau))\right], \tag{2.2}$$

where the supremum is taken over all $\mathcal{F}_t$ stopping times $\tau \in [t,T]$ and $\mathbb{E}^{t,x}$ denotes the expectation under the equivalent martingale measure Q conditioned on $X(t) = x$. In this paper we will focus on the payoff function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} (x-K)^+, & \text{call option,} \\ (K-x)^+, & \text{put option,} \end{cases} \tag{2.3}$$

where $K > 0$ is the strike price of the contract.

We recall that $V(t,x)$ defined in (2.2) is the value function of an optimal stopping problem. The following dynamic programming principle holds (see, e.g., Shiryayev [31]): For any $\varepsilon \geq 0$, let

$$\tau_\varepsilon = \tau_\varepsilon^{t,x} := \inf\left\{ s \in [t,T] \,\middle|\, V\left(s, X^{t,x}(s)\right) \leq g\left(X^{t,x}(s)\right) + \varepsilon \right\}. \tag{2.4}$$

Then τ_ε will be an ε-optimal stopping time. For any stopping time $t \leq \theta \leq \tau_\varepsilon$,

$$V(t,x) = \mathbb{E}^{t,x}\left[e^{-r(\theta-t)} V(\theta, X(\theta))\right]. \tag{2.5}$$

Choosing $\varepsilon = 0$, it is well known that τ_0 is an optimal stopping time and the process

$$M(s) := e^{-r(s-t)} V\left(s, X^{t,x}(s)\right), \qquad t \leq s \leq \tau_0 \tag{2.6}$$

is a martingale. From (2.5) one can derive the following dynamical programming principle for the optimal stopping problem (see Krylov [24]): For any stopping time $\theta \in [t,T]$, we have

$$V(t,x) = \sup_{t \leq \tau \leq T} \mathbb{E}^{t,x}\left[\mathbf{1}_{\{\tau<\theta\}} e^{-r(\tau-t)} g(X(\tau)) + \mathbf{1}_{\{\tau \geq \theta\}} e^{-r(\theta-t)} V(\theta, X(\theta))\right]. \tag{2.7}$$

By choosing $\tau = T$, we immediately get

$$V(t,x) \geq \mathbb{E}^{t,x}\left[e^{-r(\theta-t)} V(\theta, X(\theta))\right], \tag{2.8}$$

for any stopping time $\theta \in [t, T]$. Note also that by choosing $\tau = t$, we obtain $V(t,x) \geq g(x)$ (which is the so-called early exercise constraint).

As already mentioned in Section 1, the value function V defined in (2.2) (i.e., the American option value) can be found via two main methodologies.

The first is based on the formulation of Bensoussan and Lions [6, 5]. One determines V by solving the following quasi-variational inequality:

$$\begin{cases} \max\Big(\mathcal{L}_{\mathrm{BS}} v(t,x) - rv(t,x), g(x) - v(t,x)\Big) = 0, & (t,x) \in Q_T, \\ v(T,x) = g(x), & x \in [0,\infty). \end{cases} \tag{2.9}$$

To simplify the notation, we have used $\mathcal{L}_{\mathrm{BS}}$ to designate the usual linear Black and Scholes differential operator

$$\mathcal{L}_{\mathrm{BS}} = \partial_t + (r-d)x\partial_x + \frac{1}{2}\sigma^2 x^2 \partial_x^2.$$

Moreover, Q_T denotes the time-space cylinder $Q_T = [0,T) \times [0,\infty)$. Note that the quasi-variational inequality in (2.9) can be stated equivalently as

$$v \geq g, \qquad \mathcal{L}_{\mathrm{BS}} v - rv \leq 0, \qquad (v-g)(\mathcal{L}_{\mathrm{BS}} v - rv) = 0.$$

It is well known that the value function V is the unique solution (in the sense of Bensoussan and Lions [6, 5]) of the quasi-variational inequality (2.9) (see, e.g., Jaillet, Lamberton, and Lapeyre [18]). We mention also that quasi-variational inequalities and optimal stopping problems can be studied in the sense of viscosity solutions (see, e.g., [28, 29]).

We recall in the passing that the price of a European option with payoff g solves the Black and Scholes partial differential equation

$$\begin{cases} \mathcal{L}_{\mathrm{BS}} v(t,x) - rv(t,x) = 0, & (t,x) \in Q_T, \\ v(T,x) = g(x), & x \in [0,\infty). \end{cases} \tag{2.10}$$

In the second main methodology for determining V, one solves a free boundary problem formulated by McKean [25] and van Moerbeke [32] (see, e.g., [27] for an overview). Letting $x(t)$ denote the free boundary, we introduce the sets

$$\mathcal{C}(t) = \begin{cases} (0, x(t)), & \text{call option}, \\ (x(t), \infty), & \text{put option}, \end{cases} \qquad \mathcal{S}(t) = \begin{cases} [x(t), \infty), & \text{call option}, \\ (0, x(t)], & \text{put option}. \end{cases}$$

Then the free boundary problem takes the following form: Determine a function $v(t,x)$ and a free boundary $x(t)$ such that

$$\begin{cases} \mathcal{L}_{\mathrm{BS}} v(t,x) - rv(t,x) = 0, & t \in [0,T),\ x \in \mathcal{C}(t), \\ v(T,x) = g(x), & x \in [0,\infty), \\ v(t,x) = g(x), & t \in [0,T],\ x \in \mathcal{S}(t), \\ \partial_x v(t,x) = \pm 1, & t \in [0,T],\ x \in \mathcal{S}(t), \end{cases}$$

where "$\pm 1 = 1$" for the call option, "$\pm 1 = -1$" for the put option.

It is well known that the free (or optimal exercise) boundary $x(t)$ possesses the following properties for $t \in [0,T)$:

$$\begin{cases} x(t) > \max\left(\frac{r}{d}K, K\right) \text{ (call option)}, \ x(t) < \min\left(\frac{r}{d}K, K\right) \text{ (put option)}, \\ x \in \mathcal{C}(t) \Longleftrightarrow v(t,x) > g(x), \ \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) = 0, \\ x \in \mathcal{S}(t) \Longleftrightarrow v(t,x) = g(x), \ \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) < 0. \end{cases} \tag{2.11}$$

We note that if $d = 0$, an American call option is equal to a European call with the same strike price (i.e., we do not need to calculate the free boundary). Analogously, if $r = 0$, an American put is equal to a European put with the same price (i.e., we do not need to calculate the free boundary in this case either).

For later use, we end this section by stating some well known properties possessed by the value function V:

Proposition 2.1. *The value function V defined in* (2.2) *belongs to $C\left(\overline{Q_T}\right)$ and satisfies*

$$0 \leq V(t,x) \leq \begin{cases} K, & \textit{put}, \\ x, & \textit{call}, \end{cases} \qquad (t,x) \in \overline{Q_T}.$$

Proof. The continuity of V is taken from Karatzas and Shreve [21]. The lower and upper bounds on V are derived using $0 \leq g(x) \leq K$ for the put and $0 \leq g(x) \leq x$ for the call (see, e.g., Pham [29]). □

In fact, it well known that $V \in C^{1,2}(Q_T) \bigcap C\left(\overline{Q_T}\right)$ except across the free boundary where it is only continuously differentiable in the second variable. However, our mathematical framework presented in Section 4 only requires continuity of candidate solutions. In the rest of the paper we adopt the notation $C^{1,2}$, meaning the set of functions that are once continuously differentiable in t and twice continuously differentiable in x. Moreover, C denotes the set of continuous functions.

3. The semilinear Black and Scholes equation

Starting off from (2.2), we derive in this section the semilinear Black and Scholes equation for valuing American (call/put) options. The derivation is formal since the American option value V is assumed to be $C^{1,2}(Q_T) \bigcap C\left(\overline{Q_T}\right)$ regular. We dispense with this regularity requirement in the sections that follows. This non-technical section serves as a motivation for setting up the semilinear Black and Scholes equation and how to interpret it.

Note from (2.2) that the early exercise constraint holds, i.e.,

$$V \geq g \text{ on } \overline{Q_T}, \tag{3.1}$$

where g is defined in (2.3). Applying Itô's formula to the process

$$Y(s) := e^{-r(s-t)}V\left(s, X^{t,x}(s)\right), \qquad s \in [t,T],$$

yields

$$dY(s) = e^{-r(s-t)}(\mathcal{L}_{\mathrm{BS}} - r)V\left(s, X^{t,x}(s)\right)\,ds + e^{-r(s-t)}\partial_x V\left(s, X^{t,x}(s)\right)\,dW(s).$$

Using (2.8), we get $\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \le 0$ for all $(t,x) \in Q_T$. On the other hand, if $s \in [t, \tau_0]$ for τ_0 defined in (2.4) with $\varepsilon = 0$, then $Y(s) = M(s)$ where $M(s)$ is given in (2.6). Hence, $Y(s)$ is a martingale. This immediately implies that $\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) = 0$ in the continuation region. In conclusion, the following relations must hold for the value of an American option:

$$V \ge g, \qquad \mathcal{L}_{\mathrm{BS}}V - rV \le 0, \qquad (V-g)(\mathcal{L}_{\mathrm{BS}}V - rV) = 0\,.$$

From this we get:

$$\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) = 0, \qquad \text{when } V(t,x) > g(x) \text{ (continuation region)}, \tag{3.2}$$

and

$$\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \le 0, \qquad \text{when } V(t,x) = g(x) \text{ (exercise region)}. \tag{3.3}$$

In the exercise region $\mathcal{L}_{\mathrm{BS}}V - rV$ is nonpositive. However, as we will see, it is possible to derive a lower bound on $\mathcal{L}_{\mathrm{BS}}V - rV$ in this region as well. To this end, fix a point (t,x) in the exercise region. Since $V \in C^{1,2}$, $V(t,x) = g(x)$, and $V \ge g$ everywhere, we say that V *touches* g from above at (t,x), i.e., (t,x) is a local maximizer of $g - V$. Since the payoff function has a kink at $x = K$ (and hence cannot be touched from above by a $C^{1,2}$ function there), we conclude that either $x < K$ or $x > K$. In what follows, we consider the call option, see (2.3). If $x < K$, we must necessarily have that

$$\partial_t V(t,x) = 0, \qquad \partial_x V(t,x) = 0, \qquad \partial_x^2 V(t,x) \ge 0.$$

Plugging this into the equation, we get $\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \ge 0$. However, in view of (3.3), we conclude that

$$\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) = 0, \qquad \text{when } V(t,x) = g(x) \text{ and } x < K\ . \tag{3.4}$$

On the other hand, when $x > K$ we must have

$$\partial_t V(t,x) = 0, \qquad \partial_x V(t,x) = 1, \qquad \partial_x^2 V(t,x) \ge 0.$$

Plugging this into the equation, we now find $\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \ge -(dx - rK)$. However, since $(dx - rK)$ should not become negative in view of (3.3), we conclude that

$$\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \ge -(dx - rK)^+, \qquad \text{when } V(t,x) = g(x) \text{ and } x > K.$$

Summing up, we see that in the exercise region the American call option value satisfies the following inequalities:

$$-(dx - rK)^+ \le \mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \le 0, \qquad \text{when } V(t,x) = g(x). \tag{3.5}$$

Similarly, one show that in the exercise region the American put option value (2.3) satisfies

$$-(rK - dx)^+ \le \mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \le 0, \qquad \text{when } V(t,x) = g(x). \tag{3.6}$$

Using the "cash flow" function $c = c(x)$ defined in (1.3), we can state (3.5) and (3.6) as

$$-c(x) \le \mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) \le 0, \qquad \text{when } V(t,x) = g(x). \tag{3.7}$$

Remark 3.1. Let us discuss the meaning of (3.7). We restrict our discussion to the call option. To this end, suppose $d > r$ and consider (3.5). Let $x(t)$ denote the free boundary of the American call option value. If we take into account the properties (2.11) about the free boundary, we see that in the exercise region there actually holds that $V = x - K$. Plugging this into the Black and Scholes operator, we get

$$\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) = -(dx - rK), \qquad \text{when } V(t,x) = g(x). \tag{3.8}$$

In other words, (3.7) is in fact an equality. The point that we would like to stress is the following one: We are interested in a new, independent formulation of the American option valuation problem which does *not* use any a priori knowledge about the free boundary! Hence, at each point (t,x) in the exercise region ($V(t,x) = g(x)$), we cannot claim that the equation in (3.8) holds, but in general only that the inequalities in (3.7) hold (see also Remark 3.2 below). The inequalities in (3.7) (not only the equation in (3.8)) are built into the formulation that we suggest below. The analysis in this paper shows that our formulation uniquely identifies the American option value (2.2) as its solution.

Motivated by the fact that the American option value V satisfies (3.2) and (3.7), we introduce now the notions of classical sub- and and supersolutions for the American option valuation problem. We define a *classical solution* of the same problem to be a function that is simultaneously a sub- and supersolution (this coincides with (1.4)).

We say that a function $v \in C^{1,2}(Q_T) \bigcap C\left(\overline{Q_T}\right)$ is a *classical subsolution* of the American option valuation problem if $v|_{t=T} \le g$ on $[0,\infty)$ and the following inequalities hold on Q_T:

$$\begin{cases} \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \ge 0, & \text{when } v(t,x) > g(x), \\ \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \ge -c(x), & \text{when } v(t,x) = g(x), \\ \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \ge -c(x), & \text{when } v(t,x) < g(x). \end{cases} \tag{3.9}$$

Note that the American option value V is a classical subsolution whenever it is smooth enough. Since (2.2) satisfies the early exercise constraint (3.1), the last inequality in (3.9) does not matter. Indeed, it has been introduced here only for technical reasons.

Similarly, we say that a function $v \in C^{1,2}(Q_T) \bigcap C\left(\overline{Q_T}\right)$ is a *classical supersolution* of the American option valuation problem if $v|_{t=T} \ge g$ on $[0,\infty)$ and the following inequalities hold on Q_T:

$$\begin{cases} \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \le 0, & \text{when } v(t,x) > g(x), \\ \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \le 0, & \text{when } v(t,x) = g(x), \\ \mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \le -c(x), & \text{when } v(t,x) < g(x). \end{cases} \tag{3.10}$$

Note that the American option value V is a classical supersolution whenever it is smooth enough, and that the last inequality in (3.10) is again irrelevant since (3.1) holds.

To continue, we introduce the functions

$$H^*(\xi) = \begin{cases} 0, & \xi < 0, \\ 1, & \xi \ge 0, \end{cases} \qquad H_*(\xi) = \begin{cases} 0, & \xi \le 0, \\ 1, & \xi > 0. \end{cases}$$

Observe that H^* and H_* are respectively the upper and lower semicontinuous envelopes (see (1.5) for their definitions) of the Heaviside function

$$H(\xi) = \begin{cases} 0, & \xi < 0, \\ 1, & \xi \geq 0. \end{cases} \tag{3.11}$$

Of course, since H is upper semicontinuous, $H^* \equiv H$. Let us introduce the nonlinear functions $q^*, q_* : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$q^*(x,v) = c(x)H^*(g(x) - v), \qquad q_*(x,v) = c(x)H_*(g(x) - v).$$

Using q^*, q_*, we can write the classical subsolution inequality (3.9) more compactly as

$$\mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \geq -q^*(x, v(t,x)), \qquad (t,x) \in Q_T, \tag{3.12}$$

and the classical supersolution inequality (3.10) more compactly as

$$\mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) \leq -q_*(x, v(t,x)), \qquad (t,x) \in Q_T. \tag{3.13}$$

Observe that q^* and q_* are respectively the upper and lower semicontinuous envelopes of the nonlinear function $q = q(x,v)$ defined by

$$q(x,v) = c(x)H(g(x) - v). \tag{3.14}$$

Since $v \mapsto q(x,v)$ is upper semicontinuous, $q^* \equiv q$.

We now introduce the *semilinear Black and Scholes equation*

$$\mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) = -q(x, v(t,x)), \qquad (t,x) \in Q_T. \tag{3.15}$$

We augment (3.15) with the terminal condition

$$v(T,x) = g(x), \qquad x \in [0,\infty). \tag{3.16}$$

Any function $v : \overline{Q_T} \to \mathbb{R}$ that is simultaneously a classical sub- and supersolution (i.e., v satisfies (3.12) and (3.13)) is called a *classical solution* of the semilinear Black and Scholes equation (3.15). If v also satisfies the terminal condition (3.16), we call v a *classical solution* of the terminal value problem (3.15)–(3.16).

Remark 3.2. We note that $v \mapsto q(x,v)$ is a discontinuous and nonincreasing mapping. We remark that the monotonicity property of $v \mapsto q(x,v)$ is of fundamental importance for the existence and uniqueness program carried out in Section 5.

We stress that since the nonlinearity $v \mapsto q(x,v)$ is discontinuous, we can in general only interpret the semilinear Black and Scholes equation via (3.12) and (3.13), and not "pointwise" as it stands in (3.15) (even if $C^{1,2}$ solutions are sought). Let us illustrate this further by an example, which can be seen as a continuation of Remark 3.1. Suppose that we do not know the position of the free (exercise) boundary $x(t)$ for the value V of an American call option, so that the situation $\frac{r}{d}K < x(t) < K$ cannot a priori be excluded (although we know that this can never happen if we use a posteriori information (2.11) about the free boundary). Pick a point (t,x) such that $\frac{r}{d}K < x < x(t)$, which implies $V(t,x) = g(x) = 0$. Then the semilinear Black and Scholes equation (3.15) reads

$$\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) = -(dx - rK) < 0.$$

However, this is wrong since the dynamic programming principle (see (3.4)) tells us that

$$\mathcal{L}_{\mathrm{BS}}V(t,x) - rV(t,x) = 0.$$

What we would like stress here is that with our interpretation of the semilinear Black and Scholes equation (3.15) (see, e.g., (3.5)), we allow for the possibility that $\mathcal{L}_{\mathrm{BS}}V - rV = 0$ at some points in the exercise region! In some sense, it is this flexibility that allows us to carry out the well-posedness program in Section 5.

Whenever the value V of an American option belongs to $C^{1,2}(Q_T)\bigcap C\left(\overline{Q_T}\right)$, we have immediately that it is a classical solution of (3.15)–(3.16) (in the sense of (3.12) and (3.13)). We would like to dispense with the regularity assumption $V \in C^{1,2}\left(\overline{Q_T}\right)$. We will achieve this by interpreting (3.9) and (3.10) in the sense of viscosity solutions (see Crandall, Ishii, and Lions [11] and Fleming and Soner [15]). The notion of viscosity solution only requires that $V \in C(Q_T)$. Roughly speaking, we define a *viscosity solution* of the semilinear Black and Scholes equation (3.15) as a function $v : \overline{Q_T} \to \mathbb{R}$ that is simultaneously a viscosity subsolution of (3.12) and a viscosity supersolution of (3.13) (see Section 4 for details). Remarkably, this notion of viscosity solution is very much in the spirit of the one used by Ishii [16] for first order Hamilton-Jacobi equations with discontinuous Hamiltonians.

As stated in the following theorem, the terminal value problem (3.15)–(3.16) constitutes a new formulation of the American option valuation problem.

Theorem 3.1 (American option valuation problem). *The American (call and put) option valuation problem is equivalent to finding a function $v : \overline{Q_T} \to \mathbb{R}$ that satisfies* (3.16) *and the semilinear Black and Scholes equation* (3.15) *in the sense of viscosity solutions.*

This theorem is a consequence of the results in Section 5.

4. Viscosity solutions

In this section we introduce the notion of viscosity solutions for the semilinear Black and Scholes equation (3.15). As we have mentioned before, this notion is based on interpreting (3.15) in terms of (3.12) and (3.13), which are then understood in the sense of viscosity solutions. The resulting definition of a viscosity solution coincides with the one used by Ishii [16] when applied to (3.15). For a general introduction to viscosity solution theory, we refer to Crandall, Ishii, and Lions [11] and Fleming and Soner [15].

We shall need the following spaces of semicontinuous functions on $\overline{Q_T}$:

$$USC\left(\overline{Q_T}\right) = \Big\{ v : \overline{Q_T} \to \mathbb{R} \cup \{-\infty\} \,\Big|\, v \text{ is upper semicontinuous} \Big\},$$
$$LSC\left(\overline{Q_T}\right) = \Big\{ v : \overline{Q_T} \to \mathbb{R} \cup \{+\infty\} \,\Big|\, v \text{ is lower semicontinuous} \Big\}.$$

Based on (3.12) and (3.13), we introduce the following notion of viscosity solutions:

Definition 4.1. (i) A locally bounded function $v \in USC\left(\overline{Q_T}\right)$ is a *viscosity subsolution* of (3.15) if and only if $\forall \phi \in C^{1,2}\left(\overline{Q_T}\right)$ we have:

$$\begin{cases} \text{for each } (t,x) \in Q_T \text{ being a local maximizer of } v-\phi, \\ \mathcal{L}_{\mathrm{BS}}\phi(t,x) - rv(t,x) + q^*(x, v(t,x)) \geq 0. \end{cases} \tag{4.1}$$

If, in addition, $v|_{t=T} \leq g$ on $[0,\infty)$, then v is a viscosity subsolution of (3.15)-(3.16).
(ii) A locally bounded function $v \in LSC\left(\overline{Q_T}\right)$ is a *viscosity supersolution* of (3.15) if and only if $\forall \phi \in C^{1,2}\left(\overline{Q_T}\right)$ we have:

$$\begin{cases} \text{for each } (t,x) \in Q_T \text{ being a local minimizer of } v-\phi, \\ \mathcal{L}_{\mathrm{BS}}\phi(t,x) - rv(t,x) + q_*(x, v(t,x)) \leq 0. \end{cases} \tag{4.2}$$

If, in addition, $v|_{t=T} \geq g$ on $[0,\infty)$, then v is a viscosity supersolution of (3.15)-(3.16).
(iii) A function $v \in C\left(\overline{Q_T}\right)$ is a *viscosity solution* of (3.15) if and only if it is simultaneously a viscosity sub- and supersolution of (3.15). If, in addition, $v|_{t=T} = g$ on $[0,\infty)$, then v is a *viscosity solution* of the terminal value problem (3.15)-(3.16).

Remark 4.1. For notational brevity, we use from now on the terms *subsolution* and *supersolution* instead of viscosity subsolution and viscosity supersolution. Furthermore, it is well known that we can replace "local" by "strict local" or "global" or "strict global". We can also assume that the extremum value of $v-\phi$ is zero. From now on we will assume that the extremum points are *strict* and the corresponding extremum value of $v-\phi$ is *zero*.

Lemma 4.1. *Suppose v is a subsolution (supersolution) of* (3.15) *for $x > 0$. Then v is also a subsolution (supersolution) for $x \geq 0$.*

Proof. Suppose $(\bar{t}, 0)$ is a local maximum of $v - \phi$, $\phi \in C^{1,2}\left(\overline{Q_T}\right)$. We may assume that $v(\bar{t},0) - \phi(\bar{t},0) = 0$. Define the function $\psi_\varepsilon(t,x) = v(t,x) - \phi(t,x) - \frac{\varepsilon}{x}$. Let $(t_\varepsilon, x_\varepsilon)$ be a local maximum point of ψ_ε, which exists in view of the upper semicontinuity of ψ_ε and the fact that $\psi_\varepsilon(0,0) = -\infty$. Obviously, we have $x_\varepsilon > 0$. Standard arguments will reveal that the sequence $(t_\varepsilon, x_\varepsilon)$ of local maximum points of ψ_ε satisfies

$$(t_\varepsilon, x_\varepsilon) \to (\bar{t}, 0), \qquad v(t_\varepsilon, x_\varepsilon) \to v(\bar{t}, 0), \qquad \frac{\varepsilon}{x_\varepsilon} \to 0, \qquad \text{as } \varepsilon \downarrow 0. \tag{4.3}$$

For the moment, let us assume that (4.3) holds. Then, since v is a subsolution for $x > 0$,

$$q^*(x_\varepsilon, v(t_\varepsilon, x_\varepsilon)) \geq -\mathcal{L}_{\mathrm{BS}}\phi(t_\varepsilon, x_\varepsilon) + rv(t_\varepsilon, x_\varepsilon) + \left((r-d) - \sigma^2\right)\frac{\varepsilon}{x_\varepsilon}. \tag{4.4}$$

From the upper semicontinuity of q^* and (4.3), we have

$$q^*(\bar{t}, v(\bar{t}, 0)) \geq \limsup_{\varepsilon \downarrow 0} q^*(x_\varepsilon, v(t_\varepsilon, x_\varepsilon)) \geq -\mathcal{L}_{\mathrm{BS}}\phi(\bar{t}, 0) + rv(\bar{t}, 0),$$

from which it follows that the subsolution property holds. Similarly, we can prove that the supersolution property holds by replacing $-\frac{\varepsilon}{x}$ by $\frac{\varepsilon}{x}$ in the definition of ψ_ε. $\square$

To prove that the viscosity solution is unique, we shall need a formulation of sub- and supersolutions based on the so-called semijets.

Definition 4.2. For a function $v \in USC\left(\overline{Q_T}\right)$ $\left(LSC\left(\overline{Q_T}\right)\right)$, the second order *superjet* (*subjet*) of v at $(t,x) \in Q_T$, which is denoted by $\mathcal{P}^{2,+}v(t,x)$ $(\mathcal{P}^{2,-}v(t,x))$, is defined as the set of triples $(a,p,X) \in \mathbb{R}^3$ such that

$$v(s,y) \leq (\geq)\, v(t,x) + a(s-t) + p(y-x) + \frac{1}{2}X(y-x)^2 + o\left(|s-t| + (y-x)^2\right),$$

as $Q_T \ni (s,y) \to (t,x)$. The closure $\overline{\mathcal{P}}^{2,+}v(t,x)$ $(\overline{\mathcal{P}}^{2,-}v(t,x))$ is defined as the set of $(a,p,X) \in \mathbb{R}^3$ for which there exists a sequence $(t^k, x^k, p^k, X^k) \in \mathbb{R}^4$ such that

$$(t^k, x^k, v(t^k,x^k), p^k, X^k) \to (t, x, v(t,x), p, X)$$

as $k \uparrow \infty$ and $(a^k, p^k, X^k) \in \mathcal{P}^{2,+}v(t^k,x^k)$ $(\mathcal{P}^{2,-}v(t^k,x^k))$ for all k.

Recall that $(a,p,X) \in \mathcal{P}^{2,+}v(t,x)$ $(\mathcal{P}^{2,-}v(t,x))$ with $(t,x) \in Q_T$ if and only if there exists $\phi \in C^{1,2}\left(\overline{Q_T}\right)$ such that $\phi(t,x) = v(t,x)$, $\partial_t\phi(t,x) = a$, $\partial_x\phi(t,x) = p$, $\partial_x^2\phi(t,x) = X$, and $v - \phi$ has a maximum (minimum) at (t,x) (see, e.g., [15]). We therefore have following equivalent definition of sub- and supersolutions based on the semijets.

Definition 4.3. (i) A locally bounded function $v \in USC\left(\overline{Q_T}\right)$ is a subsolution of (3.15) if and only if, $\forall (t,x) \in Q_T$ and $\forall (a,p,X) \in \mathcal{P}^{2,+}v(t,x)$,

$$a + (r-d)xp + \frac{1}{2}\sigma^2 x^2 X - rv(t,x) + q^*(x, v(t,x)) \geq 0.$$

(ii) A locally bounded function $v \in LSC\left(\overline{Q_T}\right)$ is a supersolution of (3.15) if and only if, $\forall (t,x) \in Q_T$ and $\forall (a,p,X) \in \mathcal{P}^{2,-}v(t,x)$,

$$a + (r-d)xp + \frac{1}{2}\sigma^2 x^2 X - rv(t,x) + q_*(x, v(t,x)) \leq 0.$$

Remark 4.2. Thanks to the upper semicontinuity of q^* and the lower semicontinuity of q_*, the sub- and supersolution inequalities in Definition 4.3 continue to hold when the semijets $\mathcal{P}^{2,+}$ and $\mathcal{P}^{2,-}$ are replaced by their respective closures $\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,+}$.

Later we prove a uniqueness result for the viscosity solution. To this end, we need the *maximum principle for semicontinuous functions*, which is restated here in a form suitable for our application.

Theorem 4.1 (Crandall, Ishii, and Lions [10, 11])**.** *With $t,x,y \in \mathbb{R}$, let $\underline{v}(t,x), -\overline{v}(t,y)$ be (locally) upper semicontinuous functions and $\varphi(t,x,y)$ be a function that is (locally) once continuously differentiable in t and twice continuously differentiable in (x,y). Let $(t_\varphi, x_\varphi, y_\varphi)$ be a local maximum of the function*

$$(t,x,y) \mapsto \underline{v}(t,x) - \overline{v}(t,y) - \varphi(t,x,y),$$

which is assumed to be defined and upper semicontinuous in a neighborhood of $(t_\varphi, x_\varphi, y_\varphi)$. Suppose that there is a $\rho > 0$ such that for every $M > 0$ there is a constant C such that

$$\begin{cases} a \leq C \text{ if } (a,p,X) \in \mathcal{P}^{2,+}\underline{v}(t,x),\ |x - x_\varphi| + |t - t_\varphi| \leq \rho,\ |\underline{v}(t,x)| + |p| + |X| \leq M, \\ b \geq C \text{ if } (b,q,Y) \in \mathcal{P}^{2,-}\overline{v}(t,x),\ |x - x_\varphi| + |t - t_\varphi| \leq \rho,\ |\overline{v}(t,x)| + |q| + |Y| \leq M. \end{cases}$$

Then for any $\kappa > 0$ there exist two numbers $a_\varphi, b_\varphi \in \mathbb{R}$ and two symmetric 2×2 matrices X_φ, Y_φ such that

$$(a, D_x\varphi(t_\varphi, x_\varphi, y_\varphi), X_\varphi) \in \overline{\mathcal{P}}^{2,+}\underline{v}(t_\varphi, x_\varphi), \quad (b, -D_y\varphi(t_\varphi, x_\varphi, y_\varphi), Y_\varphi) \in \overline{\mathcal{P}}^{2,-}\overline{v}(t_\varphi, y_\varphi),$$

$$-\left(\frac{1}{\kappa} + \left\|D^2\varphi(t_\varphi, x_\varphi, y_\varphi)\right\|\right) I \le \begin{pmatrix} X_\varphi & 0 \\ 0 & -Y_\varphi \end{pmatrix} \le D^2\varphi(t_\varphi, x_\varphi, y_\varphi) + \kappa \left[D^2\varphi(t_\varphi, x_\varphi, y_\varphi)\right]^2,$$

and $a_\varphi - b_\varphi = \partial_t\varphi(t_\varphi, x_\varphi, y_\varphi)$. The norm of a symmetric 2×2 matrix A is defined as $\|A\| = \sup\Big\{|\langle A\xi, \xi\rangle| \,\Big|\, \xi \in \mathbb{R}^2,\ |\xi| \le 1\Big\}$.

5. Well-posedness

The purpose of this section is to prove the following well-posedness theorem for the semilinear Black and Scholes equation (3.15):

Theorem 5.1. *There exists at most one viscosity solution $v : \overline{Q_T} \to \mathbb{R}$ of the terminal value problem* (3.15)–(3.16). *Moreover, this v satisfies*

$$0 \le v(t, x) \le C_1 + C_2 x, \qquad (t, x) \in \overline{Q_T},$$

where $C_1 = 0$ and $C_2 = 1$ for the call option and $C_1 = K$ and $C_2 = 0$ for the put. Finally, v coincides with the American option value V.

The proof of this theorem will be divided into two steps. First (Theorem 5.2), we show that American option value (2.2) is a viscosity solution of (3.15)–(3.16), thereby providing the existence result. We only need the continuity and growth property of the American option value (2.2), while information about the free boundary is not used. Secondly (Theorem 5.3), we prove a comparison principle for sub- and supersolutions, which implies uniqueness of the viscosity solution.

For the existence result, the following lemma will be useful.

Lemma 5.1. *The payoff function g defined in* (2.3) *is a subsolution of* (3.15).

Proof. We prove the lemma for the call option $g(x) = (x - K)^+$. The proof for the put option is similar and it is thus omitted. We have five cases to consider. If $x > \max\left(K, \frac{r}{d}K\right)$ (Case 1), we have $g(x) = x - K$ and $q^*(x, g(x)) = dx - rK$. We plug this directly into the Black and Scholes operator and equate:

$$\mathcal{L}_{\mathrm{BS}}g(x) - rg(x) + q^*(x, g(x)) = 0.$$

If $x < \min\left(K, \frac{r}{d}K\right)$ (Case 2), we have $g(x) = 0$ and $q^*(x, g(x)) = 0$. Plugging this into the Black and Scholes operator, we trivially obtain

$$\mathcal{L}_{\mathrm{BS}}g(x) - rg(x) + q^*(x, g(x)) = 0.$$

If $K < x \leq \frac{r}{d}K$ (Case 3), which is a possible case when $d < r$, we have $g(x) = x - K$ and $q^*(x, g(x)) = 0$. We now obtain

$$\mathcal{L}_{\mathrm{BS}}g(x) - rg(x) + q^*(x, g(x)) = rK - dx \geq 0.$$

If $\frac{r}{d}K \leq x < K$ (Case 4), which is a possible case when $d > r$, we have $g(x) = 0$ and $q^*(x, g(x)) = dx - rK$. We now obtain

$$\mathcal{L}_{\mathrm{BS}}g(x) - rg(x) + q^*(x, g(x)) = dx - rK \geq 0.$$

If $x = K$ (Case 5), then there is nothing to prove, since one cannot find a test function $\phi \in C^{1,2}\left(\overline{Q_T}\right)$ such that $(x-K)^+ - \phi$ has a local maximum at $(t, x) \in Q_T$ when $x = K$. $\square$

The next theorem shows that the American option value (2.2) is a viscosity solution of the semilinear Black and Scholes equation.

Theorem 5.2. *The value function $V(t, x)$ defined in* (2.2) *is a viscosity solution of the terminal value problem* (3.15)-(3.16).

Proof. By inspection of (2.2), the terminal condition (3.16) is obviously satisfied by V. In view of this and Proposition 2.1, it remains to prove that V is a subsolution and a supersolution of the semilinear Black and Scholes equation (3.15). The proof is inspired by the heuristic derivation of (3.15) given in Section 3.

We prove first that V is a supersolution. Let $(t, x) \in Q_T$ be a minimizer of $V - \phi$ with $\phi \in C^{1,2}\left(\overline{Q_T}\right)$. Note that by Lemma 4.1 we can assume that $x > 0$. Recall that $V \geq g$ on $\overline{Q_T}$, so that

$$q_*(t, V(t, x)) = 0, \qquad \forall (t, x) \in Q_T.$$

Let θ be the exit time for the process $(s, X(s))$ from a ball with strictly positive radius and center in (t, x). From (2.8) and Itô's formula, we get

$$\begin{aligned} V(t, x) &\geq \mathbb{E}^{t,x}\left[e^{-r(\theta - t)}V(\theta, X(\theta))\right] \geq \mathbb{E}^{t,x}\left[e^{-r(\theta-t)}\phi(\theta, X(\theta))\right] \\ &= \phi(t, x) + \mathbb{E}^{t,x}\left[\int_t^\theta e^{-r(s-t)}\Big(\mathcal{L}_{\mathrm{BS}}\phi(s, X(s)) - r\phi(s, X(s))\Big)\, ds\right]. \end{aligned}$$

Hence

$$\mathbb{E}^{t,x}\left[\int_t^\theta e^{-r(s-t)}\Big(\mathcal{L}_{\mathrm{BS}}\phi(s, X_s) - r\phi(s, X(s))\Big)\, ds\right] \leq 0.$$

Dividing by $\mathbb{E}^{t,x}[\theta] > 0$ and sending $\theta \downarrow t$ we obtain the desired subsolution inequality

$$\mathcal{L}_{\mathrm{BS}}\phi(t, x) - rV(t, x) = \mathcal{L}_{\mathrm{BS}}\phi(t, x) - r\phi(t, x) \leq 0.$$

We prove next the subsolution property. To this end, let

$$\mathcal{C} = \left\{(t, x) \in Q_T \,\middle|\, V(t, x) > g(x)\right\},$$

be the hold (or continuation) region and

$$\mathcal{S} = \left\{(t, x) \in Q_T \,\middle|\, V(t, x) = g(x)\right\}$$

be the exercise (or optimal stopping) region. Let $(t,x) \in \mathcal{C}$ be a maximizer of $V - \phi$, $\phi \in C^{1,2}\left(\overline{Q_T}\right)$. Observe that

$$q^*(x, V(t,x)) = 0, \qquad \forall (t,x) \in \mathcal{C}.$$

Therefore, following the argument in the supersolution case above, this time using (2.5), we obtain the desired subsolution inequality

$$\mathcal{L}_{\mathrm{BS}}\phi(t,x) - rV(t,x) \geq 0.$$

Let $\phi \in C^{1,2}\left(\overline{Q_T}\right)$ be any test function such that $V-\phi$ has a local maximum at $(t,x) \in \mathcal{S}$. Since $V(t,x) = g(x)$ and $V \geq g$ on Q_T always holds, we conclude that $g - \phi$ has a local maximum at (t,x). Lemma 5.1 then gives

$$\mathcal{L}_{\mathrm{BS}}\phi(t,x) - rV(t,x) + q^*(x, V(t,x)) \geq 0.$$

This concludes the proof of the subsolution property, and hence the theorem. □

Our next theorem is a comparison principle. The comparison result holds in the class of semicontinuous sub- and supersolutions satisfying a natural growth condition. Besides implying uniqueness of the viscosity solution, the comparison principle is used in [7] to prove convergence of approximate solutions to the semilinear Black and Scholes equation. In proving the comparison principle, we use the (by now) standard uniqueness machinery for second order partial differential equations [11], which relies on the maximum principle for semicontinuous functions (see Theorem 4.1 herein). The proof depends fundamentally on the monotonicity property of the discontinuous nonlinearity $v \mapsto q(x,v)$.

Theorem 5.3. *Suppose* $\underline{v} \in USC\left(\overline{Q_T}\right)$ *is a subsolution of* (3.15) *and* $\overline{v} \in LSC\left(\overline{Q_T}\right)$ *is a supersolution of* (3.15)*, satisfying*

$$\underline{v}(T,x) \leq \overline{v}(T,x), \qquad x \in [0,\infty). \tag{5.1}$$

Furthermore, suppose that there exists a finite constant C *such that*

$$\underline{v}(t,x), -\overline{v}(t,x) \leq C(1+x), \qquad (t,x) \in \overline{Q_T}. \tag{5.2}$$

Then we have

$$\underline{v} \leq \overline{v} \text{ on } \overline{Q_T}. \tag{5.3}$$

Consequently, there exists at most one viscosity solution of (3.15)-(3.16).

Proof. Suppose that (5.3) holds. Let v_1 and v_2 be two viscosity solutions satisfying (5.1) and (5.2). Then (5.3) implies $v_1 \equiv v_2$ on $\overline{Q_T}$, and the uniqueness assertion is proved.

In what follows, we prove that (5.3) holds. To this end, we introduce $\overline{v}^\nu = \overline{v} + \nu(T-t)$ for $\nu > 0$. Using the monotonicity of $q(x,\cdot)$, it easy to check that $\overline{v}^\nu$ is a supersolution of

$$\mathcal{L}_{\mathrm{BS}}v(t,x) - rv(t,x) + q(x, v(t,x)) = -\nu, \qquad (t,x) \in Q_T.$$

Instead of comparing $\underline{v}$ and $\overline{v}$, we will compare $\underline{v}$ and $\overline{v}^\nu$. Sending $\nu \downarrow 0$, we obtain the desired result (5.3).

We will work towards a contradiction and suppose

$$\underline{v}\left(\overline{t}, \overline{x}\right) \geq \overline{v}^\nu\left(\overline{t}, \overline{x}\right) + 2\delta, \tag{5.4}$$

for some $(\overline{t},\overline{x}) \in \overline{Q_T}$ and $\delta > 0$. To overcome the lack of regularity of $\underline{v}, \overline{v}^\nu$, we employ the classical "doubling of variables" device [11] and look at a maximum of the function

$$\Phi(t,x,y) = \underline{v}(t,x) - \overline{v}^\nu(t,y) - \psi(x,y), \qquad (t,x,y) \in [0,T] \times [0,\infty) \times [0,\infty),$$

where the penalization function ψ takes the form

$$\psi(x,y) = \frac{\alpha}{2}|x-y|^2 + \frac{\varepsilon}{2}e^{\lambda(T-t)}\left(x^2+y^2\right),$$

for $\alpha, \lambda > 1$ and $\varepsilon \in (0,1)$. Let

$$M_\alpha = \sup_{[0,T]\times[0,\infty)\times[0,\infty)} \Phi(t,x,y),$$

From (5.2) and the upper semicontinuity of Φ, we see that $M_\alpha < \infty$ and there exists $(t_\alpha, x_\alpha, y_\alpha) \in [0,T] \times [0,\infty) \times [0,\infty)$ (suppressing the dependency on ε) such that

$$M_\alpha = \Phi(t_\alpha, x_\alpha, y_\alpha). \tag{5.5}$$

Observe that

$$M_\alpha \geq \underline{v}\left(\overline{t},\overline{x}\right) - \overline{v}^\nu\left(\overline{t},\overline{x}\right) - \varepsilon e^{\lambda(T-\overline{t})}\overline{x}^2 \geq \delta > 0,$$

for any ε that is small enough. Note that this implies

$$\underline{v}(t_\alpha, x_\alpha) \geq \overline{v}^\nu(t_\alpha, y_\alpha) + \delta, \tag{5.6}$$

for any $\alpha > 1$ and ε sufficiently small.

Using $\Phi(T,0,0) \leq \Phi(t_\alpha, x_\alpha, y_\alpha)$ and (5.2), we find

$$\frac{\varepsilon}{2}\left(x_\alpha^2 + y_\alpha^2\right) \leq \overline{v}^\nu(T,0) - \underline{v}(T,0) + \underline{v}(t_\alpha,x_\alpha) - \overline{v}^\nu(t_\alpha,y_\alpha) \leq K + 2C\left(1 + x_\alpha + y_\alpha\right),$$

which implies the existence of a finite constant C_ε (depending on ε) such that

$$x_\alpha, y_\alpha \leq C_\varepsilon.$$

From this we conclude that there exists a subsequence, still denoted by $(t_\alpha, x_\alpha, y_\alpha)$, which converges to some $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0,T] \times [0,\infty) \times [0,\infty)$ as $\alpha \uparrow \infty$ (for each fixed ε). It is classical in viscosity solution theory [11] to see that the maxima $(t_\alpha, x_\alpha, y_\alpha)$ satisfy

$$\begin{cases} x_\alpha - y_\alpha \to 0 \text{ as } \alpha \uparrow \infty \text{ (for each fixed } \varepsilon), \\ \alpha\left|x_\alpha - y_\alpha\right|^2 \to 0 \text{ as } \alpha \uparrow \infty \text{ (for each fixed } \varepsilon). \end{cases}$$

Let us now look at the special case $t_\varepsilon = T$. Note that

$$\underline{v}\left(\overline{t},\overline{x}\right) - \overline{v}^\nu\left(\overline{t},\overline{x}\right) - \varepsilon e^{\lambda(T-\overline{t})}\overline{x}^2 \leq M_\alpha \leq \underline{v}(t_\alpha,x_\alpha) - \overline{v}^\nu(t_\alpha,y_\alpha).$$

By the upper semicontinuity of $\underline{v}, -\overline{v}^\nu$ and since $\underline{v}|_{t=T} \leq \overline{v}^\nu|_{t=T}$ on $[0,\infty)$, we can send $\alpha \uparrow \infty$ and then $\varepsilon \downarrow 0$ in this inequality to obtain

$$\underline{v}\left(\overline{t},\overline{x}\right) - \overline{v}^\nu\left(\overline{t},\overline{x}\right) \leq 0.$$

This contradicts (5.4), and hence $t_\varepsilon < T$.

In what follows, we assume $t_\varepsilon < T$, so that $t_\alpha < T$ for any α sufficiently large. An application of Theorem 4.1 yields the existence of numbers $a_\alpha, b_\alpha, X_\alpha, Y_\alpha$ (again suppressing the dependency on ε) such that

$$\left(a_\alpha, \alpha\left(x_\alpha - y_\alpha\right) + \varepsilon e^{\lambda(T-t_\alpha)} x_\alpha, X_\alpha\right) \in \overline{\mathcal{P}}^{2,+} \underline{v}(t_\alpha, x_\alpha),$$
$$\left(b_\alpha, \alpha(x_\alpha - y_\alpha) - \varepsilon e^{\lambda(T-t_\alpha)} y_\alpha, Y_\alpha\right) \in \overline{\mathcal{P}}^{2,-} \overline{v}^\nu(t_\alpha, y_\alpha)$$

such that $a_\alpha - b_\alpha = -\frac{\varepsilon}{2}\lambda e^{\lambda(T-t_\alpha)}\left(x_\alpha^2 + y_\alpha^2\right)$ and the 2×2 symmetric matrix $\begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix}$ satisfies (if we choose $\kappa = \frac{1}{\alpha}$ in Theorem 4.1) the matrix inequality

$$\begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \le \left(3\alpha + 2\varepsilon e^{\lambda(T-t_\alpha)}\right) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \left(\varepsilon e^{\lambda(T-t_\alpha)} + \frac{\varepsilon^2 e^{2\lambda(T-t_\alpha)}}{\alpha}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.7}$$

By the definition of viscosity sub- and supersolutions,

$$\begin{aligned} a_\alpha + (r-d)x_\alpha\left[\alpha(x_\alpha - y_\alpha) + \varepsilon e^{\lambda(T-t_\alpha)} x_\alpha\right] \\ + \frac{1}{2}\sigma^2 x_\alpha^2 X_\alpha - r\underline{v}(t_\alpha, x_\alpha) + q^*\left(x_\alpha, \underline{v}(t_\alpha, x_\alpha)\right) \ge 0, \\ b_\alpha + (r-d)y_\alpha\left[\alpha(x_\alpha - y_\alpha) - \varepsilon e^{\lambda(T-t_\alpha)} y_\alpha\right] \\ + \frac{1}{2}\sigma^2 y_\alpha^2 Y_\alpha - r\overline{v}^\nu(t_\alpha, y_\alpha) + q_*\left(y_\alpha, \overline{v}^\nu(t_\alpha, y_\alpha)\right) \le -\nu. \end{aligned}$$

From the above two inequalities (using also (5.6)), we get

$$\begin{aligned} \nu \le -r\delta \underbrace{- \frac{\varepsilon}{2}\lambda e^{\lambda(T-t_\alpha)}\left(x_\alpha^2 + y_\alpha^2\right)}_{E_1(\alpha)} + \underbrace{(r-d)\left[\alpha(x_\alpha - y_\alpha)^2 + \varepsilon e^{\lambda(T-t_\alpha)}\left(x_\alpha^2 + y_\alpha^2\right)\right]}_{E_2(\alpha)} \\ + \underbrace{\frac{1}{2}\sigma^2 x_\alpha^2 X_\alpha - \frac{1}{2}\sigma^2 y_\alpha^2 Y_\alpha}_{E_3(\alpha)} + \underbrace{\left(q^*\left(x_\alpha, \underline{v}(t_\alpha, x_\alpha)\right) - q_*\left(y_\alpha, \overline{v}^\nu(t_\alpha, y_\alpha)\right)\right)}_{E_4(\alpha)}, \end{aligned} \tag{5.8}$$

It follows now that

$$\limsup_{\alpha\uparrow\infty} E_1(\alpha) = -\varepsilon\lambda e^{\lambda(T-t_\varepsilon)} x_\varepsilon^2, \qquad \limsup_{\alpha\uparrow\infty} E_2(\alpha) = (r-d)\varepsilon e^{\lambda(T-t_\varepsilon)} 2x_\varepsilon^2.$$

Furthermore, it is classical in viscosity solution theory [11] to use (5.7) as follows:

$$\begin{aligned} \limsup_{\alpha\uparrow\infty} E_3(\alpha) &= \limsup_{\alpha\uparrow\infty}\left[\begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix}\begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} \cdot \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix}\right] \\ &\le \limsup_{\alpha\uparrow\infty}\Bigg[\left(3\alpha + 2\varepsilon e^{\lambda(T-t_\alpha)}\right)|x_\alpha - y_\alpha|^2 \\ &\qquad + \left(\varepsilon e^{\lambda(T-t_\alpha)} + \frac{\varepsilon^2 e^{2\lambda(T-t_\alpha)}}{\alpha}\right)\left(x_\alpha^2 + y_\alpha^2\right)\Bigg] \end{aligned}$$

$$= \varepsilon e^{\lambda(T-t_\varepsilon)} 2x_\varepsilon^2.$$

It remains to estimate the "non-standard" term $E_4(\alpha)$. Choose α so large that

$$|x_\alpha - y_\alpha| \leq \frac{\delta}{2}.$$

Using this and (5.6), we get

$$\begin{aligned} g(y_\alpha) - \overline{v}^\nu(t_\alpha, y_\alpha) &= g(x_\alpha) - \overline{v}^\nu(t_\alpha, y_\alpha) + (g(y_\alpha) - g(x_\alpha)) \\ &\geq g(x_\alpha) - \overline{v}^\nu(t_\alpha, y_\alpha) - \frac{\delta}{2} \\ &\geq g(x_\alpha) - \underline{v}(t_\alpha, x_\alpha) + \frac{\delta}{2}. \end{aligned}$$

By inspection of the possible values of H^* and H_*, we see that

$$-c(y_\alpha) \leq E_4(\alpha) \leq \max\left(0, c(x_\alpha) - c(y_\alpha)\right).$$

Using the continuity of $c(\cdot)$ yields

$$\limsup_{\alpha \uparrow \infty} E_4(\alpha) \leq 0.$$

Therefore, from (5.8) and the estimates just derived, we conclude that

$$\nu \leq -r\delta + (r-d)\varepsilon e^{\lambda(T-t_\varepsilon)} 2x_\varepsilon^2 + \varepsilon e^{\lambda(T-t_\varepsilon)} 2x_\varepsilon^2 - \varepsilon\lambda e^{\lambda(T-t_\varepsilon)} x_\varepsilon^2 \leq 0,$$

if λ is chosen sufficiently large. This is the desired contradiction, and the proof of the theorem is now concluded. □

6. Early exercise premium representation

After the works by Carr, Jarrow, and Myneni [9], Jacka [17], and Kim [23] (see also [13, 26, 27]), it is well known that the price of an American option may be decomposed into the price of the corresponding European option and an early exercise premium. In this section we derive (at an informal level) this decomposition using the integral formulation of the semilinear Black and Scholes equation (3.15). Turning this around, we can view the semilinear Black and Scholes equation(3.15) as an infinitesimal (partial differential equation) version of the early exercise premium representation of the American option.

In what follows, we let $v : [0,T] \times [0,\infty) \to \mathbb{R}$ denote the unique viscosity solution of (3.15)-(3.16). First, let us define the function

$$\omega(s,y) := e^{\alpha y + \beta s} v\left(T - s/\hat{\sigma}^2, e^y\right),$$

where $\hat{\sigma}^2 = \frac{1}{2}\sigma^2$ and we have done the following change-of-variables:

$$x = e^y, \qquad t = T - s/\hat{\sigma}^2.$$

The constants α and β are defined as

$$\alpha = \frac{1}{2}\left((r-d)/\hat{\sigma}^2 - 1\right), \qquad \beta = \alpha^2 + r/\hat{\sigma}^2.$$

Straightforward calculations, using (3.15), reveal that ω satisfies the following semilinear heat equation:

$$\partial_s \omega(s,y) = \partial_y^2 \omega(s,y) + \tilde{q}(s,y,\omega), \qquad (s,y) \in \left(0, \hat{\sigma}^2 T\right] \times \mathbb{R}, \tag{6.1}$$

with the initial condition

$$\omega(y,0) = e^{\alpha y} g(e^y), \qquad y \in \mathbb{R}.$$

The reaction term $\tilde{q}$ takes the form

$$\tilde{q}(s,y,\omega) = \frac{1}{\hat{\sigma}^2} e^{\alpha y + \beta s} c(e^y) H\left(e^{\alpha y + \beta s} g(e^y) - \omega\right),$$

where $H(\cdot)$ is defined in (3.11). Denote by $p(s,y)$ the fundamental solution of (6.1), i.e.,

$$p(s,y) = \frac{\exp\left(\frac{-y^2}{4s}\right)}{\sqrt{4\pi s}}.$$

An integral formulation of the equation for $\omega(s,y)$ then looks like

$$\omega(s,y) = \int_{\mathbb{R}} \omega(z,0) p(s, y-z)\, dz + \int_0^s \int_{\mathbb{R}} \tilde{q}(\theta, z, \omega(\theta,z)) p(s-\theta, y-z)\, dz\, d\theta, \tag{6.2}$$

for $s \in [0, \hat{\sigma}^2 T]$ and $y \in \mathbb{R}$. Letting

$$\chi(\theta,z) = \mathbf{1}_{\left\{e^{\alpha z + \beta\theta} g(e^z) \geq \omega(\theta,z)\right\}},$$

the integral expression (6.2) may also be written as

$$\begin{aligned} \omega(s,y) &= \int_{\mathbb{R}} e^{\alpha z} g(e^z) p(s, y-z)\, dz \\ &\quad + \int_0^s \int_{\mathbb{R}} \chi(\theta,z) \frac{1}{\hat{\sigma}^2} e^{\alpha z + \beta\theta} c(e^z) p(s-\theta, y-z)\, dz\, d\theta. \end{aligned} \tag{6.3}$$

Recalling that $v(t,x) = e^{-\alpha y - \beta s}\omega(s,y)$ with $s = \hat{\sigma}^2(T-t)$ and $y = \ln x$, and using (6.3), we find

$$\begin{aligned} v(t,x) &= e^{-\alpha \ln x - \hat{\sigma}^2 \beta (T-t)} \omega\left(\hat{\sigma}^2(T-t), \ln x\right) \\ &= e^{-r(T-t)} e^{-\alpha \ln x - \hat{\sigma}^2 \alpha^2 (T-t)} \omega\left(\hat{\sigma}^2(T-t), \ln x\right) \\ &= e^{-r(T-t)} \int_{\mathbb{R}} e^{-\alpha(\ln x - z) - \hat{\sigma}^2\alpha^2(T-t)} g(e^z) \frac{\exp\left(-\frac{(\ln x - z)^2}{2\sigma^2(T-t)}\right)}{\sqrt{2\pi\sigma^2(T-t)}}\, dz \\ &\quad + e^{-r(T-t)} \int_0^{\hat{\sigma}^2(T-t)} \int_{\mathbb{R}} \Bigg(\chi(\theta,z) e^{-\alpha(\ln x - z) - \hat{\sigma}^2\alpha^2(T-t) + \beta\theta} \\ &\qquad\qquad \times \frac{1}{\hat{\sigma}^2} c(e^z) \frac{\exp\left(-\frac{(\ln x - z)^2}{2\sigma^2(T-t-\theta/\hat{\sigma}^2)}\right)}{\sqrt{2\pi\sigma^2\left(T-t-\theta/\hat{\sigma}^2\right)}} \Bigg)\, dz\, d\theta \end{aligned}$$

$$
\begin{aligned}
&= e^{-r(T-t)} \int_{\mathbb{R}} e^{-\alpha(\ln x - z) - \hat{\sigma}^2\alpha^2(T-t)} g(e^z) \frac{\exp\left(-\frac{(\ln x - z)^2}{2\sigma^2(T-t)}\right)}{\sqrt{2\pi\sigma^2(T-t)}}\, dz \\
&\quad + \int_0^{\hat{\sigma}^2(T-t)} \int_{\mathbb{R}} \Bigg(\chi(\theta, z) e^{-r\left(T-t-\theta/\hat{\sigma}^2\right)} \frac{1}{\hat{\sigma}^2} e^{-\alpha(\ln x - z) - \hat{\sigma}^2\alpha^2\left(T-t-\theta/\hat{\sigma}^2\right)} \\
&\qquad\qquad \times \frac{1}{\hat{\sigma}^2} c(e^z) \frac{\exp\left(-\frac{(\ln x - z)^2}{2\sigma^2(T-t-\theta/\hat{\sigma}^2)}\right)}{\sqrt{2\pi\sigma^2\left(T-t-\theta/\hat{\sigma}^2\right)}} \Bigg)\, dz\, d\theta.
\end{aligned}
$$

Denote by I_1 the first integral and I_2 the second. Introduce ξ through the change of variable

$$z = \ln x + (r - d - \hat{\sigma}^2)(T-t) + \sigma\xi.$$

Hence $dz = \sigma\, d\xi$ and I_1 becomes

$$
\begin{aligned}
I_1 &= e^{-r(T-t)} \int_{\mathbb{R}} \Bigg(e^{\alpha\left(r-d-\hat{\sigma}^2\right)(T-t) + \alpha\sigma\xi - \hat{\sigma}^2\alpha^2(T-t)} \\
&\qquad \times g\left(x e^{\left(r-d-\hat{\sigma}^2(T-t)\right)+\sigma\xi}\right) \frac{\exp\left(-\frac{\left[\left(r-d-\hat{\sigma}^2\right)(T-t)+\sigma\xi\right]^2}{2\sigma^2(T-t)}\right)}{\sqrt{2\pi\sigma^2(T-t)}} \Bigg) \sigma\, d\xi \\
&= e^{-r(T-t)} \int_{\mathbb{R}} g\left(x e^{\left(r-d-\hat{\sigma}^2\right)(T-t)+\sigma\xi}\right) \frac{\exp\left(-\frac{1}{2}\xi^2(T-t)\right)}{\sqrt{2\pi(T-t)}}\, d\xi \\
&= e^{-r(T-t)} \mathbb{E}^{t,x}\left[g\left(X(T)\right)\right],
\end{aligned}
$$

where the stochastic process $X(s)$ is defined in (2.1). Consider now I_2. Observe that

$$\mathbf{1}_{\left\{e^{\alpha z+\beta\theta} g(e^z) \geq \omega(\theta, z)\right\}} = \mathbf{1}_{\left\{e^{\alpha z+\beta\theta} g(e^z) \geq e^{\alpha z+\beta\theta} v(T-\theta/\hat{\sigma}^2, e^z)\right\}} = \mathbf{1}_{\{v(T-\theta/\hat{\sigma}^2, e^z) \leq g(e^z)\}}.$$

Making the change of variables

$$z = \ln x + \left(r - d - \hat{\sigma}^2\right)\left(T - t - \theta/\hat{\sigma}^2\right) + \sigma\xi \text{ and } u = T - \theta/\hat{\sigma}^2$$

in the integral I_2, we get

$$
\begin{aligned}
I_2 &= \int_0^{\hat{\sigma}^2} \int_{\mathbb{R}} \Bigg(e^{-r\left(T-t-\theta/\hat{\sigma}^2\right)} \mathbf{1}_{\left\{v\left(T-\theta/\hat{\sigma}^2, x e^{\left(r-d-\hat{\sigma}^2\right)\left(T-t-\theta/\hat{\sigma}^2\right)+\sigma\xi}\right) \leq g\left(x e^{\left(r-d-\hat{\sigma}^2\right)\left(T-t-\theta/\hat{\sigma}^2\right)+\sigma\xi}\right)\right\}} \\
&\qquad \times \frac{1}{\hat{\sigma}^2} c\left(x e^{\left(r-d-\hat{\sigma}^2\right)\left(T-t-\theta/\hat{\sigma}^2\right)+\sigma\xi}\right) \frac{\exp\left(-\frac{\xi^2}{2(T-t-\theta/\hat{\sigma}^2)}\right)}{\sqrt{2\pi\left(T-t-\theta/\hat{\sigma}^2\right)}} \Bigg)\, d\xi\, d\theta \\
&= \int_t^T \int_{\mathbb{R}} e^{-r(u-t)} \mathbf{1}_{\left\{v\left(u, x e^{\left(r-d-\hat{\sigma}^2\right)(u-t)+\sigma\xi}\right) \leq g\left(x e^{\left(r-d-\hat{\sigma}^2\right)(u-t)+\sigma\xi}\right)\right\}} \\
&\qquad \times c\left(x e^{\left(r-d-\hat{\sigma}^2\right)(u-t)+\sigma\xi}\right) \frac{\exp\left(-\frac{\xi^2}{2(u-t)}\right)}{\sqrt{2\pi(u-t)}}\, d\xi\, du
\end{aligned}
$$

$$= \int_t^T e^{-r(u-t)} \mathbb{E}^{t,x} \left[c\left(X(u)\right) \mathbf{1}_{\{v(u,X(u)) \le g(X(u))\}} \right] du$$
$$= \mathbb{E}^{t,x} \left[\int_t^T e^{-r(u-t)} c\left(X(u)\right) \mathbf{1}_{\{v(u,X(u)) \le g(X(u))\}} \, du \right].$$

We know that $v \ge g$. Hence

$$\mathbf{1}_{\{v(u,X(u)) \le g(X(u))\}} = \mathbf{1}_{\{v(u,X(u)) = g(X(u))\}}.$$

But $v(x,t) = g(x)$ only when (x,t) is a point in the stopping region, i.e.,

$$\mathbf{1}_{\{v(u,X(u)) = g(X(u))\}} = \mathbf{1}_{\{X(u) < x(u)\}},$$

where $x(u)$ is the free (early exercise) boundary. Therefore

$$v(t,x) = e^{-r(T-t)} \mathbb{E}^{t,x} \left[g\left(X(T)\right) \right] + \mathbb{E}^{t,x} \left[\int_t^T e^{-r(u-t)} c\left(X(u)\right) \mathbf{1}_{\{X(u) < x(u)\}} \, du \right]. \tag{6.4}$$

We recognize (6.4) as the separation of the American option price into the corresponding European option price plus an early exercise premium [9, 13, 17, 23, 26, 27].

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