Limit at Zero of the Brownian First-Passage Density

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Abstract

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion started at zero, let $g : I\!R_+ \rightarrow I\!R$ be an upper function for B satisfying $g(0) = 0$, and let

$$\tau = \inf \{ \, t > 0 \mid B_t \geq g(t) \, \}$$

be the first-passage time of B over g. Assume that g is C^1 on $\langle 0, \infty \rangle$, increasing (locally at zero), and concave (locally at zero). Then the following identities hold for the density function f of τ:

$$f(0+) = \lim_{t \downarrow 0} \frac{1}{2} \frac{g(t)}{t^{3/2}} \, \varphi\left(\frac{g(t)}{\sqrt{t}} \right) = \lim_{t \downarrow 0} \frac{g'(t)}{\sqrt{t}} \, \varphi\left(\frac{g(t)}{\sqrt{t}} \right)$$

in the sense that if the second and third limit exist so does the first one and the equalities are valid (here $\varphi(x) = (1/\sqrt{2\pi}) \, e^{-x^2/2}$ is the standard normal density). These limits can take any value in $[0, \infty]$. The method of proof relies upon the strong Markov property of B and makes use of real analysis.

* Centre for Mathematical Physics and Stochastics (supported by the Danish National Research Foundation).
Mathematics Subject Classification 2000. Primary 60J65, 60G40. Secondary 45D05, 45Q05.
Key words and phrases: Brownian motion, curved (nonlinear) boundary, first-passage time, probability density function, the upper and lower function, integral equation, strong Markov property, Kolmogorov's test, the reflection principle.

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