Examples of multivariate diffusions: time-reversibility; a Cox-Ingersoll-Ross type process

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Abstract

Concrete examples are given of multivariate diffusions, which are either time-reversible with an invariant density that can be determined explicitly, or, in the case of the Cox-Ingersoll-Ross type process, have other attractive features, including an agglomeration property and a description in terms of time-changes of a multivariate Brownian motion with a certain drift and covariance.

Key words and phrases: fundamental solution, symmetric transitions, invariant densities, Gaussian diffusions, skew product, Jacobi diffusion, time-changes.

1. Introduction

The theory of one-dimensional diffusions is well understood: one can easily determine the range of the diffusion, whether it is transient or recurrent, and if there is an invariant probability for the diffusion, simple explicit expressions are typically available. In higher dimensions the situation is much more obscure (when not considering just independent one-dimensional diffusions), and a main purpose of this paper is to present some concrete examples of multivariate diffusions that have at least some attractive analytic features – not too many examples are found in the literature. An alternative title of the paper might have been 'In search of nice diffusions'.

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In Section 2 we focus on time-reversible diffusions. In general, finding an invariant density for a multivariate diffusion (assuming this density to exist) appears quite hopeless and one would expect any kind of expression for the density to be extremely complicated and unwieldy. But as in the case of ordinary Markov chains, imposing time-reversibility makes things much easier. Apart from the examples at the end of the section, Section 2 contains a discussion of the problems arising when trying to find an invariant density in general, and how they simplify if the diffusion is *symmetric*, see Silverstein's work [11] and [12]. The discussion is intended to supplement the very readable account by Kent [7], but perhaps from a slightly different angle, leading to a simple method for verifying reversibility.

The final Section 3 deals with one particular model, which in dimension 1 is simply the classical Cox-Ingersoll-Ross process from mathematical finance. The multivariate version however was constructed with a view to its mathematical properties rather than its possible relevance to finance. The processes in the model are never reversible, in fact they do not have an invariant probability, but can be (at least in two dimensions) recurrent, and of course also transient.

2. Examples of time-reversible diffusions

Let X be a $d-$dimensional time-homogeneous diffusion moving inside an open, simply conected subset $D \subset \mathbb{R}^d$. We assume that X is the unique strong solution to a stochastic differential equation (SDE) driven by a $d-$dimensional Brownian motion B,

$$dX_t = b\left(X_t\right)\, dt + \sigma\left(X_t\right)\, dB_t \tag{2.1}$$

plus an initial condition specifying X_0, where the drift $b : D \to \mathbb{R}^d$ and the diffusion matrix $\sigma : D \to \mathbb{R}^{d\times d}$ are both smooth (at least continuous – in the examples they will be C^∞) with the squared diffusion matrix

$$C(x) := \sigma(x)\sigma^T(x)$$

satisfying that $C(x)$ is strictly positive definite for all $x \in D$. This assumption implies in particular that there are no absorbing states inside D, and that from any $x \in D$, X can move in any direction with a non-degenerate diffusion component. It is however part of our assumptions also that X never hits the boundary ∂D of D (boundary relative to the compact set $\overline{\mathbb{R}}^d$, $\overline{\mathbb{R}} = [-\infty, \infty]$) which forces some (unspecified) constraints on the behaviour of b and C near ∂D – in the examples we verify directly that X never hits ∂D. Because X can move freely, everywhere inside D without hitting ∂D, it follows that the transition densities $p\left(t, x, y\right)$ for X are conservative and form the minimal fundamental solution of the equation

$$A_x p\left(t, x, y\right) = \partial_t p\left(t, x, y\right) \quad \left(x, y \in D,\ t > 0\right), \tag{2.2}$$

A denoting the differential operator determining the infinitesimal generator for X, see (2.6) below, with the subscript x meaning that the differentiations are performed with respect to x. (For the precise formulation of the preceding statement, see the theorem by S. Itô [5], quoted as Theorem 1.1 in Kent [7]. Note that the transition densities there are with respect to the measure $\delta^{-1/2}(x)\,dx$, where

$$\delta(x) := \det\left(\tfrac{1}{2}C(x)\right) \tag{2.3}$$

i.e.

$$P\left(X_t \in B\,|X_0 = x\right) = \int_B p\,(t,x,y)\,\delta^{-1/2}(y)\,dy).$$

Remark 1. *For $d = 1$, $D =]\ell, r[$ is an open interval with $-\infty \le \ell < r \le \infty$. The reader is reminded that in order for X never to hit e.g. the left boundary point ℓ it is necessary and sufficient that (i) or (ii) be satisfied:*

(i) $S(\ell) = -\infty$,

(ii) $S(\ell) > -\infty$ *and* $\int_\ell^x (S(y) - S(\ell))\,\mu\,(y)\;dy = \infty$ *for one (hence all)* $x \in D$.

Here S is an arbitrary scale function, i.e. S has derivative

$$S'(x) = \exp\left(-\int_{x_0}^x \frac{2b(y)}{\sigma^2(y)}\,dy\right) \tag{2.4}$$

for some $x_0 \in D$, and μ is the density of the corresponding speed measure,

$$\mu(x) = \frac{1}{\sigma^2(x)S'(x)}. \tag{2.5}$$

The very nice expository paper [7] by John Kent in particular describes a simple method by which to identify time-reversible diffusions and their invariant measure. Here we shall give an alternative that is perhaps even simpler, which we then illustrate by the examples at the end of this section. First a brief discussion of invariant measures.

Let $f \in C^2(D)$. By Itô's formula

$$df(X_t) = Af(X_t)\,dt + dM_t^f,$$

where A is the differential operator

$$Af(x) = \sum_{i=1}^d b_i(x)\partial_{x_i} f(x) + \tfrac{1}{2}\sum_{i,j=1}^d C_{ij}(x)\partial^2_{x_i x_j} f(x), \tag{2.6}$$

and M^f is the continuous local martingale

$$M_t^f = \sum_{i=1}^d \int_0^t \partial_{x_i} f(X_s) \sum_{j=1}^d \sigma_{ij}(X_s)\ dB_{j,s},$$

in particular $M_0^f \equiv 0$.

Now, let $\mathcal{D}(X)$ denote the space of $C^2(D)-$functions f that are bounded and such that Af is bounded. For $f \in \mathcal{D}(X)$, since

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s)\, ds,$$

it is clear that M^f (viewed as a function of time t and $\omega \in \Omega$, the underlying probability space) is uniformly bounded on all sets $[0,T] \times \Omega$ with $0 \leq T < \infty$, hence M^f is a true martingale with expectation 0, so that if ν is the distribution of X_0,

$$E^\nu f(X_t) = \nu(f) + \int_0^t E^\nu Af(X_s)\, ds. \tag{2.7}$$

Notation. We write P^ν for the probability on Ω if X_0 has law ν and E^ν for the corresponding expectation. If $\nu = \varepsilon_{x_0}$ is degenerate at $x_0 \in D$, we simply write P^{x_0}, E^{x_0}. Of course $\nu(f) = \int_D f(x)\, \nu(dx)$.

Now suppose that X has an (necessarily unique) invariant probability μ, i.e. μ is a probability on D such that

$$E^\mu g(X_t) = \mu(g)$$

for all $t \geq 0$ and, say, all bounded and measurable $g : D \to \mathbb{R}$. Then from (2.7) with $\nu = \mu$ it follows that

$$\mu(Af) = 0 \quad (f \in \mathcal{D}(X)) \tag{2.8}$$

which is of course the standard equation for determining μ.

It is essential that in (2.8) only suitable f such as $f \in \mathcal{D}(X)$ are used: for any continuous h it is easy to solve the differential equation $Af = h$ on D and doing this for $h > 0$ would render (2.8) non-sensical. Thus

Proposition 2.1. *Suppose that for some bounded function $h : D \to \mathbb{R}$ with $h > 0$, the differential equation $Af = h$ on D has a bounded solution f. Then X does not have an invariant probability.*

The condition on h may be relaxed: if an invariant probability exists it will have an (almost everywhere) strictly positive density with respect to Lebesgue measure, hence it suffices that $h > 0$ on a subset of D of strictly positive Lebesgue measure.

We now again assume that X does have an invariant probabilty μ. Then μ has a density, also denoted by μ and using (2.8) for $f \in C_K^2(D) \subset \mathcal{D}(X)$ ($C_K^2(D)$ denoting the $f \in C^2(D)$ with compact support) together with partial integration yields

$$\int_D dx\, f(x) \left(-\sum_i \partial_{x_i} (\mu b_i)(x) + \tfrac{1}{2} \sum_{i,j} \partial^2_{x_i x_j} (\mu C_{ij})(x) \right) = 0,$$

i.e. since $C_K^2(D)$ is dense in $L^2(\ell_D)$ (with ℓ_D Lebesgue measure on D),

$$-\sum_i \partial_{x_i} (\mu b_i)(x) + \tfrac{1}{2} \sum_{i,j} \partial^2_{x_i x_j} (\mu C_{ij})(x) = 0 \quad (x \in D), \tag{2.9}$$

which is the well known differential equation satisfied by the invariant density for a diffusion.

Although, at least in many cases, (2.9) will have precisely one solution μ which is a density (see e.g. Theorem 5.1 in Kent [7]), it is a major problem if one wants to actually find μ, that (2.9) also has a host of irrelevant solutions – this together with the fact that second order partial differential equations are mostly too tough to solve anyway makes (2.9) fairly useless. We shall now discuss briefly how one may deduce integrated versions of (2.9) that, although still unsolvable for $d \geq 2$, at least have the merit of having (probably) a solution that is unique up to proportionality.

Suppose first that $d = 1$. Then (2.9) becomes

$$-(\mu b)' + \tfrac{1}{2} \left(\mu \sigma^2\right)'' = 0 \tag{2.10}$$

and by integration,

$$-\mu b + \tfrac{1}{2} \left(\mu \sigma^2\right)' = \text{constant}. \tag{2.11}$$

To find the invariant density μ one must here take the constant to be 0,

$$-\mu b + \tfrac{1}{2} \left(\mu \sigma^2\right)' = 0, \tag{2.12}$$

as will be argued shortly, and thus finds that

$$\mu(x) \propto \frac{1}{\sigma^2(x)} \exp \int_{x_0}^{x} \frac{2b(y)}{\sigma^2(y)}\, dy, \tag{2.13}$$

i.e., as is well known, μ is proportional to the density of the speed measure, see (2.5) and (2.4).

The argument that the constant in (2.11) should be 0 is simple: since $D =]\ell, r[$ is an open interval, it is clear that any $f \in C^2(D)$ such that the derivative f' has compact support belongs to $\mathcal{D}(X)$ and hence, using partial integration, (2.8) may be written

$$\int_\ell^r dx\, f'(x) \left((\mu b)(x) - \tfrac{1}{2} \left(\mu\sigma^2 \right)'(x) \right) = 0,$$

which implies (2.12) since the allowed collection of derivatives f' is dense in $L^2\left(\ell_{]\ell,r[}\right)$.

The trick used here for $d = 1$ does not apply if $d \geq 2$: for $d = 1$, if f' has compact support, f is constant to the left of that support and constant to the right of the support, but the two constants may be different. By contrast, if $d \geq 2$ and all the partial derivatives $\partial_{x_i} f$ have compact support, then f is simply constant outside a compact set K (all the partial derivatives vanish outside $D \backslash K$, and because $d \geq 2$ this set is connected), thus for some constant k, $f - k \in C_K^2(D)$ and applying (2.8) to f is the same as using $f - k$.

To arrive at an integrated version of (2.8) one may instead proceed as follows, where for simplicity we assume that $d = 2$ and in order for the method to work must assume that all b_i and all C_{ij} are bounded on D: fix some probability on $\mathbb{R}$ with compact support and a smooth (twice continuously differentiable) density ψ and distribution function Ψ. Because all b_i and C_{ij} are bounded, for any $(a_1, a_2) \in D$ and any $c > 0$ sufficiently small,

$$f(x_1, x_2) = \left(\alpha_1 + \beta_1 \Psi\left(\tfrac{x_1 - a_1}{c} \right) \right) \left(\alpha_2 + \beta_2 \Psi\left(\tfrac{x_2 - a_2}{c} \right) \right)$$

belongs to $\mathcal{D}(X)$ for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.. Applying (2.8) to f then gives

$$I + II + III + IV + V = 0 \tag{2.14}$$

where (with $x = (x_1, x_2)$),

$$\begin{aligned}
I &= \int dx_1\, dx_2\, \mu(x)\, b_1(x) \tfrac{\beta_1}{c} \psi\left(\tfrac{x_1 - a_1}{c} \right) \left(\alpha_2 + \beta_2 \Psi\left(\tfrac{x_2 - a_2}{c} \right) \right), \\
II &= \int dx_1\, dx_2\, \mu(x) \left(\alpha_1 + \beta_1 \Psi\left(\tfrac{x_1 - a_1}{c} \right) \right) b_2(x) \tfrac{\beta_2}{c} \psi\left(\tfrac{x_2 - a_2}{c} \right), \\
III &= \int dx_1\, dx_2\, \mu(x) \tfrac{1}{2} C_{11}(x) \tfrac{\beta_1}{c^2} \psi'\left(\tfrac{x_1 - a_1}{c} \right) \left(\alpha_2 + \beta_2 \Psi\left(\tfrac{x_2 - a_2}{c} \right) \right), \\
IV &= \int dx_1\, dx_2\, \mu(x) C_{12}(x) \tfrac{\beta_1}{c} \psi\left(\tfrac{x_1 - a_1}{c} \right) \tfrac{\beta_2}{c} \psi\left(\tfrac{x_2 - a_2}{c} \right), \\
V &= \int dx_1\, dx_2\, \mu(x) \tfrac{1}{2} C_{22}(x) \left(\alpha_1 + \beta_1 \Psi\left(\tfrac{x_1 - a_1}{c} \right) \right) \tfrac{\beta_2}{c^2} \psi'\left(\tfrac{x_2 - a_2}{c} \right).
\end{aligned}$$

Taking limits as $c \to 0$, using the weak convergence of the probabilities with densities $\frac{1}{c}\psi\left(\frac{x-a}{c}\right)$ towards ε_a (the probability degenerate at a) together with partial integration, while allowing $\alpha_1, \alpha_2, \beta_1, \beta_2$ to vary, eventually results in

$$\int dx_2 \, \left((\mu b_1)(a_1, x_2) - \tfrac{1}{2}\partial_{x_1}(\mu C_{11})(a_1, x_2)\right) = 0, \tag{2.15}$$

$$\int dx_1 \, \left((\mu b_2)(x_1, a_2) - \tfrac{1}{2}\partial_{x_2}(\mu C_{22})(x_1, a_2)\right) = 0 \tag{2.16}$$

and

$$\begin{aligned} & \int_{a_2} dx_2 \, \left((\mu b_1)(a_1, x_2) - \tfrac{1}{2}\partial_{x_1}(\mu C_{11})(a_1, x_2)\right) \\ & + \int_{a_1} dx_1 \, \left((\mu b_2)(x_1, a_2) - \tfrac{1}{2}\partial_{x_2}(\mu C_{22})(x_1, a_2)\right) + (\mu C_{12})(a_1, a_2) \\ = \; & 0, \end{aligned} \tag{2.17}$$

where e.g. $\int_{a_2} dx_2$ is the integral over $\{x_2 : x_2 > a_2, (a_1, x_2) \in D\}$.

It could well be argued that the system (2.15), (2.16), (2.17) of equations is even harder to solve than (2.9), but the point is that it may well be true that up to proportionality the μ solving the equations is unique. Note that (2.9) follows from (2.17) differentiating (formally) once with respect to a_1 and once with respect to a_2.

From now on, apart from the assumption that X has an invariant probability μ, make the additional assumption that X is *time-reversible*. Equivalently, subject to P^μ it holds for any $T > 0$ that $(X_{T-t})_{0 \le t \le T}$ has the same law as $(X_t)_{0 \le t \le T}$. Now, in general, under P^μ the process $(X_{T-t})_{0 \le t \le T}$ has the same law as $\left(\hat{X}_t\right)_{0 \le t \le T}$, where $\hat{X}$ is also a diffusion with invariant probability μ, $\hat{X}_0$ has distribution μ, and it is well known (see e.g. Hansen and Scheinkmann [4], Section 4, and Nelson [8] for the original result) that $\hat{X}$ has drift vector $\hat{b}$ and squared diffusion matrix $\hat{C}$ given by

$$\hat{b}_i(x) = -b_i(x) + \frac{1}{\mu(x)} \sum_j \partial_{x_j}(\mu C_{ij})(x), \quad \hat{C}_{ij}(x) = C_{ij}(x).$$

Thus X is reversible iff for all i, $\hat{b}_i \equiv b_i$, or

$$-(\mu b_i)(x) + \tfrac{1}{2} \sum_j \partial_{x_j}(\mu C_{ij})(x) = 0 \quad (1 \le i \le d, x \in D), \tag{2.18}$$

an equation that for each i resembles the integrated equation (2.12) and that for $d = 1$ reduces to that equation, confirming the well known fact that one-dimensional diffusions on an open interval that have an invariant probability are time-reversible.

Of course a multivariate diffusion will typically not be reversible. In order to find examples of reversibility, we shall now make one further simplifying assumption:

Assumption A. For all $i \neq j$, $C_{ij} \equiv 0$ on D.

If Assumption A holds, by the earlier assumption that $C(x)$ be strictly positive definite for all $x \in D$, it follows that

$$C_{ii}(x) > 0 \quad (1 \leq i \leq d, x \in D). \tag{2.19}$$

Of course examples of reversible diffusions not satisfying Assumption A may be obtained from those that do by transformation: $Y = \Phi(X)$ with $\Phi : D \to D'$ a suitably smooth bijection.

Under Assumption A, (2.18) reduces to

$$-(\mu b_i)(x) + \tfrac{1}{2}\,\partial_{x_i}(\mu C_{ii})(x) = 0 \quad (1 \leq i \leq d, x \in D). \tag{2.20}$$

Notation. If $x = (x_1, \ldots, x_d) \in D$, write $x_{\setminus i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$ and write $x = \overline{x_{\setminus i}, x_i}$ in order to express x in terms of $x_{\setminus i}$ and x_i. Also, for each $x_{\setminus i}$, write $D_{x_{\setminus i}}$ for the section

$$D_{x_{\setminus i}} = \left\{x_i : \overline{x_{\setminus i}, x_i} \in D\right\}.$$

For the main result, which we now state, let X be a diffusion satisfying Assumption A, but do not assume that X has an invariant probability.

For $a = (a_1, \ldots, a_d) \in D$, define

$$q_i^a(x) = \frac{1}{C_{ii}(x)} \exp \int_{a_i}^{x_i} \frac{2b_i\left(\overline{x_{\setminus i}, y_i}\right)}{C_{ii}\left(\overline{x_{\setminus i}, y_i}\right)}\, dy_i \tag{2.21}$$

simultaneously for all i, for all x in an open neighborhood $U_a \subset D$ of a such that for each i, $\overline{x_{\setminus i}, y_i} \in D$ whenever y_i belongs to the open interval with endpoints a_i and x_i.

Theorem 2.2. *Suppose that X satisfies Assumption A.*

(i) If the d equations (2.20) have a common strictly positive solution μ, then that solution is uniquely determined up to proportionality.

(ii) Suppose that all $b_i, C_{ii} \in C^2(D)$. Then, in order that the d equations (2.20) have a common strictly positive solution μ, it is necessary and sufficient that for all $a \in D$,

$$\partial^2_{x_i x_j}\left(\log q_i^a(x) - \log q_j^a(x)\right) = 0 \quad \left(i \neq j, x \in U_a\right). \tag{2.22}$$

(iii) If the d equations (2.20) have a common solution μ which is a density, then $\mu > 0$ on D and X is time-reversible with invariant density μ.

Note. In order to check whether a diffusion X is reversible and to find the invariant density, one first verifies (2.22) and then determines the (up to proportionality) unique solution to the system (2.20) – as will be shown in the examples, this is not difficult. The main advantage of (2.20) over the general equation (2.9) is of course that a second-order partial differential equation has been replaced by a system of first order equations, which because of Assumption A can be solved trivially, but which for diffusions with an invariant probability can be uesd only if the diffusions are also time-reversible. See also Kent's remark, [7], p. 828. As will be argued in the proof, (2.18) is precisely his fundamental system (4.3) of *balance equations.*

Proof. (i) Let $a \in D$ and for $x \in U_a$, rewrite the i'th equation from (2.20) as

$$\partial_{x_i} \log\left(\mu C_{ii}\right)(x) = \frac{2b_i}{C_{ii}}(x) \quad \left(x_i \in D_{x_{\setminus i}}\right),$$

which has the complete solution

$$\widetilde{\mu}_i(x) = R_i\left(x_{\setminus i}\right) q_i^a(x) \tag{2.23}$$

on U_a, with q_i^a given by (2.21) and R_i an arbitrary function of $x_{\setminus i}$. In particular

$$\mu(x) = R_i^\circ\left(x_{\setminus i}\right) q_i^a(x) \quad \left(x \in U_a\right)$$

for all i, and if also the $\widetilde{\mu}_i$ from (2.23) are identical and strictly positive on U_a, $\widetilde{\mu}_i \equiv \widetilde{\mu} > 0$ for all i, it follows immediately that

$$\partial_{x_i}\left(\log \mu(x) - \log \widetilde{\mu}(x)\right) = 0$$

for all i, i.e. $\widetilde{\mu} \equiv k_a \mu$ on U_a for some constant k_a. Letting a vary and patching together these identities on overlapping open sets U_a, it follows since D is connected that $\widetilde{\mu} \equiv k\mu$ on D for some constant k.

(ii) Suppose that $\mu > 0$ solves (2.20) so that for each $a \in D$ by (2.23),

$$\mu(x) = R_i\left(x_{\backslash i}\right) q_i^a(x) \quad \left(1 \le i \le d, x \in U_a\right).$$

Then, trivially

$$\partial^2_{x_i x_j} \log \mu(x) = \partial^2_{x_i x_j} \log q_i^a(x)$$

and by symmetry

$$\partial^2_{x_i x_j} \log \mu(x) = \partial^2_{x_i x_j} \log q_j^a(x)$$

and (2.22) follows. If conversely (2.22) holds, take $j = 1$, $i \neq 1$ and solve (2.22) to obtain

$$\log q_1^a(x) - \log q_i^a(x) = \phi_i\left(x_{\backslash i}\right) - \phi_1\left(x_{\backslash 1}\right) \quad \left(x \in U_a\right)$$

for arbitrary functions ϕ_i, ϕ_1. Thus

$$q_i^a(x) \exp \phi_i\left(x_{\backslash i}\right) = q_1^a(x) \exp \phi_1\left(x_{\backslash 1}\right)$$

on U_a for all i, and denoting the common function μ, which is certainly > 0, since it is of the form (2.23) for all i, it solves (2.20) on U_a. Patching together the solutions from different U_a we obtain a solution to (2.20) everywhere on D.

(iii) As noted above, p. 3, the transition densities $p\,(t,x,y)$ for X with respect to the measure $\delta^{-1/2}(x)\,dx$ are conservative and form the minimal fundamental solution to the equation (2.2) in the sense of Kent [7], the definition p. 822. To show that μ, the density solving (2.20) (which is > 0 by (ii)), is the invariant density for X and that X is reversible, it suffices to show that $p(t,x,y)$ is $v-$symmetric, where $v = \delta^{1/2}\mu$, i.e. that

$$p(t,x,y)/v(y) = p(t,y,x)/v(x) \quad (x, y \in D, t > 0)\,, \tag{2.24}$$

see Kent [7], equation (4.1): by integration

$$\begin{aligned} P^{\mu}\left(X_t \in B\right) &= \int_D dx\, \mu(x) P^x\left(X_t \in B\right) \\ &= \int_D dx \int_B dy\, \mu(x) p(t,x,y) \delta^{-1/2}(y) \end{aligned}$$

and since by (2.24)

$$\mu(x) p(t,x,y) \delta^{-1/2}(y) = \delta^{-1/2}(x) p(t,y,x) \mu(y)$$

we obtain

$$P^{\mu}\left(X_t \in B\right) = \int_B dy\, \mu(y) \int_D dx\, \delta^{-1/2}(x) p(t,y,x) = \mu(B)$$

showing that μ is invariant and by Kent [7], Theorem 6.1, that X is reversible.

It remains to establish (2.24). But by Kent [7], Theorem 4.1, $v-$symmetry holds iff the balance equations (4.3) hold,

$$\tfrac{1}{2}\sum_j C_{ij}\,\partial_{x_j} v = \widetilde{b}_i v \quad (1 \le i \le d) \tag{2.25}$$

on D, where $\widetilde{b}_i$ is the net drift (Kent [7], p. 823),

$$\widetilde{b}_i = b_i - \tfrac{1}{2}\delta^{1/2}\sum_j \partial_{x_j}\left(\delta^{-1/2}C_{ij}\right). \tag{2.26}$$

We have here given the general form of (2.25) without using Assumption A, and complete the proof by showing that if μ solves (2.18), then $v = \delta^{1/2}\mu$ solves (2.25) (and in fact, conversely).

Using (2.18) and (2.26), (2.25) becomes

$$\tfrac{1}{2}\sum_j C_{ij}\,\partial_{x_j}\left(\delta^{1/2}\mu\right) = \tfrac{1}{2}\delta^{1/2}\sum_j \left[\partial_{x_j}\left(\mu C_{ij}\right) - \delta^{1/2}\mu\,\partial_{x_j}\left(\delta^{-1/2}C_{ij}\right)\right]. \tag{2.27}$$

But on the left

$$C_{ij}\,\partial_{x_j}\left(\delta^{1/2}\mu\right) = C_{ij}\mu\,\partial_{x_j}\delta^{1/2} + C_{ij}\delta^{1/2}\,\partial_{x_j}\mu$$

and since

$$\delta^{1/2}\partial_{x_j}\delta^{-1/2} = -\delta^{-1/2}\partial_{x_j}\delta^{1/2}$$

we get on the right that

$$\begin{aligned}
& \delta^{1/2}\left[\partial_{x_j}\left(\mu C_{ij}\right) - \delta^{1/2}\mu\,\partial_{x_j}\left(\delta^{-1/2}C_{ij}\right)\right] \\
= \;& \delta^{1/2}C_{ij}\,\partial_{x_j}\mu + \delta^{1/2}\mu\,\partial_{x_j}C_{ij} - \delta^{1/2}\left(\mu\,\partial_{x_j}C_{ij} - C_{ij}\mu\delta^{-1/2}\,\partial_{x_j}\delta^{1/2}\right) \\
= \;& C_{ij}\mu\,\partial_{x_j}\delta^{1/2} + C_{ij}\delta^{1/2}\,\partial_{x_j}\mu
\end{aligned}$$

and (2.27) follows. ■

The proof of (iii) is the most difficult part of the proof of the theorem. Here are some comments on a more direct, but incomplete argument.

Notice first that if Assumption A holds and $\mu > 0$ solves (2.20), then the generator has the form

$$\begin{aligned}
Af &= \frac{1}{2\mu}\sum_i \left(\partial_{x_i}\left(\mu C_{ii}\right)\partial_{x_i} f + \mu C_{ii}\,\partial^2_{x_i x_i} f\right) \\
&= \frac{1}{2\mu}\sum_i \partial_{x_i}\left(\mu C_{ii}\,\partial_{x_i} f\right).
\end{aligned} \tag{2.28}$$

If furthermore μ is a density, in order to show that μ is invariant, it suffices to show for $f \in C_K^2(D)$ that

$$\mu(\pi_t f) = \mu(f) \tag{2.29}$$

for all $t \geq 0$, π_t denoting the transition operator

$$\pi_t f\,(x) = E^x f\,(X_t)\,.$$

Using (2.7) with $\nu = \mu$ gives

$$\mu(\pi_t f) = \mu(f) + \int_0^t \mu\,(\pi_s A f)\; ds$$

and since $\pi_s A f = A \pi_s f$ so that $\pi_s f \in \mathcal{D}(X)$ if $f \in \mathcal{D}(X)$, to deduce (2.29) it suffices to show that $\mu\,(Af) = 0$ for all $f \in \mathcal{D}(X)$. But by (2.28),

$$2\mu\,(Af) = \int_D \sum_i \partial_{x_i}\,(\mu C_{ii}\,\partial_{x_i} f)\; dx$$

and integrating the i'th term by fixing $x_{\setminus i}$ and provided (for convenience), $D_{x_{\setminus i}} = \left]\ell_{x_{\setminus i}}, r_{x_{\setminus i}}\right[$ is an open interval, one finds integrating over x_i, that $\mu\,(Af) = 0$ if

$$\lim_{x_i \uparrow r_{x_{\setminus i}}} (\mu C_{ii}\,\partial_{x_i} f)\,(x) = \lim_{x_i \downarrow \ell_{x_{\setminus i}}} (\mu C_{ii}\,\partial_{x_i} f)\,(x) = 0. \tag{2.30}$$

A proof of this would certainly exploit that X never hits ∂D and that $f \in \mathcal{D}(X)$: for $d = 1$ one needs to show e.g. that with $D =]\ell, r[$

$$\lim_{x \downarrow \ell} \left(\mu\sigma^2 f'\right)(x) = 0. \tag{2.31}$$

Now, since μ is a density, by Remark 1, $S(\ell) = -\infty$ (and $S(r) = \infty$, meaning precisely that X is recurrent), and since $\mu\sigma^2 = 1/S'$, if we define $\xi(x) = 2\mu(x)Af(x)$ it follows from (2.28) that for $\ell < x < x_0 < r$,

$$\int_x^{x_0} dy\,\xi\,(y) = \left(\mu\sigma^2 f'\right)(x_0) - \frac{f'(x)}{S'(x)}$$

whence for $\lambda \in]\ell, x_0[$,

$$\int_\lambda^{x_0} dx S'(x) \left\{\int_x^{x_0} dy\,\xi\,(y) - \left(\mu\sigma^2 f'\right)(x_0)\right\} = -\left(f(x_0) - f(\lambda)\right). \tag{2.32}$$

For $\lambda \downarrow \ell$ the right hand side stays bounded since f is bounded. Since μ is a density and Af is bounded, ξ is Lebesgue-integrable on D so

$$\lim_{x \downarrow \ell} \left\{\int_x^{x_0} dy\,\xi\,(y) - \left(\mu\sigma^2 f'\right)(x_0)\right\} = \int_\ell^{x_0} dy\,\xi\,(y) - \left(\mu\sigma^2 f'\right)(x_0) = \kappa\,(x_0) \tag{2.33}$$

exists. But since $S(\ell) = -\infty$, in order for the left hand side of (2.32) to stay bounded, it follows that $\kappa(x_0) = 0$ for all $x_0 \in D$ and then (2.31) is obtained letting $x_0 \downarrow \ell$ in (2.33). An analogous proof of (2.30) for $d \geq 2$ appears difficult.

We conclude this section with three examples of reversible diffusions. In all three examples there are explicit formulas for the conditional moments (in Example 2.4, provided they exist) which is an additional attractive feature of the models. How to find the moments is explained in the following

Remark 2. *Let X be a general diffusion just satisfying (2.1) and suppose that $\mathcal{L}$ is a finite-dimensional vector space of functions $f \in C^2(D)$ such that $Af \in \mathcal{L}$ for all $f \in \mathcal{L}$ (simplest: $\mathcal{L}$ is the one-dimensional eigenspace determined by an eigenfunction for A). If $(f_q)_{1\leq q\leq r}$ is a basis for $\mathcal{L}$, we may write*

$$Af = Gf \tag{2.34}$$

where $G \in \mathbb{R}^{r\times r}$ is a matrix of constants and f is the column vector $(f_q)_{1\leq q\leq r}$. By (2.7),

$$\pi_t f(x) = f(x) + \int_0^t G\left(\pi_s f\right)(x)\, ds \quad (x \in D) \tag{2.35}$$

provided all $f_q(X_s)$ are P^x-integrable and provided each of the local martingales

$$M_t^{f_q} = \sum_{i=1}^{d} \int_0^t \partial_{x_i} f_q(X_s) \sum_j \sigma_{ij}(X_s)\, dB_{j,s}$$

is a true martingale under each P^x. In that case (2.35) gives $\partial_t \pi_t f = G\pi_t f$ with the boundary condition $\pi_0 f = f$ so that

$$\pi_t f(x) = e^{tG} f(x) \quad (x \in D)\,. \tag{2.36}$$

The integrability conditions required for this formula to hold may be checked as follows: if X has an invariant probability μ, and all f_q are $\mu-$integrable, *since*

$$\int_D \mu\,(dx)\;\; E^x \left|f_q(X_s)\right| = \mu\left(|f_q|\right) < \infty$$

it follows (at least for $\mu-$almost all x) that $f_q(X_s)$ is P^x-integrable. Similarly, if for all functions

$$\eta_j(x) = \left(\sum_i \partial_{x_i} f_q(x)\sigma_{ij}(x)\right)^2 \quad (1 \leq j \leq d)$$

are $\mu-$integrable *one verifies that for $\mu-$almost all x the quadratic variation*

$$\left[M^{f_q}\right]_t = \sum_j \int_0^t \eta_j\left(X_s\right)\, ds$$

has finite E^x-expectation, which is enough to render M^{f_q} a true P^x-martingale.

A particularly nice case of the setup arises when $\mathcal{L}$ is the space of polynomials of degree $\leq p$ for some $p \in \mathbb{N}$. Then the invariance, $A\mathcal{L} \subset \mathcal{L}$, holds for all *$p$ provided each $b_i(x)$ is a polynomium of degree ≤ 1 and each $C_{ij}(x)$ is a polynomium of degree ≤ 2, conditions that are satisfied in all the three examples below in this section, and also by the* affine term structure models *used in finance, see Duffie and Kan [3], (where the C_{ij} are of degree ≤ 1). Thus, with the conditions satisfied, the conditional moments*

$$\pi_t\left(\prod_{i=1}^d x_i^{p_i}\right) = E^x \prod_{i=1}^d X_t^{p_i}$$

with all $p_i \in \mathbb{N}_0$ may be found from (2.36) provided they exist and the relevant local martingales are true martingales.

Note that since $A\mathbf{1} = 0$, the constant functions may always be included in $\mathcal{L}$, and it is not really required that the basis f satisfy the linear relationship (2.34) – it is sufficient that there is a vector $\mathbf{c}$ of constant functions such that $Af = \mathbf{c} + Gf$.

Example 2.3. *Let X be a $d-$dimensional Ornstein-Uhlenbeck (OU) process (homogeneous Gaussian diffusion), i.e. $D = \mathbb{R}^d$ and*

$$dX_t = (A + BX_t)\, dt + D\, dW_t$$

with $A \in \mathbb{R}^d$ the constant drift vector, $B = (b_{ij}) \in \mathbb{R}^{d\times d}$ the (constant) linear drift matrix and $D \in \mathbb{R}^{d\times d}$ the diffusion matrix, assumed to be non-singular and where as usual we write $C = DD^T$. (In this example we write W instead of B for the Brownian motion). It is well known that X has an invariant probability μ iff $\operatorname{Re}\lambda < 0$ *for all (complex) eigenvalues of B, and in that case*

$$\mu = N\left(-B^{-1}A, \Gamma\right),$$

the Gaussian distribution on $\mathbb{R}^d$ with mean vector $-B^{-1}A$ and covariance matrix Γ, which is the unique symmetric solution of the equation

$$C + B\Gamma + \Gamma B^T = 0.$$

Γ can be expressed explicitly as

$$\Gamma = \int_0^\infty e^{sB} C e^{sB^T}\, ds, \tag{2.37}$$

but only in special cases does this unpleasant integral involving matrix exponentials reduce to something simple.

We shall now see when X is reversible and therefore impose Assumption A in order to use Theorem 2.2,

$$C = \operatorname{diag}\left(\sigma_1^2, \ldots, \sigma_d^2\right)$$

with all $\sigma_i^2 > 0$ constants. (For this example, the diagonal structure is of course no restriction: for an arbitrary OU process X with C non-singular, there is a non-singular $F \in \mathbb{R}^{d\times d}$ such that $Y = FX$ is OU with a diagonal C).

Taking $a = 0$, $q_i^0 = q_i$ from (2.21) becomes

$$\begin{aligned} q_i(x) &= \frac{1}{\sigma_i^2} \exp \int_0^{x_i} \frac{2}{\sigma_i^2} \left(b_{ii} y_i + \sum_{j\neq i} b_{ij} x_j \right) dy_i \\ &= \frac{1}{\sigma_i^2} \exp \left(\frac{b_{ii}}{\sigma_i^2} x_i^2 + \frac{2}{\sigma_i^2} \sum_{j\neq i} b_{ij} x_i x_j \right). \end{aligned}$$

Using (2.22) we see that X is reversible iff for all i, j,

$$\frac{b_{ij}}{\sigma_i^2} = \frac{b_{ji}}{\sigma_j^2}, \tag{2.38}$$

equivalently $C^{-1}B = B^T C^{-1}$. Since, apart from a factor not depending on x_i, $q_i(x) = \mu(x)$, it follows directly that the (i,j)'th element of Γ^{-1} is $-2b_{ij}/\sigma_i^2$, i.e.

$$\Gamma = -\tfrac{1}{2} B^{-1} C = -\tfrac{1}{2} C \left(B^T\right)^{-1},$$

a fact also confirmed by (2.37) since (2.38) implies that e.g. $e^{sB}C = Ce^{sB^T}$.

Example 2.4. *As in the previous example, $D = \mathbb{R}^d$ and also the drift $b_i(x)$ is the same, now conveniently written as*

$$b_i(x) = a_i + \sum_j b_{ij} x_j,$$

but with Assumption A in force we now take

$$C_{ii}(x) = \alpha_i + \sum_j \gamma_{ij} x_j^2$$

with all $\alpha_i > 0$, all $\gamma_{ij} \geq 0$ and $\gamma_{ii} > 0$. The coefficients in the SDE for X then satisfies the standard Lipschitz and linear growth conditions ensuring that the equation has a unique strong solution once the initial condition X_0 is specified.

It appears that only a special structure yields reversibility. To simplify, take $a_i = 0$ and $b_{ij} = 0$ for $i \neq j$. Then

$$\partial^2_{x_i x_j} \log q_i^a(x) = \partial_{x_j} \left(-\partial_{x_i} \log C_{ii}(x) + \frac{2b_{ii}x_i}{C_{ii}(x)} \right)$$
$$= \frac{4x_i x_j}{C_{ii}^2(x)} \left(\gamma_{ii}\gamma_{ij} - b_{ii}\gamma_{ij} \right)$$

and in order for this to be symmetric in i and j (see (2.22)), it is natural that the C_{ii} are proportional functions,

$$C_{ii}(x) = \alpha_i \left(1 + \sum_j \delta_j x_j^2 \right)$$

with all $\delta_j > 0$. Thus $\gamma_{ij} = \alpha_i \delta_j$ and (2.22) is seen to hold iff there is a constant ρ such that

$$\frac{b_{ii}}{\alpha_i \delta_i} = \rho$$

for all i. Next,

$$q_i^0(x) = \frac{1}{C_{ii}(x)} \exp \int_0^{x_i} \frac{2b_{ii} y_i^2}{\alpha_i \left(1 + \delta_i y_i^2 + \sum_{j \neq i} \delta_j x_j^2 \right)} \, dy_i$$
$$= R_i \left(x_{\setminus i} \right) C_{ii}(x)^{\rho - 1}$$

which is enough to identify

$$\mu(x) \propto \left(1 + \sum_j \delta_j x_j^2 \right)^{\rho - 1}. \tag{2.39}$$

Since this is integrable iff $\rho < 1 - \frac{d}{2}$, we conclude from Theorem 2.2 that X is reversible if

$$dX_{i,t} = b_{ii} X_{i,t} \, dt + \sqrt{\alpha_i \left(1 + \sum_j \delta_j X_{j,t}^2 \right)} \, dB_{i,t} \tag{2.40}$$

with all α_i and $\delta_j > 0$ and all $b_{ii} / (\alpha_i \delta_i) = \rho < 1 - \frac{d}{2}$, and that the invariant density is then given by (2.39).

Suppose now that in (2.40) all $b_{ii} = b \in \mathbb{R}$, all $\alpha_i = \alpha > 0$ and all $\delta_j = \delta \geq 0$ (do not assume reversibility). (The case $\delta = 0$ is included for reference: if, say, $x_0 = 0$, then the X_i are (for $b \neq 0$) iid one-dimensional Ornstein-Uhlenbeck processes starting from 0, and if $b = 0, \alpha = 1$, then X is a standard $d-$dimensional

Brownian motion. For $d = 1$, a process of the type (2.40) was used by Bibby and Sørensen [1]).

Defining the radial part $R = \left(\sum X_i^2\right)^{1/2}$ and the direction vector $Y = X/R$ one finds that

$$dR = \left(bR + \tfrac{d-1}{2}\alpha RU\right)\, dt + R\sqrt{\alpha U}\, d\widetilde{B}, \tag{2.41}$$
$$dY_i = -\tfrac{d-1}{2}\alpha U Y_i\, dt + \sqrt{\alpha U}\sum_j \left(\delta_{ij} - Y_i Y_j\right) dB_j$$

where

$$U = \frac{1}{R^2} + \delta,$$

while

$$\widetilde{B} = \sum_i \int_0^\cdot Y_i\, dB_i$$

defines a standard one-dimensional Brownian motion. It follows that R is itself a diffusion and that if in addition $b = 0$, there is a skew-product *representation of X:*

$$Y_i = B_i^{\mathrm{sph}}\left(\alpha \int_0^\cdot U_s\, ds\right),$$

with B^{sph} a standard Brownian motion on the unit sphere S^{d-1} independent of R, cf. Rogers and Williams [10] (35.19) – the essentials for the classical skew-product representation of Brownian motion quoted there carries over to the example here (with the Brownian case appearing for $\delta = 0, \alpha = 1$).

Example 2.5. *In this example we start with a density μ and then look for reversible diffusions satisfying Assumption A that have μ as invariant density. Let*

$$D = \left\{x \in \mathbb{R}^d : \text{all } x_i > 0, s < 1\right\}$$

where $s = \sum x_i$, and let μ be the density for a Dirichlet distribution *on D, i.e.*

$$\mu(x) \propto \left(\prod_i x_i^{\lambda_i - 1}\right)(1-s)^{\lambda-1}, \tag{2.42}$$

where all $\lambda_i > 0$ and $\lambda > 0$.

To find the desired reversible diffusion satisfying Assumption A, we need to find b_i and C_{ii} such that

$$\partial_{x_i} \log \mu = -\partial_{x_i} \log C_{ii} + \frac{2b_i}{C_{ii}}. \tag{2.43}$$

We have

$$\partial_{x_i} \log \mu = \frac{\lambda_i - 1}{x_i} - \frac{\lambda - 1}{1 - s}$$

and taking

$$C_{ii}(x) = \gamma_i x_i (1 - s)$$

it is immediate that (2.43) is satisfied if

$$b_i(x) = \tfrac{1}{2}\gamma_i \left(\lambda_i(1 - s) - \lambda x_i\right).$$

Much the nicest case is when all $\gamma_i = \gamma > 0$, *and we are thus looking at*

$$dX_{i,t} = \tfrac{1}{2}\gamma \left(\lambda_i(1 - S_t) - \lambda X_{i,t}\right) dt + \sqrt{\gamma X_{i,t} \left(1 - S_t\right)}\, dB_{i,t}. \tag{2.44}$$

It remains to check for what values of the parameters,X *stays inside* D*! For* $d = 1$, X *is the* Jacobi diffusion, *see e.g. Karlin and Taylor [6], p. 335,*

$$dX_t = \tfrac{1}{2}\gamma \left(\lambda_1(1 - X_t) - \lambda X_t\right) dt + \sqrt{\gamma X_t \left(1 - X_t\right)}\, dB_t, \tag{2.45}$$

which is known to stay inside $]0, 1[$ *precisely when* $\lambda_1 \geq 1$ *and* $\lambda \geq 1$, *cf. Remark 1, and* μ *is then the density for a beta distribution. By analogy we claim (without proof) that for* $d \geq 2$, X *stays inside* D *if all* $\lambda_i \geq 1$ *and* $\lambda \geq 1$ *and from Theorem 2.2 it then follows that* X *is reversible with invariant density* μ *given by (2.42).*

Since D *is bounded, all conditional moments for* D *can be found by the receipt in Remark 2. Also note that* X *has the following* agglomeration property*: write*

$$\{1, \ldots, d\} = \bigcup_{k=1}^{d'} I_k$$

with the I_k *mutually disjoint and non-empty and assume that* $d' < d$. *Defining*

$$Y_k = \sum_{I_k} X_i$$

it is easily checked that Y *is a* $d'-$*dimensional process of the same type as* X,

$$dY_{k,t} = \tfrac{1}{2}\widetilde{\gamma} \left(\widetilde{\lambda}_k(1 - S_t) - \widetilde{\lambda} Y_{k,t}\right) dt + \sqrt{\widetilde{\gamma} Y_{k,t} \left(1 - S_t\right)}\, d\widetilde{B}_{k,t},$$

where $S = \sum Y_k = \sum X_i$ *and*

$$\widetilde{\gamma} = \gamma, \quad \widetilde{\lambda}_k = \sum_{I_k} \lambda_i, \quad \widetilde{\lambda} = \lambda.$$

In particular S *is a one-dimensional Jacobi diffusion with (see (2.45))* $\lambda_1 = \sum \lambda_i$ *and* γ, λ *the same as for* X.

We propose the name multivariate Jacobi diffusion *for* X *given by (2.44).*

3. A multivariate Cox-Ingersoll-Ross type process

This section is entirely devoted to one example of a diffusion which, although never reversible, has other attractive analytical features.

For $d = 1$, the (standard) Cox-Ingersoll-Ross process (CIR), Cox et al. [2] is given by

$$dX_t = (a + bX_t)\ dt + \sigma\sqrt{X_t}\, dB_t \tag{3.1}$$

where $a, b \in \mathbb{R}$ and $\sigma \geq 0$ (with $\sigma = 0$ a trivial, uninteresting case). This is a process on $]0, \infty[$ iff $2a \geq \sigma^2$, and if $\sigma > 0$ and in addition $b < 0$, X is reversible (since $d = 1$) with invariant density

$$\mu(x) \propto x^{\frac{2a}{\sigma^2}-1} \exp\left(\tfrac{2b}{\sigma^2}x\right), \tag{3.2}$$

i.e. the invariant probability is a gamma-distribution.

Recall that special cases of (3.1) arise by considering R^2 when $\delta = 0$ in (2.41) above. If in addition $\alpha = 1$, the result is of course best known as a squared Bessel process.

For $d \geq 2$ we propose the following generalization, which is quite different from the affine term structure models introduced by Duffie and Kan [3]: let $D =]0, \infty[^d$, define

$$Z = \prod_i X_i, \quad Z_{\setminus i} = \prod_{j, j \neq i} X_j$$

and consider

$$dX_{i,t} = \left(\frac{a_i}{Z_{\setminus i,t}} + b_i X_{i,t}\right)\, dt + \sqrt{\frac{X_{i,t}}{Z_{\setminus i,t}}}\, dB^\Gamma_{i,t} \quad (1 \leq i \leq d) \tag{3.3}$$

where all $a_i, b_i \in \mathbb{R}$. Here, $B^\Gamma = \left(B^\Gamma_i\right)_{i \leq i \leq d}$ denotes a Brownian motion with covariance matrix Γ, i.e. B^Γ is continuous with $B^\Gamma_0 \equiv 0$ and stationary independent increments such that for $s < t$, $B^\Gamma_t - B^\Gamma_s$ is Gaussian with mean vector 0 and covariance matrix $(t - s)\,\Gamma$. In particular, if $\Psi \in \mathbb{R}^{d \times d}$ satisfies $\Psi\Psi^T = \Gamma$, we may write $B_t = \Psi W_t$ with W a standard $d-$dimensional Brownian motion. (We need not assume that Γ is non-singular. But the case $\Gamma = 0$ is trivial since then (3.3) reduces to an ordinary differential equation, so the condition $\mathrm{rank}(\Gamma) \geq 1$ is subsumed below).

The first and immediate problem to ask is for what values of the a_i, b_i and Γ, (3.3) has a solution X that moves on D. Starting X from a point in D, the solution is well defined on $[0, \tau[$, where τ is the stopping time

$$\tau = \inf\{t : X_{i,t} = 0 \text{ for some } i\} = \inf\{t : Z_t = 0\},$$

so the problem amounts to determining those values of the parameters for which $\tau = \infty$ a.s., no matter from where in D the process starts.

Suppose for a moment that $\Gamma = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_d^2)$. Fix i and note that whenever $Z_{\setminus i,t}$ is close to some given value $z > 0$, X_i behaves like a CIR process (3.1) with $a = a_i/z$, $b = b_i$ and $\sigma = \sigma_i/\sqrt{z}$, and that this CIR-process stays strictly positive and finite at all times iff $2a_i \geq \sigma_i^2$, a condition that does not depend on z. Therefore, in this case, the condition $2a_i \geq \sigma_i^2$ for all i should suffice for making X a diffusion on D. One can do better however, and we now claim that X *is well defined as a diffusion with strictly positive and finite coordinates at all times iff* $2\overline{a} \geq \overline{\sigma}^2$, where

$$\overline{a} = \sum_i a_i, \quad \overline{\sigma}^2 = \sum_i \Gamma_{ii}.$$

The proof of this claim rests on a simple observation that in fact motivated the definition (3.3) of X. Introduce $Z_{\setminus i,j} = \prod_{k:k\neq i,j}$ for $i \neq j$ and use Itô's formula to obtain

$$\begin{aligned} dZ &= \sum_i Z_{\setminus i} \left\{ \left(\frac{a_i}{Z_{\setminus i}} + b_i X_i \right) dt + \sqrt{\frac{X_i}{Z_{\setminus i}}}\, dB_i^\Gamma \right\} \\ &\quad + \tfrac{1}{2} \sum_{i,j:i\neq j} Z_{\setminus i,j} \sqrt{\frac{X_i X_j}{Z_{\setminus i} Z_{\setminus j}}}\, d\left[B_i^\Gamma, B_j^\Gamma\right]. \end{aligned}$$

Since $d\left[B_i^\Gamma, B_j^\Gamma\right] = \Gamma_{ij}\, dt$ this reduces to

$$dZ = \left(\overline{a} + \tfrac{1}{2} \sum_{i,j:i\neq j} \Gamma_{ij} + \overline{b} Z \right) dt + \sqrt{Z} \sum_i dB_i^\Gamma$$

with

$$\overline{b} = \sum_i b_i.$$

But we may write

$$\sum_i dB_i^\Gamma = \gamma\, dB^1$$

where B^1 is a standard one-dimensional Brownian motion and

$$\gamma^2 = \sum_{i,j} \Gamma_{ij}.$$

Thus Z, which is well defined on $[0, \tau[$, on that interval behaves like a CIR process (3.1) with

$$a = \overline{a} + \tfrac{1}{2} \sum_{i,j:i\neq j} \Gamma_{ij}, \quad b = \overline{b}, \quad \sigma = \gamma, \tag{3.4}$$

and the condition $2a \geq \sigma^2$ for a CIR-process to stay strictly positive (and finite) now yields the desired necessary and sufficient condition for X to be a diffusion on D. Hence, from now on we assume that

$$2\overline{a} \geq \overline{\sigma}^2.$$

By a similar application of Itô's formula it is easy to prove that X has the following *agglomeration property:* write

$$\{1, \ldots, d\} = \bigcup_{k=1}^{d'} I_k$$

with the I_k mutually disjoint and non-empty, assume that $d' < d$ and define

$$Y_k = \prod_{I_k} X_i.$$

Then the $d'-$dimensional process $Y = (Y_k)_{1 \leq k \leq d'}$ satisfies a SDE of the form (3.3),

$$dY_k = \left(\frac{a_k^*}{Z_{\backslash k}^*} + b_k^* Y_k \right) dt + \sqrt{\frac{Y_k}{Z_{\backslash k}^*}}\, dB_k^* \quad (1 \leq i \leq d)$$

where $Z_{\backslash k}^* = \prod_{\ell \neq k} Y_\ell = Z/Y_k$ and

$$a_k^* = \sum_{I_k} a_i + \tfrac{1}{2} \sum_{i,j \in I_k : i \neq j} \Gamma_{ij}, \quad b_k^* = \sum_{I_k} b_i,$$

and B^* is a $d'-$dimensional Brownian motion with covariance matrix Γ^* given by

$$\Gamma_{k\ell}^* = \sum_{i \in I_k} \sum_{j \in I_\ell} \Gamma_{ij}.$$

We shall now comment on the long-term behaviour of X. The key observation here is that by Itô's formula

$$d \log X_i = \left(\left(a_i - \tfrac{1}{2} \Gamma_{ii} \right) \frac{1}{Z} + b_i \right) dt + \frac{1}{\sqrt{Z}}\, dB_i^\Gamma \tag{3.5}$$

which immediately shows that the process $\log X_{i,t} - b_i t$ is a time-change (see. e.g. Rogers and Williams [10], Proposition (30.10)) of a Brownian motion with drift, the same time-change applying to all coordinates X_i. More precisely, suppose that $X_0 \equiv x_0$ is a fixed, arbitrary point in D and define the strictly increasing and continuous process

$$A_t = \int_0^t \frac{1}{Z_s} \, ds$$

with the inverse

$$\rho_u = \inf \{t : A_t = u\}$$

defined for $0 \leq u < A_\infty := \int_0^\infty Z_s^{-1} \, ds \leq \infty$. Then $\rho_0 \equiv 0$, each ρ_u is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by X (or B^Γ) and by (3.5), simultaneously for all $u < A_\infty$,

$$\begin{aligned} \log X_{i,\rho_u} - \log x_{i,0} - b_i \rho_u &= \left(a_i - \tfrac{1}{2}\Gamma_{ii}\right) \int_0^{\rho_u} \frac{1}{Z_s} \, ds + \int_0^{\rho_u} \frac{1}{\sqrt{Z_s}} \, dB_{i,s}^\Gamma \\ &= \left(a_i - \tfrac{1}{2}\Gamma_{ii}\right) u + \int_0^{\rho_u} \frac{1}{\sqrt{Z_s}} \, dB_{i,s}^\Gamma. \end{aligned} \tag{3.6}$$

The last term is a local martingale with respect to the filtration $(\mathcal{F}_{\rho_u})_{u \geq 0}$, and checking the quadratic variations for each i and the cross-variations between two of the local martingales, and using Lévy's characterization of Brownian motion, Rogers and Williams [10], Theorem (33.1) or Revuz and Yor [9], Theorem 3.6, it emerges that

$$\widetilde{B}_{i,u}^\Gamma := \int_0^{\rho_u} \frac{1}{\sqrt{Z_s}} \, dB_{i,s}^\Gamma \quad (1 \leq i \leq d, u \geq 0)$$

is, provided

$$A_\infty = \infty \quad P^{x_0} - \text{a.s.} \tag{3.7}$$

a Brownian motion with covariance matrix Γ. We have shown that *if (3.7) holds, then*

$$\log X_{i,\rho_u} - \log x_{i,0} - b_i \rho_u \quad (1 \leq i \leq d, u \geq 0)$$

defines a $d-$dimensional Brownian motion with drift vector $\left(a_i - \frac{1}{2}\Gamma_{ii}\right)_{1 \leq i \leq d}$ *and covariance matrix* Γ.

Now, the scale function for Z has derivative

$$S'(x) \propto x^{-\frac{2a}{\sigma^2}} e^{-\frac{2b}{\sigma^2} x}$$

with a, b, σ as in (3.4). Therefore $S(0) = -\infty$ always, and $S(\infty) = \infty$ iff either $b < 0$ or $b = 0$ and $2a = \sigma^2$. From this it follows that if $b > 0$, then $\lim_{t \to \infty} Z_t = \infty$ $P^{x_0}-$a.s. for all $x_0 \in D$ so that Z, and therefore also X, is *transient*. If $b = 0$, $2a = \sigma^2$, Z is recurrent but has no invariant probability – Z is null recurrent. Therefore (3.7) holds, and if in addition all $b_i = 0$, $2a_i = \Gamma_{ii}$ (which is compatible with (3.4)), we see that $\log X$ is a time-changed Brownian motion with no drift and covariance Γ, in particular X is *null recurrent* if $d = 2$ and *transient* for $d > 2$.

Finally, if $b < 0$, Z has as stationary probability the gamma distribution given by (3.2), and by the ergodic theorem

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{1}{Z_s}\, ds = \begin{cases} -\frac{2b}{2a-\sigma^2} & \text{if } 2a > \sigma^2, \\ \infty & \text{if } 2a = \sigma^2 \end{cases} \tag{3.8}$$

$P^{x_0}-$a.s. for all $x_0 \in D$. But the limit in (3.8) equals

$$\lim_{u\to\infty} \frac{1}{\rho_u} \int_0^{\rho_u} \frac{1}{Z_s}\, ds = \lim_{u\to\infty} \frac{u}{\rho_u},$$

and we see from (3.6) that

$$\lim_{u\to\infty} \frac{1}{u} \log X_{i,\rho_u} = -b_i \frac{2a-\sigma^2}{2b} + \left(a_i - \tfrac{1}{2}\Gamma_{ii}\right).$$

If, for some i, the right hand side is $\neq 0$, $\log X_{i,\rho_u}$ – and therefore also $\log X_{i,t}$ – will converge to $\pm\infty$, and X_i, and therefore Z, is transient. The case remaining is when for all i,

$$b_i \frac{2a-\sigma^2}{2b} = a_i - \tfrac{1}{2}\Gamma_{ii}$$

or equivalently

$$\frac{b_i}{\bar{b}}\left(2\bar{a} - \bar{\sigma}^2\right) = 2a_i - \Gamma_{ii} \quad (1 \le i \le d), \tag{3.9}$$

and this is the only case, where there is hope that X may have an invariant probability. But for $i \neq j$ given, from (3.6) and (3.9) it follows that for some constant c,

$$\log X_{i,t} - x_{i,0} + c\left(\log X_{j,t} - x_{j,0}\right) = \widetilde{B}^{\Gamma}_{i,A_t} + c\widetilde{B}^{\Gamma}_{j,A_t}.$$

On the right is a time-changed one-dimensional Brownian motion with drift 0. Combining this with the behaviour of A_t for large t ($A_t \sim kt$ for some constant $k > 0$), it is clear that the right hand side can never be given a stationary start, hence X *never has an invariant density*. We do not know when X is null recurrent – from simulations, if (3.9) holds, this appears true if $d = 2$, but what happens if $d \ge 3$ is not known. (On the lack of an invariant density: one may of course look directly for cases where X could be reversible, using the results from Section 2. Assuming that $\Gamma = \text{diag}\left(\sigma_1^2, \ldots, \sigma_d^2\right)$ it is readily seen that (2.22) holds iff $2b_i/\sigma_i^2 = \rho$ is the same for all i, and that the only candidate for an invariant density is

$$\mu(x) \propto e^{\rho z} \prod x_i^{2a_i/\sigma_i^2},$$

where $z = \prod x_i$, which is never Lebesgue-integrable on D!)

Two final comments on the model discussed in this section: (i) we have not required Γ to be non-singular, in particular one or more B_i^Γ may vanish in which case X_i has *differentiable sample paths*; (ii) fixing $i = 1$ say and focusing on X_1 alone, one may view $1/Z_{\setminus 1}$ as a *stochastic volatility* entering the description of X_1 and the whole model as a (nicely structured) type of stochastic volatility model.

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