A String Bit Hamiltonian Approach to Two-Dimensional Quantum Gravity

B. Durhuus[a,b,1] and **C.-W. H. Lee**[a,b,2]

[a] *MaPhySto — Centre for Mathematical Physics and Stochastics*[3]
[b] *Department of Mathematics, University of Copenhagen,*
Universitetsparken 5, DK–2100 Copenhagen, Denmark.

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Abstract

Motivated by the formalism of string bit models, or quantum matrix models, we study a class of simple Hamiltonian models of quantum gravity type in two space-time dimensions. These string bit models are special cases of a more abstract class of models defined in terms of the sl_2 subalgebra of the Virasoro algebra. They turn out to be solvable and their scaling limit coincides in special cases with known transfer matrix models of two-dimensional quantum gravity.

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[1]e-mail address: durhuus@math.ku.dk
[2]e-mail address: lee@math.ku.dk

1 Introduction

So far the most productive approach to two-dimensional quantum gravity has been in terms of path integrals. Specifically, the (Euclidean) path integral formulation of Liouville gravity [1] together with its regularized versions involving random triangulations, or the equivalent matrix models (see, e.g., Ref.[2]), have permitted detailed analysis of two-dimensional quantum gravity coupled to matter fields with central charge $c \leq 1$, in particular the loop correlation functions (or Hartle-Hawking wave functions) and, in some cases, the fractal characterstics of space-time. Importantly, and as a rare instance in quantum field theory, the discrete models have provided analytic insights that presently seem out of reach of continuum methods, the most striking of which is perhaps the double scaling limit as a procedure to incorporate space-times of arbitrary topology. (See Ref.[2] and the references therein.)

One of the main motivations for studying quantum gravity in two dimensions is based on the hope that it may serve as a testing ground for ideas and methods extendable to higher dimensions. It is, indeed, straightforward to set up discrete models of quantum gravity in arbitrary dimensions in terms of random triangulations, but up to now very few analytic results have been obtained and even very basic questions are left open. A number of numerical investigations have, however, been carried out. See, e.g., Ref.[2] and the references therein.

The question arises, naturally, if there exist alternative formulations of the regularized models, or closely related models, that are better tailored for generalizations. A second, and independent, purpose of such reformulations would be to make comparisons possible with continuum approaches other than the path integral quantization, in particular canonical quantization [3]. In this paper we address this question in two dimensions and introduce a class of (Euclidean) Hamiltonian models of regularized two-dimensional quantum gravity. We do not claim to resolve the above mentioned questions, but we will show that the proposed models provide a Hamiltonian alternative to the discrete path integral (or transfer matrix) approach to a class of models introduced recently under the name *Lorentzian gravity* [4, 5], which on the other hand are closely related to the full randomly triangulated models mentioned at the beginning. More precisely, we will show that the continuum limits of the Lorentzian models can be obtained from our Hamiltonian models. Indeed, we will define and solve a more abstract class of models and introduce a formalism opening up the way for even further generalizations.

Detailed analysis of these generalized models as well as generalizations to higher dimensions is, however, outside the scope of the present paper.

Here is the basic idea of this work. Recall that the primary goal of the Hamiltonian formulation of quantum gravity is to account for the "time"-evolution of a space-like universe of fixed topology[4]. Restricting first to connected and compact space-like universes implies that the two-dimensional space-time has either the topology of a strip, corresponding to equal-time slices that are open line segments, or a cylinder with circles as equal-time slices. Since the only reparametrization-invariant quantity defined by a metric on a one-dimensional (connected) manifold is its volume, a natural way to discretise the spacial metric degree of freedom is to introduce a distance cutoff $a > 0$ and consider the equal-time slices to be polygonal lines or loops, respectively, with volume $n \cdot a$, where the integer n is the number of links in the slice. Keeping a fixed, we associate with each such space-like universe of volume $n \cdot a$ a pure quantum state $|n\rangle$, and these are assumed to form an orthogonal basis for the Hilbert space $\mathcal{T}$ of states. The Hamiltonian acting on $\mathcal{T}$ is chosen in such a way that the action couples adjacent links only.

It turns out that models of this type may conveniently be generated by a special variant of the string-bit formalism, whose basic ingredients are annihilation and creation operators that can be interpreted as annihilating or creating links in the equal-time slices. (We will explain the necessary details in Sections 2 and 3.) String bit models were originally developed as a means of regularizing string theory [6, 7]. They provide suitable frameworks for quantum chromodynamics [8] and quantum spin chain models [9], too. Our variant may be envisaged as generalized quantum spin chain models in which the numbers of "spins", i.e. links, are variable. In this sense, the relationship between Lorentzian gravity models and those string bit models equivalent to them is analogous to that between the six-vertex and the XXZ model [10]; a better understanding of one class of models will spur the study of the other.

The paper is organised as follows. In Section 2, the simplest possible Hamiltonian model in the case when space-time is a strip will be solved in the continuum limit. This model turns out to coincide with the corresponding Lorentzian gravity model considered in [5]; in Section 3, we will consider a Hamiltonian model for cylindrical space-time which is not spacially homogeneous (or cyclically symmetric). The model will be shown to reproduce, in

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[4]Note that the models considered in this paper are all within the Euclidean framework.

3

the continuum limit, the Lorentzian model with a marked link in the initial space-like slice considered in [4]; in Section 4, we will consider the cyclically symmetric version of the previously mentioned model and show that this, as well as the model in Section 2, can be obtained as special cases of a more general class of models expressed in terms of the sl_2-generators of the Virasoro algebra in a certain class of highest weight representations, the Hamiltonian being of the form

$$H = L_0 + \lambda L_1 + \lambda L_{-1}. \tag{1}$$

In Section 5, we will solve this model. In particular, both the two-loop amplitude of the continuum Liouville gravity model in Ref.[11] and the so-called p-seamed correlation functions of Ref.[5] will be obtained as special cases; finally, we will discuss briefly further generalisations and future developments in Section 6.

2   Space-time with Boundaries

In this section, we are going to consider the quantum mechanics of a space-time with the topology of a strip whose Hamiltonian is given by

$$H_{\frac{1}{2}} = \mathrm{Tr} \left[a^\dagger a + \frac{\lambda}{\sqrt{N}} (a^\dagger)^2 a + \frac{\lambda}{\sqrt{N}} a^\dagger a^2 \right] - \frac{1}{4} \bar{q}^\dagger \bar{q} - \frac{1}{4} (q^\dagger)^t q^t \tag{2}$$

for $N = \infty$. Before explicating the various terms in Eq.(2), let us briefly review what string bit models are. We will largely follow Refs.[12] and [13], *with a few modifications in definitions and notations.*

Consider an $N \times N$ matrix of creation operators. Its matrix entry is written as $a_{\mu_1}^{\dagger \mu_2}$, where μ_1 and μ_2 are row and column indices, respectively, and can take any integer values between 1 and N inclusive. The Hermitian conjugate of this matrix is an $N \times N$ matrix of annihilation operators whose matrix entries are written in the form $a_{\mu_2}^{\mu_1}$. The creation and annihilation operators satisfy the canonical commutation relations

$$\left[a_{\mu_2}^{\mu_1}, a_{\mu_4}^{\dagger \mu_3} \right] = \delta_{\mu_4}^{\mu_1} \delta_{\mu_2}^{\mu_3}.$$

All other commutators involving these operators vanish.

In addition, consider a $1 \times N$ row vector and an $N \times 1$ column vector of creation operators. Their components are written as $\bar{q}^{\dagger \mu}$ and $q_\mu^\dagger$, respectively.

Their Hermitian conjugates are an $N \times 1$ column vector and a $1 \times N$ row vector of annihilation operators. Their components take the form $\bar{q}_\mu$ and q^μ, respectively. These operators satisfy the canonical commutation relations

$$\left[\bar{q}^{\dagger\mu_1}, \bar{q}_{\mu_2}\right] = \delta^{\mu_1}_{\mu_2} \quad \text{and} \quad \left[q^\dagger_{\mu_1}, q^{\mu_2}\right] = \delta^{\mu_2}_{\mu_1}.$$

All other commutators involving them, including those involving $a^{\dagger\mu_2}_{\mu_1}$ or $a^{\mu_1}_{\mu_2}$ as well, vanish.

Let $|\Omega\rangle$ be a vacuum state annihilated by all annihilation operators, and define $\mathcal{T}_{\frac{1}{2}}$ as the Hilbert space spanned by all states of the form

$$|n\rangle_{\frac{1}{2}} = \frac{1}{N^{(n+1)/2}} \bar{q}^{\dagger\mu_1} a^{\dagger\mu_2}_{\mu_1} a^{\dagger\mu_3}_{\mu_2} \cdots a^{\dagger\mu_{n+1}}_{\mu_n} q^\dagger_{\mu_{n+1}} |\Omega\rangle$$

$$= \frac{1}{N^{(n+1)/2}} \bar{q}^\dagger (a^\dagger)^n q^\dagger |\Omega\rangle,$$

where n is a positive integer and the summation convention is adopted for all row and column indices. The choice of notation $|\ \rangle_{\frac{1}{2}}$ for vectors in $\mathcal{T}_{\frac{1}{2}}$ will be explained in Section 4. The inner product $\langle\cdot\,|\,\cdot\rangle_{\frac{1}{2}}$ on $\mathcal{T}_{\frac{1}{2}}$ is fixed uniquely by the commutation relations and $\langle\Omega|\Omega\rangle = 1$. Since [12]

$$\lim_{N\to\infty} \langle m|n\rangle_{\frac{1}{2}} = \delta_{mn},$$

the states $|1\rangle_{\frac{1}{2}}$, $|2\rangle_{\frac{1}{2}}$, ..., and so on form an orthonormal basis of $\mathcal{T}_{\frac{1}{2}}$ in the large-N limit, which is the limit we are considering. We think of $|n\rangle_{\frac{1}{2}}$ as the quantum state of a universe made up of n interior links and two boundary links.

There are various kinds of natural operators acting on $\mathcal{T}_{\frac{1}{2}}$. We define

$$\sigma^k_\ell = \frac{1}{N^{(k+\ell-2)/2}} a^{\dagger\mu_2}_{\mu_1} a^{\dagger\mu_3}_{\mu_2} \cdots a^{\dagger\nu_\ell}_{\mu_k} a^{\nu_{\ell-1}}_{\nu_l} a^{\nu_{l-2}}_{\nu_{\ell-1}} \cdots a^{\mu_1}_{\nu_1}$$

$$= \frac{1}{N^{(k+\ell-2)/2}} \operatorname{Tr}\left[(a^\dagger)^k a^\ell\right], \tag{3}$$

where k and ℓ are any positive integers, and Tr denotes the trace in index space (and not on $\mathcal{T}_{\frac{1}{2}}$). Moreover, we let

$$l^0_0 = \bar{q}^\dagger \bar{q} \tag{4}$$

and

$$r^0_0 = q^\dagger_\mu q^\mu = (q^\dagger)^t q^t, \tag{5}$$

5

where the superscript t denotes transposition. In the large-N limit [8, 12],

$$
\begin{aligned}
\sigma_\ell^k |n\rangle_{\frac{1}{2}} &= \begin{cases} 0 & \text{if } \ell > n; \\ (n - \ell + 1)|n - \ell + k\rangle_{\frac{1}{2}} + \mathcal{O}(\frac{1}{N}) & \text{if } \ell \leq n; \end{cases} \\
l_0^0 |n\rangle_{\frac{1}{2}} &= |n\rangle_{\frac{1}{2}} \\
r_0^0 |n\rangle_{\frac{1}{2}} &= |n\rangle_{\frac{1}{2}},
\end{aligned}
\tag{6}
$$

where $\mathcal{O}(1/N)$ consists of terms whose norms and whose inner products with any $|n\rangle_{\frac{1}{2}}$ are of the order of at most $1/N$. These terms are thus negligible in the large-N limit. We can see from Eq.(6) that σ_ℓ^k replaces any ℓ adjacent interior links with k adjacent interior links and annihilates $|n\rangle_{\frac{1}{2}}$ if $\ell > n$. l_0^0 and r_0^0 annihilate the left and right boundary links, respectively, and then create them back. Hence, both l_0^0 and r_0^0 effectively act as the identity operator on $\mathcal{T}_{\cong}^{\in}$, but we will nevertheless display them explicitly below.

Using Eqs.(3), (4), and (5), we can rewrite the Hamiltonian $H_{\frac{1}{2}}$ in Eq.(2) as

$$
H_{\frac{1}{2}} = \sigma_1^1 + \lambda \sigma_1^2 + \lambda \sigma_2^1 - \frac{1}{4} l_0^0 - \frac{1}{4} r_0^0,
\tag{7}
$$

where λ is a real constant. In this formula, σ_1^1 is the interior spatial volume energy term. Each interior link carries one unit of energy, and the volume energy of a state is proportional to the number of links. σ_1^2 splits any interior link into two. Since $|n\rangle_{\frac{1}{2}}$ represents n interior links, σ_1^2 maps $|n\rangle_{\frac{1}{2}}$ to $n|n+1\rangle_{\frac{1}{2}}$. On the other hand, σ_2^1 combines any two adjacent interior links into one. Since there are only $n - 1$ pairs of adjacent interior links in $|n\rangle_{\frac{1}{2}}$, σ_2^1 maps $|n\rangle_{\frac{1}{2}}$ to $(n - 1)|n\rangle_{\frac{1}{2}}$. Finally, the last two terms represent boundary volume energy, but notice that the two boundary links contribute negative energy, in total minus one half the energy of an interior link. We stress that this value of the relative size of the volume energy contributions is crucial for the existence of a continuum limit as will be seen. By now, it should have been obvious that $H_{\frac{1}{2}}$, featuring the physics of spatial homogeneity and locality, is among the simplest Hamiltonians one can conceive for a spacetime with boundaries.

We now proceed to evaluate the transition amplitudes

$$
\tilde{G}_{\frac{1}{2}}(k, l; t) = \langle l | e^{-t H_{\frac{1}{2}}} |k\rangle_{\frac{1}{2}}
$$

in the continuum limit. It is convenient to work with its generating function

$$G_{\frac{1}{2}}(x, y; t) = \sum_{k,l=1}^{\infty} x^k y^l \tilde{G}_{\frac{1}{2}}(k, l; t),$$

where x, y are complex variables. Introducing

$$\tilde{\Theta}(k, l; n) = \langle l | H_{\frac{1}{2}}^n | k \rangle_{\frac{1}{2}}$$

and its generating function

$$\Theta(x, y; n) = \sum_{k,l=1}^{\infty} x^k y^l \tilde{\Theta}(k, l; n),$$

for non-negative integers n, we have

$$G_{\frac{1}{2}}(x, y; t) \;=\; \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \Theta(x, y; n). \tag{8}$$

Note, firstly, from

$$\tilde{\Theta}(k, l; 1) = \langle l | H_{\frac{1}{2}} | k \rangle_{\frac{1}{2}} = \left(k - \frac{1}{2}\right) \delta_{lk} + \lambda k \delta_{l,k+1} + \lambda\left(k - 1\right) \delta_{l,k-1}$$

that

$$\Theta(x, y; 1) = \frac{xy(1 + \lambda x + \lambda y)}{(1 - xy)^2} - \frac{xy}{2(1 - xy)}. \tag{9}$$

Secondly, since $|1\rangle_{\frac{1}{2}}$, $|2\rangle_{\frac{1}{2}}$, $\ldots$, and so forth form an orthonormal basis of $\mathcal{T}_{\frac{1}{2}}$, we have [4]

$$\Theta(x, y; n) = \oint \frac{dz}{2\pi i z} \Theta(x, \frac{1}{z}; 1) \Theta(z, y; n - 1), \tag{10}$$

where both $\Theta(x, z^{-1}; 1)$ and $\Theta(z, y; n - 1)$ are treated as complex functions of z, and the contour encirles all singularities of $\Theta(x, z^{-1}; 1)$ but none of $\Theta(z, y; n - 1)$. Using Eq.(9), we find that Eq.(10) leads to

$$\Theta(x, y; n) = \left[-\left(\frac{1}{2} + \frac{\lambda}{x}\right) + \left(\lambda + x + \lambda x^2\right) \frac{\partial}{\partial x} \right] \Theta(x, y; n - 1).$$

7

By Eq.(8), this yields

$$\frac{\partial}{\partial t}G_{\frac{1}{2}}(x,y;t) + \left[-\left(\frac{1}{2}+\frac{\lambda}{x}\right) + \left(\lambda + x + \lambda x^2\right)\frac{\partial}{\partial x}\right]G_{\frac{1}{2}}(x,y;t) = 0. \quad (11)$$

Together with the initial condition

$$G_{\frac{1}{2}}(x,y;0) = \Theta(x,y;0) = \frac{xy}{1-xy} \quad (12)$$

this first order partial differential equation determines $G_{\frac{1}{2}}(x,y;0)$ uniquely.

We are presently only interested in evaluating the continuum limit of $G_{\frac{1}{2}}$. Singularities appear when the coefficients in the square brackets in Eq.(11) vanish, i.e., for $x = y = \pm 1$ and $\lambda = \mp\frac{1}{2}$. These are identical to the critical values found in Refs.[4] and [5], and we can apply the same scaling procedure. Hence we set

$$t = \frac{2T}{a}, \ x = e^{-Xa}, \ y = e^{-Ya}, \ \text{and} \ \lambda = -\frac{1}{2}e^{-\frac{\Lambda}{2}a^2}, \quad (13)$$

where a is the distance cutoff and T, X, Y, and Λ are finite renormalized values of t, x, y, and λ, respectively, and define the continuum continuum limit of $G_{\frac{1}{2}}$, for which we use the same notation, by

$$G_{\frac{1}{2}}(X,Y;T) = \lim_{a\to 0} aG_{\frac{1}{2}}(x,y;t) . \quad (14)$$

Substituting Eq.(13) into Eqs.(11) and (12) yields the limiting equation

$$\frac{\partial}{\partial T}G_{\frac{1}{2}}(X,Y;T) + \left(X^2 - \Lambda\right)\frac{\partial}{\partial X}G_{\frac{1}{2}}(X,Y;T) + XG_{\frac{1}{2}}(X,Y;T) = 0$$

with the initial condition

$$G_{\frac{1}{2}}(X,Y;0) = \frac{1}{X+Y}.$$

These are identical to the equations found for a model of Lorentzian gravity with boundaries in Ref.[14]. By inverse Laplace transformation of the solution with respect to X and Y, one finds (see Ref.[4]) the continuum limit of the transition amplitude expressed in terms of the physical lengths $L = k \cdot a, L' = l \cdot a$ of the two boundary components to be

$$\tilde{G}_{\frac{1}{2}}(L,L';T) = \frac{\sqrt{\Lambda}}{\sinh(T\sqrt{\Lambda})}e^{-\sqrt{\Lambda}(L+L')\coth(T\sqrt{\Lambda})}I_0\left(\frac{2\sqrt{\Lambda LL'}}{\sinh(T\sqrt{\Lambda})}\right) , \quad (15)$$

where I_0 is the zeroth modified Bessel function. Consequently, our string bit model and this particular model of Lorentzian gravity belong to the same universality class. We will come back to this model again in Section 4.

8

3   Closed, Non-homogeneous Space-time

In this section we discuss an example of a Hamiltonian model of cylindrical space-time. Conceptually, it is not the simplest such model, which we defer to the next section. However, it has the virtue of being solvable by the generating function technique of the preceding section, which is our main motivation for considering it here. The model is spatially non-homogeneous in the sense that the equal time slices have one marked link, that is a link created by a matrix of creation operators different from those creating the rest. A model of Lorentzian gravity with a marked initial loop has been studied in Ref.[4], and we will find that its continuum limit is reproduced by our model.

The Hamiltonian we consider is given by

$$
\begin{aligned}
H_c &= \operatorname{Tr}(a^\dagger a + b^\dagger b) + \frac{\lambda}{\sqrt{N}}\operatorname{Tr}(a^\dagger a^\dagger a + \frac{1}{2}a^\dagger b^\dagger b + \frac{1}{2}b^\dagger a^\dagger b \\
&\quad + a^\dagger a^2 + b^\dagger ba + b^\dagger ab)
\end{aligned}
\tag{16}
$$

for $N = \infty$. As mentioned, this quantum matrix model requires a second matrix of creation operators besides $a^\dagger$. The entries of this matrix are written as $b^{\dagger\mu_1}_{\mu_2}$, whose corresponding annihilation operator is $b^{\mu_2}_{\mu_1}$. They satisfy the same canonical commutation relations as the a-operators and commute with these.

Let $\mathcal{T}_c$ be the Hilbert space spanned by all states of the form

$$
|n\rangle_c = \frac{1}{N^{n/2}}\operatorname{Tr}\left[b^\dagger(a^\dagger)^{n-1}\right]|\Omega\rangle,
$$

where n is an arbitrary positive integer. These states form an orthonormal basis for $\mathcal{T}_c$ in the large-N limit [8, 12]:

$$
\lim_{N\to\infty}\langle m|n\rangle_c = \delta_{mn}.
$$

We consider $|n\rangle_c$ as the state of a closed universe with n links, one of which, created by $b^\dagger$, is marked. (c.f. the string bit interpretation in Refs.[6] and [7].)

The operators acting on $\mathcal{T}_c$ which are relevant to us may be written either as

$$
g^k_l = \frac{1}{N^{(k+l-2)/2}}\operatorname{Tr}\left[(a^\dagger)^k a^l\right],
\tag{17}
$$

where k and l are positive integers, or as

$$g_{l_1,l_2}^{k_1,k_2} = \frac{1}{N^{(k_1+k_2+l_1+l_2)/2}} \operatorname{Tr}\left[(a^\dagger)^{k_1} b^\dagger (a^\dagger)^{k_2} a^{l_2} b a^{l_1}\right], \tag{18}$$

where k_1, k_2, l_1, and l_2 are non-negative integers. (σ_l^k and g_l^k are the restrictions of the same operator to the open and closed string sectors, respectively. They are elements of different Lie algebras [12, 13], a fact we will see and use in Section 4.) In the large-N limit [8, 12],

$$g_l^k |n\rangle_c = \begin{cases} 0 & \text{if } l \geq n \text{ or} \\ (n-l) |n-l+k\rangle_c + \mathcal{O}(\frac{1}{N}) & \text{if } l < n, \end{cases}$$

and

$$g_{l_1,l_2}^{k_1,k_2} |n\rangle_c = \begin{cases} 0 & \text{if } l_1 + l_2 > n-1 \text{ or} \\ |n-l_1-l_2+k_1+k_2\rangle_c + \mathcal{O}(\frac{1}{N}) & \text{if } l_1 + l_2 < n. \end{cases}$$

Thus, g_l^k replaces any adjacent l unmarked links in $|n\rangle_c$ with k unmarked links, and $g_{l_1,l_2}^{k_1,k_2}$ replaces adjacent $l_1 + l_2 + 1$ links, where the (l_1+1)-th link is marked, with $k_1 + k_2 + 1$ links, where the (k_1+1)-th link is marked. Note that these operators always preserve the marked link.

Using Eqs.(17) and (18), we can paraphrase H_c in Eq.(16) as

$$H_c = g_1^1 + g_{0,0}^{0,0} + \lambda \left(g_1^2 + \frac{1}{2} g_{0,0}^{1,0} + \frac{1}{2} g_{0,0}^{0,1} + g_2^1 + g_{1,0}^{0,0} + g_{0,1}^{0,0} \right),$$

where λ is a real constant. In this formula, $g_1^1 + g_{0,0}^{0,0}$ is the volume energy term. The terms $g_1^2 + 1/2 g_{0,0}^{1,0} + 1/2 g_{0,0}^{0,1}$ implement splitting of any unmarked link into two unmarked links or splitting the marked link into a marked and an unmarked link. Finally, the terms $g_2^1 + g_{1,0}^{0,0} + g_{0,1}^{0,0}$ combine a pair of juxtaposed unmarked links into one unmarked link or combine the marked link and a juxtaposed unmarked link into the marked link. Notice that the relative constants of the terms in H_c are chosen such that its action treats the marked link in the same way as the unmarked ones. More specifically,

$$\left(g_1^1 + g_{0,0}^{0,0} \right) |n\rangle_c = n|n\rangle_c,$$

$$\left(g_1^2 + \frac{1}{2} g_{0,0}^{1,0} + \frac{1}{2} g_{0,0}^{0,1} \right) |n\rangle_c = n|n+1\rangle_c,$$

$$\left(g_2^1 + g_{1,0}^{0,0} + g_{0,1}^{0,0} \right) |n\rangle_c = n|n-1\rangle_c.$$

10

Thus this form of H_c appears to represent the most natural nearest neighboring interaction on $\mathcal{T}_c$, but notice that it is non-Hermitian.

As already mentioned, it turns out that the model so defined can be solved by the same method as that of the preceding section. Since the differences between the calculations are only minor, we will skip the details. Using the scaling conditions (13) one finds that the the continuum limit $G_c(X, Y; T)$, defined by the same procedure as in Section 2, fulfills

$$\frac{\partial}{\partial T}G_c(X, Y; T) + \left(X^2 - \Lambda\right)\frac{\partial}{\partial X}G_c(X, Y; T) + 2XG_c(X, Y; T) = 0$$

with the initial condition

$$G_c(X, Y; 0) = \frac{1}{X + Y}.$$

These equations are identical to those found for a Lorentzian gravity model in Ref.[4] with one marked link in the entrance loop. Consequently, the two continuum limits coincide, as claimed. For later reference we note that the solution in terms of the length variables is [4]

$$\tilde{G}_c(L, L'; T) = \sqrt{\frac{L}{L'}}\frac{\sqrt{\Lambda}}{\sinh(\sqrt{\Lambda}T)}e^{-\sqrt{\Lambda}(L+L')\coth(\sqrt{\Lambda}T)}I_1\left(\frac{2\sqrt{\Lambda LL'}}{\sinh(\sqrt{\Lambda}T)}\right). \quad (19)$$

4 Closed, Homogeneous Space-time and Tensor Product models

Marking a link in a boundary loop is a convenient technical device in triangulated models and the relation to the same model with no marking is simple, since the marking only gives rise to a factor equal to the length of the marked loop in the counting of triangulations. The relation is of a different kind for Hamiltonian models but, as we will immediately see, still quite straightforward.

In spatially homogeneous models, all links in an equal-time slice have identical status, meaning that only one type of creation and annihilation operators is involved. The simplest nearest neighboring Hamiltonian is then given by

$$H_1 = \text{Tr}\left[a^\dagger a + \frac{\lambda}{\sqrt{N}}(a^\dagger)^2 a + \frac{\lambda}{\sqrt{N}}a^\dagger a^2\right] \quad (20)$$

11

with $N = \infty$ and λ a real parameter. Comparing Eqs.(16) and (20), we see that removing the marked link restores not only cyclic symmetry but also Hermiticity.

The Hilbert space $\mathcal{T}_1$ on which H_1 acts is spanned by all states of the form

$$|n\rangle_1 = \frac{1}{N^{n/2}} \text{Tr}(a^\dagger)^n |\Omega\rangle,$$

where n is a positive integer. In the large-N limit [8, 12],

$$\lim_{N \to \infty} \langle m|n\rangle_1 = n\delta_{mn}.$$

Hence $|1\rangle_1$, $|2\rangle_1$, ..., and so on form an orthogonal basis for $\mathcal{T}_1$. We think of $|n\rangle_1$ as the quantum state of a closed one-dimensional universe topologically equivalent to a regular polygon with n sides (c.f. the string bit interpretation in Refs.[6] and [7]). In terms of the operators introduced in Eq.(17), the Hamiltonian H_1 can be rewritten as

$$H_1 = g_1^1 + \lambda g_1^2 + \lambda g_2^1,$$

In the large-N limit [8, 12],

$$\begin{aligned}
g_1^1 |n\rangle_1 &= n|n\rangle_1 \text{ if } n \geq 0, \\
g_1^2 |n\rangle_1 &= n|n+1\rangle_1 \text{ if } n \geq 0, \\
g_2^1 |1\rangle_1 &= 0, \text{ and} \\
g_2^1 |n\rangle_1 &= n|n-1\rangle_1 \text{if } n > 0.
\end{aligned}$$

Here g_1^1 is again a volume energy term, g_1^2 splits any link into two and g_2^1 combines any pair of adjacent links into one.

It turns out to be tricky, if not impossible, to apply the previously used generating function technique to work out the transition amplitude of this Hamiltonian in the continuum limit. Instead, we will derive it by diagonalising H. Before doing so, however, we will digress for a moment and make some observations about the underlying Lie algebras of string bit models. This will lead to the introduction of a more general and abstract class of Hamiltonian models including the one just defined as well as the model in Section 2 and, in the continuum limit, the tensor product type models of Ref.[5] as special examples.

12

It was shown in Ref.[13] that if we take $\mathcal{T}_1$ as the defining representation of the Lie algebra $\hat{C}_1$ generated by all g_l^k's, then under the identifications[5]

$$g_1^1 \leftrightarrow L_0, \ g_2^1 \leftrightarrow L_1, \ \text{and } g_1^2 \leftrightarrow L_{-1},$$

they satisfy the Lie brackets

$$
\begin{aligned}
[L_0, L_1] &= -L_1, \\
[L_0, L_{-1}] &= L_{-1}, \text{ and} \\
[L_1, L_{-1}] &= 2L_0,
\end{aligned}
\tag{21}
$$

which the reader may verify directly and easily. This Lie algebra is nothing but the sl_2 subalgebra of the Virasoro algebra. Furthermore, since

$$g_2^1 |1\rangle_1 = 0, \ g_1^1 |1\rangle_1 = |1\rangle_1 \text{ and } \langle 1|1\rangle_1 = 1,$$

$|1\rangle_1$ plays the role of the highest weight vector in the defining representation, and the highest weight $h = 1$. It should thus be natural for us to consider the model in which the Hamiltonian

$$H = L_0 + \lambda L_{-1} + \lambda L_1$$

is an element of sl_2 and acts on a certain highest weight represetation. Recall that for general $h > 0$ the representation space $\mathcal{T}_h$ is the Hilbert space spanned by the vectors

$$|n+1\rangle_h = \frac{1}{n!} L_{-1}^n |1\rangle_h, \tag{22}$$

where $|1\rangle_h$ is the highest weight vector and n is any non-negative integer. The actions of L_{-1}, L_0, and L_1 are given by

$$
\begin{aligned}
L_0 |n+1\rangle_h &= (n+h)|n+1\rangle_h \text{ if } n \geq 0, \\
L_{-1} |n+1\rangle_h &= (n+1)|n+2\rangle_h \text{ if } n \geq 0, \\
L_1 |1\rangle_h &= 0, \text{ and} \\
L_1 |n+1\rangle_h &= (n-1+2h)|n\rangle_h \text{ if } n > 0.
\end{aligned}
\tag{23}
$$

[5]There is an important difference between the way the Virasoro generators arose in Refs.[13] and [12] and the way they arise here. In those two articles, every Virasoro generator L_n was identified as a *coset* of certain elements of the Lie algebra $\hat{C}_1$ or $\hat{\Sigma}_1$, a subalgebra of $\hat{G}_{1,1}$. These cosets satisfied the Witt algebra, i.e, the classical Virasoro algebra. Therefore, the Witt algebra was a *quotient algebra* of $\hat{C}_1$ or $\hat{\Sigma}_1$. Here, on the other hand, L_{-1}, L_0 and L_1 (and nothing else) are identified with *specific elements* of $\hat{C}_1$ or a variant of $\hat{\Sigma}_1$. They turn out to form the sl_2 subalgebra of the Virasoro algebra. Therefore, sl_2 is a *subalgebra* of $\hat{C}_1$ or the variant of $\hat{\Sigma}_1$.

The inner product on $\mathcal{T}_h$ is uniquely determined by the commutation relations (21), $\langle 1|1\rangle_h = 1$, $L_0^\dagger = L_0$, and $L_1^\dagger = L_{-1}$. In particular,

$$\langle n+1|n+1\rangle_h = \frac{\Gamma(n+2h)}{n!\Gamma(2h)}. \tag{24}$$

We note that the inner product is positive-definite if and only if $h > 0$.

Next, we revisit the model in Section 2 for a spacetime with boundaries. Its Hamiltonian $H_{\frac{1}{2}}$ was given by Eq.(7). The following observation is a slight modification of the results concerning the Lie algebra $\hat{\Sigma}_1$ generated by all σ_l^k's in Ref.[12]: make the identifications[6]

$$\sigma_1^1 - \frac{1}{2}l_0^0 \leftrightarrow L_0, \ \sigma_2^1 \leftrightarrow L_1, \ \text{and} \ \sigma_1^2 \leftrightarrow L_{-1}.$$

Then the actions of L_{-1}, L_0, and L_1 on $\mathcal{T}_{\frac{1}{2}}$ satisfy the Lie brackets (21). We are thus again led to the Hamiltonian H of the form (1). Since

$$\sigma_2^1|1\rangle_{\frac{1}{2}} = 0, \ \left(\sigma_1^1 - \frac{1}{2}l_0^0\right)|1\rangle_{\frac{1}{2}} = \frac{1}{2}|1\rangle_{\frac{1}{2}} \ \text{and} \ \langle 1|1\rangle_{\frac{1}{2}} = 1,$$

$|1\rangle_{\frac{1}{2}}$ plays the role of a normalised highest weight vector.

Based on these observations, we will, in the following, consider Hamiltonians of the form (1) for arbitrary positive highest weights h. It turns out to be possible to diagonalise H for all such h and to evaluate the continuum limit of the transition amplitude. Before we perform this task in the next section, a few remarks on the interpretation of these models are in order.

As is well known from the representation theory of sl_2, its $h = 1$ highest weight representation is the symmetric tensor product of two copies of the $h = 1/2$ highest weight representation. This fact has a clear interpretation in terms of the gravity models as follows. Taking two copies of the Hamiltonian $H_{\frac{1}{2}}$, we define the symmetrised tensor product state

$$|n\rangle' = \sum_{k=1}^{n} |k\rangle_{\frac{1}{2}} \otimes |n-k+1\rangle_{\frac{1}{2}}$$

for $n \geq 1$ and regard it as representing the state of a closed polygon obtained by gluing two polygonal lines of total length n at both ends, where each of the

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[6]*ibid.*

14

four boundary links of the two polygonal lines has a negative length of $-1/4$. The action of $H_{\frac{1}{2}} \otimes 1 + 1 \otimes H_{\frac{1}{2}}$ on the space spanned by the vectors $|n\rangle$ will then be easily seen to equal to that of H for $h = 1$ under the identification $|n\rangle' = |n\rangle$. Similar remarks apply to the generators L_{-1}, L_0, and L_1.

Corresponding considerations of tensor products and gluing constructions for Lorentzian gravity models were discussed in Ref.[5], as were extensions to multiple tensor products. The latter lead to the so-called p-seamed transition amplitudes. As will also be seen from the explicit solution in the next section, these are reproduced by our algebraic model for integer values of $2h$.

5 Solution to sl_2 Gravity Model

In this section we will show how to obtain the continuum limit of the models with Hamiltonian given by Eq.(1) for arbitrary $h > 0$. We will first prove that H is diagonalisable and determine the exact energy spectrum. Then we will determine the asymptotic form of the eigenvectors close to criticality, which will enable us to extract the continuum limit.

As already remarked, $\mathcal{T}_h$ is spanned by the vectors $|1\rangle_h$, $|2\rangle_h$, $\ldots$, and so forth defined by Eq.(22). These vectors being orthogonal, it follows that states in $\mathcal{T}_h$ are given by

$$\sum_{n=1}^{\infty} a_n |n\rangle_h, \tag{25}$$

where

$$\sum_{n=1}^{\infty} n^{2h-1} |a_n|^2 < \infty \tag{26}$$

by Eq.(24). Clearly, H is an unbounded operator defined, e.g., on vectors for which the sequence a_n is rapidly decreasing.

5.1 Diagonalisation of H

We will apply a refined version of the Frobenius method used to solve a very similar quantum matrix model in Ref.[12]. Assume, for $E \geq 0$, that

$$\phi = \sum_{n=1}^{\infty} a_n |n\rangle_h \tag{27}$$

15

is an eigenstate of H. Using Eqs.(1) and (23), we may write the eigenvalue equation

$$H\phi = E\phi$$

as

$$\lambda(n + 2h)a_{n+2} + (n + h - E)a_{n+1} + \lambda n a_n = 0$$

for all non-negative values of n. (The value of the new unknown a_0 is immaterial because its coefficient is 0.) Asymptotically for large n, the equation reduces to

$$\lambda a_{n+2} + a_{n+1} + \lambda a_n \simeq 0,$$

whose solutions are of the form

$$a_n \simeq \alpha p^n + \beta p^{-n},$$

where

$$p = \frac{-1 + \sqrt{1 - 4\lambda^2}}{2\lambda} \tag{28}$$

in the continuum limit. Since $|p| < 1$ for $|\lambda| < 1/2$, it follows that a_n must asymptotically behave as p^n in order that ϕ be normalisable in accordance with Eq.(26). Hence, we set

$$a_n = b_n p^n,$$

resulting in the equation

$$\lambda(n + 2h)b_{n+2}p^2 + (n + h - E)b_{n+1}p + \lambda n b_n = 0$$

for all non-negative values of n.

In terms of the increments

$$\Delta b_n = b_{n+1} - b_n \quad \text{and} \quad \Delta^2 b_n = \Delta b_{n+1} - \Delta b_n,$$

this equation may be rewritten as

$$\lambda(n + 2h)p\Delta^2 b_n + \left[(2\lambda p + 1)n + (4h\lambda p + h - E)\right]\Delta b_n$$
$$+ (2h\lambda p + h - E)b_n = 0.$$

Introduce the ansatz [15]

$$b_n = \sum_{r=0}^{\infty} c_r n^{(r)},$$

16

where the real numbers c_r depend not on n but on r only, and the factorial polynomial $n^{(r)}$ is defined by

$$n^{(r)} = \begin{cases} n(n-1)\cdots(n-r+1) & \text{if } r > 0; \text{ and} \\ 1 & \text{if } r = 0 . \end{cases}$$

From

$$\Delta b_n = \sum_{r=0}^{\infty} r c_r n^{(r-1)} \quad \text{and} \quad \Delta^2 b_n = \sum_{r=0}^{\infty} r(r-1)c_r n^{(r-2)},$$

we then obtain the equation

$$\lambda p(r+2)(r+1)(r+2h)c_{r+2} + \left[(3r+4h)\lambda p + (r+h-E)\right](r+1)c_{r+1}$$
$$+ \left[(1+2\lambda p)(r+h) - E\right]c_r = 0 \qquad (29)$$

as a sufficient condition for the coefficients c_r to fulfill for all values of r.

For a non-negative integer R, we now set

$$E = E_R = (1 + 2\lambda p)(R + h) \qquad (30)$$

and see that we obtain a unique solution for c_r, up to constant multiples, for which $c_r = 0$ for $r > R$. The corresponding eigenstate ϕ_R is thus of the form

$$\phi_R = \sum_{n=1}^{\infty} C_R(n) p^n |n\rangle_h,$$

where $C_R(n)$ is a polynomial of degree R in n.

Since H is Hermitian, the found eigenstates form an orthogonal set. Moreover, it is a complete set, which can be seen as follows. Take any vector in $\mathcal{T}_h$ given by (25) and assume it is orthogonal to all eigenvectors ϕ_R. Since the $C_R(n)$'s span all polynomials, this means that

$$\sum_{n=1}^{\infty} a_n n^S p^n \frac{\Gamma(n+2h)}{n!\Gamma(2h)} = 0 ,$$

where S is an arbitrary non-negative integer and we have used Eq.(24). Multiplying this equation by $z^S/S!$ and summing over S give

$$\sum_{n=1}^{\infty} a_n p^n \frac{\Gamma(n+2h)}{n!\Gamma(2h)} e^{zn} = 0$$

17

for $|z| < -\log p$. Obviously, the left hand side is an analytic function of z in the half plane $\Re(z) < -\log p$ and hence vanishes there. Restricting z to the imaginary axis, we obtain a vanishing Fourier series, and consequently its Fourier coefficients vanish. This proves that $a_n = 0$ for all $n \geq 1$ and the completeness of ϕ_R for all $R \geq 0$ follows.

Thus Eq.(30) gives the whole energy spectrum. Note that in the limit (13),

$$E_R \simeq \sqrt{\Lambda}a(h + R) \to 0$$

as $a \to 0$, so the model is well behaved in this limit.

5.2   Asymptotic behaviour of eigenstates

In order to determine the continuum limit of the transition amplitude we will need the asymptotic behaviour of the polynomials $C_R(n)$ under the scaling conditions given in Eq.(13). Since n scales as a^{-1} we need to exhibit the leading behaviour of the coefficients c_r in $C_R(n)$ as $a \to 0$. Make the ansatz that c_{r+1} and ac_r are of the same order in the small-a limit. The recursion relation (29) then gives

$$c_{r+1} \simeq -\frac{R-r}{(r+1)(r+2h)}2\sqrt{\Lambda}ac_r.$$

Iterating this equation yields

$$c_r \simeq c_0\frac{(-2\sqrt{\Lambda}a)^r\Gamma(2h)}{\Gamma(r+2h)}\binom{R}{r}$$

which, owing to the scaling of $n = L \cdot a^{-1}$, yields the asymptotic form

$$C_R(n) \simeq c_0\sum_{r=0}^{R}\frac{(-2\sqrt{\Lambda}a)^r\Gamma(2h)}{\Gamma(r+2h)}\binom{R}{r}n^r, \tag{31}$$

where all summands are of order 1.

The behavior of c_0 is fixed by requiring that ϕ_R be normalised. Using the fact that ϕ_R is orthogonal to all vectors in $\mathcal{T}_h$ of the form (25) with $a_n = n^s p^n$, for $s = 0, 1, \ldots$, and $R - 1$, we get

$$\begin{aligned}
\langle\phi_R|\phi_R\rangle_h &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{R}C_R(n)c_Rm^Rp^{n+m}\langle n|m\rangle_h \\
&= \sum_{n=1}^{\infty}C_R(n)c_Rn^Rp^{2n}\frac{\Gamma(n+2h)}{n!\Gamma(2h)}.
\end{aligned} \tag{32}$$

Substituting Eq.(31) into Eq.(32) results in

$$
\begin{aligned}
c_0^{-2} &\simeq \sum_{r=0}^{R} \frac{(-2\sqrt{\Lambda}a)^{R+r}\Gamma(2h)}{\Gamma(R+2h)\Gamma(r+2h)} \left(\begin{array}{c} R \\ r \end{array}\right) \sum_{n=1}^{\infty} n^{R+r}\frac{\Gamma(n+2h)}{n!}p^{2n} \\
&\simeq \sum_{r=0}^{R} \frac{(-2\sqrt{\Lambda}a)^{R+r}\Gamma(2h)}{\Gamma(R+2h)\Gamma(r+2h)} \left(\begin{array}{c} R \\ r \end{array}\right) \sum_{n=1}^{\infty} \frac{\Gamma(n+R+r+2h)}{n!}p^{2n},
\end{aligned}
$$

where, in the last step, we have used that for k real

$$
\frac{\Gamma(n+k)}{n!n^{k-1}} \to 1 \ \ \text{as} \ \ n \to \infty \ .
$$

By the binomial theorem and the relation

$$
p^2 \simeq 1 - 2\sqrt{\Lambda}a,
$$

we finally obtain

$$
\begin{aligned}
c_0^{-2} &\simeq \sum_{r=0}^{R} \frac{(-2\sqrt{\Lambda}a)^{R+r}\Gamma(2h)}{\Gamma(R+2h)\Gamma(r+2h)} \left(\begin{array}{c} R \\ r \end{array}\right) \frac{\Gamma(R+r+2h)}{(1-p^2)^{R+r+2h}} \\
&\simeq \frac{1}{(2\sqrt{\Lambda}a)^{2h}} \sum_{r=0}^{R}(-1)^{R+r}\frac{\Gamma(R+r+2h)\Gamma(2h)}{\Gamma(R+2h)\Gamma(r+2h)} \left(\begin{array}{c} R \\ r \end{array}\right) \\
&= \frac{1}{(2\sqrt{\Lambda}a)^{2h}}\frac{R!\Gamma(2h)}{\Gamma(R+2h)} \ ,
\end{aligned}
\tag{33}
$$

where, in the last step, we have made use of the identity

$$
\sum_{r=0}^{R}(-1)^{R+r}\frac{\Gamma(R+r+2h)}{R!\Gamma(r+2h)} \left(\begin{array}{c} R \\ r \end{array}\right) = 1 \ ,
$$

which is a special case of the Chu-Vandermonde identity. (See, e.g., Ref.[16].)

5.3   The continuum limit

We are now ready to compute the continuum limit of the transition amplitude. The unnormalized transition amplitude is defined by

$$
\tilde{G}_u(L, L'; T) = \lim_{a \to 0} a^\alpha \langle \frac{L'}{a} | e^{-tH} | \frac{L}{a} \rangle_h,
\tag{34}
$$

where t and λ are given as in Eq.(13), and the exponent α is to be determined such that the limit exists. Recall from Eq.(24) that the states $|\frac{L}{a}\rangle_h$ are not normalised. On the other hand, the more natural amplitude defined in terms of the normalised states may simply be obtained from $\tilde{G}_u$ by Eq.(24); we will come back to it later.

Inserting the complete set of states $\{\phi_R\}$, we have

$$
\begin{aligned}
\tilde{G}_u(L, L'; T) &= \lim_{a\to 0} a^\alpha \sum_{R=0}^{\infty} e^{-tE_R} \langle \frac{L'}{a} | \phi_R \rangle_h \langle \phi_R | \frac{L}{a} \rangle_h \\
&\simeq \lim_{a\to 0} a^\alpha \sum_{R=0}^{\infty} e^{-2\sqrt{\Lambda}(R+h)} \langle \frac{L'}{a} \mid \phi_R \rangle_h \langle \phi_R \mid \frac{L}{a} \rangle_h .
\end{aligned}
\tag{35}
$$

Using Eqs.(31) and (33) as well as

$$
p^{\frac{L}{a}} \simeq e^{-\sqrt{\Lambda}L},
$$

we have

$$
\begin{aligned}
\langle \phi_R | \frac{L}{a} \rangle_h &= C_R \left(\frac{L}{a} \right) p^{\frac{L}{a}} \langle \frac{L}{a} | \frac{L}{a} \rangle_h \\
&\simeq a^{1-h} \left(2\sqrt{\Lambda} \right)^h \sqrt{\frac{\Gamma(R+2h)}{R!\Gamma(2h)}} L^{2h-1} \sum_{r=0}^{R} \frac{(-2\sqrt{\Lambda})^r}{\Gamma(r+2h)} \left(\begin{array}{c} R \\ r \end{array} \right) L^r e^{-\sqrt{\Lambda}L}.
\end{aligned}
$$

Inserting this expression into Eq.(35) and choosing

$$
\alpha = 2h - 2,
$$

we find that the continuum transition amplitude exists and takes the form

$$
\begin{aligned}
\tilde{G}_u(L, L'; T) = \\
(4\Lambda)^h (LL')^{2h-1} \sum_{R=0}^{\infty} \frac{\Gamma(R+2h)}{R!\Gamma(2h)} \sum_{r=0}^{R} \sum_{s=0}^{R} \frac{(-2\sqrt{\Lambda})^{r+s} L^r L'^s}{\Gamma(r+2h)\Gamma(s+2h)} \\
\cdot \left(\begin{array}{c} R \\ r \end{array} \right) \left(\begin{array}{c} R \\ s \end{array} \right) e^{-\sqrt{\Lambda}(L+L')} e^{-2(h+R)\sqrt{\Lambda}T}.
\end{aligned}
\tag{36}
$$

A priori, it is not obvious that this series is convergent for all positive values of L, L', and T. One way to see this, and simultaneously obtaining a

20

more manageable expression for $\tilde{G}_u$, is to apply an integral representation of the reciprocal Gamma function (see, e.g., Ref.[17]):

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i \beta^{x-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{e^{\beta(z+b)}dz}{(z+b)^x}. \qquad (37)$$

In this formula, x and β are arbitrary positive numbers; ϵ is real and b complex, and they satisfy $\Re(b) > \epsilon$; and the branch cut of $(z+b)^x$ lies on the negative real axis if x is not an integer. Apply Eq.(37) to $1/\Gamma(r+2h)$ with

$$x = r + 2h, \ \beta = \sqrt{\Lambda}L, \ b = 1, z = X,$$

and ϵ positive and infinitesimally small, and to $1/\Gamma(s+2h)$ with

$$x = s + 2h, \ \beta = \sqrt{\Lambda}L', \ b = 1, \ z = Y,$$

and the same ϵ in Eq.(36). The binomial theorem can then be used to perform the summation over r and s. Apply once more the binomial theorem to the sum over R, and keep the branch cut of every $[\cdots]^{2h}$ on the negative real axis. With this choice of the branch cuts, $(z_1 z_2)^{2h} = z_1^{2h} z_2^{2h}$ if both $\Re(z_1)$ and $\Re(z_2)$ are positive. Using this fact, we finally get

$$\tilde{G}_u(L,L';T) = 4^h \Lambda^{1-h} \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dY e^{\sqrt{\Lambda}L'Y} e^{-2h\sqrt{\Lambda}T}}{[Y+1-(Y-1)e^{-2\sqrt{\Lambda}T}]^{2h}}$$
$$\cdot \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dX e^{\sqrt{\Lambda}LX}}{\left[X + \frac{Y+1+(Y-1)e^{-2\sqrt{\Lambda}T}}{Y+1-(Y-1)e^{-2\sqrt{\Lambda}T}}\right]^{2h}},$$

Integrals of this type have been carried out for integer values of $2h$ in [5], but can be obtained for arbitrary values of $2h$. Indeed, apply Eq.(37) to the integration with respect to X and we immediately get

$$\tilde{G}_u(L,L';T) = \frac{4^h e^{-2h\sqrt{\Lambda}T} \sqrt{\Lambda}L^{2h-1}}{\Gamma(2h)2\pi i}$$
$$\cdot \int_{\epsilon-i\infty}^{\epsilon+i\infty} dY \frac{\exp\left[\sqrt{\Lambda}L'Y - \sqrt{\Lambda}L\frac{Y(1+e^{-2\sqrt{\Lambda}T})+(1-e^{-2\sqrt{\Lambda}T})}{Y(1-e^{-2\sqrt{\Lambda}T})+(1+e^{-2\sqrt{\Lambda}T})}\right]}{[Y(1-e^{-2\sqrt{\Lambda}T})+(1+e^{-2\sqrt{\Lambda}T})]^{2h}}.$$

Inserting the expansion

$$\exp\left[-\sqrt{\Lambda}L\frac{Y(1+e^{-2\sqrt{\Lambda}T})+(1-e^{-2\sqrt{\Lambda}T})}{Y(1-e^{-2\sqrt{\Lambda}T})+(1+e^{-2\sqrt{\Lambda}T})}\right] = \exp\left(-\sqrt{\Lambda}L\frac{1+e^{-2\sqrt{\Lambda}T}}{1-e^{-2\sqrt{\Lambda}T}}\right)$$
$$\cdot \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{4\sqrt{\Lambda}Le^{-2\sqrt{\Lambda}T}}{1-e^{-2\sqrt{\Lambda}T}}\frac{1}{Y(1-e^{-2\sqrt{\Lambda}T})+(1+e^{-2\sqrt{\Lambda}T})}\right]^n,$$

21

applying Eq.(37) again to the remaining integration, and recalling that

$$I_{2h-1}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+2h-1}}{n!\Gamma(2h+n)}$$

is the $(2h-1)$-th modified Bessel function [18], we finally obtain

$$\tilde{G}_u(L, L'; T) = \frac{(LL')^{h-\frac{1}{2}}\sqrt{\Lambda}}{\Gamma(2h)\sinh(\sqrt{\Lambda}T)} e^{-\sqrt{\Lambda}(L+L')\coth(\sqrt{\Lambda}T)} I_{2h-1}\left(\frac{2\sqrt{\Lambda LL'}}{\sinh(\sqrt{\Lambda}T)}\right).$$

Define the normalized transition function $\tilde{G}(L, L'; T)$ by normalizing $|\frac{L}{a}\rangle_h$ in Eq.(34). It follows from Eq.(24) that we have to choose $\alpha = 1$ and that $\tilde{G}(L, L'; T)$ deviates from $\tilde{G}_u(L, L'; T)$ by a factor of $(LL')^{h-\frac{1}{2}}/\Gamma(2h)$. Consequently,

$$\tilde{G}(L, L'; T) = \frac{\sqrt{\Lambda}}{\sinh(\sqrt{\Lambda}T)} e^{-\sqrt{\Lambda}(L+L')\coth(\sqrt{\Lambda}T)} I_{2h-1}\left(\frac{2\sqrt{\Lambda LL'}}{\sinh(\sqrt{\Lambda}T)}\right). \quad (38)$$

We note that if we put $h = 1/2$ in Eq.(38), we will obtain Eq.(15); for $h = 1$ we obtain Eq.(19) except for the factor $\sqrt{L/L'}$, which originates from the non-Hermiticity of H_c; if h is an integer, Eq.(38) will coincide with the propagator calculated in the proper-time gauge of two-dimensional quantum gravity in Ref.[11] with the winding number $h-1$; finally, if $2h$ is an integer, Eq.(38) will exactly be the $(2h)$-seamed correlation function in Ref.[5].

6   Discussion

We have, in this paper, investigated different Hamiltonian models of two-dimensional quantum gravity. Clearly, one open problem is the proper physical interpretation for the sl_2 gravity model with a non-integer value of $2h$. Perhaps this describe the interaction of gravity with matter, an important future problem on its own right. One could, for instance, study one-dimensional quantum spin systems whose Hamiltonians couple spin configurations of different sizes. Diagonalisation of such Hamiltonians seems to pose interesting new problems. Another possibility is to extend the class of Hamiltonians defined in this paper by exploiting the representation theory of the full Virasoro algebra instead of the sl_2 subalgebra; such models might describe the coupling between matter and gravity.

22

Finally, extension to higher dimensional Hamiltonian models of quantum gravity is an ultimate goal. It is rather straightforward to produce candidate models of nearest neighbouring type, but extracting non-trivial information from such models seems a non-trivial task. We refer to Ref.[19] for a recent work on higher dimensional Lorentzian gravity.

Acknowledgment

References

[1] V. G. Knizhnik, A. M. Polyakov, and A. A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819;
F. David, Mod. Phys. Lett. A3 (1988) 1651;
J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.

[2] J. Ambjørn, B. Durhuus, and T. Jónsson, Quantum Geometry, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1997).

[3] A. Ashtekar and R. P. Geroch, Rept. Prog. Phys. 37 (1974) 1211.

[4] J. Ambjørn and R. Loll, Nucl. Phys. B 536 (1999) 407 [hep-th/9805108].

[5] P. Di Francesco, E. Guitter, and C. Kristjansen, Nucl. Phys. B 567 (2000) 515 [hep-th/9907084].

[6] R. Giles and C. B. Thorn, Phys. Rev. D 16 (1977) 366;
C. B. Thorn, Proceedings of Sakharov Conference on Physics, Moscow (1991) pp.447–454 [hep-th/9405069].

[7] I. Klebanov and L. Susskind, Nucl. Phys. B 309 (1988) 175.

[8] C. B. Thorn, Phys. Rev. D 20 (1979) 1435.

[9] C.-W. H. Lee and S. G. Rajeev, Physical Review Letters 80 (1998) 2258 [hep-th/9711052].

[10] R. J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic Press, 1982).

[11] R. Nakayama, Phys. Lett. B 325 (1994) 347 [hep-th/9312158].

[12] C.-W. H. Lee and S. G. Rajeev, Nuclear Physics B 529 (1998) 656 [hep-th/9712090].

[13] C.-W. H. Lee and S. G. Rajeev, Journal of Mathematical Physics 39 (1998) 5199 [hep-th/9806002].

[14] P. Di Francesco, E. Guitter, and C. Kristjansen, Nucl. Phys. B 608 (2001) 485 [hep-th/0010259].

[15] M. R. Spiegel, Schaum's Outline of Theory and Problems of Calculus of Finite Differences and Difference Equations (McGraw-Hill, 1971).

[16] M. E. Larsen and E. S. Andersen, Normat 42 (1994) 116.

[17] J. W. Dettman, Applied Complex Variables (Dover Pubications, 1984).

[18] G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, 1945).

[19] J. Ambjørn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. 85 (2000) 924 [hep-th/0002050].