Long-range scattering of three-body quantum systems, II

Erik Skibsted,
Institut for Matematiske Fag and MaPhySto[1],
Aarhus Universitet,
Ny Munkegade 8000 Aarhus C,
Denmark,
skibsted@imf.au.dk

[1] Centre for Mathematical Physics and Stochastics, funded by a grant from the Danish National Research Foundation.

Contents

1 Introduction and results

This paper is the second in a series of two papers on asymptotic completeness for (generalized) three-body quantum systems with long-range interaction. Asymptotic completeness is henceforth abbreviated AC; an "effective" version of AC is the existence of the limits (1.4) and (1.5) given below. For a full account of the statement we refer the reader to our first paper [S1]. We shall in this paper prove results in the regime "$\mu \in (0, \frac{1}{2}]$" with μ measuring the decay of the "pair potentials" at infinity as in (1.2) (given below). All examples in the literature on AC for many-body systems seem so far to be restricted to μ greater than one half. Moreover Yafaev ([Y]) constructed counterexamples to AC for any $\mu \in \left(0, \frac{1}{2}\right)$ in systems of one-dimensional particles. On the other hand there are many papers in the literature on AC for two-body systems with arbitrary $\mu > 0$. We have results for two different classes of potentials: 1) One-dimensional potentials with a negative upper bound near infinity (like those considered in [S1]). 2) Potentials (in any dimension) with a positive lower bound near infinity.

We recall the basic model and a reduction scheme for AC, see [S1] for more details. We consider a finite family of subspaces $\{X_a | a \in \mathcal{F}\}$ of a finite dimensional Euclidean space X. By definition $a_{\min}, a_{\max} \in \mathcal{F}$ are given by $X_{a_{\min}} = X$ and $X_{a_{\max}} = \{0\}$, respectively, and for a and b different from $a_{\min}$ the "three-body" condition $X_a \cap X_b = \{0\}$ is imposed. The position and momentum operators on the basic Hilbert space $\mathcal{H} = L^2(X)$ are denoted by x and p, respectively. The orthogonal complement of X_a in X is denoted by X^a. The corresponding components of x and p are denoted by x_a, p_a and x^a, p^a, respectively.

The basic Hamiltonian on $\mathcal{H}$ is

$$H = \tfrac{1}{2}p^2 + V;\ V(x) = \sum_{a \in \mathcal{F}} V^a(x^a), \tag{1.1}$$

where each "pair potential" V^a is assumed to be a real-valued smooth function on X^a obeying for some $\mu > 0$ (independent of a) and all multiindices β

$$\partial_{x^a}^\beta V^a(x^a) = O\Big(|x^a|^{-\mu-|\beta|}\Big). \tag{1.2}$$

We consider the propagator $U_a(t)$ generated by $H_a(t) = H^a + \frac{1}{2}p_a^2 + I_a(t,x)$, where $I_a(t,x) = J\big(\frac{x}{t}\big)I_a(x)$ with J an arbitrary C_0^∞ cutoff function supported in $Y_a = X\backslash\big(\cup_{b\not\subset a}X_b\big)$ and with $I_a(x) = V(x) - V^a(x^a)$. (By definition $b \subset a \Leftrightarrow X^b \subset X^a$.) Clearly we have the bound

$$\partial_x^\beta I_a(t,x) = O\Big(t^{-\mu-|\beta|}\Big) \text{ uniformly in } x. \tag{1.3}$$

There exists the asymptotic energy $H^{a+} = \lim_{t\to+\infty} U_a(t)^\star H^a U_a(t)$ (understood in the strong resolvent sense). Let $\tilde{U}_a(t)$ denote the propagator generated by $H^a + \frac{1}{2}p_a^2 + I_a(t,x_a)$.

The notation $E_\Omega(D)$ denotes the spectral projection for a self-adjoint operator D corresponding to a Borel set $\Omega \subseteq \mathbf{R}$.

Now, the "effective" version of AC that we are going to address in this paper is the following statement: For all $\phi_a^+ \in E_{\{0\}}\big(H^{a+}\big)$ there exists the limit

$$\tilde{\phi}_a^+ = \lim_{t\to+\infty} \tilde{U}_a(t)^* U_a(t)\phi_a^+; \tag{1.4}$$

and for all $\tilde{\phi}_a^+ \in E_{\{0\}}(H^a)$ there exists the limit

$$\phi_a^+ = \lim_{t\to+\infty} U_a(t)^* \tilde{U}_a(t)\tilde{\phi}_a^+. \tag{1.5}$$

1.1 Negative potentials

In addition to (1.2) we shall need the following negativity condition of [S1] for a (fixed) $a \in \mathcal{F}$: For some $c, R > 0$

$$V^a(x^a) \le -c|x^a|^{-\mu},\ |x^a| \ge R. \tag{1.6}$$

Moreover we shall assume that the subspace X^a is one-dimensional. In our last set of conditions (1.7)-(1.11) stated below we identify the part of x denoted by x^a by a coordinate for this vector given by fixing a basis vector for the subspace X^a. Suppose

$$V^a(x^a) = V_1^a(x^a) + V_2^a(x^a) + V_3^a(x^a), \tag{1.7}$$

where $V_1^a(x^a)$, $V_2^a(x^a)$ and $V_3^a(x^a)$ obey (1.2),

$$\begin{aligned} &-C_r|x^a|^{-\mu-2} \le V_1^{a\prime\prime}(x^a) \le -c_r|x^a|^{-\mu-2};\ x^a \ge R,\\ &c_r, C_r > 0 \text{ and } C_r < 2^{-1}(2+\mu)^2 c_r, \end{aligned} \tag{1.8}$$

$$-C_l|x^a|^{-\mu-2} \le V_1^{a\prime\prime}(x^a) \le -c_l|x^a|^{-\mu-2};\ x^a \le -R, \quad (1.9)$$
$$c_l, C_l > 0 \text{ and } C_l < 2^{-1}(2+\mu)^2 c_l,$$

and for some $\epsilon > 0$

$$V_2^{a\prime}(x^a) = O\big(|x^a|^{-1-\mu-\epsilon}\big), \quad (1.10)$$

and

$$V_3^a(x^a) = O\Big(|x^a|^{-\frac{1+\epsilon}{\alpha}}\Big). \quad (1.11)$$

In (1.11) and henceforth $\alpha = 2(2+\mu)^{-1}$.

Now, suppose the conditions (1.2), (1.3) and (1.6) for some a with $\dim X^a = 1$; all conditions with the same (fixed) $\mu \in (0, \frac{1}{2}]$. Then one can introduce, cf. [S1],

$$\begin{aligned} &P^{a+} = P_r^{a+} + P_l^{a+},\\ &P_r^{a+} = s - \lim_{t\to+\infty} U_a(t)^* E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(x^a) U_a(t) E_{\{0\}}\big(H^{a+}\big),\\ &P_l^{a+} = s - \lim_{t\to+\infty} U_a(t)^* E_{[-t^{\alpha+\epsilon}, -t^{\alpha-\epsilon}]}(x^a) U_a(t) E_{\{0\}}\big(H^{a+}\big). \end{aligned} \quad (1.12)$$

It was proven that these limits are independent of (small) $\epsilon > 0$ and that $P^{a+} = E_{\{0\}}\big(H^{a+}\big)$. We shall show the existence of (1.4) by proving that $P^{a+} = 0$, cf. [S1].

Our main result is the following.

Theorem 1.1 *Under the conditions (1.2), (1.3), (1.6)-(1.11) with* $\dim X^a = 1$ *and* $\mu \in (0, \frac{1}{2}]$

$$E_{\{0\}}\big(H^{a+}\big) = 0; \quad (1.13)$$

in particular the existence of (1.4) holds.

The existence of (1.5) with these assumptions follows from the fact that in this case $E_{\{0\}}(H^a) = 0$, cf. [O, Theorem 2.2 p. 196]. Combined with (1.13), AC follows.

Remark 1.2 For simplicity of presentation we shall prove Theorem 1.1 with the additional assumption that $V_2^a = V_3^a = 0$. The general case may be treated in the following fashion: First we may assume $V_3^a = 0$ since V_3^a is "short-range". Next we keep V_2^a in the analysis (of Sections 2–7). We define the classical orbit in (2.3) in terms of the "dominating" term V_1^a only. Keeping track of contributions from error terms coming from V_2^a yields a weaker localization than (2.4), but strong enough for the arguments of Section 7 ((7.1) needs to be replaced by a weaker estimate).

As indicated in the above remark we devote Sections 2–7 to a proof of a slightly simplified version of Theorem 1.1. Our basic strategy is similar to one applied to a different problem, although with common spirit, in [HS]. We compare the evolution of a state $\phi_a^+ \in P_r^{a+}\mathcal{H}$ (or $\phi_a^+ \in P_l^{a+}\mathcal{H}$) with a simplified evolution in terms of a relative wave operator. Setting up this wave operator is the content of Sections 2–6. Our techniques at this point resemble at many points those applied in [HS]. In Section 7 we verify conditions of [S2] for the simplified evolution. Using the relevant result of [S2] we infer that indeed states propagated with this evolution *cannot* be localized to regions of the configuration space that the projection P_r^{a+} a priori prescribes. Consequently $\phi_a^+ = 0$. (Notice that this is a purely quantum statement; it has no analogue in classical mechanics.) The technique of Section 7 differs completely from the one applied at the similar step in [HS]. We remark that one may modify the latter technique of [HS] as to provide another approach to our second step in Section 7. However it is more complicated. Morever we remark that neither of those approaches seem to be optimal. For example we consider the conditions (1.8) and (1.9) to be "technical"; the condition (2.1) stated below should suffice. As an open problem motivated by the analysis of [S1] we mention AC for negative potentials in higher dimensions with spherical symmetry.

In Appendix A we prove bounds for classical orbits of some one-dimensional quadratic Hamilton functions, that are needed at various points in Sections 5–7.

1.2 Positive potentials

In addition to (1.2) for a $\mu \in (0, \frac{1}{2}]$ we shall need the following positivity condition for a (fixed) $a \in \mathcal{F}$: For some $\mu^+ \in [\mu, \frac{2\mu}{1-\mu})$ and $R > 0$

(1.14) $\quad V^a(x^a) \geq |x^a|^{-\mu^+}, \ |x^a| \geq R.$

Our main result is the following.

Theorem 1.3 *Under the conditions (1.2), (1.3), (1.14) with a $\mu \in (0, \frac{1}{2}]$, the limits (1.4) and (1.5) exist; in particular AC holds.*

The proof of Theorem 1.3 is given in Section 8. It is based on some energy bounds which may be viewed as modifications of results of [S1]. Conceptually and technically it is much simpler than the proof of AC for negative potentials due to the fact that there are no classical orbits (for the internal dynamics) at infinity with zero energy. To put our result and method into perspective we also give a proof of Wang's result for $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$, [W].

Obviously Theorems 1.1 and 1.3 can be combined to obtain asymptotic completeness as defined in [S1] for families of pair potentials of mixed type, each either negative or positive (at infinity) with further properties as specified in the theorems.

2 Preliminary estimates (negative potentials)

In addition to (1.2), (1.3) and (1.6) with $\dim X^a = 1$ and $\mu \in (0, \frac{1}{2}]$, we shall in Sections 2–6 need the concavity assumption

(2.1) $\quad V^{a\prime\prime}(x^a) \le 0$ for $|x^a| \ge R$.

Only in Section 7 the stronger conditions (1.8) and (1.9) are needed (to verify (7.4)). Recall that we put $V_2^a = V_3^a = 0$, cf. Remark 1.2.

We aim at showing that for any given $\phi_a^+ \in P_r^{a+}\mathcal{H}$ indeed $\phi_a^+ = 0$. Since our proof can be adapted for P_l^{a+} we then conclude (1.13).

In this section we are going to use the freedom to change $I_a(t,x)$ to the effect

(2.2) $$I_a(t,x) = I_a(t,x)F_+\Big(\tfrac{x^a}{t^{\alpha-\epsilon}}\Big) + I_a(t, x^a = 0, x_a)F_-\Big(\tfrac{x^a}{t^{\alpha-\epsilon}}\Big)$$

for $\epsilon > 0$ chosen arbitrarily small, cf. [S1, Section 5]. (Here and henceforth we adapt the notation F_+ and F_- of [S1, Definitions 2.1].) The proof of [S1, Lemma 4.5] (with $j = 1$) yields the following improved (and classically "optimal") localization (cf. the proof of [S1, (5.6)]). We remark that we are not going to use the full strength of the result; it is stated here only for completeness of presentation.

Lemma 2.1 *Let L be the solution of the initial value problem*

(2.3) $$\tfrac{d}{dt}L(t) = \sqrt{-2V^a(L(t))},\ L(0) = R;\ t \ge 0.$$

Then for all $\varepsilon > 0$ and $\phi_a^+ \in P_r^{a+}\mathcal{H}$

(2.4) $$||F_+\big(t^{\mu-\varepsilon-1}|x^a - L|\big)\phi_a^+(t)|| = o\big(t^0\big),$$

where $\phi_a^+(t) = U_a(t)\phi_a^+$.

"Proof" Given $\varepsilon > 0$ we may assume (2.2) for a small $\epsilon > 0$ obeying $\frac{3}{2}\mu\epsilon < \varepsilon$. We modify the proof of [S1, (5.6)] by introducing for small $\sigma > 0$ obeying $\frac{3}{2}\mu\epsilon < 3\sigma < \varepsilon$

(2.5) $$\begin{aligned} \alpha_0 &= \alpha + \frac{\sigma}{\mu}, \\ \gamma_1 &= \frac{\mu}{2}\alpha_0 + \sigma = \frac{\mu}{2}\alpha + \frac{3}{2}\sigma, \\ \alpha_1 &= 1 - \gamma_1 - \sigma = \alpha - \frac{5}{2}\sigma, \\ \delta &= \mu + \frac{\mu}{2}\alpha - \sigma, \\ \beta_1 &= 1 - \mu + \frac{\mu}{2}\alpha + 3\sigma. \end{aligned}$$

All requirements of the proof of [S1, (5.6)] including [S1, (4.37)] are fulfilled for the choice (2.5) with $\sigma > 0$ small enough yielding to the following statement:

$$\lim_{t\to+\infty} ||\phi_a^+(t) - \bar{B}\big(t, t^{\beta_1}\big)\phi_a^+(t)|| = 0, \tag{2.6}$$

where for parameters given by (2.5) and $K(x^a)$ being the inverse of $L(t)$

$$\begin{aligned}
&\bar{B}\Big(t, t^{\beta_1}\Big) = \bar{F}_0 F_1 F_2 F_3 F_4;\\
&\bar{F}_0 = F_-\Big(t^{-\beta_1}|t - K(x^a)|\Big),\\
&F_1 = F_+\bigg(\frac{x^a}{t^{\alpha_1}}\bigg),\ F_2 = F_-\bigg(\frac{x^a}{t^{\alpha_0}}\bigg),\ F_3 = F_+(t^{\gamma_1}p^a),\ F_4 = F_-\Big(|t^\delta H^a|\Big).
\end{aligned} \tag{2.7}$$

(In the present context we dont need [S1, (4.37)] though. We can use the fact that $\phi_a^+ \in P_r^{a+}\mathcal{H} \subseteq E_{\{0\}}\big(H^{a+}\big)\mathcal{H}$ and [S1, Lemma 2.3] to avoid a certain symmetrizing under use of [S1, (4.37)].)

$\square$

We shall need some operators r_t, b_t and $\tilde{b}_t$ which are modelled after constructions in [D] and [HS]:

$$r_t(x^a) = fr\big(f^{-1}(x^a - L)\big);\ f = t^{\frac{3}{4}},\ r(y) = \langle y\rangle = \big(1 + |y|^2\big)^{\frac{1}{2}}. \tag{2.8}$$

We compute its Heisenberg derivative (with $\mathbf{D} = \frac{d}{dt} + i[H_a(t), \cdot]$)

$$\begin{aligned}
&b_t = \mathbf{D}r_t = \frac{1}{2}\Big(r'\big(f^{-1}(x^a - L)\big)\Big(p^a - \dot{L}\Big) + h.c.\Big) + f'd_t;\\
&d_t = fr_t^{-1},\ \dot{L} = \frac{d}{dt}L(t).
\end{aligned} \tag{2.9}$$

Furthermore (with dots used again for time-derivatives)

$$\begin{aligned}
&\mathbf{D}b_t = \mathbf{D}^2 r_t = f^{-1}c_t + \ddot{f}d_t - f^{-3}e_t - r'\big(f^{-1}(x^a - L)\big)\bigg(\frac{\partial}{\partial x^a}V + \ddot{L}\bigg);\\
&c_t = P^*\bigg(\frac{d^2}{dx^{a2}}r\bigg)\big(f^{-1}(x^a - L)\big)P,\ P = p^a - \dot{L} - \frac{\dot{f}}{f}(x^a - L),\\
&e_t = 4^{-1}\bigg(\frac{d^4}{dx^{a4}}r\bigg)\big(f^{-1}(x^a - L)\big).
\end{aligned} \tag{2.10}$$

Clearly both b_t and $\mathbf{D}b_t$ are bounded relatively to p^{a2}. Also we notice the non-negativity of c_t and the uniform boundedness (with respect to t) of the terms d_t and e_t.

We may estimate using also the fact that

$$\tfrac{\partial}{\partial x^a}V + \ddot{L} = \tfrac{\partial}{\partial x^a}I_a + (x^a - L)\int_0^1 ds V^{a\prime\prime}(L + s(x^a - L)) \tag{2.11}$$

and (2.1),

$$\mathbf{D}b_t \geq -r'\big(f^{-1}(x^a - L)\big)\tfrac{\partial}{\partial x^a} I_a + C_1 \ddot{f} \geq -C_2 t^{-\min(\mu,\frac{1}{4})-1}. \tag{2.12}$$

We introduce the regularizations

$$\begin{aligned} &\tilde{b}_t = N_t^{-1} b_t N_t^{-1},\ \tilde{c}_t = N_t^{-1} c_t N_t^{-1};\\ &N_t = I + t^{-2\nu} p^{a2},\ \nu > 0. \end{aligned} \tag{2.13}$$

We also introduce

$$\begin{aligned} &h = t^{1-\rho_2} = t^{\alpha - 2\sigma};\ \rho_2 = \frac{\mu}{2}\alpha + 2\sigma,\\ &g = t^{\rho_1};\ \rho_1 = \frac{\mu}{2}\alpha + 3\sigma. \end{aligned} \tag{2.14}$$

Obviously (for future reference) $f = o(h)$ and $\rho_2 < \rho_3$.

To start out the analysis we shall use the following weaker localization than the one presented in Lemma 2.1:

$$||\phi_a^+(t) - F_-\big(h^{-1} r_t\big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.15}$$

Notice that this localization is very weak as opposed to the "optimal" one of Lemma 2.1. Similarly the following localization result is very weak; the "optimal" bound follows readily from the proof:

Lemma 2.2 *Let ϕ_a^+ be given as in Lemma 2.1. Then for all small $\sigma > 0$ (and all $\nu > 0$)*

$$||F_+\Big(g\tilde{b}_t\Big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.16}$$

Proof As in the proof of Lemma 2.1 we may assume (2.2) with $\epsilon > 0$ small. We use the proof of this lemma. By (2.6) it suffices to estimate

$$||g\tilde{b}_t \bar{B}\big(t, t^{\beta_1}\big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.17}$$

By a commutation (2.17) will follow from

$$||g\Big(p^a - \dot{L}\Big)\bar{B}\big(t, t^{\beta_1}\big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.18}$$

To show (2.18) we may insert $F_+ = F_+(4t^{\gamma_1} p^a)$ to the left and then write

$$F_+\Big(p^a - \dot{L}\Big) = F_+\Big(p^a + \dot{L}\Big)^{-1} 2(H^a - (V^a(x^a) - V(L))).$$

The contribution from H^a may by commutation be shown to be $O\big(t^{3\sigma-\delta+\mu\alpha}\big)$, while the one from the second term to the right is $O\Big(t^{6\sigma-\mu+\frac{\mu}{2}\alpha}\Big)$ due to the presence of the factor $\bar{F}_0$ and the formula

$$V^a(x^a) - V(L) = (x^a - L)\int_0^1 ds V^{a\prime}(L + s(x^a - L)).$$

$\square$

For any given $\phi_a^+ \in P_r^{a+}\mathcal{H}$ we aim at proving the existence of the limit

$$\lim_{t\to+\infty} \breve{U}_a^*(t)\phi_a^+(t), \tag{2.19}$$

for some comparison dynamics $\breve{U}_a(t)$ to be defined in Section 4 and for which a result of [S2] can be applied (to conclude $\phi_a^+ = 0$).

3 Integral estimates

In order to prove the existence of (2.19) we need certain integral estimates for the full dynamics $U_a(t)$, cf. [D] and [HS]. Henceforth we shall not use or assume (2.2). Nevertheless we are going to apply some of the parameters of (2.5). The δ is changed to

$$\delta = \mu - \sigma. \tag{3.1}$$

Lemma 3.1 *For all small enough $\sigma, \nu > 0$ (depending only on μ) and with $F(t) = F_1F_2F_3F_4$, where the factors to the right are given by (2.7) with $\alpha_0, \gamma_1,$ and α_1 given by (2.5) and δ by (3.1), and with the expectation value $\langle\cdot\rangle_\phi$ given in the state $\phi = \phi_a^+(t) = U_a(t)\phi_a^+$ for any $\phi_a^+ \in P_r^{a+}\mathcal{H}$*

$$\int_1^\infty \Big\langle G(t)^*\big(-F_-^{2\prime}\big)^{\frac12}\Big(g\tilde{b}_t\Big)\tilde{f}_t\big(-F_-^{2\prime}\big)^{\frac12}\Big(g\tilde{b}_t\Big)G(t)\Big\rangle_{\phi_a^+(t)} dt < \infty;$$

$$\tilde{f}_t = gf^{-1}\tilde{c}_t + t^{-1}I - gN_t^{-1}r'\big(f^{-1}(x^a - L)\big)F_+\left(4\frac{x^a}{t^{\alpha_1}}\right)\Big(V^{a\prime} + \ddot{L}\Big)N_t^{-1}, \tag{3.2}$$

$$G(t) = G_+(t) = F_+\big(h^{-1}r_t\big)N_t^{-1}F(t) \text{ or}$$

$$G(t) = G_-(t) = F_-\big(h^{-1}r_t\big)N_t^{-1}F(t),$$

and

$$\int_1^\infty \Big\langle H(t)^*F_-^2\Big(g\tilde{b}_t\Big)\Big(t^{-1-\sigma}\Big(2I - g\tilde{b}_t\Big) + t^{-1}I\Big)H(t)\Big\rangle_{\phi_a^+(t)} dt < \infty;$$

(3.3)

$$H(t) = H_+(t) = \big(F_+^{2\prime}\big)^{\frac12}\big(h^{-1}r_t\big)F(t) \text{ or}$$

$$H(t) = H_-(t) = \big(-F_-^{2\prime}\big)^{\frac12}\big(h^{-1}r_t\big)F(t).$$

Proof We notice that indeed the expression $\tilde{f}_t$ of (3.2) is a sum of non-negative terms, cf. (2.1) and (2.11). A similar remark is due for the integrand of (3.3).

For (3.2) we claim that the estimate with $G(t) = N_t^{-1}F(t)$ follows by considering the "propagation observable" (more precisely the uniformly bounded family of observables)

$$\Phi(t) = \big(N_t^{-1}F(t)\big)^* F_-^2\Big(g\tilde{b}_t\Big) N_t^{-1}F(t).$$

Let us compute the "leading term" coming from differentiating the middle term: We introduce the modified Heisenberg derivative $\mathbf{D}_t = \frac{d}{dt} + i\Big[\frac{1}{2}p^2 + V^a(x^a)F_+^2\Big(4\frac{x^a}{t^{\alpha_1}}\Big) + I_a(t,x), \cdot\Big]$.

$$\mathbf{D}_t F_-^2\Big(g\tilde{b}_t\Big) = -\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\Big(\mathbf{D}_t\Big(g\tilde{b}_t\Big)\Big)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big) + R_1(t). \tag{3.4}$$

Here

$$||R_1(t)|| \leq C||\Big[g\tilde{b}_t, \Big(\mathbf{D}_t\Big(g\tilde{b}_t\Big)\Big)\Big]||, \tag{3.5}$$

cf. [DG, Lemma C.4.1]. Using (3.6) given below (and concrete expressions for the derivatives involved) we readily estimate the right hand side by a constant times $g^2t^{2\nu}f^{-2}$ which is in $L^1(dt)$ for $\nu > 0$ small enough.

Obviously

$$\mathbf{D}_t\Big(g\tilde{b}_t\Big) = \dot{g}\tilde{b}_t + gN_t^{-1}(\mathbf{D}_t b_t)N_t^{-1} + 2g\mathrm{Re}\big(\big(\mathbf{D}_t N_t^{-1}\big)b_t N_t^{-1}\big). \tag{3.6}$$

The contribution from the first two terms on the right hand side of (3.6) to the Heisenberg derivative of $\Phi(t)$ is

$$T_1(t) + R_2(t);$$

$$T_1(t) = -B(t)^*\Big\{\dot{g}\tilde{b}_t + gN_t^{-1}(\mathbf{D}b_t)N_t^{-1}\Big\}B(t),\ B(t) = \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)N_t^{-1}F(t),$$

$$||R_2(t)|| \in L^1(dt).$$

Here we used that that functions $F_1 = F_+\Big(\frac{x^a}{t^{\alpha_1}}\Big)$ and any $F_-\Big(4\frac{x^a}{t^{\alpha_1}}\Big)$ have disjoint support. Commutation picks up an integrable term. (In fact $R_2(t) = O\big(t^{-\infty}\big)$.) As for the contribution from the last term on the right hand side of (3.6) we compute for a suitable real-valued $F_c \in C_0^\infty(\mathbf{R})$

$$-2gB(t)^*\mathrm{Re}\big(\big(\mathbf{D}_t N_t^{-1}\big)b_t N_t^{-1}\big)B(t) = T_2(t) + R_3(t) + R_4(t);$$

$$T_2(t) = -4\nu B(t)^* F_c\Big(g\tilde{b}_t\Big)t^{-2\nu-1}p^{a2}N_t^{-1}F_c\Big(g\tilde{b}_t\Big)B(t),$$

$$||R_3(t)|| \leq C||\Big[F_c\Big(g\tilde{b}_t\Big), t^{-2\nu-1}p^{a2}N_t^{-1}\Big]|| = O\big(gf^{-1}t^{-1}\big) = O\Big(t^{\frac{\mu}{2}\alpha - \frac{7}{4} + 3\sigma}\Big),$$

$$||R_4(t)|| \leq C||\Big[V^a(x^a)F_+^2\Big(4\frac{x^a}{t^{\alpha_1}}\Big) + I_a(t,x), N_t^{-1}\Big]N_t|| = O\Big(t^{-\nu-(1+\mu)\alpha_1}\Big).$$

Clearly it follows that $||R_3(t)||, ||R_4(t)|| \in L^1(dt)$.

By (2.10), (2.11) and (2.12)

$$T_1(t) + T_2(t) \le -B(t)^* \tilde{f}_t B(t) + R(t), \tag{3.7}$$

where $||R(t)|| \in L^1(dt)$.

Next we look at the contributions

$$T_3 = \big(N_t^{-1}F(t)\big)^* F_-^2\Big(g\tilde{b}_t\Big)\big(\mathbf{D}_t N_t^{-1}\big)F(t) + h.c. \tag{3.8}$$

and

$$T_4 = \big(N_t^{-1}F(t)\big)^* F_-^2\Big(g\tilde{b}_t\Big)N_t^{-1}\mathbf{D}F(t) + h.c.$$

to the Heisenberg derivative of $\Phi(t)$:

We compute

$$\mathbf{D}_t N_t^{-1} = \frac{d}{dt}N_t^{-1} + i\left[V^a(x^a)F_+^2\left(4\frac{x^a}{t^{\alpha_1}}\right) + I_a(t,x), N_t^{-1}\right].$$

The first term contributes to (3.8) by a term that is $O\big(t^{-1-2\nu}\big)$ (since p^{a2} may be bounded by that factor F_4). Obviously by the above bound for $R_4(t)$ the second term is integrable.

As for the term $T_4(t)$ the derivatives of the factors of F are readily handled (i.e. proven integrable) by using various estimates of [S1], cf. the proof of Lemma 2.1. Straightforward computations of commutators with the middle term $N_t^{-1}F_-^2\Big(g\tilde{b}_t\Big)N_t^{-1}$ needed when symmetrizing expressions from the derivatives of the factors F_1 and F_2 show that those contribute by integrable terms.

In combination with (3.7) we finally conclude the estimate

$$\int_1^\infty \left\langle \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\tilde{f}_t\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\right\rangle_{N_t^{-1}F(t)\phi_a^+(t)} dt < \infty. \tag{3.9}$$

To obtain (3.2) for $G(t) = G_-(t)$ it suffices by (3.9) to show the statement for $G(t) = G_+(t)$. We show the latter and (3.3) for $H(t) = H_+(t)$ in one stroke by considering propagation observable

$$\Phi(t) = G_+(t)^* F_-^2\Big(g\tilde{b}_t\Big)G_+(t).$$

We notice that

$$\mathbf{D}F_+\big(h^{-1}r_t\big) = \frac{1}{2}F_+'\big(h^{-1}r_t\big)\Big(h^{-1}\mathbf{D}r_t - \dot{h}h^{-2}r_t\Big) + h.c.$$

$$= h^{-1}b_t F_+'\big(h^{-1}r_t\big) - \frac{\dot{h}}{h}\big(h^{-1}r_t\big)F_+'\big(h^{-1}r_t\big) + O\big(h^{-2}\big),$$

tends to be negative when sandwitched by factors of $F_-\Big(g\tilde{b}_t\Big)$, cf. the proof of [D, Proposition 5.6]. The previous arguments for the contribution from $\mathbf{D}_t F_-^2\Big(g\tilde{b}_t\Big)$ applies again (this term contributes by another non-positive term). We skip the straightforward details.

To obtain (3.3) for $H(t) = H_-(t)$ we differentiate $\Phi(t) = G_-(t)^* F_-^2\Big(t^{\rho_1}\tilde{b}_t\Big)G_-(t)$ under use of similar computations as for the first estimate of (3.3), and we use (3.2) for $G(t) = G_-(t)$. $\square$

4 A simplified comparison dynamics

We introduce a comparison dynamics $\breve{U}_a(t)$ by

$$i\tfrac{d}{dt}\breve{U}_a(t) = \breve{H}_a(t)\breve{U}_a(t),\ \breve{U}_a(1) = I, \tag{4.1}$$

and

$$\breve{H}_a(t) = \frac{1}{2}p_a^2 + \breve{H}^a(t) + \breve{R}(t,x);$$
$$\breve{H}^a(t) = \frac{1}{2}p^{a2} + V^a(L) + (x^a - L)V^{a\prime}(L) + 2^{-1}V^{a\prime\prime}(L)(x^a - L)^2,$$
$$\breve{R}(t,x) = I_a(t,x) + \breve{R}^a(t,x^a), \tag{4.2}$$
$$\breve{R}^a = \breve{F}^*(x^a - L)^3\int_0^1 \frac{(1-s)^2}{2}\left(\frac{d^3}{dx^{a3}}V^a\right)(L + s(x^a - L))ds\breve{F},$$
$$\breve{F} = F_-\big(4^{-1}h^{-1}r_t\big).$$

Clearly for all $k \in \mathbf{N}\cup\{0\}$

$$\partial_{x^a}^k\breve{R}^a = O\Big(t^{-(3+\mu)\alpha}h^{3-k}\Big), \tag{4.3}$$

uniformly in x.

It is known that $\breve{U}_a(t)$ preserves the domain of $p^2 + x^2$, see [S2, Section 4].

5 Further integral estimates

We need further integral estimates for the full dynamics. To motivate those we consider the following "model Hamiltonian" $\breve{H}^a(t)$ of (4.2). Let $\breve{U}^a(t)$ denote the corresponding propagator, i.e. $i\partial_t\breve{U}^a(t) = \breve{H}^a(t)\breve{U}^a(t)$ and $\breve{U}^a(1) = I$.

We shall introduce "radiation operators" for the generator. For that we use two solutions $\alpha^+(t)$ and $\alpha^-(t)$ to the Riccati equation

$$\dot{\alpha} = -V^{a\prime\prime}(L) - \alpha^2$$

with the properties that for some $C > 1$ and all large t

$$t^{-1} \leq \alpha^{+}(t) \leq Ct^{-1},\ 0 \leq -\alpha^{-}(t) \leq Ct^{-1}. \tag{5.1}$$

(See Appendix A for an elaboration.)

In terms of these solutions we define

$$\eta^{+} = \left(p^a - \dot{L}\right) - \alpha^{+}(x^a - L),\ \eta^{-} = \left(p^a - \dot{L}\right) - \alpha^{-}(x^a - L),$$

and notice that

$$\breve{\mathbf{D}}^a \eta^{+} = -\alpha^{+}\eta^{+},\ \breve{\mathbf{D}}^a \eta^{-} = -\alpha^{-}\eta^{-}, \tag{5.2}$$

where $\breve{\mathbf{D}}^a$ here refers to the Heisenberg derivative with respect to $\breve{H}^a(t)$.

Let

$$N_t^{+} = \left(\eta^{+}\right)^2,\ N_t^{-} = \left(\eta^{-}\right)^2,$$

and

$$G(t) = F_{-}\left(N_t^{-}\right)F_{-}\left(N_t^{+}\right)F(t);$$

$$F(t) = F_{-}\left(g\tilde{b}_t\right)F_{-}\left(h^{-1}r_t\right)N_t^{-1}F_3F_4,$$

where F_3 and F_4 are given as in Lemma 3.1.

Lemma 5.1 *With $\phi_a^{+}(t)$ given as in Lemma 3.1*

$$\int_1^\infty t^{-1}\left\langle -F(t)^* F'_{-}\left(N_t^{+}\right)F(t)\right\rangle_{\phi_a^{+}(t)} dt < \infty, \tag{5.3}$$

and

$$\int_1^\infty \alpha^{-}(t)\left\langle F(t)^* F'_{-}\left(N_t^{-}\right)F(t)\right\rangle_{\phi_a^{+}(t)} dt < \infty. \tag{5.4}$$

Proof Consider for (5.3) the propagation observable

$$\Phi(t) = F(t)^* F_{-}\left(N_t^{+}\right)F(t).$$

(We shall only prove (5.3); the estimate (5.4) follows in similar manner.) Computing the Heisenberg derivative gives terms that can be treated by Lemma 3.1 after symmetrizing. (Notice that the functions F_1 and F_2 of Lemma 3.1 are one on a neighborhood of the support of the function $F_{-}\left(h^{-1}r_t\right)$ of the product $F(t)$.) To treat the contribution from the derivative of $F_{-}\left(N_t^{+}\right)$ we introduce the modified Heisenberg derivative $\breve{\mathbf{D}}_t = \frac{d}{dt} + i\left[\breve{H}_a(t), \cdot\right]$. Due to the fact that the function $F_{-}\left(h^{-1}r_t\right)$ of the product $F(t)$

and the function $\breve{F}$ of (4.2) are disjointly supported it suffices to consider $\breve{\mathbf{D}}_t F_-\big(N_t^+\big)$: We compute using (5.2)

$$\breve{\mathbf{D}}_t F_-\big(N_t^+\big) = -2\alpha^+ F_-'\big(N_t^+\big) + i\Big[\breve{R}(t,x), F_-\big(N_t^+\big)\Big].$$

Clearly the last term on the right hand side is $O\big(t^{-1-\mu}\big) + O\Big(t^{-(3+\mu)\alpha}h^2\Big)$ by (4.3), in particular integrable. We now get (5.3) by combining with (5.1).

$\square$

By (2.6), (2.15) and (2.16)

$$||\phi_a^+(t) - G(t)\phi_a^+(t)|| = o\big(t^0\big).$$

Therefore to show the existence of (2.19) it suffices to show the existence of

$$\lim_{t\to+\infty} \breve{U}_a(t)^* G(t)^* G(t)\phi_a^+(t). \tag{5.5}$$

For future reference we notice that

$$||F_+\big(4^{-1}h^{-1}r_t\big)G(t)^*G(t)|| \in L^1(dt). \tag{5.6}$$

6 Integral estimates for the comparison dynamics

We shall need the following estimates for $\breve{U}_a(t)$, cf. Lemmas 3.1 and 5.1.

Lemma 6.1 *For all small enough $\sigma, \nu > 0$ and all $\phi \in \mathcal{H}$ the following estimates hold with $\breve{\phi}_a(t) = \breve{U}_a(t)\phi$:*

$$\int_1^\infty t^{-1}\big\langle -F_-'\big(N_t^+\big)\big\rangle_{\breve{\phi}_a(t)} dt \le C||\phi||^2, \tag{6.1}$$

$$\int_1^\infty \alpha^-(t)\big\langle F_-'\big(N_t^-\big)\big\rangle_{\breve{\phi}_a(t)} dt \le C||\phi||^2, \tag{6.2}$$

$$\begin{gathered}\int_1^\infty \Big\langle H_1(t)^*\big(-F_-^{2\prime}\big)^{\frac12}\Big(g\tilde{b}_t\Big)\tilde{f}_t\big(-F_-^{2\prime}\big)^{\frac12}\Big(g\tilde{b}_t\Big)H_1(t)\Big\rangle_{\breve{\phi}_a(t)} dt \le C||\phi||^2;\\ \tilde{f}_t = gf^{-1}\tilde{c}_t + t^{-1}I - gN_t^{-1}r'\big(f^{-1}(x^a - L)\big)V^{a\prime\prime}(L)(x^a - L)N_t^{-1},\\ H_1(t) = N_t^{-1}F_-\big(N_t^-\big)F_-\big(N_t^+\big),\end{gathered} \tag{6.3}$$

$$\begin{gathered}\int_1^\infty \Big\langle H_2(t)^* F_-^2\Big(g\tilde{b}_t\Big)\Big(t^{-1-\sigma}\Big(I - g\tilde{b}_t\Big) + t^{-1}I\Big)H_2(t)\Big\rangle_{\breve{\phi}_a(t)} dt \le C||\phi||^2;\\ H_2(t) = \big(-F_-^{2\prime}\big)^{\frac12}\big(h^{-1}r_t\big)F_-\big(N_t^-\big)F_-\big(N_t^+\big),\end{gathered} \tag{6.4}$$

$$\int_1^\infty \left\langle H_3(t)^*\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\tilde{f}_t\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)H_3(t)\right\rangle_{\breve{\phi}_a(t)} dt \le C||\phi||^2;$$

(6.5)
$$\tilde{f}_t = gf^{-1}\tilde{c}_t + t^{-1}I - gN_t^{-1}r'\big(f^{-1}(x^a - L)\big)\Big(V^{a\prime} + \ddot{L}\Big)N_t^{-1},$$

$$H_3(t) = F_-\big(h^{-1}r_t\big)N_t^{-1}F_-\big(N_t^-\big)F_-\big(N_t^+\big).$$

Proof As for (6.1) and (6.2) we notice that the estimates follow from the proof of Lemma 5.1.

As for (6.3) and (6.4) we consider the observables

$$\Phi_1(t) = H_1(t)^*F_-^2\Big(g\tilde{b}_t\Big)H_1(t)$$

and

$$\Phi_2(t) = H(t)^*H(t);\ H(t) = F_-\Big(g\tilde{b}_t\Big)F_-\big(h^{-1}r_r\big)H_1(t),$$

respectively. We use the proof of Lemma 3.1, (6.1) and (6.2), the bounds

$$\sup_{t\ge1}||p^{a2}F_-\big(N_t^-\big)F_-\big(N_t^+\big)||,\ \sup_{t\ge1}||\left(\frac{x^a}{t}\right)^2F_-\big(N_t^-\big)F_-\big(N_t^+\big)|| < \infty,$$

(which compensate for energy-localization) and that $t^{\rho_1}||\partial_{x^a}\breve{R}(t,x)|| \in L^1(dt)$.

As for (6.5) we consider

$$\Phi_3(t) = H_3(t)^*F_-^2\Big(g\tilde{b}_t\Big)H_3(t).$$

To treat the contribution from $\breve{\mathbf{D}}_tF_-^2\Big(g\tilde{b}_t\Big)$ to the derivative of this observable we may replace $\breve{\mathbf{D}}_t$ by $\mathbf{D}$, cf. (5.6), and then use the proof of Lemma 3.1. For other derivatives we use (6.1)-(6.4).

□

7 Proof of $\phi_a^+ = 0$ (negative potentials)

The first step of the proof of the statement, that any given $\phi_a^+ \in P_r^{a+}\mathcal{H}$ indeed must vanish, is the following result.

Lemma 7.1 *The limit (5.5) exists.*

Proof We prove the existence of (2.19) using the integral estimates for $U_a(t)$ and $\breve{U}_a(t)$ proven in Sections 3, 5 and 6. By a support property of the factor $F(t)$ of the product $G(t)$ (used first in the proof of Lemma 5.1) it suffices to consider $G(t)^*(\mathbf{D}G(t))$, cf. (5.6): We compute the derivative of each factor of $G(t)$ and symmetrize. Then we invoke Lemmas 3.1 and 5.1 for $U_a(t)$, and (6.1), (6.2), (6.4) and (6.5) for $\breve{U}_a(t)$.

$\square$

The second step is to invoke [S2, Theorem 1.2]. We consider a state $\breve{\phi}_a^+$ for which for all $\varepsilon > 0$

$$(7.1)\qquad F_+\big(t^{\mu-\varepsilon-1}|x^a - L|\big)\breve{\phi}_a^+(t) \;=\; o\big(t^0\big),$$

where $\breve{\phi}_a^+(t) = \breve{U}_a(t)\breve{\phi}_a^+$ (cf. Lemma 2.1). We need to show that

$$(7.2)\qquad \breve{\phi}_a^+ \;=\; 0.$$

For that we introduce the asymptotic velocity

$$(7.3)\qquad \breve{x}^+ \;=\; s - C_\infty - \lim_{t\to\infty} \breve{U}_a(t)^*\Big(\tfrac{x^a - L(t)}{x^+(t)}, \tfrac{x_a}{t}\Big)\breve{U}_a(t),$$

where $x^+(t) = \exp\int_1^t \alpha^+ dt'$ with α^+ given as in (5.1) (see also Appendix A). Suppose for the moment the conditions of [S2, Theorem 1.2] so that this asymptotic velocity is well-defined and absolutely continuous w.r.t. the Lebesgue measure on X (the latter by the conclusion of the theorem). Then since $x^+(t) \geq t$ we obviously get from (7.1) that $\breve{\phi}_a^+ \in E_{\{0\}\times X_a}\big(\breve{x}^+\big)$; whence we conclude from the stated absolute continuity that indeed (7.2) holds.

Now to verify the conditions of [S2, Theorem 1.2] we notice that

$$\sup_x |\partial_x \breve{R}(t,x)| \in L^1(dt),$$

cf. (4.3) and (1.3). (This bound is sufficient for the existence of $\breve{x}^+$.) As for the second derivatives [S2, (1.6)] we use (4.3) and the bound (A.5) to estimate $\partial^2_{x^a}\breve{R}^a(t,x^a)$ as follows:

$$|\partial^2_{x^a}\breve{R}^a(t,x^a)| \leq O\big(t^{-2-2\sigma}\big) \leq x^+(t)^{-1}x^-(t)^{-1}O\big(t^{-1-2\sigma}\big),$$

uniformly in x. The condition [S2, (1.6)] follows for this second derivative since $t^{-1-2\sigma} \in L^1(dt)$. We treat $\partial^2_{x^a}I(t,x)$ and $\partial^2_{x_a}I(t,x)$ similarly.

It remains to bound

$$(7.4)\qquad |\partial_{x^a}\partial_{x_a}I(t,x)| \;\leq\; x^+(t)^{-1}h_2(t),$$

for some $h_2 \in L^1(dt)$. For that we notice that the lower bound

$$(7.5)\qquad V^{a\prime\prime}(L) \;\geq\; -\tilde{\mu}t^{-2}$$

for some $\tilde{\mu} \geq 0$ implies the bound

$$(7.6)\qquad x^+(t) \;\leq\; t^{\frac{1+\sqrt{1+4\tilde{\mu}}}{2}};$$

see Appendix A.

The following computations show that under assumption (1.8), (7.5) holds for some $\tilde{\mu} \geq 0$ with

(7.7) $\frac{1+\sqrt{1+4\tilde{\mu}}}{2} < 1 + \mu.$

Obviously we may replace (1.8) by:

(7.8) $-c_r\mu(\mu+1)|x^a|^{-\mu-2} \geq V^{a\prime\prime}(x^a);\ x^a \geq R,\ c_r > 0,$

(7.9) $V^{a\prime\prime}(x^a) \geq -C_r\mu(\mu+1)|x^a|^{-\mu-2};\ x^a \geq R,\ C_r > 0,$

and

(7.10) $C_r < 2^{-1}(2+\mu)^2 c_r.$

By integrating (7.8) (to infinity) twice we get

$$V^a(x^a) \leq -c_r|x^a|^{-\mu}.$$

Thus by the formula

$$t = \int_R^L (-2V^a(x))^{-\frac{1}{2}} dx$$

we can estimate

$$t \leq \int_R^L \left(2c_r x^{-\mu}\right)^{-\frac{1}{2}} dx \leq \left(1 + \frac{\mu}{2}\right)^{-1} (2c_r)^{-\frac{1}{2}} L^{1+\frac{\mu}{2}},$$

yielding

(7.11) $\left((2c_r)^{\frac{1}{2}}\alpha^{-1}t\right)^{\alpha} \leq L.$

Next we insert L into (7.9) and estimate the right hand side by (7.11) yielding

(7.12) $V^{a\prime\prime}(L) \geq -\frac{C_r}{c_r}\mu(\mu+1)\frac{2}{(2+\mu)^2}t^{-2}.$

Finally by combining (7.10) and (7.12) we get (7.5) for some

(7.13) $\tilde{\mu} < \mu(\mu+1).$

We notice that (7.13) implies (7.7).

By combining (7.6) and (7.7) we get

$$x^+(t) \le t^{1+\mu-\epsilon};\ \text{for some } \epsilon > 0.$$

Therefore

$$|\partial_{x^a}\partial_{x_a} I(t,x)| \le O\big(t^{-2-\mu}\big) \le x^+(t)^{-1} O\big(t^{-1-\epsilon}\big),$$

yielding (7.4).

We have verified the conditions of [S2, Theorem 1.2] and hence proved the absolute continuity of $\breve{x}^+$.

8 Positive potentials

We shall prove Theorem 1.3. We proceed somewhat more general assuming (1.2) for an arbitrary $\mu > 0$. Suppose in addition that $a \in \mathcal{F}$ is given such that for some $\mu^- > \mu$,

(8.1) $$V^a(x^a) \ge -|x^a|^{-\mu^-},\ |x^a| \ge R.$$

We define $\langle w \rangle = \big(1 + |w|^2\big)^{\frac{1}{2}};\ w \in \mathbf{C}$. Then we have the following modification of [S1, Lemma 2.2].

Lemma 8.1 *Let $\tilde\alpha \ge 0,\ \kappa, t \ge 1$ and $n \in \mathbf{N}$. Then (with the above assumptions) for all $w \in \mathbf{C}$*

(8.2) $$\begin{aligned}&||F\big(t^{-\tilde\alpha}|x^a| > 1\big)p^a(\kappa H^a - w)^{-1}||\\ &\le C_n \frac{\langle w\rangle^{\frac{1}{2}}}{|\mathit{Im} w|}\Big(t^{-\tilde\alpha\mu^-} + t^{-2\tilde\alpha} + \kappa^{-1}\Big)^{\frac{1}{2}-\left(\frac{1}{2}\right)^{n+2}}.\end{aligned}$$

Proof We proceed as in the proof of [S1, Lemma 2.2]: Let $\psi \in \mathcal{H}$ with $||\psi|| = 1$ be given; put $\tilde\psi = (\kappa H^a - w)^{-1}\psi$. Using (8.1) we obtain the following modification of [S1, (2.16)]:

(8.3) $$\begin{aligned}&||F\big(t^{-\tilde\alpha}|x^a| > 1\big)p\tilde\psi||^2 \le C_1 t^{-2\tilde\alpha}|\mathrm{Im} w|^{-2} + 4\langle H^a - V^a\rangle_{F(\cdot)\tilde\psi}\\ &\le C_2\Big(t^{-2\tilde\alpha} + t^{-\tilde\alpha\mu^-}\Big)|\mathrm{Im} w|^{-2} + C_3|\mathrm{Im} w|^{-1}||H^a F(\cdot)\tilde\psi||.\end{aligned}$$

We estimate the term $||HF(\cdot)\tilde\psi||$ on the right hand side of (8.3) as

$$\begin{aligned}||H^a F(\cdot)\tilde\psi|| &\le C_4\Bigg(t^{-2\tilde\alpha}|\mathrm{Im} w|^{-1} + t^{-\tilde\alpha}||F'(\cdot)\frac{x^a}{|x^a|}\cdot p^a\tilde\psi|| + ||F(\cdot)H^a\tilde\psi||\Bigg)\\ &\le C_5\Bigg(\frac{t^{-2\tilde\alpha}}{|\mathrm{Im} w|} + t^{-\tilde\alpha}||F'(\cdot)p^a\tilde\psi|| + \kappa^{-1}\Bigg(||F(\cdot)(\kappa H^a - w)\tilde\psi|| + C_6\frac{|w|}{|\mathrm{Im} w|}\Bigg)\Bigg)\\ &\le C_7\Bigg(\big(t^{-2\tilde\alpha} + \kappa^{-1}\big)\frac{\langle w\rangle}{|\mathrm{Im} w|} + t^{-\tilde\alpha}||F'(\cdot)p^a\tilde\psi||\Bigg),\end{aligned}$$

and insert into the right hand side of (8.3) to obtain

$$
\begin{aligned}
&||F\big(t^{-\tilde\alpha}|x^a|>1\big)p^a\tilde\psi||^2\\
&\le C_8\Big(t^{-2\tilde\alpha}+t^{-\tilde\alpha\mu^-}+\kappa^{-1}\Big)\frac{\langle w\rangle}{|\mathrm{Im}w|^2}+\sqrt{C_8}t^{-\tilde\alpha}|\mathrm{Im}w|^{-1}||F'(\cdot)p^a\tilde\psi||\\
&\le C(t)+\sqrt{C(t)}||F'(\cdot)p^a\tilde\psi||;\ C(t)=C_8\Big(t^{-2\tilde\alpha}+t^{-\tilde\alpha\mu^-}+\kappa^{-1}\Big)\frac{\langle w\rangle}{|\mathrm{Im}w|^2}.
\end{aligned}
\tag{8.4}
$$

Next we write

$$
\begin{aligned}
&f_0=||F\big(t^{-\tilde\alpha}|x^a|>1\big)p^a\tilde\psi||,\\
&f_m=||F^{(m)}\big(t^{-\tilde\alpha}|x^a|>1\big)p^a\tilde\psi||;\ m\in\mathbf{N},
\end{aligned}
$$

and notice that (8.4) may be written

$$
\begin{aligned}
&f_{m-1}^2\le C_m(t)+\sqrt{C_m(t)}f_m;\\
&C_m(t)=C_8\Big(t^{-2\tilde\alpha}+t^{-\tilde\alpha\mu^-}+\kappa^{-1}\Big)\frac{\langle w\rangle}{|\mathrm{Im}w|^2},\ m=1.
\end{aligned}
$$

Since the same bound holds for any $m\le n$ possibly upon enlarging the constant C_8, we have the bounds

$$
\begin{aligned}
&f_{m-1}^2\le C_n(t)+\sqrt{C_n(t)}f_m;\\
&C_n(t)=C_n\Big(t^{-2\tilde\alpha}+t^{-\tilde\alpha\mu^-}+\kappa^{-1}\Big)\frac{\langle w\rangle}{|\mathrm{Im}w|^2},\ m=1,\cdots,n,
\end{aligned}
\tag{8.5}
$$

with input

$$
f_n^2\ \le\ C\Big(t^{-2\tilde\alpha}+t^{-\tilde\alpha\mu^-}+\kappa^{-1}\Big)^{\frac12}\frac{\langle w\rangle}{|\mathrm{Im}w|^2}.
\tag{8.6}
$$

For the latter estimate we used (8.4); the last factor on the right hand side is estimated by [S1, (2.15)] (which obviously holds in the present context too).

We introduce $g_m=f_mC_n(t)^{-\frac12}$. Using the bound $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$ and the fact that $g_{m-1}\ \le\ \sqrt{1+g_m}$ we obtain the estimate

$$
g_m\le n-m+g_n^{2^{-(n-m)}};\ m=0,1,\cdots,n.
$$

Using this bound for $m\ =\ 0$ and (8.6) yields

$$
f_0\le C\Big(t^{-2\tilde\alpha}+t^{-\tilde\alpha\mu^-}+\kappa^{-1}\Big)^{\frac12}\frac{\langle w\rangle^{\frac12}}{|\mathrm{Im}w|}\bigg(1+\Big(t^{-2\tilde\alpha}+t^{-\tilde\alpha\mu^-}+\kappa^{-1}\Big)^{-2^{-(n+2)}}\bigg).
$$

$\square$

Lemma 8.2 *Suppose (8.1) and that for some positive* $\breve\alpha<\mu$

$$
I(t,x)\ =\ I(t,x_a)\ \textit{for}\ |x^a|\ <\ 2t^{\breve\alpha}.
\tag{8.7}
$$

Suppose $\phi_a^+ \in E_{\{0\}}\big(H^{a+}\big)$. *Then with* $\phi_a^+(t) = U(t)\phi_a^+$ *and positive* δ *and* σ *with*

$$\delta < \min\Big(2(\mu - 2\sigma), \mu - 2\sigma + \breve{\alpha}\min\Big(\tfrac{\mu^-}{2}, 1\Big)\Big), \tag{8.8}$$

$$||F\big(|t^\delta H^a| > 1\big)\phi_a^+(t)|| \le Ct^{-\sigma}. \tag{8.9}$$

Proof Following the proof of [S1, Lemma 2.3] using now Lemma 8.1 we get for all (small) $\epsilon > 0$ (by chosing n large enough)

$$\begin{aligned}
&\langle F(|\kappa H^a| > 1)\rangle_{\phi_a^+(t)} \\
&\le \int_t^\infty \kappa C_1 s^{-(1+\mu)}\Big(s^{-\tilde{\alpha}\mu^-} + s^{-2\tilde{\alpha}} + \kappa^{-1}\Big)^{\frac{1}{2}-\left(\frac{1}{2}\right)^{n+2}} ds \\
&\le C_2\kappa t^{-\left(\mu + \tilde{\alpha}\min\left(\frac{\mu^-}{2},1\right)(1-\epsilon)\right)} + C_3\kappa^{\frac{1}{2}+\epsilon}t^{-\mu}.
\end{aligned}$$

Let $\kappa = t^\delta$.

$\square$

Theorem 8.3 ([W]) *Under the conditions (1.2), (1.3) and (8.1) with* $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$ *and* $\mu^- > 2\mu^{-1}(1-\mu)$, *the limits (1.4) and (1.5) exist.*

Proof We shall only prove the existence of the limit (1.4); the existence of (1.5) can be shown completely similarly. We may assume (8.7) for any fixed $\breve{\alpha} < \mu$, cf. [E]. Let $\breve{\alpha} = \mu - \epsilon$ and $\delta = 2(1-\mu) + 5\epsilon$ for a small $\epsilon > 0$. In addition we can assume (8.8) (for σ small enough) and

$$\min\Big(\tfrac{\breve{\alpha}\mu^-}{2}, \breve{\alpha}, \tfrac{\delta}{2}\Big) > 1 - \mu + 2\epsilon. \tag{8.10}$$

By the technique used in the proof of [S1, Lemma 3.2], (8.10) and Lemmas 8.1 and 8.2 we obtain

$$\phi_a^+(t) \approx F_-\big(|t^\delta H^a|\big)F_-\big(t^{-\breve{\alpha}}|x^a|\big)F_-\big(|t^\delta H^a|\big)\phi_a^+(t) \text{ as } t \to +\infty. \tag{8.11}$$

Moreover,

$$\int_1^\infty t^{-1}|\big\langle F\big(t^\delta|H^a| \approx 1\big)\big\rangle_{\phi_a^+(t)}|dt < \infty, \tag{8.12}$$

cf. [S1, Lemma 2.4], and

$$\int_1^\infty t^{-1}|\big\langle F\big(t^{-\breve{\alpha}}|x^a| \approx 1\big)\big\rangle_{F_-(|t^\delta H^a|)\phi_a^+(t)}|dt < \infty, \tag{8.13}$$

cf. [S1, Lemma 3.4].

The analogue statements for $\tilde U_a(t)$ read with $\tilde\phi(t)=\tilde U_a(t)\tilde\phi$:

(8.14) $$\int_1^\infty t^{-1}|\langle F\big(t^\delta|H^a|\approx 1\big)\rangle_{\tilde\phi(t)}|dt\ \le\ C||\tilde\phi||^2,$$

and

(8.15) $$\int_1^\infty t^{-1}|\langle F\big(t^{-\breve\alpha}|x^a|\approx 1\big)\rangle_{F_-(|t^\delta H^a|)\tilde\phi(t)}|dt\ \le\ C||\tilde\phi||^2.$$

Substituting (8.11) in the expression

$$\tilde U_a(t)^*\phi_a^+(t),$$

and then differentiating yields integrable terms: The one with the potentials is bounded by

$$||(I_a(t,x_a)-I_a(t,x))F_-\Big(t^{-\breve\alpha}|x^a|\Big)||+||\Big[I_a(t,x),F_-\Big(|t^\delta H^a|\Big)\Big]||$$
$$=O\Big(t^{(-1-\mu+\delta)+(-1+\mu-2\epsilon)}\Big)=O\big(t^{-2\mu+3\epsilon}\big);$$

notice that the first term to the left vanishes by (8.7) and that the bound of the second term comes about by using (8.7), (8.10) and Lemma 8.1. Clearly the final bound is integrable.

The contribution from the derivative of the factor $F_-\big(t^{\epsilon-\mu}|x^a|\big)$,

$$\mathbf{D}F_-\Big(t^{-\breve\alpha}|x^a|\Big)=t^{-\breve\alpha}F_-'(\cdot)\frac{x^a}{|x^a|}\cdot p^a+O\Big(t^{-2\breve\alpha}\Big)-\frac{\breve\alpha}{t}F_-'(\cdot)(\cdot),$$

is handled as follows: Using (8.10) and Lemma 8.1 again we infer that the first term on the right hand side is $O\big(t^{-1-\epsilon}\big)$. Obviously the second term is integrable, cf. (8.10). The third term is treated by (8.13) and (8.15).

The contributions from the two factors $\frac{d}{dt}F_-\big(|t^\delta H^a|\big)$ are treated by (8.12) and (8.14) after commutations under use of (8.10) and Lemma 8.1.

□

Now to the regime $\mu\in(0,\frac12]$:

Proof of Theorem 1.3: We shall only prove the existence of the limit (1.4); the existence of (1.5) can be shown completely similarly. We pick $\epsilon>0$ so small that

(8.16) $$(1-\mu+2\epsilon)\mu^+\ \le\ 2\mu-3\epsilon.$$

Let $\breve{\alpha} = \mu - \epsilon$ and $\delta = 2\breve{\alpha}$. We can assume (8.7).

Proceeding exactly as in the beginning of the proof of Theorem 8.3 we conclude that

$$\phi_a^+(t) \approx F_-\big(t^{\mu-1-2\epsilon}|x^a|\big)F_-\big(|t^\delta H^a|\big)\phi_a^+(t) \text{ as } t \to +\infty. \tag{8.17}$$

The next result doesn't have an analogue in the previous proof. Proceeding as the proof of [S1, Lemma 3.1] we introduce for $\psi \in \mathcal{H}$

$$\tilde{\psi} = F_1F_2F_3\psi;$$

$$F_1 = F\Big(t^{-\breve{\alpha}}|x^a| > 1\Big),\ F_2 = F\big(t^{\mu-1-2\epsilon}|x^a| < 1\big),\ F_3 = F\Big(|t^\delta H^a| < 1\Big).$$

Suppose we know that $||F_1F_2F_3|| = O\big(t^{-s}\big)$, then we shall show the bound with the right hand side replaced by $O\big(t^{-s-\frac{\epsilon}{2}}\big)$, leading inductively to the conclusion that

$$||F_1F_2F_3|| = O\big(t^{-\infty}\big). \tag{8.18}$$

To do that we estimate, cf. [S1, (3.8)],

$$\begin{aligned} \langle 2H^a\rangle_{\tilde{\psi}} &= \Big\langle H^aF_1^2F_2^2 + F_1^2F_2^2H^a + \Big(t^{-\breve{\alpha}}F_1'F_2 + t^{\mu-1-2\epsilon}F_1F_2'\Big)^2\Big\rangle_{F_3\psi} \\ &\le 2t^{-\delta}||F_1F_2\Big(t^\delta H^aF_3\Big)\psi||\,||\tilde{\psi}|| \\ &\quad + 2t^{-2\breve{\alpha}}||F_1'F_2F_3\psi||^2 + 2t^{2(\mu-1-2\epsilon)}||F_1F_2'F_3\psi||^2, \end{aligned} \tag{8.19}$$

and, using here (8.16),

$$\langle 2H^a\rangle_\psi \ge Ct^{-(1-\mu+2\epsilon)\mu^+}||\tilde{\psi}||^2 \ge Ct^{-2\mu+3\epsilon}||\tilde{\psi}||^2. \tag{8.20}$$

Combining (8.19) and (8.20) leads to $||\tilde{\psi}|| \le Ct^{-s-\frac{\epsilon}{2}}||\psi||$, whence we conclude (8.18).

Combining (8.17) and (8.18) yields

$$\phi_a^+(t) \approx F_-\Big(|t^\delta H^a|\Big)F_-\Big(t^{-\breve{\alpha}}|x^a|\Big)F_-\Big(|t^\delta H^a|\Big)\phi_a^+(t).$$

Following now the last part of the proof of Theorem 8.3 we differentiate the expression

$$\tilde{U}_a(t)^*F_-\Big(|t^\delta H^a|\Big)F_-\Big(t^{-\breve{\alpha}}|x^a|\Big)F_-\Big(|t^\delta H^a|\Big)\phi_a^+(t),$$

and pick up integrable terms: The contribution with the potentials is estimated by

$$||\Big[I_a(t,x), F_-\Big(|t^\delta H^a|\Big)\Big]|| = O\Big(t^{(-1-\mu+\delta)-\left(\mu-\frac{3}{2}\epsilon\right)}\Big) = O\Big(t^{-\left(1+\frac{1}{2}\epsilon\right)}\Big) \in L^1(dt),$$

cf. (8.7) and Lemma 8.1. The one from the Heisenberg derivative of the factor $F_-\big(t^{-\breve{\alpha}}|x^a|\big)$ is clearly integrable by (8.18). The same conclusion holds for the contributions from the two factors $\frac{d}{dt}F_-\big(|t^\delta H^a|\big)$; this follows by using the bounds (8.12) and (8.14) (with the present δ) after commutations under use of (8.18) again.

$\square$

A Classical orbits

In this Appendix we shall construct solutions $\alpha^+(t)$ and $\alpha^-(t)$ to the Riccati equation

$$\dot{\alpha} = q(t)^2 - \alpha^2 \tag{A.1}$$

where $q(t)^2$ is a continuous (non-negative) function. For the example $q(t)^2 = -V^{a\prime\prime}(L)$ of Section 5, cf. (2.1), we establish (5.1) using in this case the upper bound

$$q(t)^2 \leq Ct^{-2}, \tag{A.2}$$

cf. [S1, (5.8)] (or (7.11)).

First we construct two solutions $\alpha^+(t)$ and $\alpha^-(t)$ satisfying the bounds

$$t^{-1} \leq \alpha^+(t) \text{ and } \alpha^-(t) \leq 0.$$

This will be done without using (A.2).

To find $\alpha^+(t)$ we notice that $\beta(t) = t^{-1}$ is a solution to

$$\dot{\beta} = -\beta^2.$$

Clearly we can solve (A.1) in a neighborhood of $t = 1$ with the initial condition $\alpha(1) = \beta(1) = 1$. By the standard comparison theorem (see for example [BR, Theorem 1.8]) we conclude that $\alpha(t) \geq t^{-1}$ for $t \geq 1$. Using the equation (A.1) we can readily continue $\alpha(t)$ to the whole half-axis. The obtained solution is denoted by $\alpha^+(t)$.

It remains to construct $\alpha^-(t) \leq 0$. For that we consider the Schrödinger equation

$$-x''(t) - V^{a\prime\prime}(L(t))x(t) = 0. \tag{A.3}$$

From the solution $x^+(t) = \exp\left(\int_1^t \alpha^+ dt'\right)$ we obtain another one, cf. [BR, Section 2.5], by the formula

$$x^-(t) = x^+(t)\int_t^\infty x^+(t')^{-2}dt'.$$

Let us note the following bounds

$$\begin{aligned} 0 < x^-(t) &= x^+(t)^{-1}\int_t^\infty e^{-2\int_t^{t'}\alpha^+ dt''}dt' \\ &\leq x^+(t)^{-1}\int_t^\infty e^{-2\int_t^{t'}\frac{1}{t''}dt''}dt' = x^+(t)^{-1}t \leq 1. \end{aligned} \tag{A.4}$$

From (A.3) we get $x^-(t)'' \geq 0$. Consequently if $x^-(t)'$ if positive for some t the solution $x^-(t)$ will grow at least linearly contradicting (A.4). Therefore $x^-(t)' \leq 0$ and we conclude that

$$\alpha^-(t) := \frac{x^-(t)'}{x^-(t)} \leq 0.$$

We shall now show the bounds

$$\alpha^+(t) \leq C' t^{-1} \text{ and } -C' t^{-1} \leq \alpha^-(t)$$

under the assumption (A.2).

To get the upper bound of $\alpha^+(t)$ we introduce the function

$$\beta(t) = \frac{1+\sqrt{1+4C}}{2} t^{-1}$$

which satisfies $\beta(1) \geq \alpha^+(1) = 1$ and solves

$$\dot\beta = Ct^{-2} - \beta^2.$$

By another comparison we conclude that $\alpha^+(t) \leq \beta(t)$.

To get the lower bound $\alpha^-(t) \geq -C'' t^{-1}$ we compute using the upper bound $\alpha^+(t) \leq C' t^{-1}$

$$\alpha^-(t) = \alpha^+(t) - \left(\int_t^\infty e^{-2\int_t^{t'} \alpha^+ dt''} dt' \right)^{-1}$$

$$\geq t^{-1} - \left(\int_t^\infty e^{-2\int_t^{t'} \frac{C'}{t''} dt''} dt' \right)^{-1} = -2\big(C'-1\big)t^{-1}.$$

The proof of the bounds is completed. In particular we have verified (5.1).

For an application in Section 7 we notice that the formula

$$x^+ x^- = \frac{x^+ x^-}{x^{+\prime} x^- - x^{-\prime} x^+} = \frac{1}{\alpha^+ - \alpha^-},$$

and (5.1) yield the bounds

$$(2C)^{-1} t \leq x^+(t) x^-(t) \leq t \tag{A.5}$$

for C given as in (5.1).

References

[BR] G. Birkhoff, G.C. Rota: *Ordinary differential equations*, 4. edition, New York, John Wiley & Sons, 1989.

[D] J. Dereziński: *Asymptotic completeness for N-particle long-range quantum systems*, Ann. Math., 138 (1993), 427–476.

[DG] J. Dereziński, C. Gérard: *Scattering theory of classical and quantum N-particle systems*, Texts and Monographs in Physics, Springer, Berlin Heidelberg New York, 1997.

[E] V. Enss: *Long-range scattering of two- and three-body quantum systems*, Journées Equations à Dérivées Partielles, Saint Jean de Monts, Juin, 1989, Publ. Ecole Polytechnique, Palaiseau, 1989, 1–31.

[G] C. Gérard: *Asymptotic completeness for 3–particles systems*, Invent. Math. 114 (1993), 333–397.

[HS] I. Herbst, E. Skibsted: *Quantum scattering for homogeneous of degree zero potentials: Absence of channels at local maxima and saddle points*, MaPhySto preprint 24, 1999.

[H] L. Hörmander: *The analysis of linear partial differential operators III*, Berlin, Springer-Verlag, 1985.

[O] F.W.J. Olver: *Asymptotic and special functions*, New York, Academic Press 1974.

[S1] E. Skibsted: *Long-range scattering of three-body quantum systems, I*, MaPhySto preprint.

[S2] E. Skibsted: *Asymptotic absolute continuity for perturbed time-dependent quadratic Hamiltonians*, MaPhySto preprint 12, 2001, to appear in Mathematical section of the Proceedings of the Indian Academy of Sciences.

[W] X.P. Wang: *On the three body long-range scattering problems*, Reports Math. Phys., 25 (1992), 267–276.

[Y] D. Yafaev: *New channels of scattering for three-body quantum systems with long-range petentials*, Duke Math. J., 82 no.3 (1996), 553–584.