

A NOTE ON PORTFOLIO MANAGEMENT UNDER NON-GAUSSIAN LOGRETURNS

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ABSTRACT. We calculate numerically the optimal allocation and consumption strategies for Merton's optimal portfolio management problem when the risky asset is modelled by a geometric normal inverse Gaussian Lévy process. We compare the computed strategies to the ones given by the standard asset model of geometric Brownian motion. To have realistic parameters in our studies, we choose Norsk Hydro quoted on the New York Stock Exchange as the risky asset. We find that an investor believing in the normal inverse Gaussian model puts a greater fraction of wealth into the risky asset. We also investigate the limiting investment rate when the volatility increases. We observe different behaviour in the two models depending on which parameters we vary in the normal inverse Gaussian distribution.

1. INTRODUCTION

In this paper we study Merton's classical problem [13] of optimal portfolio selection and consumption when the risky asset has non-Gaussian price increments. Based on recent results in Benth, Karlsen and Reikvam [5] (see also Bank and Riedel [2], Framstad, Øksendal, and Sulem [10], and Kallsen [12] for similar results), our objective of is to investigate from a more practical point of view the effects of using non-Gaussian models in portfolio management.

In the mathematical finance literature, a frequently used model for asset price dynamics is the so called *geometric Brownian motion*. A major concern with geometric Brownian motion as a model for asset prices is that it predicts logarithmic price changes (known as *the logreturns*) to be normally distributed. This holds on all time spans in question. Looking into stock prices, for example, empirical studies reject the normal hypothesis for daily or weekly logreturns. Monthly logreturn data, on the other hand, seem to be fitted well by a normal distribution (see, e.g., the studies made by Eberlein and Keller [8] on German data). Daily or weekly logreturns show non-Gaussian effects like *heavy tails* and *skewness*. The normal inverse Gaussian distribution has proven to be a flexible and yet simple statistical model which fit empirical logreturns on all time scales extremely well. This family of distributions was introduced by Barndorff-Nielsen [3], and later used in finance by Barndorff-Nielsen [4], Rydberg [16], Prause [15], to mention a few. Taking the normal inverse Gaussian distribution as the starting point, one is lead to a *geometric Lévy process* as the model for the stock price dynamics.

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Merton showed in [13] that the investor should keep a constant fraction of her wealth in the risky asset when the asset dynamics are modelled by geometric Brownian motion. In fact, he proved that this fraction is proportional to the expected rate of return from the stock and inverse proportional to its volatility. The question raised in this paper is how the portfolio selection and consumption is affected by changing the dynamic of the asset to a more realistic model. We will compare the optimal portfolio and consumption policies where the price of the asset follows the geometric normal inverse Gaussian Lévy process with that of geometric Brownian motion. Unfortunately, we do not have an explicit formula for the optimal portfolio selection and consumption choice in the former case. However, as is seen from the results in Benth, Karlsen, and Reikvam [5] (see also Section 3), it is still optimal to keep a constant fraction of the wealth in the risky asset. The fraction is a solution to a rather complicated *integral equation* which has to be solved numerically. We have implemented standard numerical techniques in order to find the optimal portfolio selection for concrete parameter values of the normal inverse Gaussian distribution. As for the geometric Brownian motion, the optimal consumption is given as a fraction of the wealth. In this paper we shall devote most of our time on the study of the optimal portfolio plans, and less on consumption.

To have a financially realistic comparison we have chosen to fit both models to Norsk Hydro quoted on the New York Stock Exchange. Based on these parameters, an investor believing in the normal inverse Gaussian model would allocate more of her wealth in the risky asset than advised by the geometric Brownian motion model. For risk averse investors the difference is small, however, but grows to become significant for investors willing to take greater risk. One may conclude from this that the stock is considered by the investor to be *less risky* under the normal inverse Gaussian model. By varying some of the parameters we shall see that the opposite can happen as well. The uncertainty in parameter estimates will be great for small sets of data and therefore significantly influence the investor's portfolio choices. It is of interest to study the variation of the optimal portfolio selection and consumption coming from parameter uncertainty. This is, however, outside the scope of the present paper. We can not, of course, conclude on a general basis from this study. But our findings indicate that there may be significant differences in portfolio choices when going from a normal to a non-Gaussian market dynamic which shows the importance of having realistic models for the asset dynamics.

Here is an outline of the paper: In Section 2 we introduce the two models for the stock price dynamics that will be used throughout the paper. We also present the normal inverse Gaussian distribution and some properties related to this. In Section 3 the optimal portfolio selection and consumption problem is stated together with the corresponding solutions, followed by a section where we fit the geometric Brownian motion and Lévy model to Norsk Hydro data. Sections 4 and 5 contain the main contribution of this paper, namely the investigations of portfolio and consumption choice under the two models. Different experiments and their results are presented. In Section 7 we discuss the numerical methods that were used. Finally, in Section 8 we make some concluding remarks.

2. STOCK PRICE DYNAMICS AND THE NORMAL INVERSE GAUSSIAN DISTRIBUTION

A stock price modelled by geometric Brownian motion is given by

$$S_t = S_0 \exp(\mu_{\text{gbm}} t + \sigma B_t),$$

where B_t is a standard Brownian motion and μ_{gbm} and σ is the drift and volatility parameters respectively. From here on, geometric Brownian motion will be referred to as the *standard model*.

As the alternative stock price dynamics we introduce the geometric Lévy process

$$S_t = S_0 \exp(\mu_{\text{nig}}t + L_t),$$

where L_t is a normal inverse Gaussian Lévy process, i.e., L_t has stationary and independent increments with L_1 normally inverse Gaussian distributed. The density for the (centred) normal inverse Gaussian distribution is given by

$$f(x; \alpha, \beta, \delta) = \frac{\delta \alpha}{\pi} \exp(\delta \sqrt{\alpha^2 - \beta^2} + \beta x) \frac{K_1(\alpha \sqrt{\delta^2 + x^2})}{\sqrt{\delta^2 + x^2}},$$

where K_1 is the modified Bessel function of third kind with index 1 (see [1, Ch. 9]). The parameters of the distribution centred around zero have the following meaning: α is the *steepness*, β is the *asymmetry*, and δ is the *scale* parameter. The first two moments (expectation and variance) are

$$\mathbb{E}[L_1] = \frac{\delta \beta}{\sqrt{\alpha^2 - \beta^2}}, \quad \text{Var}[L_1] = \frac{\delta \alpha^2}{(\alpha^2 - \beta^2)^{3/2}}.$$

This means that the logreturns have expectation $\mu_{\text{nig}} + \mathbb{E}[L_1]$ and variance (or volatility) $\text{Var}[L_1]$ when they are modelled as i.i.d. variables that are normally inverse Gaussian distributed. We have the following connection between the parameters in the standard and normal inverse Gaussian models

$$\mu_{\text{gbm}} = \mu_{\text{nig}} + \frac{\delta \beta}{\sqrt{\alpha^2 - \beta^2}}, \quad \sigma^2 = \frac{\delta \alpha^2}{(\alpha^2 - \beta^2)^{3/2}}.$$

In the symmetric case (i.e., when $\beta = 0$), we have $\mu_{\text{gbm}} = \mu_{\text{nig}}$ and $\sigma^2 = \delta/\alpha$.

The Lévy-Kintchine representation of L_t is

$$L_t = \xi t + \int_0^t \int_{\mathbb{R} \setminus \{0\}} z \tilde{N}(dt, dz), \quad \xi = \mathbb{E}[L_1],$$

where $N(dt, dz)$ is a Poisson random measure and $\tilde{N}(dt, dz) = N(dt, dz) - dt \times n(dz)$ is the compensated Poisson random measure associated to L_t . The Lévy measure of L_t is

$$n(dz) = \frac{\delta \alpha}{\pi |z|} \exp(\beta z) K_1(\alpha |z|) dz.$$

It follows from the Lévy-Kintchine representation that the normal inverse Gaussian Lévy process is a pure jump process. Furthermore, L_t is a martingale with respect to its own filtration when $\mathbb{E}[L_1] = \xi = 0$, which is equivalent to $\beta = 0$. We will exclusively consider this case in our numerical study of Merton's problem for the normal inverse Gaussian distribution.

3. MERTON'S PORTFOLIO MANAGEMENT PROBLEM

3.1. The stochastic control problem. Let $X_t^{c, \pi, x}$ denote the investor's wealth at time t with consumption rate $c = (c_t)$ and portfolio selection $\pi = (\pi_t)$. The initial wealth is given as $X_0^{c, \pi, x} = x \geq 0$. In the standard model, the dynamics of the wealth process reads

$$(3.1) \quad dX_t^{c, \pi, x} = \left[r + \left(\mu_{\text{gbm}} + \frac{1}{2} \sigma^2 - r \right) \pi_t \right] X_t^{c, \pi, x} dt - c_t dt + \sigma \pi_t X_t^{c, \pi, x} dW_t,$$

while for the normal inverse Gaussian model it reads

$$(3.2) \quad dX_t^{c,\pi,x} = [r + (\hat{\mu}_{\text{nig}} - r)\pi_t] X_t^{c,\pi,x} dt - c_t dt + \pi_t X_{t-}^{c,\pi,x} \int_{\mathbb{R} \setminus \{0\}} (e^z - 1) \tilde{N}(dt, dz),$$

where

$$\hat{\mu}_{\text{nig}} = \mu_{\text{nig}} + \xi + \int_{\mathbb{R} \setminus \{0\}} (e^z - 1 - z\mathbf{1}_{|z|<1}) n(dz).$$

Denote by \mathcal{A}_x the set of all admissible consumption plans and portfolio allocations when the initial wealth is x . More precisely, $(c, \pi) \in \mathcal{A}_x$ if the following conditions hold:

- (A.1) c_t is a positive and adapted process such that $\int_0^t \mathbb{E}[c_s] ds < \infty$ for all $t > 0$.
- (A.2) π_t is adapted, right-continuous with left limits and with values in $[0, 1]$.
- (A.3) c_t is such that $X_t^{c,\pi,x} \geq 0$ almost surely for all $t \geq 0$.

Note that we have assumed $\pi_t \in [0, 1]$, thus ruling out borrowing of money in the bank and short-selling of stocks. In the standard framework, one can allow π to be bigger than 1, meaning that the investor borrows a fraction $\pi - 1$ of her wealth in the bank in order to speculate in the stock. Here one assumes that the interest is the same for borrowing and investing money. Similarly, one can allow for short-selling of stocks, which means that π may be negative. For different parameter choices, one may obtain optimal (constant) fractions being either greater than one or negative. It is important to notice that the wealth remains nonnegative in the standard model also for these cases of optimal strategies.

The situation in the normal inverse Gaussian model is more complicated than in the standard model. Since the normal inverse Gaussian process has jumps of all sizes (between $-\infty$ and $+\infty$), the wealth is nonnegative if and only if the jump term

$$X_{t-} \pi_{t-} \int_{\mathbb{R} \setminus \{0\}} (e^z - 1) \tilde{N}(dt, dz)$$

has the property $\pi(e^z - 1) \geq -1$ (see, e.g., Protter [15]). But this implies $\pi \in [0, 1]$ since $e^z - 1 > -1$ and z is arbitrary.

If we insist on allowing borrowing of money in the bank, for example, we must impose certain restrictions on the jumps of the Lévy process. Assume for instance that $\pi \in [0, \bar{\pi}]$, where $\bar{\pi}$ is some constant strictly greater than one. In order for the wealth to stay positive for all consumption policies we need $z \geq \ln(1 - \frac{1}{\bar{\pi}})$, which implies that the Lévy process cannot have jumps smaller than $\ln(1 - \frac{1}{\bar{\pi}})$ (note that $\ln(1 - \frac{1}{\bar{\pi}}) < 0$). This rules out the possibility to model the stock price dynamics with the normal inverse Gaussian distribution. However, it is possible to use a truncated version of it.

The investor is assumed to derive her utility from the present consumption rate with a HARA (Hyperbolic Absolute Risk Aversion) shaped utility function. The stochastic control problem consists of finding optimal portfolio allocation and consumption strategies which optimize the expected discounted utility of consumption over an infinite investment horizon, i.e., to find π^* and c^* such that

$$(3.3) \quad V(x) = \sup_{(c,\pi) \in \mathcal{A}_x} \mathbb{E} \left[\int_0^\infty e^{-\eta t} \frac{c_t^\gamma}{\gamma} dt \right] = \mathbb{E} \left[\int_0^\infty e^{-\eta t} \frac{(c_t^*)^\gamma}{\gamma} dt \right],$$

where $\eta > 0$ is the discount factor. In (3.3), $1 - \gamma \in (0, 1)$ is known as the *risk aversion* coefficient. It should be observed that the functional in (3.3) implicitly depends on the portfolio allocation π through the consumption strategy c . Due to condition (A.3) this is a

control problem with a state space constraint. If τ is the first exit time of the wealth process from the set $(0, \infty)$, it is easily seen that

$$(3.4) \quad V(x) = \sup_{(c,\pi) \in \mathcal{A}_x} \mathbb{E} \left[\int_0^\tau e^{-\eta t} \frac{c_t^\gamma}{\gamma} dt \right].$$

After the wealth process has reached zero, it will remain there forever. Therefore consumption (and hence utility) will be zero from this time on and it is sufficient to consider the control problem up to the stopping time τ , i.e., (3.3) and (3.4) coincide.

3.2. The Hamilton-Jacobi-Bellman equation. Using the optimality principle of dynamic programming, the Hamilton-Jacobi-Bellman (HJB) equations for the two models are easily derived. For the standard model we have the well-known HJB (nonlinear differential) equation

$$(3.5) \quad \begin{aligned} \eta v(x) &= F_{\text{gbm}}(x, v_x, v_{xx}) + G(v_x) + r x v_x(x), \quad \forall x > 0, \\ v(0) &= 0, \end{aligned}$$

where the nonlinear functions F_{gbm}, G take the form

$$\begin{aligned} F_{\text{gbm}}(x, v, v_x, v_{xx}) &= \max_{\pi \in [0,1]} \left[\frac{1}{2} \sigma^2 \pi^2 x^2 v_{xx}(x) + \left(\mu_{\text{gbm}} + \frac{1}{2} \sigma^2 - r \right) \pi x v_x(x) \right], \\ G(v_x) &= \max_{c \geq 0} \left[-c v_x(x) + \frac{c^\gamma}{\gamma} \right]. \end{aligned}$$

Let us introduce the so-called Merton constant

$$(3.6) \quad k_{\text{gbm}}(\gamma) = \gamma \left(r + \frac{(\mu_{\text{gbm}} + \frac{1}{2} \sigma^2 - r)^2}{2\sigma^2(1-\gamma)} \right).$$

Under the condition

$$(3.7) \quad \eta - k_{\text{gbm}}(\gamma) > 0,$$

it is well-known that the unique solution of this equation is $V_{\text{gbm}}(x) = K_{\text{gbm}} x^\gamma$, where

$$K_{\text{gbm}} = \frac{1}{\gamma} \left(\frac{1-\gamma}{\eta - k_{\text{gbm}}(\gamma)} \right)^{1-\gamma}.$$

Moreover, $V_{\text{gbm}}(x)$ coincides with the value function, see Merton [13] for details.

In the normal inverse Gaussian case, the HJB equation is actually a nonlinear integro-differential equation [5]:

$$(3.8) \quad \begin{aligned} \eta v(x) &= F_{\text{nig}}(x, v_x, n(dz)) + G(v_x) + r x v_x(x), \quad \forall x > 0, \\ v(0) &= 0, \end{aligned}$$

where the nonlinear function F_{nig} takes the form

$$\begin{aligned} &F_{\text{nig}}(x, v, v_x, n(dz)) \\ &= \max_{\pi \in [0,1]} \left[(\hat{\mu}_{\text{nig}} - r) \pi x v_x(x) + \int_{\mathbb{R} \setminus \{0\}} \left(v(x + \pi x(e^z - 1)) - v(x) - \pi x v_x(x)(e^z - 1) \right) n(dz) \right]. \end{aligned}$$

In [5] (see also the discussion below), it is proved that the function

$$(3.9) \quad V_{\text{nig}}(x) = K_{\text{nig}} x^\gamma, \quad K_{\text{nig}} = \frac{1}{\gamma} \left(\frac{1-\gamma}{\eta - k_{\text{nig}}(\gamma)} \right)^{1-\gamma}$$

is the unique solution of (3.8) and that it coincides with the value function (3.3). Here,

(3.10)

$$k_{\text{nig}}(\gamma) = \max_{\pi \in [0,1]} \left\{ \gamma(r + (\hat{\mu}_{\text{nig}} - r)\pi) + \int_{\mathbb{R} \setminus \{0\}} \left((1 + \pi(e^z - 1))^\gamma - 1 - \gamma\pi(e^z - 1) \right) n(dz) \right\},$$

with the assumption

$$(3.11) \quad \eta - k_{\text{nig}}(\gamma) > 0.$$

Among other things, condition (3.11) ensures that the value function (3.3) is finite and non-negative. Contrary to k_{gbm} , it is not clear whether k_{nig} is non-negative and non-decreasing for $\gamma \in [0, 1]$.

Remark. We remark that the main topic in [5] is the study of Merton's problem with Hindy-Huang-Kreps intertemporal preferences. The investor's utility is derived through an averaging of present and past consumption. For the special case of HARA utility, we are able to produce an explicit solution to the control problem for the normal inverse Gaussian case. This solution is similar to the one in the normal Gaussian case, which has been calculated by Hindy and Huang [11]. We refer to [5] for further details and references to relevant literature. In the passing, we mention that in a future paper we will compare numerically our results with theirs for this more general optimization problem.

3.2.1. *Viscosity solutions.* For the technically oriented reader, we mention that the method of analysis in [5] relies on the newly developed theory of viscosity solutions of HJB equations. The survey paper by Crandall, Ishii, Lions [7] provides a good overview of this theory, see also the book by Fleming and Soner [9] for applications to stochastic control problems.

As it turns out, the HJB equation is a consequence of the dynamic programming principle and one expects the value function to satisfy this equation [9]. However, due to degeneracy as well as market imperfections such as trading constraints (see **(A.3)**) and transaction costs, to mention only a few, the value function might not satisfy the HJB equation in the classical sense, that is, the value function might not possess all the continuous derivatives occurring in the HJB equation and thus not satisfy this equation pointwise everywhere. It therefore becomes important to relax the notion of classical solution of HJB equations so as to allow functions that are not necessarily smooth as (generalized) solutions. This has been achieved successfully by the introduction of the notion of viscosity solution which allows merely continuous functions to be solutions of fully nonlinear first and second order HJB equations.

As already mentioned, the HJB equation (3.8) associated with our control problem is a nonlinear integro-differential equation which contains a non-local operator with a highly singular Lévy measure $n(dz)$. Although the HJB equation (3.8) contains only first order derivatives, if we insist on interpreting (3.8) in the classical sense, we have to consider twice continuously differentiable functions because of the (singular) Lévy measure $n(dz)$, see [5]. We point out that it is *not* easy to show directly that the value function (3.3) is twice continuously differentiable, although we can prove quite easily that it is continuous and sublinearly growing [5]. However, if we interpret (3.8) in the viscosity sense, it is sufficient to consider continuous functions, and one can indeed show that the value function (3.3) is a viscosity solution of (3.8). Moreover, one can prove that there exists only one viscosity solution of the integro-differential equation (3.8) which is continuous and sublinearly growing, see [5] for details.

In [5], we prove that the candidate solution (3.9) is a viscosity solution of (3.8). This can be done by simply inserting the expression for $V_{\text{nig}}(x)$ into (3.8). Then, thanks to our

characterization of the value function (3.3) as the unique viscosity solution of (3.8), we conclude that the candidate solution $V_{\text{nig}}(x)$ must coincide with the value function and hence the latter is obtained in closed form. We recall that such verification of a candidate solution is usually done by a so-called verification theorem stating that if the HJB equation has a unique classical solution, then it must coincide with the value function, see [9] for further details. As already mentioned, for our problem it is difficult to show directly that the value function (3.3) is smooth, and thus this classical verification technique is not easy to apply to the problem at hand, a fact that clearly demonstrates the power of the viscosity solution approach.

3.3. Optimal portfolio allocation. In the standard model (see, e.g., Merton [13]), the optimal portfolio allocation strategy is known to be

$$(3.12) \quad \pi_{\text{gbm}} = \frac{\mu_{\text{gbm}} + \frac{1}{2}\sigma^2 - r}{\sigma^2(1 - \gamma)}.$$

From Benth, Karlsen, and Reikvam [5], we know that the optimal allocation strategy in the normal inverse Gaussian case, which is denoted by π_{nig} , solves the integral equation

$$(3.13) \quad \hat{\mu}_{\text{nig}} - r + \int_{\mathbb{R} \setminus \{0\}} ((1 + \pi(e^z - 1))^{\gamma-1} - 1)(e^z - 1) n(dz) = 0.$$

Equation (3.13) tells us that also for the normal inverse Gaussian model the optimal allocation is given as a fixed (time independent) fraction of wealth. Note that the left-hand side of (3.13) is decreasing as a function in π . This is seen from its derivative

$$(\gamma - 1) \int_{\mathbb{R} \setminus \{0\}} (1 + \pi(e^z - 1))^{\gamma-2} (e^z - 1)^2 n(dz),$$

which is always negative since $\gamma \in (0, 1)$. Thus, if the parameters of the problem are such that

$$\hat{\mu}_{\text{nig}} > r$$

and

$$\hat{\mu}_{\text{nig}} < r + \int_{\mathbb{R} \setminus \{0\}} (1 - e^{(1-\gamma)z}) (e^z - 1) n(dz),$$

there exists a unique portfolio strategy $\pi_{\text{nig}} \in [0, 1]$ solving (3.13). Unfortunately, it is hard to see directly from (3.12) and (3.13) which of the two models giving the highest fraction of wealth to be invested in the risky stock. Depending on the parameters in the problem, we shall later see that π_{nig} can be both below and above the fraction π_{gbm} . We have developed numerical routines for calculating π_{nig} . These routines will be described in Section 7.

3.4. Optimal consumption plans. The optimal consumption rate in the standard model is known to be

$$(3.14) \quad c_{\text{gbm}} = \frac{\eta - k_{\text{gbm}}(\gamma)}{1 - \gamma},$$

see, e.g., Merton [13]. As shown in Benth, Karlsen and Reikvam [5], the normal inverse Gaussian model gives an optimal consumption plan which is a constant fraction of wealth: Namely, we have

$$(3.15) \quad c_{\text{nig}} = \frac{\eta - k_{\text{nig}}(\gamma)}{1 - \gamma},$$

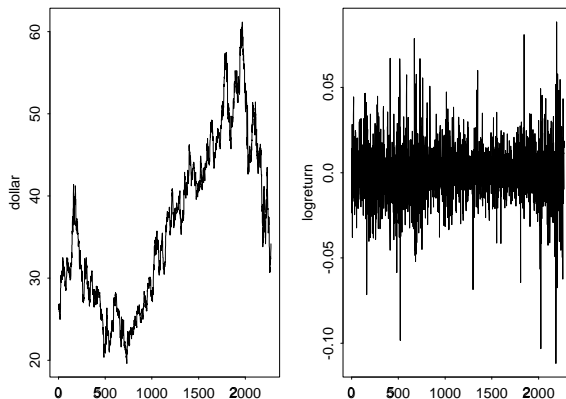


FIGURE 1. Daily quotes and logreturns for Norsk Hydro.

We have developed numerical routines for calculating $k_{\text{nig}}(\gamma)$ to obtain c_{nig} , see Section 7 for a description of these routines.

4. ANALYSIS OF THE LOGRETURN DATA

Our investigations are based on a data series of daily closing prices for Norsk Hydro on the New York Stock Exchange. The data series contained a total of 2274 prices, lasting from January 2, 1990 until December 31, 1998. The empirical logreturns are derived from the price series through the transformation $y_i = \ln(x_{i+1}/x_i)$, $i = 1, \dots, 2273$, where $\{x_i\}$ are the daily prices. In Figure 1 we can see how the value of Norsk Hydro has developed over the period of the dataset. Also plotted in the same figure is the logreturns of Norsk Hydro. We have fitted both the standard and the normal inverse Gaussian model to the logreturn data.

To estimate the drift and volatility in the standard model, we used maximum likelihood to obtain $\mu_{\text{gbm}} = 0.000101$ and $\sigma = 0.0166$. For the normal inverse Gaussian model, we simplified our considerations to the symmetric case, i.e., when $\beta = 0$, which implies $\mu := \mu_{\text{gbm}} = \mu_{\text{nig}}$. To perform maximum likelihood estimation for the normal inverse Gaussian distribution one needs highly sophisticated numerical optimization routines. It is outside the scope of this paper to develop such numerical routines, and we resort to the much simpler (but also highly unstable) method of moments. The parameters α and δ can be estimated through the second (m_2) and fourth (m_4) empirical moments by

$$(4.1) \quad \alpha = \sqrt{\frac{3m_2}{m_4 - 3m_2^2}}, \quad \delta = \alpha m_2.$$

In general one is not even guaranteed that the data is viable (e.g., that the expression under the square root sign of α is positive) for these equations. However, it turned out that the method of moments worked fine for Norsk Hydro. The estimated parameters are $\alpha = 51.7$ and $\delta = 0.0143$. Note that $\delta/\alpha \approx \sigma^2$.

The empirical logreturn density is plotted together with both the fitted normal and normal inverse Gaussian densities in Figure 2. We used standard routines from S-PLUS (see e.g. Venables and Ripley [17]) to find the empirical density. We have used a log-scale on the

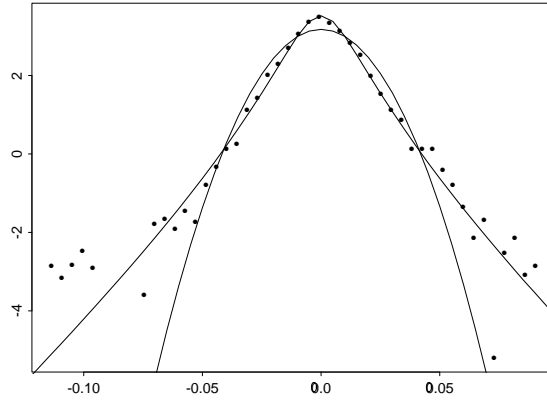


FIGURE 2. Log-density of daily Norsk Hydro data together with the fitted normal and normal inverse Gaussian distributions. The empirical log-density is plotted with dots, while the normal is the thin line and the normal inverse Gaussian the fatter line.

y - axis to demonstrate that the normal distribution fails to fit the tails of the logreturn data. The normal distribution will go like a parabola which we see heavily underestimates the tail uncertainty compared to the empirical distribution. The normal inverse Gaussian distribution explains much better the tail behaviour of the data. We also see that the normal inverse Gaussian distribution reflects the data better around the centre, i.e., there is a higher probability for price movements around the mean than described by the normal distribution. This plot confirms visually that the method of moments worked successfully for Norsk Hydro.

In order to fit into the framework of [5], the Lévy measure $n(dz)$ (see Section 2) must satisfy some integrability conditions. For the normal inverse Gaussian model $\alpha > 1$ is shown to be a sufficient condition, which is clearly satisfied for Norsk Hydro.

5. OPTIMAL PORTFOLIO ALLOCATION FOR NORSK HYDRO

We will in this section calculate the optimal portfolio allocation strategy for different risk aversion coefficients γ and interest rates. Two different interest rates will be chosen. The first is “far below” the expected rate of return on the stock, while the second is roughly the same. Note that the interest rates we use are *not* necessarily relevant for the NYSE-market, but simply chosen for the sake of illustration. In order to investigate the sensitivity on model parameters, we will consider varying volatility (i.e., varying σ).

In the standard model the expected rate of return for an investment in Norsk Hydro is $\mu + \frac{1}{2}\sigma^2 = 0.00023878$, or 5.9695% annually if we assume there are 250 trading days in a year. For the normal inverse Gaussian model the rate of return will be $\hat{\mu} = 0.00023923$, or 5.9807% annually. The normal inverse Gaussian model gives a slightly expected rate of return.

5.1. Experiment 1: The case of “low” annual interest rate. We first consider an annual interest rate of 5%, i.e., $r = 0.0002$. In the calculations we let the risk aversion coefficient γ vary between 0.1 and 0.95 with a step length of 0.05. The results are listed for the standard and the normal inverse Gaussian models in Table 1, Appendix A. We see that for $\gamma > 0.85$ the optimal strategy is to invest all the money in the stock. Even though

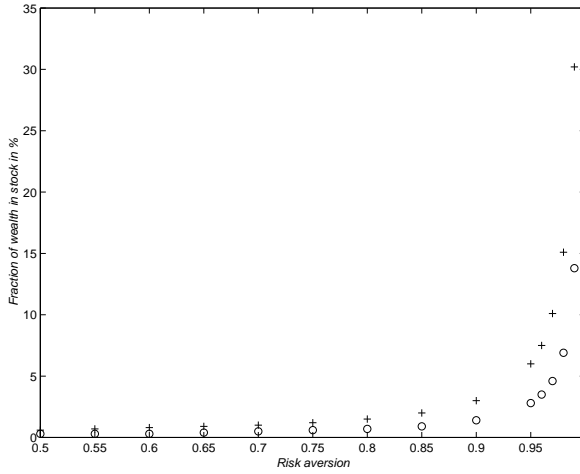


FIGURE 3. Investment profile as a function of risk aversion γ for the normal inverse Gaussian and standard cases. Here 'o' denotes the standard case, while '+' denotes the normal inverse Gaussian case. Annual interest rate is 5.96%.

the difference between the two models are marginal, it is noteworthy that the normal inverse Gaussian investor consistently place more money in the stock than the “standard” investor. Since there is a big difference between the interest rate of the market and the expected stock returns, the investor will in both models have an incentive to place money in the stock. As seen from the numerical calculations, even investors with a very low tolerance towards risk will place as much as 15% of their wealth in shares.

5.2. Experiment 2: The case of “high” annual interest rate. Let the annual interest rate be 5.96% (i.e., $r = 0.000238$), which is considerably closer to the expected rate of return for Norsk Hydro stocks in the standard model than in Experiment 1. In this case we calculated the optimal allocation strategies for γ between 0.5 and 0.95 with a step length of 0.05. For γ below 0.5, the obtained values for the optimal portfolio allocation π were very small and thus suppressed. The results are listed for both models in Table 1 Appendix A. Note that for γ close to one, we get a significant difference between the normal inverse Gaussian investor and standard. To really emphasise the difference, we also simulated the allocation strategies for γ between 0.96 and 0.99 with step length 0.01. The normal inverse Gaussian investor will put more than twice as much of her wealth in the risky stock than the “standard” investor when she has a high tolerance towards risk (γ close to one). Observe that for the highest level $\gamma = 0.99$, the normal inverse Gaussian investor will put nearly 1/3 of her wealth into Norsk Hydro shares, while the “standard” investor puts the more conservative fraction of 15%. In Figure 3 we have plotted the optimal fraction of wealth in stock against the risk aversion factor γ .

5.3. Experiment 3: Varying volatility. From the formula for π_{gbm} we see that when the volatility σ tends to infinity, π_{gbm} tends to the fixed number $\frac{1}{2(1-\gamma)}$. For $\gamma \in (0, \frac{1}{2})$ this number will be between 0 and 1, and the investor will in the standard model put a fraction of her wealth close to $\frac{1}{2(1-\gamma)}$ when investing in a highly volatile stock. We want to investigate numerically if the same is true for the normal inverse Gaussian investor, and if so, does it

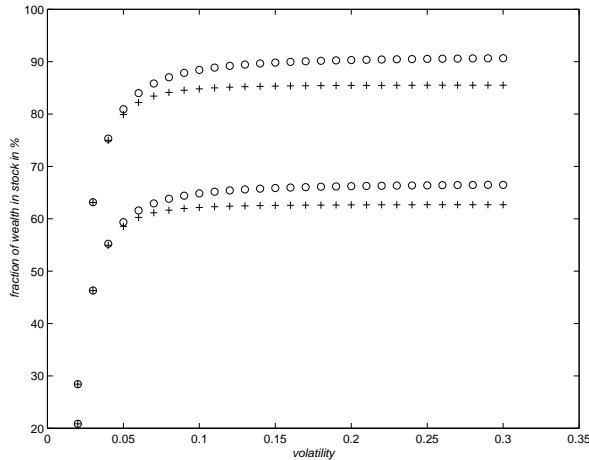


FIGURE 4. The investment profiles for increasing volatility with $\gamma = 0.25$ and $\gamma = 0.45$ in Experiment 3b. The normal inverse Gaussian case is plotted with '+' and the standard case with 'o'.

tend to the same fraction. It turns out that the conclusion depends on whether we increase the scale parameter or decrease the steepness of the normal inverse Gaussian distribution.

We assume that the standard model and the normal inverse Gaussian model have the same variance, i.e., $\sigma^2 = \frac{\delta}{\alpha}$. In this experiment we let the volatility σ increase from 0.02 to 0.3 with a step length of size 0.01. The drift μ is chosen similar to the drift for the Norsk Hydro stock. For instance, note that $\sigma = 0.1$ is an unrealistic case since the expected return in the standard model is above 100% annually. Motivated from the results in Experiment 2 we choose the annual interest rate to be 5.96%. The risk aversion coefficients are set to $\gamma = 0.25$ and $\gamma = 0.45$. In the former case π_{gbm} will converge to $2/3$ when σ goes to infinity, while in the latter the limit is 0.9091. In the normal inverse Gaussian model we can vary both the steepness and the scale in order to increase σ . We have thus split Experiment 3 into (two sub) Experiments 3a and 3b. In Experiment 3a we let α be fixed and δ vary according to the relation $\delta = \alpha\sigma^2$. In Experiment 3b, δ is held fixed while α vary according to $\alpha = \frac{\delta}{\sigma^2}$.

The results from the two subexperiments are listed in Tables 2 and 3, Appendix A. Experiment 3a shows that the normal inverse Gaussian model has the same convergence rate towards the limits $2/3$ and 0.9091 as the standard model. The two models give in fact (nearly) identical allocation strategies for the chosen sequence of σ 's. Experiment 3b, on the other hand, indicates that the normal inverse Gaussian model converges to *different* limits than $2/3$ and 0.9091, see Figure 4. When $\gamma = 0.25$ the limit is close to the number 62.7%, while for $\gamma = 0.45$ it is 85.5%.

It is worth noticing that in Experiment 3b the normal inverse Gaussian investor will put *less* money in the risky asset than the ‘‘standard’’ investor. In Experiments 1 and 2 the opposite were the case, and in Experiment 3a the models gave nearly identical results. The shape triangle (see, e.g., Rydberg [16]) may shed some light on the differences between Experiment 3a and 3b: Introduce the parameters (χ, ζ) by

$$\zeta = (1 + \delta\sqrt{\alpha^2 - \beta^2})^{-1}, \quad \chi = \frac{\beta\zeta}{\alpha}.$$

The domain of variation of the parameters (χ, ζ) is $0 \leq |\chi| < \zeta < 1$, thus motivating the name *shape triangle*. The coordinates for a normal distribution is $(0, 0)$, while for a centered normal inverse Gaussian they are $(0, (1 + \alpha\delta)^{-1})$. Note that Norsk Hydro has coordinates $(0, 0.575)$ in the shape triangle, thus being far from a normal distribution. In Experiment 3a, α is fixed and $\delta = \alpha\sigma^2$ with σ increasing. We see that $\alpha\delta = \alpha^2\sigma^2 \rightarrow \infty$ as $\sigma \rightarrow \infty$, and hence

$$(0, (1 + \alpha\delta)^{-1}) \rightarrow (0, 0) \text{ as } \sigma \rightarrow \infty.$$

This shows that in Experiment 3a the normal inverse Gaussian model will converge to the standard model and therefore reaching the same investment rate in the limit. When δ is fixed, we have $\alpha = \delta/\sigma^2$ and $\alpha\delta = \delta^2/\sigma^2 \rightarrow 0$ as $\sigma \rightarrow \infty$. Hence

$$(0, (1 + \alpha\delta)^{-1}) \rightarrow (0, 1) \text{ as } \sigma \rightarrow \infty,$$

and we see that the normal inverse Gaussian distribution converges to a symmetric Cauchy distribution, which is on the opposite end of the normal distribution in the shape triangle. This explains the different limits of the normal inverse Gaussian model in Experiment 3b.

6. OPTIMAL CONSUMPTION PLANS FOR NORSK HYDRO

We calculate the optimal consumption plans for the Norsk Hydro stock, that is, we find the fractions of wealth which the investor should take out for consumption in each of the two models. Since we are working with an infinite time horizon, one expects the consumption ratio to be very small, which indeed is confirmed by our numerical results.

The consumption ratio is in both models dependent on the discounting factor η . We need to choose one η such that condition (3.7) is satisfied. In this experiment, which we call Experiment 4, we have chosen η to be 6% annually. Condition (3.7) is then met for all γ up to 0.99 at least. The annual interest rate is 5.96%. Furthermore, the risk aversion constant γ is ranging between 0.1 and 0.95 with step length 0.05. The results are listed in Table 4, Appendix A. We have in addition calculated the consumption ratios for $\gamma = 0.99$. As can be seen from the calculations, both models give very low consumption fractions ranging from 0.024% to 0.0453%. We observe that the normal inverse Gaussian model suggests a slightly bigger consumption rate than the standard.

7. DISCUSSION OF THE NUMERICAL METHODS

It is seen from equations (3.13) and (3.15) that in order to calculate π_{nig} and c_{nig} we must solve an integral equation and perform three integrations over $\mathbb{R} \setminus \{0\}$ with respect to the Lévy measure $n(dz)$. We have used standard numerical techniques to solve these problems approximately.

To calculate π_{nig} from equation (3.13) we used the method of bisection (see e.g. [6]). The bisection approach turned out to be more efficient for our purposes than the usually preferred Newton's method. Newton's method will move very slowly since the derivative of the function in question is considerably bigger than the function itself. Thus, very small moves are made in each iteration of Newton's method. We iterated the bisection method 14 times, which means that the distance between the approximated solution $\tilde{\pi}_{\text{nig}}$ and the exact was less than 0.0001. In each iteration we calculated two integrals over $\mathbb{R} \setminus \{0\}$ with respect to $n(dz)$. The numerical integration was performed using Simpson's rule with 20000 equidistant points between -0.5 and 0.5 , leaving out the origin (see, e.g., [6]). By simple numerical testing we found this subpartition of $\mathbb{R} \setminus \{0\}$ to have a sufficient degree of accuracy and efficiency for our

problem. We point out, however, that no rigorous error analysis has been carried through for the numerical integration procedure.

After having found an approximation to the optimal portfolio allocation, the calculation of c_{nig} involves only integration, as is seen from (3.15). Again we approximated the integrals by Simpson's rule using 20000 equidistant points between -0.5 and 0.5 , leaving out the origin.

Our algorithms were implemented in Matlab, allowing us to use the built-in Bessel functions in the numerical integration procedure. The algorithms we used were not optimized in any way, and thus computationally demanding and slow. It was not the intention of this study to develop fast algorithms. However, the need for efficient and accurate algorithms are substantial, since we are dealing with functions which have most of their mass concentrated near zero and with a singularity at zero. These issues must be treated effectively and accurately if one wants to develop general software for solving Merton's problem in the normal inverse Gaussian case.

8. CONCLUSIONS

Based on the mathematical equations derived by Benth, Karlsen, and Reikvam [5], we have implemented numerical routines which enabled us to solve Merton's problem when the risky asset is modelled as a geometric normal inverse Gaussian Lévy process. The Norsk Hydro stock quoted on the New York Stock Exchange was taken to be the risky asset, and we compared the normal inverse Gaussian model to the geometric Brownian motion model. It was demonstrated that an investor believing in the normal inverse Gaussian model would consistently invest more in Norsk Hydro shares than a "standard" investor. Due to an infinite investment horizon, the optimal consumption plans in the two models became very low. The normal inverse Gaussian model assigned a slightly higher fraction of wealth to be taken out as consumption than the normal Gaussian model.

When there was a big difference between the risk-free interest rate and the rate of return on Norsk Hydro shares, the two models gave almost identical investment strategies. We observed, on the other hand, significant differences when the interest rate and rate of return were close. The normal inverse Gaussian investor would allocate more than twice as much of her fortune in the risky asset compared to the standard investor in our example. In fact, for high risk aversion factors (i.e., γ close to one) the normal inverse Gaussian investor will put nearly 30% of her wealth in Norsk Hydro shares, while the standard model assigns only 15%. This may be explained by the fact that the normal inverse Gaussian model predicts a higher rate of return than Merton.

We also studied the effect of increasing volatility. For the normal inverse Gaussian model, both scale and steepness can be varied to produce increasing volatility. When we increased the scale, both models converged to the same investment strategy. This was as expected since the normal inverse Gaussian distribution will converge to the normal distribution. For decreasing steepness, however, the normal inverse Gaussian distribution converges to a Cauchy distribution, which resulted in a significantly *lower* limiting investment rate than the standard model.

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APPENDIX A: TABLES OF RESULTS

All the numbers are rounded to one decimal of accuracy except in the table for Experiment 4, where we have used 4 decimals of accuracy. When we say “> 100” in the tables, we mean that the numerical calculation gave a figure above 100%, which means that we are advised to place all our money in the risky asset.

γ	π_{nig} in %	π_{gbm} in %	π_{nig} in %	π_{gbm} in %
0.10	15.8	15.6	-	-
0.15	16.7	16.6	-	-
0.20	17.7	17.6	-	-
0.25	18.9	18.8	-	-
0.30	20.3	20.1	-	-
0.35	21.8	21.7	-	-
0.40	23.6	23.5	-	-
0.45	25.7	25.6	-	-
0.50	28.4	28.1	0.6	0.3
0.55	31.5	31.3	0.7	0.3
0.60	35.4	35.2	0.8	0.3
0.65	40.5	40.2	0.9	0.4
0.70	47.2	46.9	1.0	0.5
0.75	56.7	56.3	1.2	0.7
0.80	70.9	70.4	1.5	0.7
0.85	94.6	93.8	2.0	0.9
0.90	>100	>100	3.0	1.4
0.95	>100	>100	6.0	2.8
0.96	-	-	7.5	3.5
0.97	-	-	10.1	4.6
0.98	-	-	15.1	6.9
0.99	-	-	30.2	13.8

TABLE 1. Results from Experiment 1 and 2. The last two columns relates to Experiment 2.

σ	π_{nig}		π_{gbm}	
	$\gamma = 0.25$	$\gamma = 0.45$	$\gamma = 0.25$	$\gamma = 0.45$
0.02	20.8	28.4	20.9	28.5
0.03	46.3	63.2	46.3	63.2
0.04	55.2	75.3	55.2	75.3
0.05	59.4	80.9	59.3	80.9
0.06	61.6	84.0	61.6	84.0
0.07	63.0	85.8	62.9	85.8
0.08	63.8	87.0	63.8	87.0
0.09	64.4	87.9	64.4	87.8
0.10	64.9	88.4	64.8	88.4
0.11	65.2	88.9	65.1	88.8
0.12	65.4	89.2	65.4	89.2
0.13	65.6	89.5	65.6	89.4
0.14	65.8	89.7	65.7	89.6
0.15	65.9	89.8	65.8	89.8
0.16	66.0	90.0	66.0	89.9
0.17	66.0	90.1	66.0	90.0
0.18	66.1	90.2	66.1	90.1
0.19	66.2	90.2	66.2	90.2
0.20	66.2	90.3	66.2	90.3
0.21	66.3	90.4	66.3	90.3
0.22	66.3	90.4	66.3	90.4
0.23	66.3	90.5	66.3	90.4
0.24	66.4	90.5	66.3	90.5
0.25	66.4	90.5	66.4	90.5
0.26	66.4	90.6	66.4	90.5
0.27	66.4	90.6	66.4	90.6
0.28	66.5	90.6	66.4	90.6
0.29	66.5	90.6	66.4	90.6
0.30	66.5	90.7	66.5	90.6

TABLE 2. Results from Experiment 3a.

σ	π_{nig}		π_{gbm}	
	$\gamma = 0.25$	$\gamma = 0.45$	$\gamma = 0.25$	$\gamma = 0.45$
0.02	20.8	28.4	20.9	28.5
0.03	46.3	63.1	46.3	63.2
0.04	54.9	75.0	55.2	75.3
0.05	58.6	79.9	59.3	80.9
0.06	60.3	82.2	61.6	84.0
0.07	61.1	83.4	62.9	85.8
0.08	61.7	84.1	63.8	87.0
0.09	62.0	84.5	64.4	87.8
0.10	62.2	84.8	64.8	88.4
0.11	62.3	85.0	65.1	88.8
0.12	62.4	85.1	65.4	89.2
0.13	62.5	85.2	65.6	89.4
0.14	62.5	85.3	65.7	89.6
0.15	62.6	85.3	65.8	89.8
0.16	62.6	85.4	66.0	89.9
0.17	62.6	85.4	66.0	90.0
0.18	62.6	85.4	66.1	90.1
0.19	62.6	85.4	66.2	90.2
0.20	62.7	85.5	66.2	90.3
0.21	62.7	85.5	66.3	90.3
0.22	62.7	85.5	66.3	90.4
0.23	62.7	85.5	66.3	90.4
0.24	62.7	85.5	66.3	90.5
0.25	62.7	85.5	66.4	90.5
0.26	62.7	85.5	66.4	90.5
0.27	62.7	85.5	66.4	90.6
0.28	62.7	85.5	66.4	90.6
0.29	62.7	85.5	66.4	90.6
0.30	62.7	85.5	66.5	90.6

TABLE 3. Results from Experiment 3b.

γ	c_{nig} in %	c_{gbm} in %
0.10	0.0241	0.0240
0.15	0.0241	0.0240
0.20	0.0241	0.0240
0.25	0.0242	0.0241
0.30	0.0242	0.0241
0.35	0.0243	0.0241
0.40	0.0244	0.0241
0.45	0.0245	0.0241
0.50	0.0247	0.0242
0.55	0.0249	0.0242
0.60	0.0252	0.0242
0.65	0.0257	0.0243
0.70	0.0264	0.0244
0.75	0.0277	0.0245
0.80	0.0300	0.0246
0.85	0.0350	0.0249
0.90	0.0371	0.0254
0.95	0.0386	0.0270
0.99	0.0453	0.0396

TABLE 4. Results from Experiment 4.

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