BARRIER OPTIONS AND TOUCH-AND-OUT OPTIONS UNDER REGULAR LÉVY PROCESSES OF EXPONENTIAL TYPE

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Abstract. We derive explicit formulas for barrier options of European type and touch-and-out options assuming that under a chosen equivalent martingale measure the stock returns follow a Lévy process from a wide class, which contains Brownian Motions (BM), Normal Inverse Gaussian Processes (NIG), Hyperbolic Processes (HP) and Truncated Lévy Processes (TLP), and any finite mixture of independent BM, NIG, HP and TLP. In contrast to the Gaussian case, for a barrier option, a rebate must be specified not only at a barrier but for all values of the stock the other side of the barrier, the reason being that trajectories of a non-Gaussian Lévy process are discontinuous. We consider options with the constant or exponentially decaying rebate, and options which pay a fixed rebate when the first barrier has been crossed but the second one has not. We obtain pricing formulas by solving corresponding boundary problems for the generalized Black-Scholes equation. We use the connection between the resolvent and the infinitesimal generator of the process, the representation theorem for analytic semigroups, the Wiener-Hopf factorization method and the theory of pseudo-differential operators.

Key words: Lévy processes, European barrier options, touch-and-out options, Wiener-Hopf factorization

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Contents

1. Introduction

Various aspects of pricing of barrier options have been considered in a number of papers and books, see e.g. Rubinstein and Reiner (1991), Wilmott et al (1995), Musiela and Rutkowski (1997) and the bibliography there, but to the best of our knowledge only Gaussian processes have been allowed.

In the paper, we consider the case when the returns $X_t = \ln S_t$ on the stock S_t follow a Lévy process from a wide class of processes, which we introduced in Boyarchenko and Levendorskiĭ (1999, 2000a) under the name Generalized Truncated Lévy Processes. In a recent paper Barndorff-Nielsen and Levendorskiĭ (2000), where a generalization of the class for Feller processes is developed, a new name: "Regular Lévy processes of exponential type" (RLPE) is suggested, and so we will use the new name.[1].

If the Lévy process is neither the Brownian Motion nor the Poisson process, the market is incomplete. According to the modern martingale approach to option pricing (Delbaen and Schachermayer (1994)), arbitrage-free prices can be obtained as expectations under any equivalent martingale measure (EMM), which is absolutely continuous w.r.t. to the historic measure. We assume that the riskless rate $r > 0$ is fixed, and EMM $\mathbf{Q}$ is chosen so that under $\mathbf{Q}$, X is a RLPE, and we derive explicit formulas for the prices of barrier options on the stock with one fixed barrier and touch-and-out options. In forthcoming papers, we will consider cases of time-dependent barriers and double barrier options (the latter are considered in e.g. Geman and Yor (1996)).

Notice that in contrast to the Gaussian case, the rebate (if any) must be specified not only at the barrier but for all values of the stock the other side of the barrier, the reason being that trajectories of a non-Gaussian Lévy process are discontinuous. In particular, we calculate the value of an option with the constant or exponentially decaying rebate; our general formulas give also explicit formulas for options which pay a fixed rebate when the first barrier has been crossed but the second barrier (situated farther than the first one) has not. We also consider touch-and-out options; they can be considered as barrier options with the constant rebate and 0 terminal payoff, so the treatment is essentially the same (and even simpler).

The class of regular Lévy processes of exponential type contains, in particular, Brownian Motions (BM), Normal Inverse Gaussian Processes (NIG), Hyperbolic Processes (HP), Truncated Lévy Processes (TLP) and any finite mixture of independent BM, NIG, HP and TLP. Not only BM, but the other mentioned processes as well have been widely used to describe the behavior of stock prices in real financial markets:

HP were constructed and used by Eberlein and co-authors (Eberlein and Keller (1995), Eberlein et al (1998), Eberlein and Prause (1999)) hyperbolic distributions were constructed by Barndorff-Nielsen (1977));

NIG were introduced by Barndorff-Nielsen (1998) and used to model German stocks by Barndorff-Nielsen and Jiang (1998); in Eberlein and Prause (1998) and Eberlein (1999), Generalized Hyperbolic Processes were constructed, which contained NIG and HP as subclasses;

TLP constructed by Koponen (1995) were used for modeling in real financial markets by Bouchaud and Potter (1997), Cont et al (1997) and Matacz (1997); a simple generalization of this family was constructed in Boyarchenko and Levendorskiĭ (1999, 2000a) (the generalization was needed since probability distribution of Koponen's family have tails of

[1]The second author thanks A.N.Shiryaev for pointing out that the old name was non-informative

the same rate of exponential decay whereas in real financial markets, the left tail is usually much fatter). Earlier, non-infinitely divisible truncations of stable Lévy distributions had been constructed and used to model the behavior of the Standard & Poor 500 Index by Mantegna and Stanley (1994, 1997).

In the name of the class under consideration, "Of exponential type" means that tails of PDF are exponentially decaying, and "regular" indicates that generators of these processes enjoy very favorable features from the point of view of the theory of pseudo-differential operators (PDO); roughly speaking, regular Lévy processes are the best class one can find if the Brownian Motion is not available. (We recall the definition of PDO in Section 2; for basic facts of the theory of PDO, see Eskin (1973) and Taylor (1981).) This is important since PDO-technique is very powerful. We applied it in Boyarchenko and Levendorskiǐ (2000a, 2000b), where we obtained explicit analytical formulas for perpetual American options, showed that the smooth fit principle failed in some cases, and suggested a substitute for it. Later, Mordecki (2000) has obtained pricing formulas for perpetual American puts and calls by using the probabilistic technique, but without explicit analytic formulas for processes observed in Financial Markets; his method allows one neither to notice the failure of the smooth fit principle nor suggest a substitute for it.

By using the Dynkin formula, we reduced the optimal stopping problem to a free boundary problem, and to solve the latter, we used the Wiener-Hopf factorization technique in the form of Eskin (1973). In this paper, we use the relation between the resolvent of a strongly measurable strong Markov process and its infinitesimal generator to reduce the pricing problem to the corresponding boundary problem for the generalized Black-Scholes equation; the latter is solved by means of the Wiener-Hopf factorization technique, the representation theorem for analytical semigroups Yosida (1964) and the theory of PDO, and in the end we obtain explicit pricing formulas for barrier options and touch-and-out options.

Our technique cannot be directly applied to Variance Gamma Processes used by Madan and co-authors in a series of papers during 90th (see Madan (1999), Madan et al (1998) and the bibliography there).

Notice that if X is a process of any of classes listed above, it belongs to the same class under the Esscher transform of the historic measure, which was used e.g. by Madan and co-authors and Eberlein and co-authors. In Boyarchenko and Levendorskiǐ (1999a) we have shown that if X is a RLPE, then it remains a regular Lévy process of exponential type under EMM from a wide class. It justifies our standing assumption below that X is a RLPE under a chosen EMM.

The plan of the paper is as follows. In Section 2, we reduce the pricing problem of a contingent claim to the corresponding boundary problem for the Generalized Black-Scholes equation, and give the schemes of the solution of these problems for some barrier options and touch-and-out options. Notice that this part is valid for any strongly measurable strong Markov process, and constructions in the rest of the paper can be modified and used in the case of Lévy-like Feller processes introduced in Barndorff-Nielsen and Levendorskiǐ (2000), the difference being that here the infinitesimal generator of the (Lévy) process is a PDO with the constant symbol (i.e. the symbol depends only on the dual variable), and the symbol of a Lévy-like Feller process is a PDO with the non-constant symbol (i.e. with a non-trivial dependence on the state variable).

In Section 3, we give the definition of regular Lévy processes of exponential type and examples, and consider the action of the "generalized Black-Scholes operator" in Sobolev spaces. In Section 4, we prove the Wiener-Hopf factorization formula with the parameter and give formulas (in terms of PDO) for the solutions of the boundary problems which are needed in Sections 5–7, where we explicitly calculate prices of down-and-out barrier options without the rebate (Section 5), down-and-out barrier options with the rebate and the touch-and-out put (Section 6), and up-and-out options and the touch-and-out call (Section 7). Section 8 concludes, and in Section 9, we prove some auxiliary technical estimates.

2. Pricing of Contingent Claims and Boundary Problems for Generalized Black-Scholes Equation

2.1. **Reduction to boundary problems for the Generalized Black-Scholes equation.** Consider a model market of a bond yielding the riskless rate of return $r > 0$, and a stock, which price at time t is denoted by $S_t = \exp X_t$. We assume that $X = \{X_t\}$ is a Lévy process under a chosen equivalent martingale measure $\mathbf{Q}$. Let $L_X^{\mathbf{Q}}$ be the infinitesimal generator of the transition semigroup of $\{X_t\}$ under $\mathbf{Q}$. Consider a contingent claim; its price at time t we denote by $f(t, X_t)$. Denote by $\mathcal{C}$ the continuing observation region for the claim; e.g. for a down-and-out call option with the expiry date T and the barrier $H = e^h$, $\mathcal{C} = [0, T) \times (h, +\infty)$. By the analogy with the initial Merton-Black-Scholes approach, we are going to derive an equation (generalized Black-Scholes equation), which the function f obeys on $\mathcal{C}$, and by adding appropriate boundary conditions, which specify a given claim, we obtain a well-posed problem. By solving the problem, we find $f(t, X_t)$, the price of the contingent claim. Though the set-up is similar to the initial one, the technique differs significantly at some steps since we no longer live in the Gaussian world; in particular, it is simpler to use not the Itø-Meyer formula but the relation between the generator of the process and the resolvent, and it is necessary to use the Wiener-Hopf factorization method and the representation theorem for analytical semigroups. At the same time, the technique we use here produces answers in the Gaussian case as well.

Introduce a two-dimensional process $\tilde{X}_t = (t, X_t)$ on the state space $\tilde{E} = [0, T] \times \mathbf{R}$; its generator is $\tilde{L} = \partial_t + L_X^{\mathbf{Q}}$. Set $\tilde{E}^0 := \tilde{E} \setminus \mathcal{C}$, and notice that for $\tilde{X}_t \in \tilde{E}^0$, the value $f(\tilde{X}_t)$ of the contingent claim is specified by the contract; denote it $\tilde{g}(\tilde{X}_t)$, and for $\tilde{X}_t \in \mathcal{C}$, set $\tilde{g}(\tilde{X}_t) = 0$. Let τ_0 be the hitting time of $\tilde{E}^0$. If the contingent claim is a local martingale under $\mathbf{Q}$, we must have

$$f = \tilde{R}_r(r\tilde{g}),$$

where

$$(\tilde{R}_r\tilde{g})(\tilde{x}) = E^{\mathbf{Q}}\left[\int_0^\infty e^{-rt}\tilde{g}(\tilde{X}_{t\wedge\tau_0})dt \mid \tilde{X}_0 = \tilde{x}\right]$$

is the resolvent of the process $\tilde{X}$ stopped at τ_0. We will see that for touch-and-out options and barrier options with the sufficiently regular rebate, e.g. the constant rebate or exponentially decaying one, the following conditions hold

(i) $\tilde{g}$ is non-negative;

(ii) there exists a pointwise non-decreasing sequence $\{\tilde{g}_n\}$ of sufficiently regular functions converging pointwise to $\tilde{g}$;

(iii) "sufficiently regular" means that $f_n = \tilde{R}_r(r\tilde{g}_n)$ belongs to $C_0(\tilde{E})$ and satisfies

$$(r - \tilde{L})f_n = \tilde{g}_n. \tag{2.1}$$

Eq. (2.1) means that f_n is a solution to the "generalized Black-Scholes equation"

$$(\partial_t + L_X^{\mathbf{Q}} - r)f(t,x) = 0, \quad \forall\ (t,x) \in \mathcal{C}, \tag{2.2}$$

of the class $C_0([0,T] \times \mathbf{R})$, satisfying the boundary condition

$$f(t,x) = \tilde{g}_n(t,x), \quad \forall\ (t,x) \in \tilde{E}^0. \tag{2.3}$$

We will find the explicit form of the solution f_n of the problem (2.2)–(2.3) for each type of the touch-and-out options and some types of barrier options (provided $\tilde{g}_n$ is sufficiently regular)–other types can be considered similarly–and after that find

$$f(0,x) = \lim_{n\to\infty} f_n(0,x). \tag{2.4}$$

Equality (2.4) follows from (i)–(iii) and the Lebesgue theorem.

To explain the logic of the auxiliary constructions in Sections 3 and 4, here we consider informally: first, the down-and-out call option with the constant rebate, the case without the rebate being a special case, and then the touch-and-out put option; other types of barrier options and the touch-and-out call option can be considered similarly.

Consider a down-and-out call option with the time-independent barrier H, the strike price K and the terminal date T. The payoff at expiry equals $\max\{S_T - K, 0\}$, provided that $S_t = \exp X_t$ never falls below $H = e^h$ during the life of the option. If S_t ever reaches H or falls below it, then either the option becomes worthless: $f(t, X_t) = 0$ if $t \le T$ and $X_t \le h$, or an option owner is entitled to some rebate $g^r(t, X_t)$. Notice that unlike in the Gaussian case, we must specify f not only at the barrier but everywhere below the barrier as well, the reason being that the trajectories of the process are no longer continuous. Without loss of generality, we may assume that $H = 1$ and hence, $h = \ln H = 0$.

Thus, $\tilde{g}(t,x)$ is given by

$$\tilde{g}(T,x) = g^T(x) := \max\{e^x - K, 0\}, \quad x > 0, \tag{2.5}$$

$$\tilde{g}(t,x) = g^r(t,x), \quad x \le 0,\ t \in [0,T]. \tag{2.6}$$

For simplicity, we assume that $g^r(t,x) = g^r_0 e^{\beta x}$, where $\beta \ge 0$ (more general rebates can also be considered). Construct a pointwise increasing sequence $\{\chi_n\} \subset C_0^\infty((0,+\infty))$, converging pointwise to $\mathbf{1}_{(0,+\infty)}$, set $q = \max\{\ln K, 0\}$, $g^T_n(x) = g^T(x)\chi_n(q+x)$, $g^r_n(x) = g^r(t,x)\chi_n(-x)$, and define the approximating sequence $\{\tilde{g}_n\}$ by (2.5) and (2.6) with g^T_n and g^r_n in the RHS, respectively. Clearly, $g^T_n \in C_0^\infty((0,+\infty))$ and $g^r_n \in C_0^\infty([0,T]\times(-\infty,0))$, therefore $\tilde{g}_n$ is as regular as one may only wish. In particular, $f_n := \tilde{R}r\tilde{g}_n$ is continuous and bounded, and hence, it is the unique continuous bounded solution of the problem (2.2)–(2.3) (see e.g. Breiman (1968), p.342). It remains to find any $f_n \in C_0(\mathbf{R})$ (in the sense: find an analytic expression), which solves the problem (2.2)–(2.3), for all n, and compute the limit (2.4).

Write the boundary condition (2.3) as the pair of conditions

$$f(T,x) = g^T_n(x), \quad x > 0; \tag{2.7}$$

$$f(t,x) = g^r_n(x), \quad t \in [0,T],\ x \le 0, \tag{2.8}$$

and look for the solution to the problem (2.2), (2.7)–(2.8) in the form

$$f_n(t,x) = g^r_n(x) + u_n(x) + v_n(T-t,x), \tag{2.9}$$

where $u_n \in C_0(\mathbf{R})$ solves the problem

$$-(r - L^{\mathbf{Q}}_X)u(x) = (r - L^{\mathbf{Q}}_X)g^r_n(x), \quad x > 0, \tag{2.10}$$

$$u(x) = 0, \quad x \le 0, \tag{2.11}$$

and $v \in C_0([0,T]\times\mathbf{R})$ solves the problem

$$(\partial_\tau + r - L^{\mathbf{Q}}_X)v(\tau,x) = 0, \quad \tau > 0,\ x > 0; \tag{2.12}$$

$$v(0,x) = g^T_n(x) - u_n(x), \quad x > 0; \tag{2.13}$$

$$v(\tau,x) = 0, \quad \tau \ge 0,\ x \le 0. \tag{2.14}$$

Remark 2.1. Notice that the constructions above are valid for any strongly measurable strong Markov process.

The problem (2.10)–(2.11) is an analog of the Dirichlet problem for an elliptic differential operator (in the Gaussian set-up, this is the Laplacian perturbed by an operator of the first order) on a half-axis, and the problem (2.12)–(2.14) is an analog of the Cauchy problem for a parabolic operator, with the Dirichlet boundary condition. In the case of a non-gaussian Lévy process, the elliptic part, $A := r - L^{\mathbf{Q}}_X$, is not a differential operator but an integro-differential operator (another name: pseudo-differential operator – PDO). The standard technique of the theory of differential operators is no longer applicable, and the adequate technique is the Wiener-Hopf factorization; to study the problem for the parabolic equation, one needs the representation theorem for analytic semigroups Yosida (1964), and the Wiener-Hopf factorization with the parameter. All these auxiliary constructions and results are collected in Sections 3 and 4.

The touch-and-out put option is essentially the barrier down-and-out call but with the zero terminal payoff and the constant rebate $g_r(t,x) = 1$. To be more specific, the problems for the touch-and-out put option with the strike price $K = 1$ can be obtained by letting in the constructions above $g^T(x) = 0, g^r(t,x) = 1$.

Before proceeding further, we make an important remark on the last step, namely, the calculation of the limit. u_n and v_n, hence, f_n will be found with the help of the theory of PDO and the Wiener-Hopf factorization, hence, by using the Fourier transform; the resulting formula involves oscillating integrals (which do not converge absolutely), and so the passage to the limit is non-trivial. To simplify this problem, we construct the approximating sequences so that they converge in appropriate Sobolev spaces (the definitions and basic properties are listed in Section 3), and general boundedness theorems on the action of PDO in the scale of Sobolev spaces can be applied to show that the limit of the sequence f_n, call it temporarily F, exists in the sense of the theory of generalized functions, and can be defined by exactly the same expression as f_n, with $\tilde{g}$ substituted for $\tilde{g}_n$. Moreover, by using the Sobolev embedding theorem, we will be able to prove that F is continuous on $\mathcal{C}$. Since we know that f_n is non-decreasing sequence of continuous functions, converging pointwise to f, f is its limit in the sense of generalized functions. Hence, $f = F$, and to finish the calculation of the price of the down-and-out call option, it remains to calculate oscillating integrals in formulas involving PDO. We write them down and simplify to certain extent in Sections 5–7; further simplifications leading to more effective computational procedures are possible but they are very lengthy.

The same procedure applies to options of other types.

2.2. **The Generalized Black-Scholes equation as a pseudo-differential equation.** Recall that the characteristic exponent of a Lévy process under a measure $\mathbf{Q}$ is defined by $E^{\mathbf{Q}}[e^{i\xi X_t}] = e^{-t\psi^{\mathbf{Q}}(\xi)}$ (for basic definitions and results of the theory of Lévy processes, see e.g. Bertoin (1996) and Sato (1999)). In our previous papers, we used the definition $E^{\mathbf{Q}}[e^{-i\xi X_t}] = e^{-t\psi^{\mathbf{Q}}(\xi)}$, since in the theory of PDO, the standard definition of the Fourier transform $\hat{u}$ of a function u is

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x) dx; \tag{2.15}$$

this lead to the uncomfortable appearance of the minus sign in many places, and so we decided to switch to the definition of the characteristic exponent common in Probability Theory; but we use (2.15) as the definition of the Fourier transform.

By using the integro-differential representation of $L_X^{\mathbf{Q}}$:

$$L_X^{\mathbf{Q}} f(x) = \frac{\sigma^2}{2} f''(x) + bf'(x) + \int_{-\infty}^{+\infty} (f(x+y) - f(x) - f'(x) y \mathbf{1}_{[-1,1]}(y)) F(dy),$$

where $(\sigma^2, b, F(dy))$ is the characteristic triplet of X_t, and the Lévy-Khintchine formula

$$\psi^{\mathbf{Q}}(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\le 1}(y)) F(dy),$$

we obtain that $L_X^{\mathbf{Q}}$ acts on oscillating exponents as follows:

$$(-L_X^{\mathbf{Q}})e^{ix\xi} = \psi^{\mathbf{Q}}(\xi)e^{ix\xi}.$$

By using the Fourier inversion formula and this equality, we conclude that for a sufficiently regular u,

$$(-L_X^{\mathbf{Q}})u(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\psi^{\mathbf{Q}}(\xi)\hat{u}(\xi)d\xi.$$

This means that $-L_X^{\mathbf{Q}}$ is a pseudo-differential operator with the symbol $\psi^{\mathbf{Q}}(\xi)$:

$$-L_X^{\mathbf{Q}} = \psi^{\mathbf{Q}}(D).$$

Recall that a pseudo-differential operator with the symbol $a = a(x,\xi)$ is defined by

$$a(x,D)u(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}a(x,\xi)\hat{u}(\xi)d\xi. \tag{2.16}$$

When the symbol is independent of the state variable, x, one writes $a(D)$ and calls a a PDO with the constant symbol.

Now we can rewrite the generalized Black-Scholes equation in variables $\tau = T - t, x$ as follows

$$\partial_\tau f + (r + \psi^{\mathbf{Q}}(D_x))f = 0. \tag{2.17}$$

Remark 2.2. If X belongs to the class of Lévy-like Feller processes introduced in Barndorff-Nielsen and Levendorskiǐ (2000), we obtain $-L_X^{\mathbf{Q}} = \psi^{\mathbf{Q}}(x, D_x)$; it is a PDO with the non-constant symbol $\psi^{\mathbf{Q}}(x,\xi)$.

Properties of a pseudo-differential equation (2.17) strongly depending on the properties of the symbol $r+\psi^{\mathbf{Q}}(\xi)$, we can proceed further only after a class of characteristic exponents (equivalently, a class of Lévy processes) is specified.

3. Regular Lévy Processes of Exponential Type and main properties of the Generalized Black-Scholes equation

3.1. **Definition of regular Lévy processes of exponential type.** In Boyarchenko and Levendorskiǐ (1999), we have shown that for wide classes of Lévy processes X used in empirical studies of financial markets, characteristic exponents (both under a historic measure and under EMM from wide classes) satisfy the conditions of the following definition.

Definition 3.1. Let there exist constants $c > 0$, $\nu \in (0, 2]$, $\nu' < \nu$, $\mu \in \mathbf{R}$, $\lambda_- < 0 \le \lambda_+$, and C such that

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \tag{3.1}$$

where ϕ admits the analytic continuation from $\mathbf{R}$ into a strip $\Im\xi \in (\lambda_-, \lambda_+)$, and the continuous extension up to the boundary of the strip, and satisfies the following two estimates: for all ξ in a strip $\Im\xi \in [\lambda_-, \lambda_+]$

$$|\phi(\xi) - c|\xi|^\nu| \le C\langle\xi\rangle^{\nu'}, \tag{3.2}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, and for any $[\lambda'_-, \lambda'_+] \subset (\lambda_-, \lambda_+)$ and all ξ in a strip $\Im k \in [\lambda'_-, \lambda'_+]$

$$|\phi'(\xi)| \le C\langle\xi\rangle^{\nu-1}, \tag{3.3}$$

where C depends on $[\lambda'_-, \lambda'_+]$ but not on ξ.

We say that X is a regular Lévy process of order ν and exponential type $[\lambda_-, \lambda_+]$.

Remark 3.2. a) We have modified the definition from Boyarchenko and Levendorskiĭ (1999, 2000a) in order to allow for a diffusion component, simplify a bound (3.2), and allow for the left tail to decay slower than exponentially. A bound (3.3) is introduced in order to obtain uniform estimates for the resolvent in Section 3.

b) In order that the stock itself can be priced under EMM $\mathbf{Q}$, $\psi^{\mathbf{Q}}(-i)$ must be well-defined, and hence, we must have $\lambda_- \le -1$.

c) If necessary for applications, one can generalize (3.2):

$$\phi(\xi) \sim c_\pm|\xi|^\nu + O(|\xi|^{\nu'}),$$

as $\Re\xi \to \pm\infty$ in the strip, where $\Re c_\pm \ge 0$. This generalization allows for a significant asymmetry in the central part of PDF. If $\Re c_\pm > 0$, all the results below holds, only formulas for exponents $\kappa_\pm$ and the factor d in the construction of the factors in the Wiener-Hopf factorization formula in Section 4 change (see the proof of Theorem 6.1 in Eskin (1973)).

Example 3.1. A model class of NIG can be described by characteristic exponents of the form

$$\psi(\xi) = -i\mu\xi + c[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}],$$

where $\alpha > |\beta| > 0$. Clearly, (3.1)–(3.3) hold with $\nu = 1$, $\nu' = 0$, and $\lambda_- = -\alpha + \beta, \lambda_+ = \alpha + \beta$. Thus, NIG are processes of order 1.

Example 3.2. Hyperbolic Processes are also processes of order 1. In the symmetric case, a hyperbolic process can be defined by

$$E^{\mathbf{Q}}[e^{i\xi X_1}] = \frac{\alpha}{K_1(\alpha\delta)} \frac{K_1(\delta\sqrt{\alpha^2+\xi^2})}{\sqrt{\alpha^2+\xi^2}},$$

where K_1 is the modified Bessel function of third kind and order 1, and $\alpha, \delta > 0$.

NIG and HP can be obtained from pure diffusions by subordination Barndorff-Nielsen (1998) and Eberlein (1999), which has a natural economic interpretation: the Brownian Motion in the random "business time" – see e.g. general discussion in Geman et al. (1998) (for different processes).

Example 3.3. Truncated Lévy processes of Koponen's (1995) family provide examples of processes of order $\nu \in (0,2), \nu \ne 1$ with $-\lambda_- = \lambda_+$; a generalization of this family constructed in Boyarchenko and Levendorskiĭ (1999, 2000a) provides examples of processes of order $\nu \in [0,2)$ with arbitrary λ_-, λ_+. This is important since for processes in real financial markets, the left tails are fatter than the right ones, and Koponen's family can contain processes with asymmetric PDF only when PDF are asymmetric in the central part as well, whereas PDF observed in real financial markets are approximately symmetric in the central part.

If $\nu \in (0,2), \nu \neq 1$, $c > 0$, then for a TLP X of order ν, ψ is of the form

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu].$$

Clearly, (3.1)–(3.3) holds; an example satisfying not (3.2) but its modification in Remark 3.1 c) is

$$\psi(\xi) = -i\mu\xi + d_+\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu] + d_-[(-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu],$$

where $d_+ \neq d_-$ are positive.

Example 3.4. If in Examples 3.1–3.3, we add a diffusion component or consider a pure diffusion, we obtain a process of order 2.

Clearly, any finite mixture of independent RLPE is a RLPE.

Remark 3.3. a) In Boyarchenko and Levendorskiǐ (1999), we used a definition, which regarded Variance Gamma Processes (VGP) as RLPE of order 0. Notice that our constructions below do not apply to VGP, and this is the reason why we exclude VGP here.

b) A convenient feature of a class of HP is its closedness under the Esscher transform, and the same holds for NIG, VGP and TLP.

In the next lemma, an important property of the characteristic exponent of a RLPE is derived.

Lemma 3.4. *Let (3.1) and (3.2) hold. Then there exist $\omega_- < 0 \le \omega_+$ and $\delta > 0$ such that*

$$\Re\psi^{\mathbf{Q}}(\xi + i\sigma) + r \ge \delta, \quad \forall\ \xi \in \mathbf{R},\ \sigma \in [\omega_-, \omega_+]. \tag{3.4}$$

Proof. Set $M_1^{\mathbf{Q}}(\sigma) = \int_{-\infty}^{+\infty} e^{-\sigma x} p_1^{\mathbf{Q}}(x)dx$. By differentiating twice, we conclude that $M_1^{\mathbf{Q}}$ is convex, and clearly, $M_1^{\mathbf{Q}}(0) = 1 < e^r$. Hence, there exist $\omega_- < 0 \le \omega_+$ and $\delta > 0$ such that for all $\sigma \in [\omega_-, \omega_+]$, $M_1^{\mathbf{Q}}(\sigma) \le e^{r-\delta}$.

Now, for any $\xi \in \mathbf{R}$, and these σ,

$$\exp(-\Re\psi^{\mathbf{Q}}(\xi + i\sigma)) = |\exp(-\psi^{\mathbf{Q}}(\xi + i\sigma))| =$$
$$= \left| \int_{-\infty}^{+\infty} e^{i\xi x - \sigma x} p_1^{\mathbf{Q}}(x)dx \right| \le \int_{-\infty}^{+\infty} e^{-\sigma x} p_1^{\mathbf{Q}}(x)dx,$$

therefore (3.4) holds. Notice that if $\lambda_+ > 0$, we can choose $\omega_+ > 0$. □

Main properties (3.1)–(3.4) of the symbol $a(\xi) = r + \psi^{\mathbf{Q}}(\xi)$ of the stationary part of the generalized Black-Scholes operator in the LHS of (2.17) having been stated, we can study its action in appropriate scales of normed spaces; this is a necessary component of the theory of boundary problems for PDO.

In the following three subsections, we list main standard results of the theory of PDO (see e.g. Eskin (1973), Ch.3–4). The reader should be aware of the following systematic differences: the monograph Eskin (1973) is chosen as a reference book on PDO since in many respects its exposition is simpler than in later monographs on the subject but it uses the different definition of the Fourier transform, which has become obsolete in the theory of PDO. In the result, to establish the correspondence between the results in Eskin (1973) and their counterparts here, the lower half-plane of the complex plane must be replaced with the upper one and visa versa, etc.

3.2. **Action of PDO in the Sobolev spaces on R.** We use the following standard notation: $\mathcal{S}(\mathbf{R})$ denotes the space of C^∞-functions decaying at the infinity faster any power of $|x|$, together with all their derivatives. The topology in $\mathcal{S}(\mathbf{R})$ is defined by a system of seminorms

$$||u||_{\mathcal{S};s,N} = \sup_{k\le s}\sup_{x\in\mathbf{R}} |u^{(s)}(x)\langle x\rangle^N|,$$

where $N, s \ge 0$ are integers. By Lemma 2.1 in Eskin (1973), the Fourier transform defined by (2.15) is an isomorphism of $\mathcal{S}(\mathbf{R})$.

A functional ϕ over $\mathcal{S}(\mathbf{R})$ is called linear if for any $\alpha_1, \alpha_2 \in \mathbf{C}$ and $f_1, f_2 \in \mathcal{S}(\mathbf{R})$,

$$(\phi, \alpha_1 f_1 + \alpha_2 f_2) = \overline{\alpha_1}(\phi, f_1) + \overline{\alpha_2}(\phi, f_2).$$

The space of continuous linear functionals in $\mathcal{S}(\mathbf{R})$ is denoted by $\mathcal{S}'(\mathbf{R})$. Its elements are called distributions or generalized functions. A $\phi \in \mathcal{S}'(\mathbf{R})$ is called regular, if it can be identified with a locally integrable function F, growing not faster than a polynomial at the infinity:

$$(\phi, f) = \int_{-\infty}^{+\infty} F(x)\bar{f}(x)dx.$$

The action of the Fourier transform $\mathcal{F}$ in $\mathcal{S}'(\mathbf{R})$ is defined by duality

$$(\mathcal{F}\phi, \mathcal{F}f) = 2\pi(\phi, f), \quad \forall\ f \in \mathcal{S}(\mathbf{R}).$$

It is a continuous operator in $\mathcal{S}'(\mathbf{R})$.

Definition 3.5. Let s be real. A generalized function u belongs to the Sobolev space $H^s(\mathbf{R})$ if and only if the norm

$$||u||_s = \left(\int_{-\infty}^{+\infty} \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 dk\right)^{1/2} \tag{3.5}$$

is finite.

¿From (3.5), one easily deduces several simple but important properties of the Sobolev spaces.

Lemma 3.6. *a) $H^0(\mathbf{R}) = L_2(\mathbf{R})$;*

b) for $s > s'$, $H^s(\mathbf{R})$ is continuously embedded in $H^{s'}(\mathbf{R})$;

c) if $s \ge 0$ is an integer, an equivalent norm in $H^s(\mathbf{R})$ can be defined by

$$||u||'_s = \left(\sum_{m\le s} ||D^m u||^2_{L_2}\right)^{1/2}.$$

If f is a regular functional defined by a locally integrable function $f(x)$, supp$f$ is the complement to the maximal open set on which $f(x) = 0$ a.e. Let $J \subset \mathbf{R}$ be an open set. $C_0^\infty(J)$ denotes the space of C^∞-functions with the support in J, and $\mathcal{S}(J)$ denotes the closure of $C_0^\infty(J)$ in $\mathcal{S}(\mathbf{R})$.

Lemma 3.7. *(Eskin (1973), Theorem 4.1.) For any s, $C_0^\infty(\mathbf{R})$ is a dense subset of $H^s(\mathbf{R})$.*

Definition 3.8. Let $m \in \mathbf{R}$. We write $a \in S^m(\mathbf{R})$ if there exists C such that for all $\xi \in \mathbf{R}$,

$$|a(\xi)| \le C\langle\xi\rangle^m. \tag{3.6}$$

For $a \in S^m(\mathbf{R})$, one defines the action of a PDO $A = a(D)$ by (2.16).

Theorem 3.9. *Let $m, s \in \mathbf{R}$, and let $a \in S^m(\mathbf{R})$. Then $a(D) : H^s(\mathbf{R}) \to H^{s-m}(\mathbf{R})$ is bounded, with the norm bounded by a constant C in (3.6).*

Proof. Under the Fourier transform, the action of PDO $a(D)$ becomes the multiplication by $a(\xi)$, hence (3.5) and (3.6) gives the necessary estimate. □

The next definition and theorem (the Sobolev embedding theorem – see e.g. Eskin (1973), Theorem 4.3) show that C_0– estimates can be derived from the ones in the scale of Sobolev spaces. This observation allows us to work in a simpler scale of spaces though it is possible to develop a similar theory for action of PDO in Hölder spaces; this requires additional restrictions on a class of symbols (see e.g. Taylor (1981)).

Definition 3.10. Let $s \ge 0$ be an integer. $C^s(\mathbf{R})$ denotes the space of functions continuous together with all the derivatives up to order s, and $C_0^s(\mathbf{R})$ denotes its subspace consisting of functions vanishing at the infinity together with all the derivatives up to order s.

Theorem 3.11. *Let $s > 1/2$, and $0 \le s' < s - 1/2$. Then the embedding $H^s(\mathbf{R}) \subset C_0^{s'}(\mathbf{R})$ is continuous.*

Definition 3.12. We say that a is elliptic, if and only if $a \in S^m(\mathbf{R})$ and there exists $c > 0$ such that for all ξ,

$$|a(\xi)| \ge c\langle\xi\rangle^m. \tag{3.7}$$

Remark 3.13. Usually one says that a is elliptic if (3.7) is satisfied outside some compact.

Theorem 3.14. *Let $m, s \in \mathbf{R}$, and let $a \in S^m(\mathbf{R})$ be elliptic. Then $a(D) : H^s(\mathbf{R}) \to H^{s-m}(\mathbf{R})$ is invertible, with the (bounded) inverse $a(D)^{-1}$.*

Proof. It follows from (3.7), that $a^{-1} \in S^{-m}(\mathbf{R})$, hence by Theorem 3.9, $a(D)^{-1} : H^{s-m}(\mathbf{R}) \to H^s(\mathbf{R})$ is bounded. Under the Fourier transform, the action of PDO $a(D)$ becomes the multiplication by $a(\xi)$, hence $a(D)$ and $a(D)^{-1}$ are mutual inverses. □

3.3. **Properties of the elliptic part of the Generalized Black-Scholes equation as an operator on R.** If $\nu \ge 1$ or $\mu = 0$, then from (3.1), (3.2) and (3.4), we conclude that there exist $C_1, c_1 > 0$ such that $a(\xi) := r + \psi^{\mathbf{Q}}(\xi)$ satisfies, for all $\xi \in \mathbf{R}$,

$$\Re a(\xi) \ge c_1\langle\xi\rangle^\nu; \tag{3.8}$$

and

$$|\Im a(\xi)/\Re a(\xi)| \le C_1. \tag{3.9}$$

Consider $A = r + \psi^{\mathbf{Q}}(D)$ as an unbounded operator in $H^0(\mathbf{R}) = L_2(\mathbf{R})$ with the domain $H^\nu(\mathbf{R})$. ¿From (3.8), it follows that $\Re A$ is positive definite: for any $u \in H^s(\mathbf{R})$,

$$(\Re Au, u)_0 \geq c_1||u||_0^2, \quad \forall\, u \in L_2(\mathbf{R}),$$

and from (3.9), $(\Re A)^{-1}\Im A$ is bounded. This means that A is a strongly elliptic PDO, and therefore (2.17) is an analogue of the parabolic equation. If $\nu \in (0, 1)$ and $\mu \neq 0$, then one can reduce (2.17) to a parabolic equation by changing coordinates $x' = x + \mu\tau$ but this spoils the time-independent boundary for barrier options and touch-and-out options. This observation means, in particular, that in cases of time-dependent barriers, processes of the order $\nu \in [1, 2]$ are more tractable than the ones of the order $\nu \in (0, 1)$.

3.4. **Action of PDO in weighted Sobolev spaces on a half-line.** Let $J \subset \mathbf{R}$ be an open set. We say that f is a distribution on J, if f is a continuous linear functional in $\mathcal{S}(J)$. The space of continuous linear functionals in $\mathcal{S}(J)$ is denoted by $\mathcal{S}'(J)$. Let $f \in \mathcal{S}'(\mathbf{R})$. The functional $f_J \in \mathcal{S}'(J)$ is called the restriction of f on J if $(f, u) = (f_J, u)$ for all $u \in \mathcal{S}(J)$. The restriction operator will be denoted p_J, hence, $f_J = p_J f$. Since $\mathcal{S}(J)$ is a closed subspace of $\mathcal{S}(\mathbf{R})$, any $f \in \mathcal{S}'(J)$ admits an extension $lf \in \mathcal{S}'(\mathbf{R})$ (clearly, an extension is non-unique).

We say that f equals zero on J, if $f_J = 0$. Since $C_0^\infty(J)$ is dense in $\mathcal{S}(J)$, $f_J = 0$ iff $(f, u) = 0$ for all $u \in C_0^\infty(J)$. Let J be the maximal open set on which $f = 0$. Then the complement to J is denoted by supp$f$ and called the support of f. If f is a regular functional, this definition coincide with the one given earlier. $\overset{o}{H}{}^{s}(\mathbf{R}_\pm)$ denotes a subspace of $H^{s,\gamma}(\mathbf{R})$ consisting of distributions f with supp$f \subset \overline{\mathbf{R}_\pm}$.

Lemma 3.15. *(Eskin (1973), Lemmas 4.2 and 4.3).* $\overset{o}{H}{}^{s}(\mathbf{R}_\pm)$ *is equal to the closure of* $C_0^\infty(\mathbf{R}_\pm)$ *in* $H^s(\mathbf{R})$.

Theorem 3.16. *(Eskin (1973), Theorem 4.4). a) If* $a \in S^m(\mathbf{R})$ *admits the analytic continuation into the lower half-plane* $\Im\xi < 0$ *and satisfies an estimate (3.6) in the closed half-plane* $\Im\xi \leq 0$*, then* $a(D) : \overset{o}{H}{}^{s}(\mathbf{R}_+) \to \overset{o}{H}{}^{s-m}(\mathbf{R}_+)$ *is bounded.*

b) If $a \in S^m(\mathbf{R})$ *admits the analytic continuation into the upper half-plane* $\Im\xi > 0$ *and satisfies an estimate (3.6) in the closed half-plane* $\Im\xi \geq 0$*, then* $a(D) : \overset{o}{H}{}^{s}(\mathbf{R}_-) \to \overset{o}{H}{}^{s-m}(\mathbf{R}_-)$ *is bounded.*

Denote by θ_+ (resp., θ_-) the-multiplication-by-$\mathbf{1}_{[0,+\infty)}(x)$ (resp., $\mathbf{1}_{(-\infty,0]}(x)$) operator. Clearly, they are well-defined on $\mathcal{S}(\mathbf{R})$.

Theorem 3.17. *(Eskin (1973), Theorem 5.1) For* $|s| < 1/2$, $\theta_\pm$ *admits a unique continuous extension* $\theta_\pm : H^s(\mathbf{R}) \to \overset{o}{H}{}^{s}(\mathbf{R}_\pm)$.

Lemma 3.18. *(Eskin (1973), Lemma 5.4) For* $|s| < 1/2$*, any function* $f \in H^s(\mathbf{R})$ *admits a unique representation* $f = f_+ + f_-$*, where* $f_\pm \in \overset{o}{H}{}^{s}(\mathbf{R}_\pm)$*, and* $f_\pm = \theta_\pm f$.

Definition 3.19. Let $J \subset \mathbf{R}$ be an open set. One writes $f \in H^s(J)$ if f is a distribution on J, which admits an extension $lf \in H^s(\mathbf{R})$. The norm in $H^s(J)$ is defined by

$$||f||_{J;s} = \inf ||lf||_s,$$

where infimum is taken over all extensions $lf \in H^s(\mathbf{R})$.

Set $p_\pm = p_{\mathbf{R}_\pm}$. It maps $H^s(\mathbf{R})$ onto $H^s(\mathbf{R}_\pm)$.

Theorem 3.20. *(Eskin (1973), Lemma 4.6) a) If $a \in S^m(\mathbf{R})$ admits the analytic continuation into the lower half-plane $\Im\xi < 0$ and satisfies (3.6) in the closed half-plane, then the operator*

$$H^s(\mathbf{R}_-) \ni u \mapsto p_- a(D) lu \in H^{s-m}(\mathbf{R}_-)$$

is well-defined and bounded.
b) If $a \in S^m$ admits the analytic continuation into the upper half-plane $\Im\xi > 0$ and satisfies (3.6) in the closed half-plane, then the operator

$$H^s(\mathbf{R}_+) \ni u \mapsto p_+ a(D) lu \in H^{s-m}(\mathbf{R}_+)$$

is well-defined and bounded.

Lemma 3.21. *(Eskin (1973), Lemma 4.5.) If $s \geq 0$ is an integer, an equivalent norm in $H^s(\mathbf{R}_\pm)$ can be defined by*

$$||u||'_{\mathbf{R}_\pm,s} = \left(\sum_{j\leq s} ||D^j u||^2_{L_2(\mathbf{R}_\pm)}\right)^{1/2}. \tag{3.10}$$

The following theorem is a special case of general interpolation theorems (see e.g. Triebel (1978), Section 2.10.1); it is valid for other scales, $H^s(\mathbf{R})$ in particular, but here we need only this special case.

Theorem 3.22. *For any $m > 0$, $L_2(\mathbf{R}_\pm) = H^0(\mathbf{R}_\pm)$ and $H^m(\mathbf{R}_\pm)$ form an interpolation pair, and for any $s \in (0,1)$, $H^{sm}(\mathbf{R}_\pm)$ is the interpolation space: $H^{sm}(\mathbf{R}_\pm) = [H^0(\mathbf{R}_\pm), H^m(\mathbf{R}_\pm)]_s$. This means, in particular, that*

$$||u||_{H^{sm}(\mathbf{R}_\pm)} \leq C_s \left(||u||_{L_2(\mathbf{R}_\pm)}\right)^{1-s} \left(||u||_{H^m(\mathbf{R}_\pm)}\right)^s,$$

where C_s is independent of $u \in H^m(\mathbf{R}_\pm)$.

$e_\pm u$ denotes the-extension-by-zero operator of regular functionals from $\mathbf{R}_\pm$: $e_\pm u(x) = 0$ for all $\pm x < 0$. ¿From Theorem 3.17 and Lemma 3.18, the following lemma is immediate.

Lemma 3.23. *Let $|s| < 1/2$. Then*
$e_\pm$ extends from $p_\pm \mathcal{S}(\mathbf{R})$ to a bounded operator $e_\pm : H^s(\mathbf{R}_\pm) \to H^s(\mathbf{R})$;
$e_\pm(H^s(\mathbf{R}_\pm))$ can be identified with $\overset{o}{H}{}^s(\mathbf{R}_\pm)$, and
$p_\pm\left(\overset{o}{H}{}^s(\mathbf{R}_\pm)\right)$ can be identified with $H^s(\mathbf{R}_\pm)$.

If the Fourier transform $\hat{u}$ of a regular functional $u \in H^s(\mathbf{R}_+)$ is given, one can try to find u by using formally the integration by part: for $x > 0$

$$\int_{-\infty}^{+\infty} e^{ix\xi}\hat{u}(\xi)d\xi = -x^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}D_\xi\hat{u}(\xi)d\xi.$$

If one ends with the absolutely converging integral, one gets a formula for u. This procedure is called a regularization of oscillatory integrals; we will use it on the final stage of the calculation of the prices of options in Sections 5–7.

4. Generalized Black-Scholes equation on a half-axis

In this Section, we solve problems (2.10)–(2.11) and (2.12)–(2.14) when X is a regular Lévy process of order $\nu \in (0, 2]$ and the exponential type $[\lambda_-, \lambda_+]$, where $\lambda_- \le -1 < 0 \le \lambda_+$. We modify some constructions and results of Ch. 6, 7 in Eskin (1973).

4.1. **The Wiener-Hopf factorization.** Factorization of $a(\xi) = r + \psi^{\mathbf{Q}}(\xi)$ can be done for any Lévy process (see e.g. Theorem 45.1 in Sato (1999)) though without the explicit formulas for the factors; in Boyarchenko and Levendorskiĭ (2000a, 2000b), explicit formulas are derived for any RLPE, and by using them, one can explicitly solve the problem (2.10)–(2.11). To be able to apply the representation theorem for analytical semigroups and solve the problem (2.12)–(2.14), one needs certain estimate for the resolvent, which can be obtained only under the following additional condition on μ in (3.1):

$$\text{if } \nu \in \ (0,1), \quad \text{then } \mu = 0. \tag{4.1}$$

¿From now on, we add (4.1) to the list of standing assumptions (3.1)–(3.3) on the process X. For $\theta \in (0, \pi)$, set $\Sigma_\theta = \{\lambda \in \mathbf{C} \mid \arg\lambda \in [-\theta, \theta]\}$, and let $\omega_- < 0 \le \omega_+$ be the same as in Lemma 3.1.

Lemma 4.1. *There exists $c_1 > 0$ and $\theta \in (\pi/2, \pi)$ such that if $\Im\xi \in [\omega_-, \omega_+]$, and $\lambda \in \Sigma_\theta$,*

$$|\lambda + a(\xi)| \ge c_1(1 + |\lambda| + |\xi|^\nu). \tag{4.2}$$

Proof. Fix $C_1 > 0$ and $\epsilon > 0$, and consider domains

$$U^-(C_1, \epsilon) = \{(\lambda, \xi) \mid |\lambda| \le C_1(1 + |\xi|^\nu),\ \arg\lambda \in (-\pi/2 - \epsilon, \pi/2 + \epsilon),\ \Im\xi \in [\omega_-, \omega_+]\},$$
$$U^+(C_1) = \{(\lambda, \xi) \mid |\lambda| \ge C_1(1 + |\xi|^\nu),\ \lambda \in \mathbf{C},\ \Im\xi \in [\omega_-, \omega_+]\}.$$

On $U^+(C_1)$, it suffices to prove (4.2) without $|\xi|$ in the RHS. From (3.1), (3.2), (3.4) and (4.1) it follows that there exists C_0 such that

$$|a(\xi)| \le C_0(1 + |\xi|)^\nu,$$

and hence if C_1 is sufficiently large, we obtain $|\lambda + a(\xi)| \ge |\lambda|/2$; thus, any θ fits. On $U^-(C_1, \epsilon)$, it suffices to prove (4.2) without $|\lambda|$ in the RHS. ¿From (3.1), (3.2), (3.4) and (4.1) it follows that for any C_1 we can find sufficiently small ϵ and $c_1 = c_1(C_1, \epsilon)$ such that for indicated (λ, ξ),

$$\Re(\lambda + a(\xi)) \ge c_1(1 + |\xi|^\nu).$$

Hence, (4.2) holds with $\theta = \pi/2 + \epsilon$. □

Fix a branch of ln by a requirement: $\ln y$ is real for $y > 0$, set $\epsilon_0 = 1 - \omega_- + \omega_+$ and $\Lambda_\pm(\lambda,\xi)^s = (\epsilon_0 + |\lambda|^{1/\nu} \mp i\xi)^s = \exp[s\ln(\epsilon_0 + |\lambda|^{1/\nu} \mp i\xi)]$, and choose $d > 0$ and $\kappa_-, \kappa_+ \in \mathbf{R}$ so that

$$B(\lambda,\xi) := d^{-1}\Lambda_+(\lambda,\xi)^{-\kappa_+}\Lambda_-(\lambda,\xi)^{-\kappa_-}(\lambda + a(\xi)) \tag{4.3}$$

satisfies for all $\lambda \in \Sigma_\theta$, $\xi \in \mathbf{R}$ and $\sigma \in [\omega_-, \omega_+]$

$$\lim_{\xi\to\pm\infty} B(\lambda, \xi + i\sigma) = 1, \tag{4.4}$$

and $b(\lambda, \xi + i\sigma) = \ln B(\lambda, \xi + i\sigma)$ is well-defined for these λ, ξ, σ.

Choices of d, κ_+ and κ_- depending on properties of B, hence on ν, μ and c in (3.1)–(3.2), we have to consider two cases:

1) If $\nu \in (0, 2], \nu \neq 1$, we set $d = c, \kappa_- = \kappa_+ = \nu/2$;

2) If $\nu = 1$, we set $d = (c^2 + \mu^2)^{1/2}, \kappa_\pm = 1/2 \pm \pi^{-1}\arctan(\mu/c)$.

In the first case, (4.4) immediately follows from (3.1)–(3.2), and if $\nu = 1$, then the simplest way to prove (4.4) is to check that $\ln B(\lambda, \xi + i\sigma) \to 0$ as $\xi \to \pm\infty$:

$$\lim_{\xi\to\pm\infty} \ln B(\lambda, \xi + i\sigma) = \pm\frac{\pi i}{2}\kappa_+ \mp \frac{\pi i}{2}\kappa_- + \ln\frac{c \mp i\mu}{(c^2 + \mu^2)^{1/2}} =$$

$$= \pm(\kappa_+ - \kappa_-)\frac{\pi i}{2} \mp i\arctan\frac{\mu}{c} = 0.$$

Lemma 4.2. *For any $\lambda \in \Sigma_\theta$ and $\sigma \in (\omega_-, \omega_+)$, the winding number around the origin of the curve $\{B(\lambda, \xi + i\sigma) \mid -\infty < \xi < +\infty\}$ is zero:*

$$(2\pi)^{-1}\int_{\xi=-\infty}^{\xi=+\infty} d\arg B(\lambda, \xi + i\sigma) = 0. \tag{4.5}$$

Proof. Due to (4.4) and (4.3), the LHS in (4.5) is an integer. From (4.2), $B(\lambda, \xi) \neq 0\ \forall\ \lambda \in \Sigma_\theta$ and ξ in a strip $\Im\xi \in [\omega_-, \omega_+]$, hence this integer is independent of $\lambda \in \Sigma_\theta$ and $\sigma \in [\omega_-, \omega_+]$. With $\lambda = 0$ and $\Im\xi = \sigma$, the last factor in (4.3) assumes values in a half-plane $\Re z > 0$ by (3.4), and the same is true of the product of the first three factors, since the first one is positive, $\Lambda_-(\lambda,\xi)$ and $\Lambda_+(\lambda,\xi)$ assume values in the same half-plane but in different quadrants, and $0 < \kappa_\pm \leq 1$. Hence, for all ξ in a strip $\Im\xi \in [\omega_-, \omega_+]$, $-\pi < \arg B(0, \xi) < \pi$, and therefore, (4.5) holds. $\square$

Under condition (4.5), $b(\lambda, \xi) := \ln B(\lambda, \xi)$ is well-defined on $\Sigma_\theta \times \{\xi \mid \Im\xi \in [\omega_-, \omega_+]\}$. Next, for real ξ, $\sigma > \omega_-$ and $\sigma_1 \in (\omega_-, \sigma)$, we set

$$b_+(\lambda, \xi + i\sigma) = -(2\pi i)^{-1}\int_{-\infty + i\sigma_1}^{+\infty + i\sigma_1} \frac{b(\lambda, \eta)}{\xi + i\sigma - \eta}d\eta, \tag{4.6}$$

and for real ξ, $\sigma < \omega_+$ and $\sigma_2 \in (\sigma, \omega_+)$, we set

$$b_-(\lambda, \xi + i\sigma) = (2\pi i)^{-1}\int_{-\infty + i\sigma_2}^{+\infty + i\sigma_2} \frac{b(\lambda, \eta)}{\xi + i\sigma - \eta}d\eta. \tag{4.7}$$

By the Cauchy theorem, $b_\pm(\lambda, \xi + i\sigma)$ are independent of choices of σ_1 and σ_2.

It follows from (3.1), (3.2), (4.4) and (4.5), that there exist $C, \rho > 0$ such that for all η in a strip $\Im\eta \in [\omega_-, \omega_+]$,

$$|b(\lambda, \eta)| \le C(1 + |\eta|)^{-\rho},$$

where C depends on λ but not on η (and $\rho > 0$ is independent of λ and η). Hence, the integrals in (4.6) and (4.7) converge, and $b_\pm(\lambda, \xi)$ is well-defined and holomorphic in a half-plane $\pm\Im\xi > \pm\omega_\mp$. In Section 8, we will prove the following lemma.

Lemma 4.3. *For any $[\omega'_-, \omega'_+] \subset (\omega_-, \omega_+)$, there exists $C > 0$ such that*

$$|b_+(\lambda, \xi)| \le C, \quad \forall\ \lambda \in \Sigma_\theta, \Im\xi \ge \omega'_-, \tag{4.8}$$

and

$$|b_-(\lambda, \xi)| \le C, \quad \forall\ \lambda \in \Sigma_\theta, \Im\xi \le \omega'_+. \tag{4.9}$$

By the residue theorem, for $\omega_- < \sigma_1 < \sigma < \sigma_2 < \omega_+$,

$$b_+(\lambda, \xi + i\sigma) + b_-(\lambda, \xi + i\sigma) = -(2\pi i)^{-1} \left(\int_{-\infty+i\sigma_1}^{+\infty+i\sigma_1} - \int_{-\infty+i\sigma_2}^{+\infty+i\sigma_2} \right) \frac{b(\lambda, \eta)}{\xi + i\sigma - \eta} d\eta =$$
$$= b(\lambda, \xi + i\sigma).$$

Hence, $B_\pm = \exp b_\pm$ satisfy $B = B_+ B_-$ on $\Sigma_\theta \times \{\xi \mid \Im\xi \in (\omega_-, \omega_+)\}$, and if we set

$$A_-(\lambda, \xi) = \Lambda_-(\lambda, \xi)^{\kappa_-} B_-(\lambda, \xi),\ A_+(\lambda, \xi) = d\Lambda_+(\lambda, \xi)^{\kappa_+} B_+(\lambda, \xi),$$

then for $\lambda \in \Sigma_\theta,\ \Im\xi \in (\omega_-, \omega_+)$,

$$\lambda + a(\xi) = A_+(\lambda, \xi) A_-(\lambda, \xi). \tag{4.10}$$

Lemma 4.4. *a) For any $\lambda \in \Sigma_\theta$, $A_+(\lambda, \xi)$ is holomorphic in the half-plane $\Im\xi > \omega_-$, admits the continuous extension up to the boundary of the half-plane, and satisfies an estimate*

$$c(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_+} \le |A_+(\lambda, \xi)| \le C(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_+}, \tag{4.11}$$

where $C, c > 0$ are independent of $\lambda \in \Sigma_\theta$ and ξ in the half-plane $\Im\xi \ge \omega_-$;
b) For any $\lambda \in \Sigma_\theta$, $A_-(\lambda, \xi)$ is holomorphic in the half-plane $\Im\xi < \omega_+$, admits the continuous extension up to the boundary of the half-plane, and satisfies an estimate

$$c(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_-} \le |A_-(\lambda, \xi)| \le C(1 + |\lambda|^{1/\nu} + |\xi|)^{\kappa_-}, \tag{4.12}$$

where $C, c > 0$ are independent of $\lambda \in \Sigma_\theta$ and ξ in the half-plane $\Im\xi \le \omega_+$;
c) for all $\lambda \in \Sigma_\theta$ and ξ in a strip $\omega_- \le \Im\xi \le \omega_+$, (4.10) holds;
d) factors in (4.10) are uniquely defined by properties a) and b), up to scalar multiples, depending on λ.

Proof. Fix $[\omega'_-, \omega'_+] \subset (\omega_-, \omega_+)$, and prove a)–c) for $\lambda \in \Sigma_\theta$ and ξ with $\Im\xi \in [\omega'_-, \omega'_+]$. Clearly, $\Lambda_\pm(\lambda, \xi)^{\kappa_\pm}$ satisfy a) and b), and since $b_\pm$ are holomorphic and bounded on the same set due to (4.8)–(4.9), a) and b) are proven; (4.10) has already been proven.

To prove a)–c) in the full generality, we notice that $a(\xi)$ is continuous on the strip $\Im\xi \in [\omega_-, \omega_+]$, and hence $A_+(\lambda, \xi)$ admits the continuous extension on $\Sigma_\theta \times \{\xi \mid \Im\xi \ge$

$\omega_-\}$ by $A_+(\lambda,\xi) = (\lambda + a(\xi))/A_-(\lambda,\xi)$, and $A_-(\lambda,\xi)$ admits the continuous extension on $\Sigma_\theta \times \{\xi \mid \Im\xi \le \omega_+\}$ by $A_-(\lambda,\xi) = (\lambda + a(\xi))/A_+(\lambda,\xi)$; then (4.10) holds for $\lambda \in \Sigma_\theta$, $\Im\xi \in [\omega_-, \omega_+]$ by construction. (4.11) and (4.12) for these λ and ξ follows from (3.1)–(3.2) and (3.4) and from the already proven (4.11) and (4.12) for $\lambda \in \Sigma_\theta$, $\Im\xi \in [\omega'_-, \omega'_+]$.

To prove d), fix λ, and suppose, $\lambda + a(\xi) = A'_+(\lambda,\xi)A'_-(\lambda,\xi)$ is another factorization with the same properties. Then $A'_+(\lambda,\xi)/A_+(\lambda,\xi)$ (resp., $A'_-(\lambda,\xi)/A_-(\lambda,\xi)$) is holomorphic in the upper half-plane $\Im\xi > 0$ (resp., the lower half-plane $\Im\xi < 0$), and continuous up to the boundary. Both functions are bounded and non-zero, and coincide on $\mathbf{R}$. Hence, the analytic continuation of any of them is a bounded holomorphic function on $\mathbf{C}$. By the Liouville theorem, it must be constant. □

4.2. **Solution of the problem (2.10)–(2.11).** Choose s so that

$$-1/2 < s - \kappa_- < 1/2, \tag{4.13}$$

write (2.10) as

$$p_+ a(D)u = -p_+ a(D) g_n^r, \tag{4.14}$$

and look for a solution to (4.14) in $\overset{o}{H}{}^s(\mathbf{R}_+)$, by considering $g_n^r \in C_0^\infty(\mathbf{R}_-)$ as an element of $H^s(\mathbf{R})$. Then by Theorem 3.9, $a(D)g_n^r \in H^{s-\nu}(\mathbf{R})$, and we can represent (4.14) in the form

$$a(D)u = -a(D)g_n^r + f_-,$$

where $f_- \in \overset{o}{H}{}^{s-\nu}(\mathbf{R}_-)$. By applying $A_+(0,D)^{-1}$ and using (4.10), we obtain

$$A_-(0,D)u_n = -A_-(0,D)g_n^r + A_+(0,D)^{-1}f_-. \tag{4.15}$$

Since A_+ satisfies (4.11) and $f_- \in \overset{o}{H}{}^{s-\nu}(\mathbf{R}_-)$, we can apply Theorem 3.16 and obtain $A_+(0,D)^{-1}f_- \in \overset{o}{H}{}^{s-\kappa_-}(\mathbf{R}_-)$; and since $u \in \overset{o}{H}{}^s(\mathbf{R}_+)$ and A_- satisfies (4.12), Theorem 3.16 gives that the LHS in (4.15) belongs to $\overset{o}{H}{}^{s-\kappa_-}(\mathbf{R}_+)$. Due to (4.13), Lemma 3.18 is applicable. By multiplying (4.15) first by θ_+, and then by $A_-(0,D)^{-1}$, we find

$$-A_-(0,D)u_n = \theta_+ A_-(0,D)g_n^r,$$

and

$$u_n = -A_-(0,D)^{-1}\theta_+ A_-(0,D)g_n^r.$$

¿From (4.12) and Theorem 3.16, it follows that $u_n \in \overset{o}{H}{}^s(\mathbf{R}_+)$. Notice that we may choose $s > 1/2$; then from Theorem 3.11, $u_n \in C_0(\mathbf{R})$, and by the definition of the space $\overset{o}{H}{}^s(\mathbf{R}_+)$, $\mathrm{supp} u_n \subset [0, +\infty)$, hence (2.11) is satisfied. Further, by using Lemma 3.18 and the identification of Lemma 3.23,

$$u_n = p_+ u_n = -p_+ A_-(0,D)^{-1}\theta_+ A_-(0,D)g_n^r =$$
$$= -p_+ g^r + p_+ A_-(0,D)^{-1}\theta_- A_-(0,D)g_n^r,$$

and since $p_+g^r = 0$, we obtain

$$u_n = p_+A_-(0,D)^{-1}\theta_-A_-(0,D)g^r_n. \tag{4.16}$$

Thus, (4.16) gives the solution to the problem (2.10)–(2.11), and the argument above shows that the map

$$\overset{o}{H}{}^s(\mathbf{R}_-) \ni g^r_n \mapsto u_n \in \overset{o}{H}{}^s(\mathbf{R}_+)$$

is bounded. To proceed further, we need to construct a sequence $g^r_n \in C^\infty_0(R_-)$, which converges to g^r in the topology of $\overset{o}{H}{}^s(\mathbf{R}_-)$, not only pointwise: $g^r_n(x) \uparrow g^r(x), \forall\ x < 0$. This can be done iff the rebate g^r is exponentially decaying, i.e. $\beta > 0$. We fix non-decreasing $\phi \in C^\infty(\mathbf{R})$ with the properties $\phi(x) = 0, x \le 1$, $\phi(x) = 1, x \ge 2$, and set $\chi_n(x) = \phi(nx)(1 - \phi(x/n))$, $g^r_n(x) = \chi_n(-x)g^r(x)$. Clearly, $g^r_n(x) \uparrow g^r(x), \forall\ x < 0$. Since g^r exponentially decays at the infinity, straightforward calculations with the help of (3.10) show that

$$n||g^r_n - g^r||^2_{L_2(\mathbf{R}_+)} \le C,$$

and

$$n^{-1}||D(g^r_n - g^r)||^2_{L_2(\mathbf{R}_+)} \le C_1,$$

where $C, C_1 > 0$ are independent of n. By using the interpolation theorem (Theorem 3.22), we find that $g^r_n \to g^r$ in the topology of $H^s(\mathbf{R}_-)$, provided $s \in [0, 1/2)$. On the strength of Lemma 3.23, g^r can be identified with $e_-g^r \in \overset{o}{H}{}^s(\mathbf{R}_-)$ and $g^r_n \to g^r$ in the topology of $\overset{o}{H}{}^s(\mathbf{R}_-)$, hence we can pass to the limit in (4.16) and obtain

$$u = p_+A_-(0,D)^{-1}\theta_-A_-(0,D)g^r. \tag{4.17}$$

There is a small technical problem: $g^r \in \overset{o}{H}{}^s(\mathbf{R}_-)$ only if $s < 1/2$, and the approximating sequence can be constructed to converge in the topology of this space, hence the general argument does not give $u \in C_0(\mathbf{R})$. Still, this can be verified easily when an analytic formula for u is derived in Section 6.

If the rebate is constant, i.e. $\beta = 0$, we replace g^r with $g^{r,\sigma}(x) = g^r_0 e^{\sigma x}$, where $\sigma > 0$, denote u calculated from (4.17) with $g^{r,\sigma}$ instead of g^r by u^σ, notice that $g^{r,\sigma}(x) \uparrow g^r(x)$, $\forall\ x < 0$, as $\sigma \downarrow 0$, and hence, $u(x) = \lim_{\sigma\downarrow 0} u^\sigma(x)$, and calculate the limit. Finally, if $\lambda_+ > 0$ and hence $\omega_+ > 0$ (and in many studies of financial markets λ_+ is shown to be not only positive but large – see e.g. Barndorff-Nielsen and Jiang (1998)), we can calculate u for any bounded rebate without resorting to the last limiting procedure. Namely, take any $\sigma \in (0, \omega_+)$, and notice that:

(i) $g^{r,\sigma}$ decays exponentially at $-\infty$, and hence $g^{r,\sigma}_n \to g^{r,\sigma}$ in $\overset{o}{H}{}^s(\mathbf{R}_-)$, for any $s < 1/2$;
(ii) $e^{\sigma x}A_\pm(\lambda, D)e^{-\sigma x} = A_\pm(\lambda, D + i\sigma)$, and $e^{\sigma x}\psi^{\mathbf{Q}}(D)e^{-\sigma x} = \psi^{\mathbf{Q}}(D + i\sigma)$;
(iii) symbols $\psi^{\mathbf{Q},\sigma}(\xi) := \psi^{\mathbf{Q}}(\xi + i\sigma)$ and $A^\sigma_\pm(0,\xi) := A_\pm(0, \xi + i\sigma)$ satisfy the same conditions as in the case $\sigma = 0$, with the strips $[\lambda_- + \sigma, \lambda_+ + \sigma]$ and $[\omega_- + \sigma, \omega_+ + \sigma]$ instead of $[\lambda_-, \lambda_+]$ and $[\omega_-, \omega_+]$.

Hence,

$$u_n = e^{-\sigma x}p_+A_-(0, D + i\sigma)^{-1}\theta_-A_-(0, D + i\sigma)g^{r,\sigma}_n,$$

and we can pass to the limit and obtain

$$u = e^{-\sigma x} p_+ A_-(0, D + i\sigma)^{-1} \theta_- A_-(0, D + i\sigma) e^{\sigma x} g^r.$$

Notice that we can write the last formula as (4.17), with the usual understanding that in the formula for the Fourier transform implicit in the notation $A_-(\lambda, D)g^r$, the integration w.r.t. the dual variable ξ is not over the real line but over the line $\Im\xi = \sigma$.

4.3. **Solution of the problem (2.12)–(2.14).** Choose s satisfying (4.13) and the following condition

$$s > 1/2 > s - \kappa_- > s - \nu > -1/2; \tag{4.18}$$

if $\nu < 2$, then $\kappa_+ = \nu - \kappa_- \in (0, 1)$, and this choice is possible. The case $\nu = 2$ will be considered in Remark 4.6 below. Consider $p_+ a(D)$ as an unbounded operator A^s in $H^{s-\nu}(\mathbf{R}_+) = \overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$ with the domain $\overset{o}{H}{}^{s}(\mathbf{R}_+)$, and the problem (2.12)–(2.14) as the Cauchy problem for an ordinary differential equation with the operator coefficient:

$$w'(\tau) + A^s w(\tau) = 0, \quad \tau > 0, \tag{4.19}$$

$$w(0) = G_n. \tag{4.20}$$

Here the initial data $G_n := g_n^T - u_n \in \overset{o}{H}{}^{s'}(\mathbf{R}_+)$, for any $s' > 1/2$, and the unknown w is a continuous (vector)-function on $[0, +\infty)$, assuming values in $\overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$, $w(\tau) \in \mathcal{D}(A^s)$ and $w'(\tau) \in \overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$, for any $\tau > 0$. Notice that due to the choice $s > 1/2$ and on the strength of Theorem 3.11, $\mathcal{D}(A^s) \subset C_0(\mathbf{R})$.

We are going to construct the resolvent $(\lambda + A^s)^{-1}$ and show that

$$||(1 + |\lambda|)(\lambda + A^s)^{-1}|| \le C, \tag{4.21}$$

where C is independent of $\lambda \in \Sigma_\theta$. (4.21) means that a condition (III) in Section 10, Chapter IX of Yosida (1964) is satisfied, and therefore, all the results of this Section 10 hold. In particular, A^s is the infinitesimal generator of the strongly continuous semigroup T^s, the representation theorem for the analytic semigroups is applicable, and for the solution of the problem (4.19)–(4.20), an explicit formula obtains:

$$w(\tau) = (2\pi i)^{-1} \int_{\mathcal{L}_\theta} e^{\tau\lambda} (\lambda + A^s)^{-1} G_n d\lambda. \tag{4.22}$$

Here $\mathcal{L}_\theta$ is the contour $\lambda = \lambda(\sigma), -\infty < \sigma < +\infty$, where $\arg\lambda(\sigma) = -\theta$ for $\sigma < 0$, and $\arg\lambda(\sigma) = \theta$ for $\sigma > 0$. The formula (4.22) will be used to solve the pricing problems for barrier options and touch-and-out options.

Theorem 4.5. *Let (4.18) hold. Then*

a) $\lambda + A^s$ is invertible, with the inverse given by

$$(\lambda + A^s)^{-1} = A_-(\lambda, D)^{-1} \theta_+ A_+(\lambda, D)^{-1}; \tag{4.23}$$

b) $(\lambda + A^s)^{-1} : \overset{o}{H}{}^{s-\nu}(\mathbf{R}_+) \to \overset{o}{H}{}^{s}(\mathbf{R}_+)$ is bounded uniformly in $\lambda \in \Sigma_\theta$;

c) for the norm of $(\lambda + A^s)^{-1}$, as an operator in $\overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$, the estimate (4.21) holds.

Proof. First, we show that for λ fixed, R_λ, defined by the RHS in (4.23), is a bounded operator from $\overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$ into $\overset{o}{H}{}^{s}(\mathbf{R}_+)$, uniformly in $\lambda \in \Sigma_\theta$, and as an operator in $\overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$, it satisfies

$$||(1 + |\lambda|)R_\lambda|| \le C, \quad \forall\ \lambda \in \Sigma_\theta. \tag{4.24}$$

Since $\kappa_- + \kappa_+ = \nu$, we have

$$(1 + |\lambda|)R_\lambda = (1 + |\lambda|)^{\kappa_-/\nu} A_-(\lambda, D)^{-1}\theta_+(1 + |\lambda|)^{\kappa_+/\nu} A_+(\lambda, D)^{-1}.$$

Since (4.11)–(4.12) hold, Theorems 3.16 and 3.17 give that operators

$$\theta_+A_+(\lambda, D)^{-1} : \overset{o}{H}{}^{s-\nu}(\mathbf{R}_+) \to \overset{o}{H}{}^{s-\kappa_-}(\mathbf{R}_+), \quad A_-(\lambda, D)^{-1} : \overset{o}{H}{}^{s-\kappa_-}(\mathbf{R}_+) \to \overset{o}{H}{}^{s}(\mathbf{R}_+)$$

are bounded, with the norms admitting estimates uniform in $\lambda \in \Sigma_\theta$; and as operators in $\overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$, both $(1 + |\lambda|)^{\kappa_+/\nu}\theta_+A_+(\lambda, D)^{-1}$ and $(1 + |\lambda|)^{\kappa_-/\nu} A_-(\lambda, D)^{-1}$ are bounded uniformly in $\lambda \in \Sigma_\theta$.

This proves (4.24) – and b)–c) as well, provided a) is proved.

Now we check that R_λ is the right inverse to $\lambda + A^s$. By using (4.10), we obtain

$$(\lambda + A^s)R_\lambda = p_+A_+(\lambda, D)\theta_+A_+(\lambda, D)^{-1}.$$

By Theorem 3.9, for any $f \in \overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$, $A_+(\lambda, D)^{-1}f \in H^{s-\kappa_-}(\mathbf{R})$, and

$$\theta_+A_+(\lambda, D)^{-1}f = A_+(\lambda, D)^{-1}f - \theta_-A_+(\lambda, D)^{-1}f, \tag{4.25}$$

where $\theta_-A_+(\lambda, D)^{-1}f \in \overset{o}{H}{}^{s-\kappa_-}(\mathbf{R}_-)$ due to the choice (4.13), Lemma 3.18 and Theorem 3.17. ¿From Theorem 3.16,

$$A_+(\lambda, D)\theta_-A_+(\lambda, D)^{-1}f \in \overset{o}{H}{}^{s-\nu}(\mathbf{R}_-),$$

and therefore by applying θ_+, we obtain 0. It follows that if we apply $\theta_+A_+(\lambda, D)$ to (4.25), we obtain f. This proves that R_λ defines the right inverse.

Similarly, we show that R_λ is the left inverse:

$$A_-(\lambda, D)^{-1}\theta_+A_+(\lambda, D)^{-1}(\lambda + A^s) =$$

$$= A_-(\lambda, D)^{-1}\theta_+A_+(\lambda, D)^{-1}\theta_+(\lambda + a(D)) =$$

(here we identify $p_+(\lambda + a(D)) = \theta_+(\lambda + a(D))$, which is admissible since $|s - \nu| < 1/2$)

$$= A_-(\lambda, D)^{-1}\theta_+A_-(\lambda, D) - A_-(\lambda, D)^{-1}\theta_+A_+(\lambda, D)^{-1}\theta_-(\lambda + a(D)).$$

For $f \in H^s(\mathbf{R})$, by Theorems 3.9 and 3.17 (and due to the choice $|s-\nu| < 1/2$), we obtain $\theta_-(\lambda + a(D))f \in \overset{o}{H}{}^{s-\nu}(\mathbf{R}_-)$; by Theorem 3.16, $A_+(\lambda, D)^{-1}\theta_-(\lambda + a(D))f \in \overset{o}{H}{}^{s-\kappa_-}(\mathbf{R}_-)$, and since $s - \kappa_- > -1/2$, Lemma 3.18 is applicable. It gives $\theta_+A_+(\lambda, D)^{-1}\theta_-(\lambda + a(D))f = 0$. By applying Theorems 3.16 and 3.17, we obtain

$$A_-(\lambda, D)^{-1}\theta_+A_-(\lambda, D) = A_-(\lambda, D)^{-1}A_-(\lambda, D) = I,$$

which finishes the proof that R_λ is the left inverse. □

Remark 4.6. If a process X contains a diffusion component, then $\nu = 2, \kappa_\pm = 1$, and therefore (4.18) cannot be satisfied. Notice however, that we can manage without (4.18) if we choose s so that $|s-\nu| < 1/2$, and define A^s as an unbounded operator in $\overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$ with the domain

$$\mathcal{D}(A^s) = \{A_-(0,D)^{-1}\theta_+ A_+(0,D)^{-1}u \mid u \in \overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)\}.$$

If we choose $s-\nu > 1/2-\kappa_+ = -1/2$, an analysis of the first part of the proof of Theorem 4.5 gives $\mathcal{D}(A^s) \subset \overset{o}{H}{}^{\kappa_-+1/2-\rho}(\mathbf{R}_+)$ for any $\rho > 0$. Since $\kappa_- > 0$, we can choose $\rho > 0$ so that $\kappa_- + 1/2 - \rho > 1/2$.

Thus, in the statement of Theorem 4.5, only b) needs a slight change: for any $\rho > 0$, there exists $C > 0$ such that for any $\lambda \in \Sigma_\theta$,

$$||(\lambda + A^s)^{-1} : \overset{o}{H}{}^{s-\nu}(\mathbf{R}_+) \to \overset{o}{H}{}^{\kappa_-+1/2-\rho}(\mathbf{R}_+)|| \le C,$$

and the proof changes in the evident manner.

As in the case $\nu < 2$, $\mathcal{D}(A^s) \subset C_0(\mathbf{R})$, by Theorem 3.11.

By substituting (4.23) into (4.22), we obtain the solution of the problem (2.12)–(2.14), and it remains to compute the limits

$$v_1(\tau,x) := \lim_{n\to+\infty} (2\pi i)^{-1} \int_{\mathcal{L}_\theta} e^{\tau\lambda}(\lambda + A^s)^{-1}u_n(x)d\lambda, \tag{4.26}$$

and

$$v_2(\tau,x) := \lim_{n\to+\infty} (2\pi i)^{-1} \int_{\mathcal{L}_\theta} e^{\tau\lambda}(\lambda + A^s)^{-1}g_n^T(x)d\lambda. \tag{4.27}$$

As we have shown in the previous subsection, in the case of the exponentially rebate, $u_n \to u$ in $\overset{o}{H}{}^{s-\nu}(\mathbf{R}_+)$, for any $s < 1/2+\nu$, and hence by Theorem 4.5 and Remark 4.6, there exist $s < 1/2+\nu, \epsilon > 0$ and C such that

$$||(\lambda + A^s)^{-1}(u_n - u)||_{1/2+\epsilon} \le C||u_n - u||_{s-\nu}, \quad \forall\ \lambda \in \Sigma_\theta. \tag{4.28}$$

For any $\alpha > 0$, there exists $C_\alpha > 0$ such that for all $\tau \ge \alpha$,

$$\int_{\mathcal{L}_\theta} |e^{\lambda\tau}d\lambda| \le C_1, \tag{4.29}$$

and we derive from (4.28) and (4.29), that for any $0 < \alpha < \beta$, the limit (4.26) exists in the sense of $C([\alpha,\beta]; \overset{o}{H}{}^{1/2+\epsilon}(\mathbf{R}_+))$, hence in the sense of $C_0([\alpha,\beta]\times\mathbf{R})$, and the limit is given by

$$v_1(\tau,x) = (2\pi i)^{-1} \int_{\mathcal{L}_\theta} e^{\tau\lambda}(\lambda + A^s)^{-1}u(x)d\lambda. \tag{4.30}$$

If the rebate does not decay at the infinity, but is bounded, e.g. constant, we choose $\sigma > 0$, replace g^r with $g^{r,\sigma}$, find u^σ, calculate v_1^σ from (4.30) with u^σ in the RHS, and pass to the limit as $\sigma \downarrow 0$. If $\lambda_+ > 0$, we choose any $\sigma \in (0, \omega_+)$, and use (4.30) with the understanding that when we insert (4.23) into (4.30) and apply the formula (2.16) to $A_+(\lambda, D)^{-1}u$, we calculate $\hat{u}(\xi)$ on the line $\Im\xi = \sigma$, and integrate over this line in the equation (2.16), which defines the action of a PDO (cf. the argument in the end of the previous subsection).

If the terminal payoff g^T is bounded, we can calculate v_2 by using similar trick: we choose any $\gamma \in (\omega_-, 0)$, define

$$(\lambda + A^{s,\gamma})^{-1} := A_-(\lambda, D + i\gamma)^{-1}\theta_+ A_+(\lambda, D + i\gamma)^{-1},$$

and notice that $g^{T,\gamma}(x) = g^T(x)e^{\gamma x}$ exponentially decays as $x \to +\infty$, and hence exactly the same argument as with the derivation of (4.30) shows that

$$e^{\gamma x}(\lambda + A^s)^{-1}g_n^T(x) = (\lambda + A^{s,\gamma})^{-1}g_n^{T,\gamma}(x)$$

converges to

$$e^{\gamma x}(\lambda + A^s)^{-1}g^T(x) = (\lambda + A^{s,\gamma})^{-1}g^{T,\gamma}(x),$$

and therefore v_2 can be calculated as follows

$$v_2(\tau, x) = (2\pi i)^{-1}\int_{\mathcal{L}_\theta} e^{\tau\lambda}(\lambda + A^s)^{-1}g^T(x)d\lambda,$$

with the understanding that $A_+(\lambda, D)^{-1}g^T$ in the definition of $(\lambda + A^s)^{-1}$ is calculated by using (2.16) with the integration over the line $\Im\xi = \gamma$.

If the terminal payoff grows as $e^{-\omega_- x}$ as $x \to +\infty$ or faster, we cannot choose γ with the desired properties, and the construction above must be modified. First, assume that $\lambda_- < -1$ (this condition is satisfied in many empirical studies of financial markets; it means that the rate of the exponential decay of the density of positive jumps is larger than 1), choose any $\gamma \in (\lambda_-, -1)$, and for $C > 0$ and $\theta \in (\pi/2, \pi)$, set $\Sigma_{C,\theta} = \{\lambda \mid |\lambda| \geq C,\ \arg\lambda \in [-\theta, \theta]\}$.

Lemma 4.7. *There exist $C, c > 0$ and $\theta \in (\pi/2, \pi)$ such that if $\Im\xi \in [\lambda_-, \lambda_+]$, and $\lambda \in \Sigma_{C,\theta}$, then (4.2) holds.*

Proof. Use (3.2) and modify the proof of Lemma 4.1 in an evident fashion. □

After that, for $\lambda \in \Sigma_{C,\theta}$, we can repeat word by word all the constructions and proofs above, with $\mathcal{L}_{C,\theta}$, the boundary of $\Sigma_{C,\theta}$, instead of $\mathcal{L}_\theta$, the boundary of Σ_θ; the representation theorem for analytic semigroups applies for the modified contour. In the result, we obtain

$$v_2(\tau, x) = (2\pi i)^{-1}\int_{\mathcal{L}_{C,\theta}} e^{\tau\lambda}(\lambda + A^s)^{-1}g^T(x)d\lambda, \tag{4.31}$$

with the understanding that $A_+(\lambda, D)^{-1}g^T$ is calculated by using (2.16) with the integration over the line $\Im\xi = \gamma$.

When u, v_1 and v_2 are found from (4.17), (4.30) and (4.31), we calculate $v = v_2 - v_1$, pass to the limit in (2.9), and find for $t < T$ and $x > 0$:

$$f(t,x) = u(x) + v_2(T-t,x) - v_1(T-t,x). \tag{4.32}$$

By using (4.32), we will calculate prices of barrier options with the lower barrier and the touch-and-out put option; prices of barrier options with the upper barrier and the touch-and-out call option are calculated by using an analog of (4.32), which can be obtained by making the change of variables $x \mapsto -x$ and replacing in all the constructions the signs "+" and "-" with "-" and "+", respectively.

5. Pricing of down-and-out options without the rebate

5.1. **The down-and-out call option without the rebate: the case of the strike less than or equal to the barrier.** Consider the down-and-out call with the barrier H and the strike price $K \le H$; we normalize H to 1. If during the life of the option the price of the stock reaches H or falls below it, the option expires worthless, but if the price stays above the barrier until the expiry date, T, an option owner obtains $g^T(X_T) = e^{X_T} - K$. Assuming that $\lambda_- < -1$ (the case $\lambda_- = -1$ will be considered in the end of the subsection), we choose $\gamma \in (\lambda_-, -1)$, $C > 0$ and $\theta \in (\pi/2, \pi)$ such that (4.2) holds on $\Sigma_{C,\theta}$, defined before Lemma 4.7, and apply (4.31). Since for $\Im\eta = \gamma$,

$$\widehat{g^T}(\eta) = \int_0^{+\infty} e^{-iz\eta}(e^z - K)dz = \frac{1}{i\eta - 1} - \frac{K}{i\eta} = \frac{1}{i(\eta+i)} - \frac{K}{i\eta},$$

we obtain, for $y > 0$:

$$A_+(\lambda, D)^{-1}g^T(y) = (2\pi)^{-1}\int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{iy\eta}A_+(\lambda,\eta)^{-1}\left[\frac{1}{i(\eta+i)} - \frac{K}{i\eta}\right]d\eta =$$

the integral converges absolutely due to (4.11), and the integrand is holomorphic and has two poles in the half-plane $\Im\eta > \lambda_-$, therefore, by choosing any $\gamma_1 > 0$ and applying the residue theorem, we continue

$$= e^y A_+(\lambda, -i)^{-1} - KA_+(\lambda, 0)^{-1} +$$

$$+(2\pi)^{-1}\int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1} e^{iy\eta}A_+(\lambda,\eta)^{-1}\left[\frac{1}{i(\eta+i)} - \frac{K}{i\eta}\right]d\eta.$$

In the last term, the integrand admits the bound via

$$Ce^{-\gamma_1 y}|\eta|^{-1-\kappa_+},$$

where $\kappa_+ > 0$ (see (4.11)), hence in the limit $\gamma_1 \to +\infty$ the integral vanishes. Thus, for $y > 0$

$$A_+(\lambda, D)^{-1}g^T(y) = e^y A_+(\lambda, -i)^{-1} - KA_+(\lambda, 0)^{-1}.$$

Apply the definition of PDO to $A_-(\lambda, D)^{-1}\theta_+ A_+(\lambda, D)^{-1}g^T$, substitute into (4.31) and then in (4.32), and take into account that the contributions u and v_1 coming from $g^r = 0$ are zero as well; the result is

$$f(t,x) = (2\pi i)^{-1}\int_{\mathcal{L}_{C,\theta}} e^{\lambda\tau}(2\pi i)^{-1}\times$$

$$\times\int_{-\infty+i\gamma}^{+\infty+i\gamma}\frac{\exp[ix\eta]}{A_-(\lambda,\eta)}\left[\frac{1}{A_+(\lambda,-i)(\eta+i)} - \frac{K}{A_+(\lambda,0)\eta}\right]d\eta d\lambda.$$

By simplifying, we obtain the pricing formula

$$f(t,x) = F_1(t,x) - KF_0(t,x), \tag{5.1}$$

where

$$F_\beta(t,x) := -(2\pi)^{-2}\int_{\mathcal{L}_{C,\theta}} e^{\lambda\tau}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\frac{\exp[ix\eta]d\eta d\lambda}{A_-(\lambda,\eta)A_+(\lambda,-i\beta)(\eta+i\beta)}, \tag{5.2}$$

for any $\gamma \in (\lambda_-, -\beta)$.

In the case of $\lambda_- = -1$, consider a portfolio of one down-and-out call option long and one share of the stock short; if $f_0(t, X_t)$ is the price of the portfolio at time t, then $f(t, X_t) = f_0(t, X_t) + \exp X_t$. Since f and e^x satisfy the generalized Black-Scholes equation, f_0 also does. The terminal condition for f_0 is $-K$, and its contribution to the price f_0 is $-KF_0(t,x)$. (This time the payoff is bounded, and hence any $\lambda_- < 0$ causes no problem, $\lambda_- = -1$ in particular). In addition, there appears non-zero "rebate": $g^r(x) = -g^{r,1}(x)$, where $g^{r,\beta}(x) := e^{\beta x}$. The contribution coming from the rebate $g^{r,\beta}$ is calculated in Section 6; it equals $u^\beta(x) - v_1^\beta(T-\tau, x)$, where u^β is given by (6.5), and v_1^β by (6.8). To sum up, if $\lambda_- = -1$, we replace (5.1) with

$$f(t,x) = e^x - u^1(x) + v_1^1(T-t,x) - KF_0(t,x). \tag{5.3}$$

5.2. **The down-and-out call option without the rebate: the case of the strike greater than the barrier.** The set-up is as in Subsection 5.1, but this time $K > H(=1)$. We have $g^T(X_T) = (e^X_T - K)_+$, and if $\lambda_- < -1$, we take $\gamma_1 \in (\lambda_-, -1)$ and calculate for η on the line $\Im\eta = \gamma_1$:

$$\widehat{g^T}(\eta) = \int_{\ln K}^{+\infty} e^{-iz\eta}(e^z - K)dz =$$

$$= K^{1-i\eta}/(-i\eta) - K^{1-i\eta}/(1-i\eta) = -K^{1-i\eta}/(\eta(\eta+i)).$$

Now set $q = \ln K$, choose γ so that

$$\lambda_- < \gamma < \gamma_1 < -1, \tag{5.4}$$

and calculate, for $x > 0$:

$$A_-(\lambda, D)^{-1}\theta_+ A_-(\lambda, D)^{-1}g^T(x) =$$

$$= (2\pi)^{-1}\int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{ix\xi}A_-(\lambda,\xi)^{-1}\int_0^{+\infty} e^{-iy\xi}(2\pi)^{-1}\int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1} e^{iy\eta}A_+(\lambda,\eta)^{-1}\widehat{g^T}(\eta)d\xi dy d\eta.$$

(The integral is understood as an iterated one). From (5.4) and (4.11), the inner double integral converges absolutely. By applying the Fubini theorem and integrating w.r.t. y first:

$$\int_0^{+\infty} e^{i(-\xi+\eta)y}dy = i(\eta-\xi)^{-1},$$

and then substituting in (4.31)–(4.32) and simplifying, we obtain

$$f(t,x) = \frac{K}{(2\pi)^3}\int_{\mathcal{L}_{C,\theta}}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1}\frac{\exp[\lambda\tau+ix\xi-i\eta q]d\eta d\xi d\lambda}{A_-(\lambda,\xi)(\xi-\eta)A_+(\lambda,\eta)\eta(\eta+i)}, \tag{5.5}$$

where γ,γ_1 satisfy (5.4). If $\lambda_- = -1$, we construct the same portfolio as in the end of Subsection 5.1; it has the same "rebate" $g^r(x) = -e^x$, and the terminal payoff $-K + g^{T0}$, where $g^{T0}(x) = (K-e^x)_+\mathbf{1}_{(0,+\infty)}(x)$ is the terminal payoff of the down-and-out put with the same strike, expiry date and barrier. Hence,

$$f_{\text{down, call}}(t,x) = f_{\text{down, put}}(t,x) + e^x - u^1(x) + v_1^1(T-t,x) - KF_0(t,x), \tag{5.6}$$

where F_0 is given by (5.2), u^β by (6.5), v_1^β by (6.8), and $f_{\text{down, put}}$ by (5.10) below. Notice that (5.6) is derived for the case of down-and-out options without the rebate; if the rebate is specified, the evident modification is needed.

5.3. **The down-and-out put option without the rebate: the case of the strike greater than the barrier.** Consider the down-and-out put with the barrier H and the strike price $K > H$; we normalize H to 1. If during the life of the option the price of the stock reaches H or falls below it, the option expires worthless, but if the price stays above the barrier until the expiry date, T, an option owner obtains $g^T(X_T) = \theta_+(X_T)(K-e^{X_T})_+$. We have $g^T = -g_1^T + g_2^T$, where $g_1^T(x) = \theta_+(x)(e^x-K)$ and $g_2^T(x) = \theta_+(x)(e^x-K)_+$. Since g_1^T and g_2^T are the payoffs in Subsections 5.1 and 5.2, respectively (the condition $K \le H$ in Subsection 5.1 is used only to conclude that the payoff is equal to $\theta_+(x)(e^x-K)$), we obtain in the case $\lambda_- < -1$, that the price $f(t,x)$ here is given as the difference of the RHS in (5.5) and (5.1), i.e. if the strike is greater than the barrier,

$$f_{\text{down, put}}(t,x) = f_{\text{down, call}}(t,x) - F_1(t,x) + KF_0(t,x). \tag{5.7}$$

Thus, (5.7) is an analog of the put-call parity for the down-and-out puts and calls, where $e^x - K$ is replaced with $F_1(t,x) - KF_0(t,x)$, the price of the down-and-out option with the barrier $H = 1$, zero rebate and the terminal payoff $e^{X_T} - K$. By substituting (5.1) and (5.5) in the RHS of (5.7), we obtain the pricing formula for the put.

In the case $\lambda_- = -1$, (5.3) must be used instead of (5.1), but the analog of (5.5) in the case $\lambda_- = -1$ is (5.6), which uses the price of the put. Hence, we need to derive the formula for the latter independently.

Set $q = \ln K$. For the put, the Fourier transform

$$\widehat{g^T}(\eta) = \int_0^{\ln K} e^{-iz\eta}(K-e^z)dz = \tag{5.8}$$

$$= (K^{1-i\eta} - K)/(-i\eta) - (K^{1-i\eta} - 1)/(1 - i\eta)$$

is holomorphic on $\mathbf{C}$. Choose γ and γ_1 so that

$$\lambda_- < \gamma < \gamma_1 < 0, \tag{5.9}$$

calculate, for $x > 0$,

$$(\lambda + A^s)^{-1} g^T(x) = A_-(\lambda, D)^{-1} \theta_+ A_+(\lambda, D)^{-1} g^T(x),$$

as in Subsection 5.2, and substitute into (4.30); the result is

$$f_{\text{down, put}}(t, x) = (2\pi)^{-3} \int_{\mathcal{L}_{C,\theta}} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1} \frac{\exp[\lambda\tau + ix\xi] \widehat{g^T}(\eta) d\eta d\xi d\lambda}{A_-(\lambda, \xi)(\eta - \xi) A_+(\lambda, \eta)}, \tag{5.10}$$

where $\widehat{g^T}$ is given by (5.8), and γ, γ_1 satisfy (5.9).

6. Down-and-out options with the rebate and touch-and-out options

6.1. **The down-and-out calls and puts: the case of exponentially decaying or constant rebate.** Now we assume that when the price of the stock reaches the barrier or falls below it, an option owner is entitled to the rebate $g^r(X_T) = g_0^r e^{\beta X_T}$, where $g_0^r > 0$, $\beta \geq 0$. It follows from (4.32), that in all cases of down-and-out options considered in Section 5, we need to add the same contribution $u(x) - v_1(T - t, x)$ coming from the rebate. To calculate it, we need the following lemma which will be proved in Appendix.

Lemma 6.1. *For any $[\omega'_-, \omega'_+] \subset (\omega_-, \omega_+)$, any $s = 0, 1, \dots$ and any $\epsilon > 0$, there exists $C_{s\epsilon} = C_{s\epsilon}(\omega'_-, \omega'_+)$ such that*

a) for all ξ in the half-plane $\Im\xi \geq \omega'_-$,

$$|A_+^{(s)}(0, \xi)| \leq C_{s\epsilon} \langle\xi\rangle^{\kappa_+ - 1 + \epsilon}; \tag{6.1}$$

b) for all ξ in the half-plane $\Im\xi \leq \omega'_+$,

$$|A_-^{(s)}(0, \xi)| \leq C_{s\epsilon} \langle\xi\rangle^{\kappa_- - 1 + \epsilon}. \tag{6.2}$$

Now we calculate, for a model function $g^{r,\beta}(x) := \mathbf{1}_{(-\infty,0)}(x) e^{\beta x}$, where $\beta > 0$, for $y < 0$,

$$A_-(0, D) g^{r,\beta}(y) =$$

$$= (2\pi)^{-1} \int_{-\infty}^{+\infty} d\eta e^{iy\eta} A_-(0, \eta) \int_{-\infty}^{0} e^{-iz\eta} e^{\beta z} dz =$$

$$= (2\pi)^{-1} \int_{-\infty}^{+\infty} d\eta e^{iy\eta} A_-(0, \eta) (\beta - i\eta)^{-1} =$$

we want to apply (6.2) but if $\lambda_+ = 0$ and $\omega_+ = 0$, we cannot apply it on the real axis, so we take $\sigma \in (-\beta, 0)$, and shift the line of integration

$$= (2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} d\eta e^{iy\eta} A_-(0, \eta) (\beta - i\eta)^{-1}.$$

(This equality can be justified in the sense of generalized functions). Assume that $\nu < 2$ and hence, $\kappa_- \in (0,1)$ (the case $\nu = 2$ and $\kappa_- = 1$ will be considered later). Then on the strength of (6.2), which we apply with $\epsilon > 0$ such that $\kappa_- - 1 + \epsilon < 0$, we can integrate by part by using $e^{iy\eta} = -iy^{-1}\partial_\eta(e^{iy\eta})$, show that the integral can be understood as

$$\lim_{M,L\to+\infty} \int_{-M+i\sigma}^{L+i\sigma}$$

and the residue theorem can be applied to shift the line of integration in order to obtain, for any $\sigma_1 < -\beta$,

$$A_-(0,D)g^{r,\beta}(y) = A_-(0,-i\beta)e^{\beta y} + (2\pi)^{-1} \int_{-\infty+i\sigma_1}^{+\infty+i\sigma_1} d\eta e^{iy\eta} A_-(0,\eta)(\beta - i\eta)^{-1}.$$

By integrating by part in the last integral, we obtain the integrand, which admits a bound via $C_\epsilon e^{-\sigma_1 y}\langle\eta\rangle^{-2+\kappa_- +\epsilon}$. Hence, we can pass to the limit $\sigma_1 \to -\infty$, and show that the last integral is zero. Thus, the result is

$$A_-(0,D)g^{r,\beta}(y) = A_-(0,-i\beta)g^{r,\beta}(y). \tag{6.3}$$

If $\nu = 2$ and $\kappa_- = 1$, we can use (3.2), analyze the construction of $A_-(0,\xi)$, and show that

$$A_-(0,D) = -iD + A'_-(0,D), \tag{6.4}$$

where $A'_-(0,\xi)$ satisfies the same estimates as $A_-(0,\xi)$ in the case $\nu < 2$, with some $\kappa_-' < 1$. Hence, (6.3) holds for A'_-. Clearly, $-iDe^{\beta y} = -i(-i\beta)e^{\beta y}$, and from (6.4) we conclude that (6.3) holds for A_- as well.

By using (6.3), we obtain for u defined from (4.17) with the model rebate, $g^{r,\beta}$,

$$u^\beta(y) = \frac{A_-(0,-i\beta)}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp[iy\xi]d\xi}{A_-(0,\xi)(\beta - i\xi)}. \tag{6.5}$$

To obtain an answer for the case $\beta = 0$, we notice that since for any $x < 0$, $e^{\beta x} \uparrow 1$ as $\beta \downarrow 0$, it is the limit of the RHS in (6.5), as $\beta \downarrow 0$:

$$u^0(y) = \lim_{\beta\to+0} \frac{A_-(0,-i\beta)}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp[iy\xi]d\xi}{A_-(0,\xi)(\beta - i\xi)}.$$

To calculate the limit, we assume that there exist $\rho > 0$ such that as $\xi \to 0$ in the lower half-plane $\Im\xi \le 0$,

$$\psi^{\mathbf{Q}}(\xi) = O(|\xi|)^\rho. \tag{6.6}$$

Certainly, if 0 is inside the strip $\Im\xi \in (\lambda_-, \lambda_+)$, (6.6) is satisfied for any RLPE; for model classes of RLPE, we can check it by inspection even when 0 is on the boundary of the strip. ¿From (4.11) and (4.10), it follows that $A_-(0,\xi)$ also satisfies (6.6). Since $A_-(0,\xi)$ is holomorphic in the lower half-plane, continuous up to the boundary of the half-plane, and satisfies the estimate (4.12), we can use the residue theorem and obtain

$$u^0(y) = A_-(0,0)\left[\frac{1}{2}(A_-(0,0))^{-1} + \text{v.p.}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp[iy\xi]d\xi}{A_-(0,\xi)(-i\xi)}\right].$$

By simplifying, we arrive at

$$u^0(y) = \frac{1}{2} + \frac{A_-(0,0)i}{2\pi}\int_0^{+\infty}\left[\frac{\exp[iy\xi]}{A_-(0,\xi)} - \frac{\exp[-iy\xi]}{A_-(0,-\xi)}\right]\xi^{-1}d\xi. \tag{6.7}$$

By multiplying (6.5) or (6.7) (depending on β: is it positive or 0) by g_0^r, we obtain u. It remains to substitute u into (4.30). Since u is bounded, we take γ and γ_1 satisfying (5.9), and then exactly the same calculations as in Subsection 5.2 give

$$v_1(\tau,x) = (2\pi)^{-3}\int_{\mathcal{L}_{C,\theta}}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1}\frac{\exp[\lambda\tau+ix\xi]\hat{u}(\eta)d\eta d\xi d\lambda}{A_-(\lambda,\xi)(\eta-\xi)A_+(\lambda,\eta)}$$

(cf. (5.5)). If $\beta > 0$, we can take $\gamma_1 \in (-\beta, 0)$, and then from (6.5), for η on the line $\Im\eta = \gamma_1$, calculate $\hat{u}^\beta(\eta) = g_0^r A_-(0,-i\beta)/[A_-(0,\eta)(\beta - i\eta)]$, and the corresponding $v_1 = v_1^\beta$:

$$v_1^\beta(\tau,x) = \frac{1}{(2\pi)^3}\int_{\mathcal{L}_{C,\theta}}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1}\frac{\exp[\lambda\tau+ix\xi]A_-(0,-i\beta)d\eta d\xi d\lambda}{A_-(\lambda,\xi)(\eta-\xi)A_+(\lambda,\eta)A_-(0,\eta)(\beta-i\eta)}. \tag{6.8}$$

Hence, from (6.8) and (6.5), (4.32) we conclude that the price of the option with the rebate is

$$f_r(t,x) = f(t,x) + g_0^r[u^\beta(x) - v_1^\beta(T-t,x)], \tag{6.9}$$

where f is the price of the corresponding option without the rebate, u^β is given by (6.5), and v_1^β by (6.8).

A formula for the case $\beta = 0$ can be obtained by passing to the limit $\beta \downarrow 0$ in (6.8) similarly to (6.7); if $\lambda_+ > 0$, we can obtain instead of the complicated formula (6.7) and its analogue for v_1^0, simpler formulas. Namely, if $\lambda_+ > 0$, then $\omega_+ > 0$ as well, and we choose γ, γ_1 satisfying

$$\lambda_- < \gamma < \gamma_1 < \omega_+, \quad \gamma_1 > 0, \tag{6.10}$$

instead of (5.9), and obtain (6.9) with

$$u^0(y) = \frac{A_-(0,0)}{-2\pi i}\int_{-\infty+i\gamma_1}^{+\infty+i\gamma_1}\frac{\exp[iy\eta]d\eta}{A_-(0,\eta)\eta}, \tag{6.11}$$

and v_1^0 given by (6.8) with $\beta = 0$ and γ, γ_1 satisfying (6.10).

6.2. **Touch-and-out put option.** To make comparison with the results above easier, we denote the strike price by H, and without loss of generality, we assume that $H = 1$, so that $h = \ln H = 0$. If at any moment t up to the expiry date, T, the price of the stock, S_t, reaches H or falls below it, an option owner can exercise the option and obtain 1. Clearly, it is optimal to exercise the option the first instant when $S_t \leq H$ or equivalently, $X_t \leq 0$. This translates into a boundary condition for $f(t, X_t)$, the price of the touch-and-out put option:

$$f(t,x) = 1, \quad t \leq T, x \leq 0.$$

If at the expiry date the price of the stock satisfies $S_T > H$, the option expires worthless, and hence the terminal condition is

$$f(T,x) = 0, \quad x > 0.$$

In a region $t < T, x > 0$, f obeys the generalized Black-Scholes equation.

This means that f is exactly the contribution from the rebate $g^r(x) = 1$, $x \le 0$, computed above, and hence, if $\lambda_+ > 0$, say,

$$f(t,x) = u^0(x) - v_1^0(T-t,x),$$

where u^0 is given by (6.11), v_1^0 is given by (6.8) with $\beta = 0$ and γ, γ_1 satisfy (6.10).

6.3. **Up-and-in options.** As in the gaussian case, the standard no-arbitrage considerations show that the price of the up-and-in call (put) equals the price of the European call (put) minus the price of the down-and-out call (put) with the same expiry date, strike price and barrier.

7. Up-and-out barrier options and the touch-and-out call option

We start with a general remark, which allows one to obtain the pricing formula for any up-and-out option from the (already obtained) pricing formula for the corresponding down-and-out option, under a different process and measure. The correspondence is established as follows.

Let X be a Lévy process under EMM $\mathbf{Q}$, let $\psi^{\mathbf{Q}}$ be its characteristic exponent, and consider a contingent claim with the barrier H normalized to 1, the terminal payoff g^T, and the rebate g^r. Notice that now boundary conditions for $f(t, X_t)$, the price of an up-and-out option or the touch-and-out call option are

$$f(T,x) = g^T(x), \quad x < 0;$$

$$f(t,x) = g^r(x), \quad x \ge 0,\ t \le T.$$

Set $\mathcal{C} = [0,T) \times (-\infty, 0)$, $\tilde{E} = [0,T] \times \mathbf{R}$, and $\tilde{E}^0 := \tilde{E} \setminus \mathcal{C}$; $\mathcal{C}$ and $\tilde{E}^0$ here are reflections of $\mathcal{C}$ and $\tilde{E}^0$ in Section 2 w.r.t. the axis $x = 0$.

Let $\tilde{X}_t = (t, X_t)$ be the two-dimensional process on $\tilde{E}$, and $\tilde{g}$ be the generalized payoff constructed from g^r and g^T by analogy with Section 2. Let τ_0 be the hitting time of $\tilde{E}$; then

$$f(0,x) = E^{\mathbf{Q}}\left[\int_0^{+\infty} e^{-rt} r\tilde{g}(\tilde{X}_{t\wedge\tau_0})dt \mid \tilde{X}_0 = (0,x)\right]. \tag{7.1}$$

Introduce the new process $X' = -X$, measure $\mathbf{Q}'$ by

$$E_x^{\mathbf{Q}'}[u(X'_t)] = E_{-x}^{\mathbf{Q}}[e^{X_t} u(-X_t)], \tag{7.2}$$

sets $\mathcal{C}' = [0,T) \times (0,+\infty)$ and $\tilde{E}^{0\prime} = \tilde{E} \setminus \mathcal{C}'$, and functions

$$G^r(x) = e^x g^r(-x),\ G^T(x) = e^x g^T(-x). \tag{7.3}$$

It is straightforward to check that if $\mathbf{Q}$ is an EMM for the market of the riskless bond with the rate of return r and the stock $S_t = \exp X_t$, then $\mathbf{Q}'$ is an EMM for the market of the riskless bond with the rate of return r and the stock $S'_t = \exp X'_t$.

Let τ'_0 be the hitting time of $\tilde{E}^{0\prime}$, and $\tilde{G}$ be defined by G^r and G^T as in Section 2.

By making the change of variables $x \mapsto -x$ in (7.1), we obtain, for $x > 0$:

$$f(0,-x) = E^{\mathbf{Q}}\left[\int_0^{+\infty} e^{-rt} r\tilde{g}(\tilde{X}_{t\wedge\tau_0})dt \mid \tilde{X}_0 = (0,-x)\right] =$$

$$= E^{\mathbf{Q}}\left[\int_0^{+\infty} e^{-rt}e^{X_t}\left(e^{X'_t} r\tilde{g}(-\tilde{X}'_{t\wedge\tau'_0})\right) dt \mid \tilde{X}_0 = (0,-x)\right] =$$

$$= E^{\mathbf{Q}'}\left[\int_0^{+\infty} e^{-rt} r\tilde{G}(\tilde{X}'_{t\wedge\tau'_0})dt \;|\tilde{X}'_0 = (0,x)\right].$$

Denote by $f(\mathbf{Q}, g^r, g^T; t, x)$ the price of the contingent claim with the expiry date T, barrier $H = 1$, the rebate g^r and the terminal payoff g^T, at time t, conditioned on $X_t = x$; X is a Lévy process under EMM $\mathbf{Q}$. We have proved that

$$f(\mathbf{Q}, g^r, g^T; t, x) = f(\mathbf{Q}', G^r, G^T; t, -x) \tag{7.4}$$

By using (7.4) and formulas of Sections 5–6, we can obtain pricing formulas for up-and-out options and the touch-and-out call option.

Formally, we must use formulas in Sections 5–6 with factors $A'_\pm(\lambda, \xi)$, calculated for $\lambda + r + \psi^{\mathbf{Q}'}(\xi)$ instead of $\lambda + r + \psi^{\mathbf{Q}}(\xi)$ but we can use the ready formulas for $\lambda + r + \psi^{\mathbf{Q}}(\xi)$ by using the following observations.

¿From (7.2),

$$e^{-t\psi^{\mathbf{Q}'}(\xi)} = E^{\mathbf{Q}'}[e^{i\xi X'_t}] = E^{\mathbf{Q}}[e^{X_t - i\xi X_t}] = e^{-t\psi^{\mathbf{Q}}(-\xi - i)},$$

hence,

$$\psi^{\mathbf{Q}'}(\xi) = \psi^{\mathbf{Q}}(-\xi - i).$$

We see that $\psi^{\mathbf{Q}'}$ satisfies the conditions (3.1)–(3.3) on the strip $[-\lambda_+ - 1, -\lambda_- - 1]$, where $-\lambda_+ - 1 \le -1 < 0 \le -\lambda_- - 1$, and we may set

$$A'_\pm(\lambda, \xi) = A_\mp(\lambda, -\xi - i). \tag{7.5}$$

As a simple example, for the up-and-out put option with the strike $K > H = 1$, without the rebate, $g^T(x) = \mathbf{1}_{(-\infty,0)}(K - e^x)$, $g^r = 0$, and from (7.3), $G^T(x) = K\mathbf{1}_{(0,+\infty)}(e^x - K^{-1})$ and $g^r = 0$. This is K times the terminal payoff for the down-and-out call option with the strike $K^{-1} < 1$, and from (5.1)–(5.2) and (7.4)–(7.5), we obtain

$$f(t,x) = (2\pi)^{-2}\int_{\mathcal{L}_{C,\theta}}\int_{-\infty+i\gamma}^{+\infty+i\gamma} \frac{\exp[ix\eta]}{A_+(\lambda, -\eta - i)}\left[\frac{1}{A_-(\lambda, -i)\eta} - \frac{K}{A_-(\lambda, 0)(\eta + i)}\right] d\eta d\lambda, \tag{7.6}$$

for any $\gamma \in (-\lambda_+ - 1, -1)$. Clearly, (7.6) is applicable if $\lambda_+ > 1$; in many empirical studies of real financial markets this condition is satisfied. If one needs the result for the case $\lambda_+ = 0$, one can either use the limiting procedure described in Sections 4 and 6 or, if the parametrized family of processes, for which the given X is a member, is available, one can simply calculate the limit of the RHS in (7.6), as $\lambda_+ \downarrow 0$.

8. Conclusion

We suggested a general procedure of the computation of the price of a contingent claim for Lévy processes, and applied it to barrier options and touch-and-out options under regular Lévy processes of exponential type. The procedure is based on the interplay between two limiting procedures applied to the price $f(t, X_t)$ of the contingent claim. We represent f as the resolvent $\tilde{R}r\tilde{g}$, where the terminal payoff $\tilde{g}$ is understood in the generalized sense, as the terminal value of $f(\tilde{X}_t)$, and $\tilde{X}_t = (t, X_t)$ is a process on the state space $[0, T] \times \mathbf{R}$.

The first limiting procedure is in the stochastic integral $f = \tilde{R}r\tilde{g}$: we replace $\tilde{g}$ by smooth functions $\tilde{g}_n$ with the compact support such that $\tilde{g}_n \uparrow \tilde{g}$ pointwise. This allows one to reduce the problem of the computation of f to the problem of the computation of $f_n = \tilde{R}r\tilde{g}_n$. $\tilde{g}_n$ being regular, we can use the connection between the resolvent and the infinitesimal generator of the process, and look for f_n as the solution of the corresponding boundary problem for the generalized Black-Scholes equation (Section 2). The latter being a pseudo-differential one, we apply the standard tools in the theory of boundary problems for pseudo-differential equations, namely, theorems on the action of PDO in the scale of the Sobolev spaces in the whole space (here, line) and in a half-space (here, half-line) – this is Section 3 – and the Wiener-Hopf factorization method and the representation theorem for analytical semigroups (Section 4). The formulas for f_n being found, we use the second limiting procedure, in the sense of the topology of appropriate Sobolev spaces, to show that the limit exists in the sense of the generalized functions, and is given by the same explicit formulas (Section 4). By inspection, we see that the limit is a continuous function, hence it coincides with f.

Thus, we have an explicit formula for f but in terms of the action of PDO, which are defined as oscillatory integrals. To simplify the analytical expression for f, in Sections 5–6, we use the integration by part in oscillatory integrals, which define the action of PDO, and the simplest tools of Complex Analysis: the Residue Formula and the Cauchy theorem. We find explicit formulas for down-and-out barrier options without the rebate, with exponentially decaying rebate and for the case of the double barrier, and for the touch-and-out call option. Further simplifications (from the point of view of the numerical calculations, not the length of the resulting formulas) are possible, but they are much more involved.

In Section 7 we introduce a general procedure of obtaining formulas for contingent claims with the upper barrier from formulas for options with the lower barrier. As an illustration, we write down an explicit formula for the up-and-out put option; the formulas for other types of up-and-out options can be written also quite easily.

9. Appendix

Proof of Lemma 4.3. We prove (4.6); (4.7) is proven similarly. By making an appropriate change of variables, we may assume that $\sigma > 0 = \sigma_1 > \omega_-$.

By using (3.2), (3.3) and (4.2), we easily obtain the following estimates

$$|B(\lambda, \eta)| \le C; \tag{9.1}$$

$$|B(\lambda, \eta) - 1| \le C_1(1 + |\lambda| + |\eta|^{\nu'})/(1 + |\lambda| + |\eta|^{\nu}); \tag{9.2}$$

$$|\partial_\eta B(\lambda, \eta)/B(\lambda, \eta)| \le C_2(1 + |\lambda|^{1/\nu} + |\eta|)^{-1}, \tag{9.3}$$

where C, C_1 and C_2 are independent of $\lambda \in \Sigma_\theta$ and η in a strip $\Im\eta \in [\omega_-, \omega_+]$, as well as all constants below. Set $K = (|\lambda| + 1)^{1/\nu}$, and for each pair (λ, ξ), introduce intervals $J_j \subset \mathbf{R}$:

$$J_1 = \{\eta \mid |\eta - \xi| \le K\}, \quad J_2 = \{\eta \mid |\eta - \xi| > K,\ |\eta| \le K\},$$
$$J_3 = \{\eta \mid |\eta - \xi| \ge |\eta|,\ |\eta| > K\}, \quad J_4 = \{\eta \mid K < |\eta - \xi| < |\eta|,\ |\eta| > K\}.$$

By using the mean value theorem and (9.3), we obtain

$$\frac{b(\lambda, \eta)}{\xi + i\sigma - \eta} = \frac{b(\lambda, \xi)}{\xi + i\sigma - \eta} + R(\lambda, \xi, \eta, \sigma), \tag{9.4}$$

where

$$|R(\lambda, \xi, \eta, \sigma)| \le C_3(1 + |\lambda|)^{-1/\nu}. \tag{9.5}$$

Since

$$\left| \int_{-K}^{K} \frac{d\eta}{i\sigma - \eta} \right| = \left| \ln \frac{-K - i\sigma}{K - i\sigma} \right| \le 2\pi,$$

we deduce from (9.4)–(9.5) and (9.1)

$$\left| \int_{J_1} \frac{b(\lambda, \eta) d\eta}{\xi + i\sigma - \eta} \right| \le 2\pi \ln C + C_3 \int_{|\eta - \xi| \le K} (1 + |\lambda|)^{-1/\nu} d\eta = C_4. \tag{9.6}$$

To prove the following estimate, only (9.1) is needed:

$$\left| \int_{J_2} \frac{b(\lambda, \eta) d\eta}{\xi + i\sigma - \eta} \right| \le C_5 \int_{|\eta| \le K} (1 + |\lambda|)^{-1/\nu} d\eta = C_6. \tag{9.7}$$

Further, we infer from (9.2) that b admits an estimate of the same form as $B - 1$, and using this estimate on J_3, we obtain

$$\left| \int_{J_3} \frac{b(\lambda, \eta) d\eta}{\xi + i\sigma - \eta} \right| \le C_7 \int_{|\eta| \ge K} \frac{1 + |\lambda| + |\eta|^{\nu'}}{|\eta|(1 + |\lambda| + |\eta|^{\nu})} d\eta. \tag{9.8}$$

By changing variables $\eta = K\eta'$, we see that the RHS in (9.8) is bounded uniformly in $\lambda \in \Sigma_\theta$, $\xi \in \mathbf{R}, \sigma > 0$.

Since $\nu' \in [0, \nu)$, a function

$$f(s) = (1 + |\lambda| + s^{\nu'})/(1 + |\lambda| + s^{\nu})$$

is decreasing on $[0, +\infty)$, and therefore, we deduce from (9.2) an estimate, for $\eta \in J_4$,

$$|b(\lambda, \eta)| \le C_8(1 + |\lambda| + |\xi - \eta|^{\nu'})/(1 + |\lambda| + |\xi - \eta|^{\nu}). \tag{9.9}$$

¿From (9.9),

$$\left| \int_{J_4} \frac{b(\lambda, \eta) d\eta}{\xi + i\sigma - \eta} \right| \le C_8 \int_{|\xi - \eta| \ge K} \frac{1 + |\lambda| + |\xi - \eta|^{\nu'}}{|\xi - \eta|(1 + |\lambda| + |\xi - \eta|^{\nu})} dl, \tag{9.10}$$

and the change of variables $\eta = \xi + K\eta'$ shows that the RHS in (9.10) is bounded uniformly in $\lambda \in \Sigma_\theta$.

By gathering bounds (9.6)–(9.8) and (9.10), we obtain (4.6).

Lemma 4.3 has been proved.

Proof of Lemma 6.1. We prove (6.1); (6.2) is proved similarly. Since $(\epsilon_0 - i\xi)^{\kappa_+}$ satisfies (6.1), and $A_+(0, \xi) = d(\epsilon_0 - i\xi)^{\kappa_+} \exp b_+(0, \xi)$, it suffices to prove that for any $s = 1, 2, \dots$ and $\epsilon > 0$,

$$|b_+^{(s)}(0, \xi)| \le C_{s\epsilon} \langle \xi \rangle^{-1+\epsilon}. \tag{9.11}$$

It follows from (3.3), (4.2), (4.4) and (4.5), that there exists C such that for all η in a strip $\Im\eta \in [\omega_-, \omega_+]$,

$$|b(0, \eta)| = |\partial_\eta B(0, \eta)/B(0, \eta)| \le C(1 + |\eta|)^{-1},$$

and differentiating under the integral sign in (4.6), and then integrating by part, we obtain an estimate

$$|b_+^{(s)}(0, \xi)| \le C_{1s} \int_{-\infty}^{+\infty} (1 + |\eta|)^{-1}(1 + |\xi - \eta|)^{-s} d\eta. \tag{9.12}$$

Introduce $J_1 = \{\eta \in \mathbf{R} \mid |\xi| \le |\eta|/2\}$, $J_2 = \{\eta \in \mathbf{R} \mid |\eta|/2 \le |\xi| \le 2|\eta|\}$, $J_3 = \{\eta \in \mathbf{R} \mid |\xi| \ge 2|\eta|\}$. We have

$$\int_{J_1} (1 + |\eta|)^{-1}(1 + |\xi - \eta|)^{-s} d\eta \le$$

$$\le C\langle \xi \rangle^{-1+\epsilon} \int_{-\infty}^{+\infty} (1 + |\eta|)^{-1-\epsilon} d\eta \le C_{1\epsilon} \langle \xi \rangle^{-1+\epsilon},$$

$$\int_{J_2} (1 + |\eta|)^{-1}(1 + |\xi - \eta|)^{-s} d\eta \le$$

$$\le C\langle \xi \rangle^{-1+\epsilon} \int_{-\infty}^{+\infty} (1 + |\xi - \eta|)^{-1-\epsilon} d\eta \le C_{2\epsilon} \langle \xi \rangle^{-1+\epsilon},$$

and

$$\int_{J_3} (1 + |\eta|)^{-1}(1 + |\xi - \eta|)^{-s} d\eta \le$$

$$\le C\langle \xi \rangle^{-1+\epsilon} \int_{-\infty}^{+\infty} (1 + |\eta|)^{-1-\epsilon} d\eta \le C_{3\epsilon} \langle \xi \rangle^{-1+\epsilon}.$$

By gathering these three estimates, we obtain (9.12) and (9.11).

Lemma 6.1 has been proved.

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