FELLER PROCESSES OF NORMAL INVERSE GAUSSIAN TYPE

O.E. BARNDORFF-NIELSEN * AND S.Z. LEVENDORSKIĬ†

*MaPhySto[1], University of Aarhus, Ny Munkegade,
DK-8000 Aarhus, Denmark
e-mail: oebn@imf.au.dk

†Department of Mathematics, Rostov State University of Economics
B. Sadovaya 69, Rostov-on-Don, 344007, Russia
e-mail: leven@ns.rnd.runnet.ru

Abstract. We consider construction of Normal Inverse Gaussian (NIG) (and some related) Lévy processes from the probabilistic viewpoint and from the one of the theory of pseudo-differential operators, and then we introduce and analyse natural generalisations of these constructions. The resulting Feller processes are somewhat similar to the NIG Lévy process but may, for instance, possess mean-reverting features. Possible applications to Financial Mathematics are discussed, and approximations to solutions of corresponding generalisations of the Black-Scholes equation are derived.

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Contents

1. Introduction

The theory of Pseudo-Differential Operators (PDO) offers many possibilities for more refined modelling of the dynamics of financial markets and for the determination of corresponding financial procedures, in particular the pricing of derivatives. The present paper explores some of these possibilities in connection to processes of normal inverse Gaussian type.

Normal Inverse Gaussian processes, or NIG processes for short, is a term used here as a general nomer for a class of stochastic processes introduced and studied in Barndorff-Nielsen (1995, 1997, 1998a, 1998b) and Barndorff-Nielsen and Shephard (2001), see also Eberlein (1999), Prause (1999), Tompkins and Hubalek (2000), Barndorff-Nielsen and Prause (2001). As discussed in the papers cited and in references given there, the family of NIG (normal inverse Gaussian) distributions and the NIG processes, which are constructed around the NIG family, have been found to provide accurate modelling of a great variety of empirical findings in the physical sciences and in financial econometrics.

The simplest NIG process is the NIG Lévy process, definable as the subordination of Brownian motion with drift β by the inverse Gaussian subordinator with parameters δ and γ. After addition of a linear drift μt, the resulting subordinated process X_t has cumulant function

$$\ln \mathrm{E}\{e^{i\xi X_1}\} = i\mu\xi - \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}] \tag{1.1}$$

where $\alpha = \sqrt{\beta^2 + \gamma^2}$.

In the present paper, relying on the theory and techniques of Pseudo-Differential Operators, we introduce and study a new type of NIG processes, which are Feller processes obtained by allowing one or more of the parameters δ, β and γ in (1.1) to depend on the actual state x of the process.

NIG-like Feller processes having been constructed, we discuss (in Section 4) pricing of derivative securities under such processes. More specifically, we consider a natural analogue of the Black-Scholes equation for option pricing and develop approximate pricing formulae for European options.

The necessary background material on Pseudo-Differential Operators is summarised in the Appendix (Section 6).

In the future publications we plan to consider applications to pricing of interest rate derivative products. To treat the latter, special classes of pseudo-differential operators arising in the study of degenerate elliptic differential equations are needed (see the review Levendorskiĭ and Paneah (1994) and the monograph Levendorskiĭ (1993)).

2. Some preliminary considerations

2.1. Pseudo-differential approach to the construction of NIG Lévy processes. Let ψ be the characteristic exponent of a Lévy process X_t, i.e. $E[e^{i\xi X_t}] = e^{-t\psi(\xi)}$. By using the Lévy-Khintchine formula, one easily computes an action of the infinitesimal generator, L, of X_t on oscillating exponents:

$$Le^{ix\xi} = -\psi(\xi)e^{ix\xi},$$

and from the Fourier inversion formula, a formula for the action of L on a sufficiently regular function u follows:

$$(-L)u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\xi}\psi(\xi)\hat{u}(\xi)d\xi, \tag{2.1}$$

where $\hat{u}$ is the Fourier image of u. Equation (2.1) means that $-L$ is a pseudo-differential operator (PDO) with the symbol $\psi(\xi)$: one says that A is a pseudo-differential operator with the symbol $a(x,\xi)$ and writes $A = a(x, D)$ if A acts as

$$Au(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\xi}a(x,\xi)\hat{u}(\xi)d\xi \tag{2.2}$$

on functions from the space $\mathcal{S}(\mathbf{R}^n)$ of C^∞ functions rapidly decaying at infinity together with their derivatives. A pseudo-differential approach can be very useful because it provides many powerful techniques for solving various boundary problems for many classes of operators, and one of the aims of the paper is to give some examples of application of these techniques.

It is interesting to note that if one wants to construct a non-Gaussian process with semi-heavy (i.e. exponentially decaying) tails[2] of the density functions, whose generator looks as simple as possible and possesses the most tractable properties from the point of view of the theory of pseudo-differential operators, then one naturally guesses the formula for the infinitesimal operator of the NIG Lévy process. Here is a list of observations which naturally lead to NIG Lévy:

(i) the generator of the Brownian motion is $\frac{\sigma^2}{2}\Delta$, but the tails of a Gaussian distribution decay faster than an exponential function;

(ii) stable non-Gaussian Lévy processes have generators of the form $L = -\delta|-\Delta|^{\nu/2}$, where $\delta > 0$ and $\nu \in (0,2)$, and their symbols are non-smooth at the origin: $-\delta|\xi|^\nu$, which leads to polynomial decay of the tails of the density functions;

(iii) the tails of the density functions observed in financial markets, in turbulence and in many other fields of study (see references quoted above) usually have exponential decay; this means that the symbols of generators must be not only smooth but holomorphic in a strip of the form $\Im\xi \in (\lambda_-, \lambda_+)$, where $\lambda_- < 0 < \lambda_+$ (in the multi-dimensional case, in a tube domain $\Im\xi \in U$, where $U \in \mathbf{R}^n$ is an open set containing 0).

The natural candidate for a generator with the property (iii) is

$$L = -\delta[(\alpha^2 - \Delta)^{\nu/2} - \alpha^\nu], \tag{2.3}$$

where $\alpha > 0$, $\delta > 0$, and if we want an asymmetric version, we simply shift the strip:

$$L = -\delta[(\alpha^2 - (\beta + iD)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}], \tag{2.4}$$

where $\alpha > 0$, $\delta > 0$, $\alpha > |\beta|$, and $D = -i\partial$ is the standard notation in the theory of PDO; it has the symbol ξ (The last terms in (2.3) and (2.4) are needed to ensure that $L \cdot 1 = 0$, which is necessary for a process without killing). From results on subordinated processes (see Subsection 2.3), it follows that (2.3) and (2.4) really define generators of a Lévy process.

The simplest of the fractional powers is the square root, i.e. (2.3) and (2.4) with $\nu = 1$, and if we add a drift, (2.4) becomes the generator of NIG Lévy, cf. Section 1. Recall that the characteristic exponent of the $NIG(\alpha, \beta, \mu, \delta)$ Lévy process is determined by (1.1).[3]

Operators with symbols of the form (1.1) are nice from the point of view of the theory of pseudo-differential operators. The same is true of generators of many other Lévy processes which are used by various groups of researchers in empirical studies of financial markets. For the discussion of common features of these families of Lévy processes, see Boyarchenko

[2]Specifically, by semiheavy or exponentially decaying tails we mean that the probability distribution has a density which behaves, for $x \to \pm\infty$, as

$$const. \, |x|^{\rho_\pm} \exp(-\sigma_\pm |x|)$$

for some $\rho_+, \rho_- \in \mathbf{R}$ and $\sigma_+, \sigma_- > 0$.

[3]In Subsection 2.3 we shall briefly consider a class of Lévy processes, the normal tempered stable (NTS) processes, whose infinitesimal generators are of the form (2.4) for $\nu \in (0,2)$, thus generalising the NIG Lévy.

and Levendorskiǐ (1999, 2000b, 2000c), where a general class containing all these processes is defined. We shall refer to these as $RLPE$ processes (regular Lévy processes of exponential type).[4] The Lévy density of a $RLPE$ can be characterized by the polynomial singularity at the origin:

$$F(dx) \sim c|x|^{-\nu-1}dx, \quad x \to 0,$$

where $\nu \geq 0$, and exponential decay as $x \to \pm\infty$; ν is called the order of the $RLPE$. If $\nu > 0$, ν coincides with the order of the infinitesimal generator in the sense of the theory of PDO.

NIG Lévy processes are especially tractable since the symbols given by (1.1) belong to a basic class of *classical symbols* (see Hörmander (1985)) even having the additional property of being holomorphic in a strip (tube domain). Notice that in the monograph of Jacob (1996), where a pseudo-differential approach to the study of stochastic processes is strongly advocated, a class of PDO with non-smooth symbols (model examples being generators of Lévy stable processes) is considered; this leads to more difficult situations having much less relevance to applications in Financial Mathematics and Physics.[5]

2.2. **On possible approaches to construction of NIG-like Feller processes.** The aim of this paper is to consider generalizations of the constructions of NIG Lévy described above. In the probabilistic language, we want to consider Feller processes such that for any fixed x, the exponent ψ in the equality $E^x[e^{i\xi(X_t-x)}] = e^{-t\psi(x,\xi)}$ is a characteristic exponent of NIG Lévy type, but with one or more of the parameters depending on x; in the language of the theory of PDO, we want to consider processes whose generators are PDO with "non-constant" symbols, i.e. symbols depending not only on the dual variable ξ but on x as well. A natural way from the probabilistic viewpoint is to consider processes obtained by subordination from general diffusion processes; this leads to the necessity of calculation of the symbol of the generator. The simplest variant is to consider processes whose generators are PDO's with symbols $-a(x,\xi)$ given by the RHS in (1.1) with μ, δ, α and β depending on x:

$$a(x,\xi) = -i\mu(x)\xi + \delta(x)[(\alpha(x)^2 - (\beta(x)+i\xi)^2)^{1/2} - (\alpha(x)^2 - \beta(x)^2)^{1/2}], \tag{2.5}$$

and in the paper we consider and compare these constructions. One can try various types of x-dependence, fairly weird ones including; we consider relatively simple versions which may have useful applications.

[4] In the above-mentioned papers the term Generalized Truncated Lévy processes ($GTLP$) was suggested for the processes. However, we find that the name $GTLP$ is not very informative and choose instead to speak of regular Lévy processes of exponential type.

[5] There is one useful property which generators of NIG and other $RLPE$ processes fail to have: the transmission property, Boutet de Monvel (1971), or a weaker smoothness-in-a-half-space property, Eskin (1973), which simplifies the treatment of boundary value problems. This property ensures that a solution to a "sufficiently good" boundary value problem, e.g. the Dirichlet problem, with smooth data is smooth up to the boundary.

Recall the roles played by the steepness parameter, α, and the asymmetry parameter, β: $\alpha-\beta$ describes the rate of exponential decay of the right tail of the density function, and $\alpha+\beta$ describes the decay of the left tail. (The scale parameter, δ, plays essentially the same role as the (square root of) the variance in Gaussian models). In other words, the larger the value of $\alpha-\beta$ the smaller the probability of large positive jumps, and the larger the value of $\alpha+\beta$ the smaller the probability of large negative jumps. If we want to reproduce a mean-reverting effect then at high levels of x, we should have probability of large positive jumps smaller than that of negative jumps; at small levels of x the probability of large positive jumps should be larger than that of negative jumps. Thus, $(\alpha(x)-\beta(x))/(\alpha(x)+\beta(x))$ must be increasing. If we want to have a less volatile behaviour for small levels of x, then we may either decrease the scale parameter δ (which is in direct analogy with a standard device in Gaussian modelling) or increase α. For instance, one may expect that if for some $c>0, \rho>1$,

$$\alpha(x)+\beta(x) \geq cx^{-\rho}, \quad x>0,$$

then the trajectories of the process never reach 0 from above. Processes of this type are suitable for interest rate modelling purposes. We consider such processes in a subsequent publication.

In this paper, we consider the simpler case of bounded α, β, δ, μ, assuming that δ and $\alpha \pm \beta$ are bounded away from 0. In addition, we assume that these parameters are smooth and their derivatives are small. The simplest example which we have in mind, and which is capable of reproducing the mean-reverting effect, is the case of constant δ and α, and β given by

$$\beta(x) = -\frac{2\chi}{\pi}\arctan(\epsilon(x-x_0)) + \beta_0, \tag{2.6}$$

where $\epsilon>0, \chi>0$, and $|\beta_0 \pm \chi| < \alpha$. The derivatives of β admit estimates

$$|\beta^{(s)}(x)| \leq C_s\epsilon^s, \quad s=0,1,\dots, \tag{2.7}$$

where the C_s are independent of ϵ, and if we assume that ϵ is small, we can obtain asymptotic solutions in terms (roughly speaking) of a power series in ϵ.

Another example obtains when ϵ in (2.6) is not small but $\epsilon_1 = \chi/\alpha$ is small; in this case, ϵ_1 plays the part of ϵ above, and if both ϵ and ϵ_1 are small then an asymptotic solution in terms of a series in the still smaller parameter $\epsilon\epsilon_1$ is obtained.

By using a characterization theorem for certain operators satisfying the positive maximum principle (Courrége (1966b)) and the pseudo-differential operators technique, one can show that a general theorem due to Hille-Yosida-Ray (see Courrége (1966a), Ethier and Kurtz (1986), Jacob (1996)) is applicable to PDO with the symbol (2.5) and α, β, δ specified as above (see Subsection 3.1 for details), and therefore the PDO $-a(x,D)$ is the infinitesimal generator of a Feller semigroup on C_0, the space of continuous functions vanishing at infinity.

If a small parameter is present then, as we shall show, the two constructions of NIG-like Feller processes give processes with approximately equal generators.

2.3. **Subordination and TS and NTS Lévy processes.** A *subordinator* is a Lévy process T such that $T(t) \in (0, \infty)$ for all $t > 0$. For a subordinator T the function $\kappa(\xi) = -\ln \mathrm{E}\{\exp(-\xi T(1))\}$ is often referred to as the *Laplace exponent* of T.

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{[0,\infty)}, P)$, suppose that on this space there is defined a Lévy process X and a subordinator T, with X and T independent, and let $Y = X \circ T$, i.e. $Y_t = X_{T(t)}$. Then Y is also a Lévy process termed the subordination of X by T, and X is called the *subordinand*. In this case the characteristic exponents ϕ and ψ of Y and X are related by

$$\phi(\xi) = \kappa(\psi(\xi)) \tag{2.8}$$

In particular, if X is Brownian motion with drift β then ϕ is of the form

$$\phi(\xi) = \kappa(\frac{1}{2}\xi^2 - i\beta\xi) \tag{2.9}$$

Now, let $p(x; \nu, \delta)$ denote the probability density function of the positive $\nu/2$-stable law with cumulant transform $-\delta(2\theta)^{\nu/2}$, $0 < \nu < 2$, and let $p(x; \nu, \delta, \gamma)$ denote the exponentially tempered version of $p(x; \nu, \delta)$ defined by

$$p(x; \nu, \delta, \gamma) = e^{\delta\gamma^\nu} p(x; \nu, \delta) e^{-\frac{1}{2}\gamma^2 x} \tag{2.10}$$

The distribution with density (2.10) ($\nu \in (0, 2), \delta > 0, \gamma > 0$) will be referred to as a *tempered stable law* and we shall denote it by $TS(\nu, \delta, \gamma)$. The Lévy density of $TS(\nu, \delta, \gamma)$ is

$$u(x) = \delta \frac{\nu 2^{\nu/2-1}}{\Gamma(1-\nu/2)} x^{-1-\nu/2} e^{-\frac{1}{2}\gamma^2 x} \tag{2.11}$$

Next, let X denote a random variable of the form $X \overset{law}{=} \mu + \beta T + \sqrt{T}\varepsilon$ where $T \sim TS(\nu, \delta, \gamma)$, $\varepsilon \sim N(0, 1)$ and T is independent of ε. We then say that X follows the *normal tempered stable law* $NTS(\nu, \gamma, \beta, \mu, \delta)$.

An *NTS Lévy process* is a stochastic process Y such that Y_1 follows an NTS distribution, with parameters $\nu, \gamma, \beta, \mu, \delta$, say. By the above discussion, such a process is, except for a linear drift μt, the subordination of Brownian motion with drift β by the $TS(\nu, \delta, \gamma)$ subordinator T, i.e. the subordinator with $T(1)$ distributed according to $TS(\nu, \delta, \gamma)$. The process Y has then, by (2.9), characteristic exponent

$$\phi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}]; \tag{2.12}$$

Cf. the discussion at (2.4).

3. Constructions of NIG-like Feller process via Pseudo-Differential Operators

3.1. **Naive construction (NIG-type generators with state-dependent parameters).** We use the following standard notation. Set $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. For a multi-index

$\alpha \in \mathbf{Z}_+^n$, set

$$|\alpha| = \sum_j \alpha_j,\ \partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n},\ D^\alpha = (-i)^{|\alpha|}\partial^\alpha,\ \alpha! = \alpha_1!\cdots\alpha_n!,$$

and for multi-indices α, β and a function $a = a(x,\xi)$, let $a_{(\beta)}^{(\alpha)} := \partial_\xi^\alpha D_x^\beta a$.

Definition 3.1. Let $U \in \mathbf{R}^n$ be an open domain whose closure contains the origin, and let $m \in \mathbf{R}$.

We write $a \in S^m(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$ if the following two conditions are satisfied[6]

(i) $a \in C^\infty(\mathbf{R}_x^n \times \mathbf{R}_\xi^n)$, together with all its derivatives, admits the analytic continuation w.r.t. ξ into a tube domain $\Im\xi \in U$

(ii) for any multi-indices α, β, the derivative $a_{(\beta)}^{(\alpha)}$ admits the continuous extension up to the boundary of U, and satisfies estimates

$$|a_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}, \tag{3.1}$$

where the constants $C_{\alpha\beta}$ are independent of $(x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n + i\bar{U})$.

Since in many cases the holomorphicity of a is not required, we shall use this definition with $U = \emptyset$ by letting $\bar{U} = \{0\}$. For this U, we recover the standard Hörmander class $S_{1,0}^m(\mathbf{R}^n \times \mathbf{R}^n)$.

Definition 3.2. Let $m \in (0,2]$, $a \in S^m(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$ and, for each fixed $x \in \mathbf{R}^n$, suppose $a(x,\xi)$ is the characteristic exponent of a Lévy process. Then we write $a \in FLS^m(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$.

The following fact is standard in the theory of PDO.

Theorem 3.3. *Let (3.1) be valid on $\mathbf{R}^n \times \mathbf{R}^n$, i.e. $a \in S^m(\mathbf{R}^n \times (\mathbf{R}^n + i\{0\}))$.*

Then a PDO $a(x,D)$ maps $\mathcal{S}(\mathbf{R}^n)$ into itself, continuously.

The next fact is a variation of many similar statements in the theory of PDO; for the proof, see Appendix.

Theorem 3.4. *Let (3.1) be valid on $\mathbf{R}^n \times \mathbf{R}^n$, and let there exist $c > 0$ such that on $\mathbf{R}^n \times \mathbf{R}^n$*

$$\Re a(x,\xi) + 1 \ge c\langle\xi\rangle^m. \tag{3.2}$$

Then there exists $\lambda > 0$ such that $a(x,D) + \lambda : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ is invertible.

Theorem 3.5. *Let $m \in (0,2]$ and $a \in FLS^m(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$.*

Then a PDO $a(x,D)$ has a closed extension $(A, D(A)), D(A) \subset C_0(\mathbf{R}^n)$, such that $-A$ is the generator of a Feller semigroup on $C_0(\mathbf{R}^n)$.

Proof. By the theorem due to Hille-Yosida-Ray (see e.g. Courrége (1966a), Ethier and Kurtz (1986), Jacob (1996)) it is necessary and sufficient to verify the following three conditions:

(i) $\mathcal{D}(a(x,D)) := \mathcal{S}(\mathbf{R}^n)$ is dense in C_0;

(ii) $a(x,D)$ satisfies the positive maximum principle on $\mathcal{D}(a(x,D))$;

[6]The more standard notation would be $S_{1,0}^m(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$; we have omitted the lower indices to simplify the notation.

(iii) for some $\lambda > 0$, the range of $a(x, D) + \lambda$ is dense in $C_0(\mathbf{R}^n)$.

Since $\mathcal{S}(\mathbf{R}^n)$ is dense in $C_0(\mathbf{R}^n)$, (i) is fulfilled, and if we take Theorem 3.4 into account, we obtain (iii). Finally, (ii) is a special case of a theorem due to Courrége (1966b); for a pseudo-differential version of this result, see Jacob (1996). □

Notice that in Courrége (1966b) the smoothness of the symbol was not required, and an analogue of Theorem 3.5 can be established for much nastier symbols than here (see Jacob (1996)). We use the simplest conditions suitable for our purposes.

Definition 3.6. Let the conditions of Theorem 3.5 hold.

Then we call a Feller process X with generator $-A$ a Feller-Lévy process (on $\mathbf{R}^n$) of order m and exponential type $\bar{U}$ and write $X \in FLP^m(\mathbf{R}^n; \bar{U})$. If we do not want to specify the order m and/or U, we say that X is a regular Feller-Lévy process of exponential type.

If the symbol a of A is independent of x, we call the process X a Lévy process (on $\mathbf{R}^n$) of order m and exponential type $\bar{U}$; if we do not specify the order m and/or the exponential type $\bar{U}$, we say that X is a regular Lévy process of exponential type.

In the sequel, we identify $a(x, D)$ with its extension, A.

By choosing a particular Lévy process as a starting point, we can obtain special classes. In the following definition, we use NIG Lévy as a model, and we consider the simplest case possible; more involved versions can also be considered.

Definition 3.7. (NIG-like Feller processes in 1D) Let $\mu, \delta, \alpha, \beta \in C_b^\infty(\mathbf{R})$, δ and α be positive, μ and β real-valued, and let there exist $C, c > 0$ such that for all x,

$$\delta(x) > c, \quad \alpha(x) - |\beta(x)| > c, \quad c \le \mu(x) \le C. \tag{3.3}$$

Let a be defined by (2.5).

Then we call a Feller process X with the generator $-a(x, D)$ a NIG-like Feller process.

Notice that it is a process of order 1 and exponential type $[\lambda_-, \lambda_+]$ for any $\lambda_- < \lambda_+$ satisfying

$$\sup_x\{-\alpha(x) - \beta(x)\} < \lambda_- < 0 < \lambda_+ < \inf_x\{\alpha(x) - \beta(x)\}. \tag{3.4}$$

Condition (3.4) can be satisfied due to (3.3).

Definition 3.8. (Multi-dimensional NIG-like Feller processes) Let $\mu, \alpha, \beta \in C_b^\infty(\mathbf{R}^n; \mathbf{R}^n)$, $\delta \in C^\infty(\mathbf{R}^n; \mathbf{R}_{++})$, and let there exist $c > 0$ and open sets $U, V \subset \mathbf{R}^n$ such that $\{0\} \subset \bar{U} \subset V$; for all x, $\delta(x) > c$; and

$$(\alpha(x) - \beta(x) + i\xi, \alpha(x) + \beta(x) - i\xi) \notin \bar{\mathbf{R}}_-, \quad \forall\ (x, \xi) \in \mathbf{R}^n \times (\mathbf{R}^n + iV).$$

Here $(\cdot, \cdot)$ is a bilinear form in $\mathbf{C}^n$, which extends the standard scalar product in $\mathbf{R}^n$.

Let a be defined by

$$\begin{aligned} a(x, \xi) \ &= \ -i\mu(x)\xi + \delta(x)[(\alpha(x) - \beta(x) + i\xi, \alpha(x) + \beta(x) - i\xi)^{1/2} \\ &\qquad -(\alpha(x) - \beta(x), \alpha(x) - \beta(x))^{1/2}]. \end{aligned}$$

Then we call a Feller process X with the generator $-a(x, D)$ a NIG-like Feller process.

This is a process of order 1 and exponential type $\bar{U}$.

3.2. **Constructions via infinitesimal generators: subordination of semigroups of operators.** We shall use the following theorem due to Phillips (see Theorem 32.1 in Sato (1999)).

Theorem 3.9. *Let $\{Z_t : t \geq 0\}$ be a subordinator with Lévy measure ρ and drift β_0, and let λ^t denote the law of Z_t. Let $\{P_t : t \geq 0\}$ be a strongly continuous semigroup of linear operators on a Banach space B with infinitesimal generator L. Define*

$$Q_t f = \int_{[0,\infty)} P_s f \lambda^t(ds), \quad f \in B. \tag{3.5}$$

Then $\{Q_t : t \geq 0\}$ is a strongly continuous contraction semigroup of linear operators on B. Denote its infinitesimal generator by $-A$. Then $\mathfrak{D}(L)$ is the core of A, and

$$-Af = \beta_0 Lf + \int_{(0,\infty)} (P_s f - f)\rho(ds). \tag{3.6}$$

We apply Theorem 3.9 with P_t being the transition semigroup of a diffusion process with the infinitesimal operator L,

$$Lf(x) = \frac{1}{2}(\sigma^2(x)\partial, \partial)f(x) + (b(x), \partial)f(x) + c(x)f(x), \tag{3.7}$$

where $\sigma^2 \in C_b^\infty(\mathbf{R}^n; \mathrm{End}\mathbf{R}^n)$ is a positive definite (uniformly in x) symmetric matrix, $b \in C_b^\infty(\mathbf{R}^n; \mathbf{R}^n)$, $c \in C_b^\infty(\mathbf{R}^n)$, c non-positive; and with the subordinator having Laplace exponent $\kappa(u) = (d^2 + u)^{\nu/2} - d^\nu$, where $\nu \in (0, 2)$, and $d > 0$. This means, in particular, that $\beta_0 = 0$, and the Laplace exponent and the Lévy measure, ρ, are related by

$$\kappa(w) = \int_0^\infty (1 - e^{-ws})\rho(ds). \tag{3.8}$$

To show that A is a PDO and calculate its symbol, we notice that (3.6) can be written as

$$\begin{aligned} -Af &= \int_0^\infty (\exp(sL) - 1)\rho(ds) \\ &= \kappa(-L) = (d^2 - L)^{\nu/2} - d^\nu. \end{aligned} \tag{3.9}$$

Since $d > 0$ and $\Re L$ is a non-positive elliptic operator, $d^2 - \Re L$ is a positive-definite elliptic operator, and, therefore, fractional powers of $d^2 - L$ are well-defined. Moreover, in the theory of PDO there is a well-known result about complex powers of elliptic PDO which states that these powers are PDO, and an asymptotic expansion of the symbol is provided. For PDO on compact manifolds, the result is due to Seeley (1967); an analogue for rather general classes of PDO on $\mathbf{R}^n$ was obtained in Hyakawa and Kumano-go (1971) (see also the monograph Grubb (1996)); and in the case $\bar{U} = \{0\}$, the result in Grubb (1996) suffices for our simple class; in the case of a general U, a straightforward modification of constructions in Grubb (1996) is needed.

Notice, however, that the "NIG-generating" Laplace exponent (the case $\nu = 1$) is so simple that no general result on the asymptotic expansion of the symbol is required: one can easily calculate an asymptotic expansion of the symbol, $b = b(x, \xi)$, of the square root of an elliptic PDO $d^2 - L(x, D)$ with positive-definite real part, by using the Ansatz

$$b(x,\xi) \sim b_1(x,\xi) + b_0(x,\xi) + b_{-1}(x,\xi) + \cdots, \tag{3.10}$$

where $b_j \in S^j(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$, and the composition theorem, one of the basic tools of the theory of PDO. As the result, we obtain the following theorem.

Theorem 3.10. *Let $U \subset \mathbf{R}^n$ be an open set whose closure contains 0 and such that*

$$d^2 - \Re L(x,\xi) > 0 \quad \forall\ (x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}). \tag{3.11}$$

Then

a) $\kappa(-L) + d$ is a PDO with the symbol of the class $S^1(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$, and the symbol admits an asymptotic expansion (3.10) in the sense that for any $N > 0$,

$$r_N = b - \sum_{j=0}^{N-1} b_{1-j} \in S^{1-N}(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U})), \tag{3.12}$$

where

$$b_1(x,\xi) = (d^2 - L(x,\xi))^{1/2}; \tag{3.13}$$

$$b_0(x,\xi) = -\frac{1}{2} b_1(x,\xi)^{-1} \sum_{|\alpha|=1} b_1^{(\alpha)}(x,\xi) b_{1(\alpha)}(x,\xi); \tag{3.14}$$

b) if the coefficients of L depend on a small parameter $\epsilon > 0$ and for any s derivatives of order s admit an estimate via $C_s\epsilon^s$, where C_s is independent of ϵ, then uniformly in $\epsilon > 0$,

$$\epsilon^{j-1} b_j \in S^j(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U})), \quad \epsilon^{-N} r_N \in S^{1-N}(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U})); \tag{3.15}$$

c) let the coefficients of L be of the class $C_b^\infty(\mathbf{R}^n)$ and $d > 0$ be large; then (3.12)–(3.15) are valid with a small parameter d^{-1} instead of ϵ;

d) if the coefficients satisfy the condition in b) and d is a large parameter, then (3.12)–(3.15) are valid with a small parameter ϵ/d instead of ϵ.

For the proof and inductive formulae for $b_j, j < 0$, see Appendix.

Theorem 3.10 means, in particular, that NIG-like Feller processes constructed by subordination from diffusions with the uniformly elliptic $-L$, where L is the generator of the diffusion, are NIG-like processes in the sense of the "naive" definition. Moreover, if we use the symbol $a(x,\xi) = (d^2 - L(x,\xi))^{1/2} - d$ to construct a NIG-like process, and if the coefficients of L are slowly varying and/or d is large, we obtain a process whose generator differs insignificantly from $-\kappa(-L)$, the generator of the process obtained via subordination.

4. Applications for Financial Mathematics

4.1. **Generalised Black-Scholes equation for contingent claims under NIG-like Feller processes.** Let $r > 0$ be the riskless rate, let $S_t = \exp X_t$ be the price process of a stock, where X_t is a NIG-like Feller process with the generator $-a(x, D)$, and let $f(t, X_t)$ be the price of a contingent claim. In Boyarchenko and Levendorskiǐ (2000c) it is explained how to reduce the problem of the calculation of $f(t, X_t)$ to the corresponding boundary problem for a generalised Black-Scholes equation

$$(\partial_t + L - r))f(t, x) = 0,$$

where L is the infinitesimal generator; the reduction is valid for any strongly measurable strong Markov process. In our case, the generalised Black-Scholes equation (backward Kolmogorov equation) is

$$(\partial_t - (r + a(x, D_x))f(t, x) = 0. \tag{4.1}$$

Prices of many derivative securities can be found by solving corresponding boundary problems for the equation (4.1); for the case of regular Lévy processes of exponential type, when $a(x, D) = \psi(D)$ is a PDO with constant symbol, some of these problems have been solved in Boyarchenko and Levendorskiĭ (2000a, 2000b, 2000c) (for perpetual American options, barrier options and touch-and-out options).

4.2. **Pricing of European options under NIG-like Feller processes obtained via subordination.** Let T be the expiry date and $g(X_T)$ the terminal payoff. Let Q_t be a strongly continuous semigroup constructed as in Theorem 3.9. Then by solving equation (4.1) subject to the terminal condition $f(T, x) = g(x)$, we obtain

$$f(t, x) = e^{-\tau r}(Q_\tau g)(x),$$

for $\tau = T - t > 0$. By applying (3.5), we find

$$f(t, x) = e^{-\tau r} \int_{[0,\infty)} P_s g(x) \lambda^\tau(ds). \tag{4.2}$$

Thus, if P_s is the generator of a diffusion, we can obtain a pricing formula for the contingent claim by finding first $P_s g$, all $s > 0$, which differs by a factor e^{sr} from the price $\phi(T - s, x)$ of the contingent claim with the expiry date T and the same payoff g:

$$P_s g(x) = e^{sr} \phi(T - s, x),$$

under a diffusion process, and then integrating $e^{-r\tau} P_s g(x)$ w.r.t. the measure $\lambda^\tau(ds)$ to obtain

$$f(t, x) = \int_{[0,\infty)} e^{(s-\tau)r} \phi(T - s, x) \lambda^\tau(ds).$$

Recall that one should choose the parameters of the model so that the EMM-requirement

$$e^x = e^{-r\tau} \int_{[0,\infty)} P_s e^x \lambda^\tau(ds), \quad \forall \tau > 0 \tag{4.3}$$

be met.

4.3. **Pricing of European options under NIG-like Feller processes obtained by the naive approach.** The pricing formula is evident

$$f(t,x) = e^{-\tau r} Q_\tau g(x), \tag{4.4}$$

where Q_τ is the transition semigroup of a NIG-like Feller process with the generator $-a(x,D)$; but in order to apply it, we need to obtain an explicit formula for Q_τ. That can be derived from the representation theorem for analytic semigroups (see e.g. Yosida (1964)) similarly to Boyarchenko and Levendorskiĭ (2000c), where the transition semigroup was computed for the case of an operator of a boundary problem for a generator of NIG and some other Lévy processes, in order to compute the rational prices of barrier options and touch-and-out options.

In order that the RHS in (4.2) and (4.4) be finite, some regularity conditions on g are needed, and these conditions are related to the domain U in the definition of the generator of a NIG-like Feller process; for instance, in 1D when U is an interval (λ_-, λ_+), it suffices to assume that g is piecewise continuous and satisfies estimates

$$g(x) \le Ce^{-\omega_\pm x}, \quad \forall \pm x > 0, \tag{4.5}$$

with $\lambda_- < \omega_- < \omega_+ < \lambda_+$. In the case of European puts and calls, it amounts to the restriction on λ_-, λ_+: $\lambda_- < -1 < 0 < \lambda_+$. Due to the put-call parity, it suffices to calculate the price of the European put, for which $g(x) = (K - e^x)_+$, where K is the strike price. Thus, below, we discuss the pricing of the European put.

We consider a NIG-like Feller process in 1D with the generator $-a(x,D)$, where a is defined by (1.1), and $\mu, \delta, \alpha, \beta$ satisfy (3.3) and (3.4). The EMM-requirement reduces to

$$r + a(x, -i) = 0. \tag{4.6}$$

The construction of the transition semigroup starts with

Lemma 4.1. *Let $\mathcal{B}$ be either the Sobolev space $H^s(\mathbf{R})$, where $s \in \mathbf{R}$, or the Hölder space $C^s(\mathbf{R})$, where s is a positive non-integer.*

Then there exist $C_0 > 0$ and $\theta \in (\pi/2, \pi)$ such that if $arg\lambda \in [-\theta, \theta]$, then

$$\| \lambda(\lambda + r + a(x,D))^{-1} \|_{\mathcal{B}\to\mathcal{B}} \le C_0. \tag{4.7}$$

Proof. Fix large C and consider first the region $\Sigma_{C,\theta} = \{\lambda \mid |\lambda| \ge C, \ \arg\lambda \in [-\theta, \theta]\}$. Since

$$a(x,\xi) \sim -i\mu\xi + \delta|\xi|, \quad \text{as } \xi \to \pm\infty,$$

for C fixed, we can easily find $\theta \in (\pi/2, \pi)$ and C_1 such that for all $\lambda \in \Sigma_{C,\theta}$ and all $(x,\xi) \in \mathbf{R} \times (\mathbf{R} + i\{0\})$,

$$|\lambda(\lambda + r + a(x,\xi))^{-1}| \le C_1. \tag{4.8}$$

By using (4.8), and arguing as in the proof of Theorem 3.4 (cf. also Theorem 6.17), we obtain that (4.7) holds for $\lambda \in \Sigma_{C,\theta}$, with some C_0 independent of these λ.

To prove that for C fixed, there exist $\theta \in (\pi/2, \pi)$ such that (4.7) holds for λ satisfying $|\lambda| \le C$ and $\arg\lambda \in [-\theta, \theta]$ it suffices to notice that since $-a(x,D)$ is a generator of the Feller process, $r + \Re a(x,D)$ is positive-definite.

Remark 4.1. Since the boundedness theorem is valid for much more general spaces, the estimate (4.7) and the formulas and estimates below are also valid in these spaces.

The estimate (4.7) means that the representation theorem for the semigroup with the generator $-r-a(x,D)$ is applicable (see e.g. Yosida (1964)), and the following formula is valid:

$$\exp[-\tau(r+a(x,D))]g = e^{-r\tau}Q_\tau g$$

$$= (2\pi i)^{-1}\int_{\mathcal{L}_\theta} e^{\lambda\tau}(\lambda+r+a(x,D))^{-1}g d\lambda. \tag{4.9}$$

Here $\mathcal{L}_\theta$ is the contour $\lambda=\lambda(\sigma), -\infty<\sigma<+\infty$, where $\arg\lambda(\sigma)=-\theta, \sigma<0$, $\arg\lambda(\sigma)=\theta, \sigma>0$.

By using an asymptotic expansion of the symbol $R_\lambda(x,\xi)$ of the resolvent $(\lambda+r+a(x,D))^{-1}$:

$$R_\lambda(x,\xi) \sim (\lambda+r+a(x,\xi))^{-1}+\cdots \tag{4.10}$$

(for the proof and the full asymptotic expansion, see Appendix), we can compute the RHS in (4.9) in the form of a series. Since

$$(2\pi i)^{-1}\int_{\mathcal{L}_\theta} e^{\lambda\tau}(\lambda+r+a(x,\xi))^{-1}d\lambda = \exp[-\tau(r+a(x,\xi))],$$

the leading term in the asymptotic expansion of the RHS in (4.9) is

$$(2\pi)^{-1}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \exp[ix\xi-\tau(r+a(x,\xi))]\hat{g}(\xi)d\xi,$$

where $\sigma\in(\lambda_-,\lambda_+)$ is chosen so that $\hat{g}$ is well-defined on the line $\Im\xi=\sigma$.

Consider the put with the strike price normalized to 1: $K=1$. We must take $\sigma\in(0,\lambda_+)$, and compute

$$\begin{aligned}
\hat{g}(\xi) &= \int_{-\infty}^{+\infty} e^{-iy\xi}(1-e^y)_+dy \\
&= \int_{-\infty}^{0}(e^{-iy\xi}-e^{(1-i\xi)y})dy \\
&= \frac{1}{(-i\xi)(1-i\xi)} \\
&= -\frac{1}{\xi(\xi+i)}.
\end{aligned}$$

To sum up: a formula for the leading term for the price of the European put with the strike price normalized to 1 is

$$f(t,x) = -\frac{1}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma}\frac{\exp[ix\xi-\tau(r+a(x,\xi))]}{\xi(\xi+i)}d\xi+\cdots. \tag{4.11}$$

In the Appendix, we give formulae for the next terms of the asymptotic expansion (4.11), and show that the first term is already fairly small: if the symbol $a(x,\xi)$ depends on parameters (ϵ, d) and satisfies estimates

$$|a^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha\beta}\epsilon^{|\beta|}d^{1-|\alpha|-\min\{1,|\beta|\}} \tag{4.12}$$

for any α, β, where $C_{\alpha\beta}$ is independent of (ϵ, d), then the first omitted term in (4.11) admits an estimate via $C\tau^2\epsilon$, where C is independent of (τ, ϵ, d), and the next term is less still: of order $\tau^2\epsilon^2/d$.

Notice that the leading term looks exactly as in the case of the option pricing under Lévy processes, only when we calculate $f(t,x)$ do we use the characteristic exponent depending on x. Thus, the leading term can be calculated as easily as in the case of Lévy processes.

Remark 4.2. Consider the NIG-like Feller process defined by the symbol (1.1) with constant δ, parameters μ, α, β being bounded and having bounded derivatives, and satisfying

$$\alpha_0 := \min \alpha(x) >> \beta^0 := \max |\beta(x)|. \tag{4.13}$$

Then we have (4.12) with $\epsilon = 1$ and $d = \alpha_0$.

If the parameters α, β and μ depend on a small parameter $\epsilon > 0$ in such a way that any of their derivatives of order s admits an estimate via $C_s\epsilon^s$, where C_s is independent of ϵ, then we have (4.12), and the effective small parameter is ϵ/d, i.e. the second omitted term in (4.11) is very small. The first omitted term in (4.11) (see (6.37)) is more involved than the leading term but since it is small – near expiry, very small indeed – it can be calculated with relatively large error, which simplifies the task of the development of appropriate computational procedures.

5. Discussion and conclusions

We have used two approaches for construction of a class of Feller processes, generalizing a class of Lévy processes, in particular the Normal Inverse Gaussian Lévy processes: via subordination, and by describing a class of PDO to which the generators belong; the processes themselves have been constructed by using the representation theorem for analytic semigroups. We have shown that the class of NIG-like Feller processes obtained by the first approach is a subclass of the NIG-like Feller processes obtained by the second approach, and have discussed applicability of both types of processes to option pricing.

A model for shocks based on the subordination approach may seem more natural from the probabilistic viewpoint, but it may be not so easy to fit it to the data so that the EMM-requirement (4.3) is met. So, our observation that symbols of generators obtained under the two approaches differ little if the derivatives of the coefficients of a subordinated diffusion are small can be used to justify the usage of the naive approach. The EMM-requirement for the lattter approach is easy to satisfy, and one can obtain approximate analytic pricing formulae, which are computationally as easy (or difficult) as the pricing formulae for the corresponding Lévy process.

The symbols and the pricing formulae differ little even if derivatives are not small but (4.13) holds. Notice that empirical studies in, inter alia, Barndorff-Nielsen and Jiang (1998) allow one to expect that (4.13) usually holds, and hence the pricing formula (4.11) can be used.

One may use another construction of NIG-like processes: suppose, that there is a coordinate system in which the process under consideration is a NIG Lévy. In this case, the natural question is what are the properties of the generator written in an initial coordinate system, in particular, what does its symbol look like; this question can easily be answered by applying the change-of-variables formula in the theory of PDO.

Finally, we note that results, similar to the ones in Section 3 and 4, can be obtained for other Lévy-like Feller processes obtained as modifications of, say, Hyperbolic Processes, used by Eberlein and coauthors (see Eberlein and Keller (1995), Eberlein et al (1998), Eberlein and Raible (1999), Eberlein (1999) and the bibliography there), or Truncated Lévy Processes of the Koponen family (Koponen (1995)) used by Bouchaud and coauthors (see e.g. Cont et al (1997)) or a simple generalization of this family discussed in Boyarchenko and Levendorskiĭ (1999, 2000a). The generalization is needed since Koponen's family does not capture the different rates of exponential decay in the left and right tails of the density function, typically observed in Financial Markets.

6. Appendix. Main facts of the theory of PDO and auxiliary technical results

6.1. **Basics of the theory of PDO with symbols of the class $S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.** We start with this by now quite classical case since results for PDO with symbols of the class $S^m(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$ can be deduced quite easily from the results for $S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$. Most of the results below, with the exception of the boundedness theorem for the Hölder spaces, can be found in Grubb (1996); however, the form and proofs of some results, which we give below, do not necessarily coincide with the proofs in that monograph.

6.1.1. *Asymptotic summation and the composition theorem.* The following notation and theorem simplify many formulations in the theory of PDO.

Definition 6.1. We say that $a \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ admits an asymptotic expansion

$$a \sim \sum_{j=0}^{+\infty} a_j, \tag{6.1}$$

if for any N

$$r_N := a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n). \tag{6.2}$$

Remark 6.1. In the theory of PDO, it is often convenient to use a weaker form of (6.2): $r_N \in S^{m-M}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, where $M \to +\infty$ as $N \to +\infty$, but we will not need this form here.

For large $|\xi|$, the error term $r_N(x,\xi)$ is relatively small, and in the case of symbols depending on a parameter (see the next subsection), it is relatively small uniformly in (x,ξ), so one can regard (6.1) as an approximate equality.

Theorem 6.2. *(Asymptotic summation). Let $a_j \in S^{m-j}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), j = 0, 1, \dots$. Then there exists $a \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ such that (6.1) holds.*

On the space $\mathcal{S}(\mathbf{R}^n)$, the action of PDO $a(x, D)$ with the symbol $a \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ is defined by (2.2), and the proof of Theorem 3.3, which is the next step in the systematic construction of the theory of PDO, is a straightforward exercise. By duality, the action of $a(x, D)$ extends on $\mathcal{S}'(\mathbf{R}^n)$, the space of continuous linear functionals ϕ on $\mathcal{S}(\mathbf{R}^n)$:

$$\langle a(x, D)\phi, f\rangle = \langle \phi, \bar{a}(x, D)f\rangle, \quad \forall\ f \in \mathcal{S}(\mathbf{R}^n).$$

This extension is a continuous map from $\mathcal{S}'(\mathbf{R}^n)$ into itself, and the standard agreement in the theory of PDO is to denote this map by the same symbol $a(x, D)$; the same applies to its restriction on any subspace $L \subset \mathcal{S}'(\mathbf{R}^n)$.

The next basic fact and the most important tool in the theory is the composition theorem.

Theorem 6.3. *Let $a \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ and $b \in S^{m'}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.*

Then $C = a(x, D)b(x, D)$ is a PDO with the symbol $c \in S^{m+m'}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, which admits an asymptotic expansion

$$c(x, \xi) \sim \sum_{|\alpha|\geq 0} (\alpha!)^{-1} a^{(\alpha)}(x, \xi) b_{(\alpha)}(x, \xi), \tag{6.3}$$

in the sense that for any integer $N > 0$,

$$r_N = c - \sum_{0\leq|\alpha|\leq N-1} (\alpha!)^{-1} a^{(\alpha)} b_{(\alpha)} \in S^{m+m'-N}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n). \tag{6.4}$$

Definition 6.4. We write $a \in IS^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ iff $a \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ is invertible and admits a bound

$$|a(x, \xi)^{-1}| \leq C\langle\xi\rangle^{-m}, \tag{6.5}$$

where C is independent of (x, ξ).

We call a an elliptic symbol, and $A = a(x, D)$ an elliptic operator.

Remark 6.2. The standard definition of an elliptic symbol requires (6.5) outside a compact set in ξ-space. We use (6.5), since it suffices for our purposes and admits a trivial generalization for the case of operators depending on a parameter.

6.1.2. *Parametrix and Approximate Square Root.* The following definition formalizes an intuitive notion of an approximate inverse.

Definition 6.5. We say that $B = b(x, D)$ is a parametrix of $A = a(x, D)$ if and only if

$$AB = I + t_1(x, D), \quad BA = I + t_2(x, D), \tag{6.6}$$

where $t_j \in S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^n) := \cap_{m\in\mathbf{R}} S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.

Theorem 6.6. *(Parametrix construction) Let $a \in IS^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$. Then $A = a(x, D)$ has a parametrix $B = b(x, D)$, where $b \in S^{-m}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ admits an asymptotic expansion*

$$b \sim b_{-m} + b_{-m-1} + \cdots, \tag{6.7}$$

where

$$b_{-m} = 1/a \in S^{-m}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), \tag{6.8}$$

and $b_{-m-s} \in S^{-m-s}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), s = 1, 2, \dots$ *are defined inductively by*

$$b_{-m-s} = -a^{-1} \sum_{j<s, j+|\alpha|=s} (\alpha!)^{-1} a^{(\alpha)} (b_{-m-j})_{(\alpha)}. \tag{6.9}$$

Proof. Step 1. We construct b such that

$$Ab(x, D) = I + t(x, D), \tag{6.10}$$

and similarly one can construct $b'(x, D)$ such that

$$b'(x, D)A = I + t'(x, D), \tag{6.11}$$

where $t, t' \in S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^n)$. By mimicking the standard argument on the equality of the left and right inverses, one deduces that (6.11) holds with $b(x, D)$ (and generally, with different $t' \in S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^n)$).

Step 2. From (6.5), we deduce that $b_{-m} := 1/a \in S^{-m}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, and by substituting the asymptotic expansion (6.7) with yet unknown $b_{-m-s} \in S^{-m-s}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), s = 1, 2, \dots$ (for any sequence of such $b_{-m-s}, s \geq 1$, b is defined by Theorem 6.2), and by using the composition theorem we obtain (6.10), where $t \in S^{-1}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ admits the asymptotic expansion

$$t \sim \sum_{s\geq 1} t_s, \tag{6.12}$$

with $t_s \in S^{-s}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), s \geq 1$, are given by

$$t_s = \sum_{j+|\alpha|=s} (\alpha!)^{-1} a^{(\alpha)} (b_{-m-j})_{(\alpha)} =$$

$$= ab_{-m-s} + \sum_{j<s, j+|\alpha|=s} (\alpha!)^{-1} a^{(\alpha)} (b_{-m-j})_{(\alpha)}. \tag{6.13}$$

Define $b_{-m-s}, s \geq 1$, by (6.9). Then by induction, we find $b_{-m-s} \in S^{-m-s}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, and the RHS in (6.13) is zero. Hence, t given by (6.12) is of the class $S^{-\infty}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, and $b(x, D)$ is a parametrix of $a(x, D)$. □

Theorem 6.7. *(Asymptotic square root) Let* $a \in IS^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ *satisfy*

$$a(x, \xi) \notin \bar{R}_-, \quad \forall\ (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n. \tag{6.14}$$

Then there exists $b \in S^{m/2}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ *such that*

$$b(x, D)^2 = a(x, D) + t(x, D), \tag{6.15}$$

where $t \in S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^n)$, *and* b *admits an asymptotic expansion*

$$b \sim \sum_{j=0}^{+\infty} b_{m/2-j}, \tag{6.16}$$

where

$$b_{m/2} := a^{1/2} \in S^{m/2}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), \tag{6.17}$$

and $b_{m/2-s} \in S^{m/2-s}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, $s \geq 1$, *are defined inductively by*

$$b_{m/2-s} = -\frac{1}{2} b^{-1}_{m/2} \sum_{j,k<s, |\alpha|+k+j=s} (\alpha!)^{-1} (b_{m/2-k})^{(\alpha)} (b_{m/2-j})_{(\alpha)}. \tag{6.18}$$

Proof. Fix a branch of ln by the requirement $\ln a \in \mathbf{R}$ for $a > 0$, and set

$$b_{m/2}(x,\xi) = a(x,\xi)^{1/2} := \exp[\frac{1}{2} \ln a(x,\xi)].$$

Under assumption (6.14), the symbol $b_{m/2}$ is well-defined, and by using (6.5), one can easiliy check that it belongs to $S^{m/2}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$. Fix any sequence of $b_{m/2-s} \in S^{m/2-s}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, $s \geq 1$, define $b \in S^{m/2}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ by (6.16), by using Theorem 6.2, and calculate the symbol of $b(x,D)^2$ by using the composition theorem. We obtain that it admits an asymptotic expansion

$$b^2_{m/2} + \sum_{s \geq 1} t^s, \tag{6.19}$$

where

$$t^s = \sum_{|\alpha|+k+j=s} (\alpha!)^{-1} (b_{m/2-k})^{(\alpha)} (b_{m/2-j})_{(\alpha)} =$$

$$= 2 b_{m/2-s} b_{m/2} + \sum_{j,k<s, |\alpha|+k+j=s} (\alpha!)^{-1} (b_{m/2-k})^{(\alpha)} (b_{m/2-j})_{(\alpha)}. \tag{6.20}$$

If we define $b_{m/2-s}$, $s \geq 1$, by (6.18), we obtain that the RHS in (6.20) vanishes, and (6.19) gives (6.15). □

6.1.3. *Boundedness theorem.* One also needs a boundedness result for PDO in Banach spaces embedded in $\mathcal{S}'(\mathbf{R}^n)$. The most popular and easy such spaces are L_2-based Sobolev spaces $H^s(\mathbf{R}^n)$, since in this case estimates for norms can easily be obtained by means of the Fourier transform (see e.g. Eskin (1973), Taylor (1981) and Grubb (1996)). For the theory of stochastic processes, Hölder spaces $C^s(\mathbf{R}^n)$ are more appropriate. Until relatively recent times, only local Hölder estimates were available (see e.g. Taylor (1981)); in Yamazaki (1986)[7], global estimates are obtained for Besov spaces $B^s_{p,q}$, $p,q \in (0,\infty]$, and results, which are necessary for us, follow as special cases, since

for $s \in \mathbf{R}$, $B^s_{2,2}$ is the Sobolev space $H^s(\mathbf{R}^n)$;

for $s > 0$, $B^s_{\infty,\infty}(\mathbf{R}^n)$ is the Hölder-Zygmund space $C^s(\mathbf{R}^n)$,

and finally,

for non-integer $s > 0$, $C^s(\mathbf{R}^n)$ is the Hölder space, with the norm defined by

[7]The second author thanks Gerd Grubb for bringing this paper into attention

for $s \in (0,1)$:

$$||u||_{C^s} = ||u||_{L_\infty} + \sup_{x\in\mathbf{R}^n} \sup_{|h|\le 1} |h|^{-s}|u(x+h) - u(x)|;$$

for $s = m + s'$, where $m > 0$ is an integer and $s' \in (0,1)$,

$$||u||_{C^s} = \sum_{|\alpha|\le m} ||D^\alpha u||_{C^{s'}}.$$

Theorem 6.8. *Let $p, q \in (0,\infty]$, $m, s, \in \mathbf{R}$, and $a \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.*

Then $a(x,D) : B^s_{p,q}(\mathbf{R}^n) \to B^{s-m}_{p,q}(\mathbf{R}^n)$ is bounded, and its norm admits an estimate

$$||a(x,D)|| \le C \sup_{|\alpha|+|\beta|\le N} \sup_{(x,\xi)\in\mathbf{R}^n\times\mathbf{R}^n} \langle\xi\rangle^{|\alpha|-m}|a^{(\alpha)}_{(\beta)}(x,\xi)|, \tag{6.21}$$

where the constants C and N depend on s, m, p, q and n but not on $a \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.

6.1.4. *The inverse operator and the square root.*

Theorem 6.9. *Let $a \in IS^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, and let for some s, $A = a(x,D) : H^s(\mathbf{R}^n) \to H^{s-m}(\mathbf{R}^n)$ be invertible.*

Then the inverse is a PDO of the class $S^{-m}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, and its symbol admits the same asymptotic expansion (6.7) as the parametrix.

Corollary 6.10. *Under the conditions of Theorem 6.9, A is an invertible operator in $\mathcal{S}(\mathbf{R}^n)$, and for any r, $A : H^r(\mathbf{R}^n) \to H^{r-m}(\mathbf{R}^n)$ is invertible.*

Proof. $\mathcal{S}(\mathbf{R}^n) \subset H^r(\mathbf{R}^n)$, for any r, hence both A and A^{-1} maps $\mathcal{S}(\mathbf{R}^n)$ into itself. Therefore, they are mutual inverses as operators in $\mathcal{S}(\mathbf{R}^n)$. Since $\mathcal{S}(\mathbf{R}^n) \subset H^r(\mathbf{R}^n)$ densely, for any r, and $A^{-1} : H^{r-m}(\mathbf{R}^n) \to H^r(\mathbf{R}^n)$ is bounded, we conclude that A^{-1} is the inverse to $A : H^r(\mathbf{R}^n) \to H^{r-m}(\mathbf{R}^n)$. □

Remark 6.3. Corollary 6.10 is valid not for the scale $H^s(\mathbf{R}^n)$ only but for many other scales of spaces as well.

Theorem 6.11. *Let the conditions of Theorem 6.7 hold, and let $\Re a(x,D)$ be positive-definite, and $\Im a(x,D)(\Re a(x,D))^{-1}$ be bounded.*

Then there exists a PDO $B = b(x,D)$ such that $B^2 = A$, and its symbol $b \in S^{m/2}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ admits the same asymptotic expansion (6.16) as an approximate square root.

6.2. **Operators depending on parameters.** Consider a PDO with the symbol depending on a parameter (or parameters). If the dependence is nice, one can exploit it and obtain useful results, e.g. prove the invertibility of the operator for large (or small, depending on the situation) values of the parameter(s), and derive approximate formulae for the inverse. As it turns out, the simplest way to realize this possibility is to repeat the whole sequence of definitions and theorems from the very beginning by explicitly specifying the dependence on the parameter(s). The proofs themselves can be repeated word by word, without any change except in the notation. For similar results about operators with a parameter, see e.g. Grubb (1996).

We consider the case when there are two parameters – one, call it ϵ, describing the possible smallness of the derivatives w.r.t. x, and the other, call it d, which describes the smallness of the derivatives w.r.t. ξ. The case when only derivatives w.r.t. x (respectively ξ) are small obtains by fixing d (respectively ϵ).

So, let $\mathcal{E} \subset (0,1]$ and $\mathcal{D} \subset [1,+\infty)$ be the sets on which ϵ and d live; we assume that the statement $\epsilon/d \to 0$ makes sense on the set $\mathcal{E} \times \mathcal{D}$, and we regard a pair (ϵ, d) as a parameter.

The starting point is a natural generalization of the class $S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.

Definition 6.12. Let $m \in \mathbf{R}$. We write $a \in S^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$ if $a = a(\epsilon, d; x, \xi)$ satisfies estimates

$$|a^{(\alpha)}_{(\beta)}(\epsilon, d; x, \xi)| \le C_{\alpha\beta}\epsilon^{|\beta|}(d + |\xi|)^{m-|\alpha|}, \tag{6.22}$$

for all multi-indices α, β, where the constants $C_{\alpha\beta}$ are independent of $\epsilon \in \mathcal{E}, d \in \mathcal{D}, (x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$.

Sometimes we will use this definition with $(\mathcal{E} \times \mathcal{D})_{\epsilon_0} = \{(\epsilon, d) \in \mathcal{E} \times \mathcal{D} \mid \epsilon/d < \epsilon_0\}$ instead of $\mathcal{E} \times \mathcal{D}$.

Definition 6.13. We say that $a \in S^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$ admits an asymptotic expansion (6.1) if for any N

$$r_N := a - \sum_{j=0}^{N-1} a_j \in \epsilon^N S^{m-N}_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n). \tag{6.23}$$

Theorem 6.14. *(Asymptotic summation). Let $\epsilon^{-j}a_j \in S^{m-j}_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$, $j = 0, 1, \dots$*
Then there exists $a \in S^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$ such that (6.1) holds in the sense (6.23).

Theorem 6.15. *Let $a \in S^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$ and $b \in S^{m'}_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$.*
Then $C = a(x,D)b(x,D)$ is a PDO with the symbol $c \in S^{m+m'}_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$, which admits an asymptotic expansion (6.3) in the sense that for any integer $N > 0$,

$$r_N = c - \sum_{0 \le |\alpha| \le N-1} (\alpha!)^{-1} a^{(\alpha)} b_{(\alpha)} \in \epsilon^N S^{m+m'-N}_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n). \tag{6.24}$$

Definition 6.16. We write $a \in IS^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$ iff $a \in S^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$, and there exists $\epsilon_0 > 0$ such that if $\epsilon/d < \epsilon_0$ then $a(\epsilon, d; x, \xi)$ is invertible and admits a bound

$$|a(\epsilon, d, x, \xi)^{-1}| \le C(d^2 + |\xi|^2)^{-m/2},$$

where C is independent of (ϵ, d, x, ξ) with $\epsilon/d < \epsilon_0$.

We call $A = a(\epsilon, d; x, D)$ an elliptic operator with a parameter.

In the sequel, we omit arguments ϵ, d, and write $a(x,D)$.

Theorem 6.17. *Let $a \in IS^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$, and $s, s-m > 0$ be positive non-integers. There exists $\epsilon_1 > 0$ such that if $\epsilon/d < \epsilon_1$, then $A = a(x,D) : C^s(\mathbf{R}^n) \to C^{s-m}(\mathbf{R}^n)$ is invertible, and its inverse B is a PDO, whose symbol $b \in S^{-m}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_1}; \mathbf{R}^n \times \mathbf{R}^n)$ admits* *the asymptotic expansion (6.7) in the sense of Definition 6.13, where*

$$b_{-m} = 1/a \in S^{-m}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_1}; \mathbf{R}^n \times \mathbf{R}^n),$$

and

$$b_{-m-s} \in \epsilon^s S^{-m-s}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_1}; \mathbf{R}^n \times \mathbf{R}^n), \quad s \geq 1,$$

are determined by (6.9).

Proof. Define $b_{-m} = 1/a$, $B^0 = b_{-m}(x, D)$. Since $a \in IS^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$, there exists $\epsilon_0 > 0$ such that $b_{-m} \in S^{-m}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_0}; \mathbf{R}^n \times \mathbf{R}^n)$, and by applying Theorem 6.15, we obtain

$$AB^0 = I + t_1(x, D), \quad B^0 A = I + t_2(x, D), \tag{6.25}$$

where $t_j \in \epsilon S^{-1}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_0}; \mathbf{R}^n \times \mathbf{R}^n)$. Hence, $(d/\epsilon)t_j \in S^0_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ uniformly in $(\epsilon, d) \in (\mathcal{E} \times \mathcal{D})_{\epsilon_0}$, and by the boundedness theorem, there exists C such that for all $(\epsilon, d) \in (\mathcal{E} \times \mathcal{D})_{\epsilon_0}$,

$$||(d/\epsilon)t_1(x, D)||_{C^{s-m} \to C^{s-m}} \leq C, \tag{6.26}$$

and

$$||(d/\epsilon)t_2(x, D)||_{C^s \to C^s} \leq C. \tag{6.27}$$

When $\epsilon/d < \epsilon_1 := \min\{\epsilon_0, 1/C\}$, we conclude from (6.25)–(6.27) that AB^0 and B^0A are invertible in $C^{s-m}(\mathbf{R}^n)$ and $C^s(\mathbf{R}^n)$, respectively. By the boundedness theorem, $B^0 : C^{s-m}(\mathbf{R}^n) \to C^s(\mathbf{R}^n)$ is bounded, and hence, $B^0(1 + t_1(x, D))^{-1}$ and $(1 + t_2(x, D))^{-1}B^0$ are bounded right and left inverses to $A : C^s(\mathbf{R}^n) \to C^{s-m}(\mathbf{R}^n)$.

An analogue of Theorem 6.9 for operators with parameter(s) is valid: a parametrix B has the symbol $b \in S^{-m}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_0}; \mathbf{R}^n \times \mathbf{R}^n)$ can be constructed exactly as in Theorem 6.6, and A^{-1} is a PDO with the symbol of the class $S^{-m}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_1}; \mathbf{R}^n \times \mathbf{R}^n)$.

They satisfy (6.6) with $t_j \in S^{-\infty}((\mathcal{E} \times \mathcal{D})_{\epsilon_0}; \mathbf{R}^n \times \mathbf{R}^n) := \cap_{m \in \mathbf{R}} S^m_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_0}; \mathbf{R}^n \times \mathbf{R}^n)$. By multiplying the left equality in (6.6) by A^{-1} from the right and using Theorem 6.15, we find $A^{-1} - B = t(x, D)$, where $t \in S^{-\infty}((\mathcal{E} \times \mathcal{D})_{\epsilon_1}; \mathbf{R}^n \times \mathbf{R}^n)$.

This proves that an asymptotic expansion for the symbol of the parametrix is an asymptotic expansion for the symbol of the inverse. □

Remark 6.4. Theorem 6.17 is valid for other spaces as well; for instance, in the scale of Sobolev spaces $H^s(\mathbf{R}^n)$, it is valid without the restrictions on s and m.

Theorem 6.18. *Let $\Re a \in IS^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$ be positive, and $\Im a \in S^m_{1,0}(\mathcal{E} \times \mathcal{D}; \mathbf{R}^n \times \mathbf{R}^n)$.*

Then there exist $\epsilon_1 > 0$ and $\tilde{b} \in S^{m/2}_{1,0}((\mathcal{E} \times \mathcal{D})_{\epsilon_1}; \mathbf{R}^n \times \mathbf{R}^n)$ such that

$$\tilde{b}(x, D)^2 = a(x, D), \tag{6.28}$$

and $\tilde{b}$ admits the asymptotic expansion (6.16) in the sense of Definition 6.13, with the terms given by (6.17)–(6.18).

Proof. Exactly the same argument as in the proof of Theorem 6.7 allows us to construct $b \in S^m_{1,0}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n)$, which satisfies (6.15) with $t \in S^{-\infty}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n)$; (6.16) is understood in the sense of Definition 6.13, and all the terms are given by (6.17)–(6.18). By using a straightforward modification of the result on fractional powers in Grubb (1996), we obtain that if $\epsilon_0 > 0$ is sufficiently small and $(\epsilon, d) \in (\mathcal{E}\times\mathcal{D})_{\epsilon_0}$, then $B = a(x,D)^{1/2}$ is well-defined, its symbol $\tilde b$ belongs to $S^{m/2}_{1,0}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n)$, and $\tilde b - a^{1/2} \in \epsilon S^{m/2-1}_{1,0}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n)$. It follows that

$$\tilde b = b + k, \tag{6.29}$$

where $k \in \epsilon S^{m/2-1}_{1,0}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n)$. Suppose, that for some $j \geq 1$,

$$k \in \epsilon^j S^{m/2-j}_{1,0}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n), \tag{6.30}$$

and show that then (6.30) holds with $j+1$ instead of j; this will finish the proof of the Theorem.

By calculating $\tilde b(x,D)^2$ with the help of Theorem 6.15 and taking (6.28)–(6.30) and (6.15) into account, we find that

$$T := b(x,D)k(x,D) + k(x,D)b(x,D)$$

is a PDO with the symbol $t \in \epsilon^{j+1} S^{m-j-1}_{1,0}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n)$.

Apply the parametrix of $b(x,D)$ to $T = t(x,D)$, and use Theorem 6.15; the result is that $2k \in \epsilon^{j+1} S^{m/2-j-1}_{1,0}((\mathcal{E}\times\mathcal{D})_{\epsilon_0}; \mathbf{R}^n\times\mathbf{R}^n)$, and the verification is complete. □

6.3. **Operators with symbols holomorphic in a tube domain.** Let $U \subset \mathbf{R}^n$ be an open set such that its closure contains the origin: $\bar U \ni \{0\}$, and consider the classes $S^m(\mathbf{R}^n\times(\mathbf{R}^n + i\bar U)) = S^m_{1,0}(\mathbf{R}^n\times(\mathbf{R}^n + i\bar U))$ introduced in Definition 3.1. If we work with the same spaces as in Subsection 6.1, we can repeat all the statements there word by word by simply adding $+i\bar U$ in the notation of each class of symbols. This means that we make no use of the fact that symbols admit holomorphic continuation into the tube domain, and if we consider European options with bounded payoffs, e.g. European puts, we may do it and lose no essential information.

When the payoff is exponentially growing at infinity, as is the case with European calls, appropriate spaces are spaces with exponential weights, and one can consider the action of PDO in such spaces if and only if their symbols admit analytic continuation into an appropriate tube domain. For PDO of the class $S^m_{1,0}(\mathbf{R}^n\times(\mathbf{R}^n + i\bar U))$, the part of $\mathcal{S}(\mathbf{R}^n)$ is played by

$$\mathcal{S}(\mathbf{R}^n; \bar U) = \{u \mid e^{(x,y)}u \in \mathcal{S}(\mathbf{R}^n),\ \forall\, y \in \bar U\},$$

with a natural system of seminorms, and the following theorem can be trivially deduced from Theorem 3.3 by using the equality

$$\begin{aligned} e^{(x,y)}a(x,D)u &= \left(e^{(x,y)}a(x,D)e^{-(x,y)}\right)e^{(x,y)}u \\ &= a(x, D+iy)e^{(x,y)}u. \end{aligned}$$

Theorem 6.19. *Let $a \in S^m_{1,0}(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$. Then $a(x,D)$ is a continuous operator in $\mathcal{S}(\mathbf{R}^n;\bar{U})$.*

As in Subsection 6.1, we define by duality the action of $a(x,D)$ in $\mathcal{S}'(\mathbf{R}^n;\bar{U})$, the space of continuous linear functionals over $\mathcal{S}(\mathbf{R}^n;-\bar{U})$. The next step, namely, the action in Banach subspaces of $\mathcal{S}'(\mathbf{R}^n;\bar{U})$, is not very difficult but technically more involved.

To make the main idea clear, we restrict ourselves to the 1D-case, when $U = (\lambda_-,\lambda_+)$ is an interval, and $\lambda_- \le -1 < 0 \le \lambda_+$. The multi-dimensional case can be treated similarly.

Definition 6.20. Let $\omega_- \le \omega_+$ and $s > 0$ be non-integer. $C^s(\mathbf{R};[\omega_-,\omega_+])$ denotes the space of functions with the finite norm

$$||u||_{C^s(\mathbf{R};[\omega_-,\omega_+])} = ||(e^{\omega_- x} + e^{\omega_+ x})u||_{C^s(\mathbf{R})}. \tag{6.31}$$

Similarly, one defines Sobolev (and Besov) spaces with weight.

Theorem 6.21. *Let $a \in S^m_{1,0}(\mathbf{R} \times (\mathbf{R} + i[\lambda_-,\lambda_+]))$, $[\omega_-,\omega_+] \subset [\lambda_-,\lambda_+]$, and $s, s-m$ be positive non-integers.*
Then $a(x,D) : C^s(\mathbf{R};[\omega_-,\omega_+]) \to C^{s-m}(\mathbf{R};[\omega_-,\omega_+])$ is bounded, and its norm admits an estimate

$$||a(x,D)|| \le C \sup_{|\alpha|+|\beta|\le N} \sup_{(x,\xi)\in\mathbf{R}\times(\mathbf{R}+i[\omega_-,\omega_+])} \langle\xi\rangle^{|\alpha|-m}|a^{(\alpha)}_{(\beta)}(x,\xi)|, \tag{6.32}$$

where constants C and N depend on s,m and ω_-,ω_+ but not on $a \in S^m_{1,0}(\mathbf{R} \times (\mathbf{R} + i[\lambda_-,\lambda_+]))$.

Proof. For any $u \in C^s(\mathbf{R};[\omega_-,\omega_+])$

$$||a(x,D)u||_{C^s(\mathbf{R};[\omega_-,\omega_+])} \le$$

$$\le ||e^{\omega_- x}a(x,D)u||_{C^s(\mathbf{R})} + ||e^{\omega_+ x}a(x,D)u||_{C^s(\mathbf{R})} =$$

$$= ||(e^{\omega_- x}a(x,D)e^{-\omega_- x})e^{\omega_- x}u||_{C^s(\mathbf{R})} + ||(e^{\omega_+ x}a(x,D)e^{-\omega_+ x})e^{\omega_+ x}u||_{C^s(\mathbf{R})} =$$

By using the equality

$$e^{\omega_\pm x}a(x,D)e^{-\omega_\pm x} = a(x,D+i\omega_\pm),$$

and Theorem 6.8, we continue

$$= ||a(x,D+i\omega_-)e^{\omega_- x}u||_{C^s(\mathbf{R})} + ||a(x,D+i\omega_+)e^{\omega_+ x}u||_{C^s(\mathbf{R})} \le$$

$$\le C\left(||e^{\omega_- x}u||_{C^s(\mathbf{R})} + ||e^{\omega_+ x}u||_{C^s(\mathbf{R})}\right),$$

where C can be chosen in the form of the RHS in (6.32). The sum in the brackets defines the norm equivalent to the norm (6.31), which finishes the proof of the Theorem. □

6.4. **Proofs of auxiliary technical results.** *Proof of Theorem 3.4.* If $m \le 0$, then $a(x,D)$ is bounded in $L_2(\mathbf{R}^n)$, therefore for λ large enough, $\lambda + a(x,D)$ is invertible in $L_2(\mathbf{R}^n)$. By Corollary 6.10, it is invertible in $\mathcal{S}(\mathbf{R}^n)$.

Now let $m > 0$. Due to (3.2), for λ large enough, $b(\lambda;x,\xi) = (\lambda + a(x,\xi))^{-1}$ is well-defined on $\mathbf{R}^n \times \mathbf{R}^n$. By using the composition theorem, it is straightforward to show that

$$b(\lambda;x,D)(\lambda + a(x,D)) = I + t(\lambda;x,D), \tag{6.33}$$

where t satisfies: for any pair of multi-indices (α,β),

$$\lim_{\lambda\to+\infty} \sup_{\mathbf{R}^n\times\mathbf{R}^n} |t^{(\alpha)}_{(\beta)}(\lambda;x,\xi)|\langle\xi\rangle^{|\alpha|} = 0. \tag{6.34}$$

It follows from the boundedness theorem and (6.34) that if λ is sufficiently large, the RHS in (6.33) is an invertible operator in $L_2(\mathbf{R}^n)$. Hence, for these λ, $a(x,D)+\lambda : H^s(\mathbf{R}^n) \to L_2(\mathbf{R}^n)$ has a bounded left inverse $(I + t(\lambda;x,D))^{-1}b(\lambda;x,D)$. Similarly, it has a bounded right inverse, hence it is invertible. By Corollary 6.10, $a(x,D)+\lambda$ is invertible in $\mathcal{S}(\mathbf{R}^n)$.

Theorem 3.4 has been proven.

Proof of Theorem 3.10. a) We have $a(x,\xi) := d^2 - L(x,\xi) \in S^2_{1,0}(\mathbf{R}^n \times (\mathbf{R}^n + i\bar{U}))$ for any U, and if U satisfies (3.11), we see that (6.14) holds for all $(x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n + i\bar{U})$. Hence, the analogues of Theorems 6.11 and 6.7 for PDO with symbols holomorphic in a tube domain are valid, which gives not only (3.12)–(3.14) but the other terms of the asymptotic expansion of the symbol as well (see (6.16)–(6.18)).

b)–d) are special cases of the analogues of Theorems 6.11 and 6.7 for PDO depending on a parameter.

Theorem 3.10 has been proven.

The omitted terms of the asymptotic expansion (4.10) can be easily inferred from Theorems 6.9 and 6.6, by using (6.7)–(6.11) with $\lambda + r + a(x,\xi)$ instead of $a(x,\xi)$. For instance, the RHS of (4.10) with two terms (the only case which may have some relevance to practice) is

$$(\lambda + r + a(x,\xi))^{-1} + (\lambda + r + a(x,\xi))^{-3} \sum_{|\alpha|=1} a^{(\alpha)}(x,\xi)a_{(\alpha)}(x,\xi) + \cdots \tag{6.35}$$

By substituting (6.35) into (4.9) we can obtain the next terms in (4.11). In particular, the first omitted term is equal to

$$\frac{i}{4\pi^2}\int_{\mathcal{L}_\theta}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[\lambda\tau + ix\xi]\sum_{|\alpha|=1} a^{(\alpha)}(x,\xi)a_{(\alpha)}(x,\xi)}{\xi(\xi+i)(\lambda + r + a(x,\xi))^3} d\xi d\lambda, \tag{6.36}$$

where $\sigma \in (0,\lambda_+)$. The integral converges absolutely, hence we can apply the Fubini theorem and change the order of integration. After that we can transform the contour $\mathcal{L}_\theta$ by pushing it to the left. There is only one pole of order three, and when it is crossed, by the residue theorem the term

$$-\frac{\tau^2}{4\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma}\frac{\exp[ix\xi-\tau(r+a(x,\xi))]\sum_{|\alpha|=1}a^{(\alpha)}(x,\xi)a_{(\alpha)}(x,\xi)}{\xi(\xi+i)}d\xi \tag{6.37}$$

appears. After that we can push the contour to infinity. Since the integrand in (6.36) admits a bound via

$$C(x,\xi)e^{\Re\lambda\tau}|\lambda|^{-1},$$

we obtain 0 in the limit. If the derivatives of the symbol admit estimates (4.12), the integrand in (6.37) admits an estimate via

$$C\epsilon\exp(-\tau|\xi|)|\xi|^{-2}.$$

Since for $\sigma\neq 0$ fixed,

$$\int_{-\infty+i\sigma}^{+\infty+i\sigma}\exp(-\tau|\xi|)|\xi|^{-2}d\xi\leq C_1,$$

where C_1 is independent of τ, we conclude that the expression in (6.37), which is the first omitted term in (4.11), is of order $\tau^2\epsilon$. Similarly, the next term comes from an expression (6.37) with $\sum_{|\alpha|=2}$, and hence is of order $\tau^2\epsilon^2/d$, as claimed at the end of Section 4. All the next terms can be calculated as well, but they are relatively small w.r.t. the first omitted term, and hence of no practical significance.

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