On the existence of optimal controls for a singular stochastic control problem in finance

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Abstract. We prove existence of optimal investment-consumption strategies for an infinite horizon portfolio optimization problem in a Lévy market with intertemporal substitution and transaction costs. This paper complements our previous work [4], which established that the value function can be uniquely characterized as a constrained viscosity solution of the associated Hamilton-Jacobi-Bellman equation (but [4] left open the question of existence of optimal strategies). In this paper, we also give an alternative proof of the viscosity solution property of the value function. This proof exploits the existence of optimal strategies and is consequently simpler than the one proposed in [4].

1. Introduction

We prove existence of optimal controls for the singular stochastic control problem studied in Benth, Karlsen, and Reikvam [4] (see also [1, 2, 3] for related problems). An optimal consumption-investment problem over an infinite investment horizon in a market consisting of one risky asset (stock) and one risk-free asset (bank) is considered. The dynamics of the risky asset follows a geometric-type Lévy motion, generalizing the classical Black & Scholes model. Proportional transaction costs are incurred when selling or buying assets. In addition, following Hindy and Huang [9], the investor derives utility from an average of present and past consumption.

In Benth, Karlsen, and Reikvam [4], the value function of this optimization problem was characterized as the unique constrained viscosity solution [6] of the associated Hamilton-Jacobi-Bellman equation. Due to the singular controls and the Lévy dynamics of the risky asset, the Hamilton-Jacobi-Bellman equations takes the form of a second-order *integro-differential* variational inequality.

The existence of optimal controls was not addressed in [4]. To prove existence of an optimal control, we shall here use the convex analysis techniques described in Ekeland and Temam [10] together with the martingale methods of Cvitanić and Karatzas [7]. The main problem we are facing is the infinite investment horizon, which leads to unbounded controls. However, from Benth, Karlsen, and Reikvam

Benth is partially sponsored by MaPhySto - Centre for Mathematical Physics and Stochastics (University of Aarhus, Denmark), funded by a grant from the Danish National Research Foundation. Reikvam is supported by the Norwegian Research Council (NFR) under grant 118868/410.

[5], we have explicit bounds on the expected growth of the controls, which enables us to introduce a time-weighted L^2 - space. This space provides the starting point for using the martingale methods in [7].

We end this paper by giving an alternative proof of the result in [4] stating that the value function is a constrained viscosity solution of the associated Hamilton-Jacobi-Bellman equation. Compared with the proof proposed in [4], the present proof is simpler since we exploit the existence of an optimal strategies.

2. Formulation of the control problem

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ be a filtered complete probability space satisfying the usual hypotheses. We consider a single investor who divides her wealth between one risk-free asset (bank account) paying a fixed interest rate $r > 0$ and a risky asset (stock). We denote by $B(t)$ the amount of money the investor has in the bank account and $S(t)$ the amount of money the investor has in the stock, at time $t \geq 0$. We assume that the holdings of the investor follow the dynamics

$$\begin{cases} B(t) = b_0 - C(t) + \displaystyle\int_0^t rB(s)\,ds - (1+\lambda)L(t) + (1-\mu)M(t), \\ S(t) = s + \displaystyle\int_0^t aS(s)\,ds + \int_0^t \sigma S(s)\,dW(s) \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}\backslash\{0\}} \eta(z)S(s-)\,\tilde{N}(ds,dz) + L(t) - M(t), \end{cases} \tag{1}$$

where $a, \sigma > 0$ are constants, $C(t)$ is the cumulative consumption up to time t, $L(t)$ is the cumulative value of the shares *bought* up to time t, $M(t)$ is the cumulative value of the shares *sold* up to time t, and $\mu \in [0,1]$ and $\lambda \geq 0$ are the proportional transaction costs of respectively selling and buying shares from the stock. We assume $\mu + \lambda > 0$. In addition, $W(s)$ is a standard Brownian motion and $\tilde{N}$ is a compensated Poisson random measure independent of W with Lévy measure $\ell(dz)$. The function $\eta(z)$ is assumed to be Borel measurable on $\mathbb{R}\backslash\{0\}$ with the property $\eta(z) > -1$ to ensure that the stock holdings remains positive as long as we are not short of stocks. In addition, we require the following integrability conditions on the Lévy measure:

$$\int_{|z|<1} (\eta(z))^2\,\ell(dz) < \infty, \quad \int_{|z|\geq 1} |\eta(z)|\,\ell(dz) < \infty.$$

We assume throughout the paper that the expected rate of return a of the stock is greater than or equal to the risk-free interest rate r.

Introduce the process of average past consumption,

$$dY(t) = \beta\,dC(t) - \beta Y(t)\,dt, \qquad \beta > 0, \tag{2}$$

which has the explicit solution

$$Y(t) = y\mathrm{e}^{-\beta t} + \beta \mathrm{e}^{-\beta t} \int_{[0,t]} \mathrm{e}^{\beta s}\, dC(s).$$

The investor will derive utility from this average, rather than directly from present consumption [9].

The market considered here does not allow short-selling of stocks nor borrowing of money in the bank. In other words, the amount of money allocated in the bank account and the stocks must stay nonnegative. Hence the domain for the control problem is

$$\mathcal{D} = \Big\{ x = (b, s, y) \in \mathbb{R}^3 \mid x > 0 \Big\}.$$

We refer to $\Pi = (C, L, M)$ as a policy for investment and consumption if Π belongs to the set $\mathcal{A}_x$ of *admissible controls*. For $x \in \overline{\mathcal{D}}$, we say that $\Pi \in \mathcal{A}_x$ if the following conditions hold:

(C.1) $C(t), L(t), M(t)$ are adapted, nondecreasing, and right-continuous with left limits. Moreover, $C(0-) = M(0-) = L(0-) = 0$.

(C.2) The state process $X(t) = (B(t), S(t), Y(t))$ is a solution to the stochastic differential equations (1) and (2) and respects the state-space constraint $X(t) = X^{\Pi}(t) \in \overline{\mathcal{D}}$ for all $t \geq 0$.

Note that $0 \in \mathcal{A}_x$. The objective of the investor is to maximize her expected utility over an infinite investment horizon. The functional to be optimized is

$$\mathcal{J}(x; \Pi) = \mathbb{E}\Big[\int_0^{\infty} \mathrm{e}^{-\delta t} U(Y^{\Pi}(t))\, dt \Big], \quad x \in \overline{\mathcal{D}},$$

where U is the investor's utility function and $\delta > 0$ is the discount factor. We introduce the following assumptions on the utility function:

(U.1) $U(z)$ is a continuous, nondecreasing, and concave function on $[0, \infty)$ with $U(0) = 0$.

(U.2) There exist $\gamma \in (0,1)$ and constant $K > 0$ such that $U(z) \leq K(1+z)^{\gamma}$ for all $z \in [0, \infty)$.

Define the value function of the optimization problem to be

$$V(x) = \sup_{\Pi \in \mathcal{A}_x} \mathcal{J}(x; \Pi), \qquad x \in \overline{\mathcal{D}}. \tag{3}$$

Our singular stochastic control problem is to find an optimal control $\Pi^* \in \mathcal{A}_x$ such that

$$V(x) = \mathcal{J}(x; \Pi^*), \qquad x \in \overline{\mathcal{D}}. \tag{4}$$

Sections 3 and 4 are devoted to the existence of Π^*, while Section 5 is devoted to a brief discussion of the Hamilton-Jacobi-Bellman equation satisfied (in a suitable sense) by the value function (3).

In what follows (Sections 3 and 4), the point $x \in \overline{\mathcal{D}}$ will always be considered as fixed and not explicitly mentioned anymore.

3. Some estimates on the control and state processes

We recall some results from [5] which will be needed in Section 4. The first result states that the set of admissible controls is uniformly bounded in $L^2(P)$. Moreover, we can control the growth in time.

Proposition 3.1. *For every $t \geq 0$, the controls are uniformly bounded in $L^2(P)$:*

$$\sup_{\Pi \in \mathcal{A}_x} \mathbb{E}\big[C_t^2 + L_t^2 + M_t^2\big] \leq K e^{kt},$$

where K is a positive constant and

$$k = 2r + 2(a-r)(2K_1^2 + K_2) + 2K_2^2\Big(\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} (\eta(\xi))^2 \ell(d\xi)\Big),$$

for $K_1 = (x_0 + (1+\lambda)x_1)/(\lambda+\mu)$ and $K_2 = (1+\lambda)/(\mu+\lambda)$.

Proof. This result is proven in [5], where the rate of the exponential growth is explicitly calculated. □

We have a similar uniform bound on the averaging process Y_t^Π:

Proposition 3.2. *For every $t \geq 0$,*

$$\sup_{\Pi \in \mathcal{A}_x} \mathbb{E}\big[(Y_t^\Pi)^2\big] \leq K(y + e^{kt}),$$

where K is a positive constant and k is as in Proposition 3.1.

Proof. This follows from the fact that $Y_t \leq y + C_t$ and the estimate on $\mathbb{E}[C_t^2]$. □

Via the growth of V proven in [4, Cor. 3.5], we have that the value function is well-defined, that is, the following result holds:

Proposition 3.3. *There exists a positive constant K such that*

$$0 \leq V(x) \leq K\big(1 + |x|\big)^\gamma.$$

4. Existence of optimal controls

In this section we prove existence of an optimal control. Let $\alpha > k$, for the k given in Prop. 3.1, and introduce the measure $m_\alpha(dt) = e^{-\alpha t}\, dt$ on $\mathbb{R}_+$. Notice that

$$\int_{\mathbb{R}_+} e^{ct}\, m_\alpha(dt) < \infty, \qquad \forall c \leq k. \tag{5}$$

Define the the (weighted Hilbert) space

$$\mathcal{H}_x^\alpha = \Big\{0 \leq H \in L^2(m_\alpha \otimes P) \,:\, \exists \Pi \in \mathcal{A}_x \text{ s. t. } H \leq Y^\Pi,\, m_\alpha \otimes P \text{ - a. e.}\Big\}.$$

The next two lemmas show that $\mathcal{H}_x^\alpha$ is a non-empty, bounded, convex, and closed subspace of $L^2(m_\alpha \otimes P)$:

Lemma 4.1. *$\mathcal{H}_x^\alpha$ is a non-empty, bounded, and convex subspace of $L^2(m_\alpha \otimes P)$.*

Proof. Since $Y_t^\Pi \geq 0$ for all $\Pi \in \mathcal{A}_x$, it follows that $0 \in \mathcal{H}_x^\alpha$. The uniform exponential bound of Y^Π in Prop. 3.2 proves that $\mathcal{H}_x^\alpha$ is a norm-bounded subspace of $L^2(m_\alpha \otimes P)$. Let $H^1, H^2 \in \mathcal{H}_x^\alpha$ be such that $H^1 \leq Y^{\Pi^1}$ and $H^2 \leq Y^{\Pi^2}$. Define the control $\Pi := \theta\Pi^1 + (1-\theta)\Pi^2$ for $\theta \in (0,1)$. From the proof of Prop. 5.1 in [4], it follows that $\Pi \in \mathcal{A}_x$. Moreover, by uniqueness of paths, $Y^\Pi = \theta Y^{\Pi^1} + (1-\theta)Y^{\Pi^2}$, and hence $H := \theta H^1 + (1-\theta)H^2 \leq Y^\Pi$. This proves the convexity of $\mathcal{H}_x^\alpha$. □

Lemma 4.2. *$\mathcal{H}_x^\alpha$ is closed in $L^2(m_\alpha \otimes P)$.*

Proof. Choose a sequence H^n in $\mathcal{H}_x^\alpha$ which converges to H in $L^2(m_\alpha \otimes P)$. We can associate a control Π^n to each H^n such that $H^n \leq Y^{\Pi^n}$. Moreover, by the uniform bound on the controls (see Section 3) and (5), we know that $\{\Pi^n\}_n$ is uniformly bounded in $L^1(m_\alpha \otimes P)$, and, hence by the Dunford-Pettis compactness criterion, there exists $\Pi \in L^1(m_\alpha \otimes P)$ and a subsequence, also denoted Π^n, such that $\Pi^n \to \Pi$ weakly in $L^1(m_\alpha \otimes P)$. The question is whether Π is an admissible control or not. Following the arguments of Karatzas and Shreve [11, Section 4], we can prove the existence of a version of Π, still denoted by Π, for which $\Pi \in \mathcal{A}_x$ and $\Pi^n \to \Pi$ weakly. In addition, as in Cvitanić and Karatzas [7, Appendix A], we can show that $B^{\Pi^n} \to B^\Pi$, $S^{\Pi^n} \to S^\Pi$, and $Y^{\Pi^n} \to Y^\Pi$ weakly. Since $H^n \leq Y^{\Pi^n}$ for every n, we conclude $H \leq Y^\Pi$, thereby proving the closedness of $\mathcal{H}_x^\alpha$. □

Define the functional $\mathcal{I} : \mathcal{H}_x^\alpha \to \mathbb{R}$ to be

$$\mathcal{I}(H) = -\mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(H_s)\, ds\Big].$$

Lemma 4.3. *If $\delta > \alpha/2$, then $\mathcal{I}$ is proper, convex, and lower-semicontinuous with respect to the $L^2(m_\alpha \otimes P)$ - norm.*

Proof. From the growth property of V in Prop. 3.3 it follows that $\mathcal{I}$ is proper. Furthermore, since U is assumed to be concave, $\mathcal{I}$ is obviously convex.

To prove the lower semicontinuity of $\mathcal{I}$, we modify slightly the argument of Cvitanić and Karatzas. Let H^n be a sequence in $\mathcal{H}_x^\alpha$ that converges to H in $L^2(m_\alpha \otimes P)$. From the sublinear growth of U, there exist positive constants a and b such that $a + bH - U(H) \geq 0$ for all $H \in \mathcal{H}_x^\alpha$. Fatou's lemma gives us

$$\begin{aligned}\mathbb{E}\Big[\int_0^\infty e^{-\delta t}\Big(a + bH_t - U(H_t)\Big)\, dt\Big] &\leq \liminf_{n\to\infty} \mathbb{E}\Big[\int_0^\infty e^{-\delta t}\Big(a + bH_t^n - U(H_t^n)\Big)\, dt\Big] \\ &= \mathbb{E}\Big[\int_0^\infty e^{-\delta t}\big(a + bH_t\big)\, dt\Big] + \liminf_{n\to\infty} -\mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(H_t^n)\, dt\Big].\end{aligned}$$

The last step holds true due to the assumption $\delta > \alpha/2$ since, from Hölder's inequality,

$$\begin{aligned}\Big|\mathbb{E}\Big[\int_0^\infty e^{-\delta t} H_t\, dt\Big] - \mathbb{E}\Big[\int_0^\infty e^{-\delta t} H_t^n\, dt\Big]\Big| &\leq \mathbb{E}\Big[\int_0^\infty e^{-\delta t}|H_t - H_t^n|\, dt\Big] \\ &\leq \Big(\int_0^\infty e^{-(2\delta-\alpha)t}\Big)^{1/2} \mathbb{E}\Big[\int_0^\infty e^{-\alpha t}|H_t - H_t^n|^2\, dt\Big]^{1/2} \to 0.\end{aligned}$$

In conclusion,

$$-\mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(H_t)\,dt\Big] \le \liminf_{n\to\infty} -\mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(H_t^n)\,dt\Big],$$

and hence the lower-semicontinuity of $\mathcal{I}$ follows. □

Remark 4.4. *We mention that if the utility function $U(\cdot)$ is Hölder continuous, then one can prove, under a slightly different assumption on δ, that the functional $\mathcal{I}$ is continuous (and not merely semicontinuous).*

Set

$$\mathcal{V}(x) = -\inf_{H\in\mathcal{H}_x^\alpha} \mathcal{I}(H).$$

From Ekeland and Temam [10] we can conclude that there exists a $H^*\in\mathcal{H}_x^\alpha$ such that $\mathcal{V}(x) = -\mathcal{I}(H^*)$. We claim that Π^* associated to H^* is an optimal control, that is,

$$V(x) = \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(Y_t^{\Pi^*})\,dt\Big].$$

First, observe that since $Y^\Pi\in\mathcal{H}_x^\alpha$ for every $\pi\in\mathcal{A}_x$, $\mathcal{A}_x$ can be naturally imbedded in $\mathcal{H}_x^\alpha$, which implies $V(x)\le\mathcal{V}(x)$. We next claim that $H^* = Y^{\Pi^*}$ $m_\alpha\otimes P$ - a.e. If not, $H^* < Y^{\Pi^*}$ on a set with positive measure. But this contradicts the optimality of H^* since $Y^{\Pi^*}\in\mathcal{H}_x^\alpha$ and U is nondecreasing. Hence,

$$V(x)\le\mathcal{V}(x) = \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(H_t^*)\,dt\Big] = \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(Y_t^{\Pi^*})\,dt\Big] \le V(x).$$

Summing up, we have proven the following main theorem:

Theorem 4.5. *Suppose $\delta > \alpha/2$. For each $x\in\overline{\mathcal{D}}$, there exists an optimal control $\Pi^*\in\mathcal{A}_x$ for the singular stochastic control problem* (3) *such that* (4) *holds.*

Remark 4.6. *Theorem 4.5 may easily be generalized to hold for n risky assets, as is the set-up in* [4].

5. The Hamilton-Jacobi-Bellman equation

Thm. 4.5 is only of theoretical interest since it says nothing about the structure of the optimal strategies or how we can compute them. A natural way to compute the optimal strategies is via the dynamic programming method, which is based on Bellman's principle of dynamic programming:

Proposition 5.1 (Dynamic programming principle)**.** *For any stopping time τ and $t\ge 0$, the value function satisfies*

$$V(x) = \sup_{\Pi\in\mathcal{A}_x} \mathbb{E}\Big[\int_0^{t\wedge\tau} e^{-\delta s}U(Y^\Pi(s))\,ds + e^{-\delta(t\wedge\tau)}V(X^\Pi(t\wedge\tau))\Big]. \tag{6}$$

Thanks to the dynamic programming principle, the value function (3) can be associated with the so-called Hamilton-Jacobi-Bellman equation, which is the infinitesimal version of (6). For $x = (b, s, y) \in \overline{\mathcal{D}}$, define a second order degenerate elliptic integro-differential operator $\mathcal{A}$ by

$$\mathcal{A}v = -\beta y v_y + rbv_b + asv_s + \frac{1}{2}\sigma^2 s^2 v_{ss} + \int_{\mathbb{R}\setminus\{0\}} \Big(v(b, (1+\eta(z))s, y) - v(x) - \eta(z) s v_s(x) \Big)\, n(dz).$$

The Hamilton-Jacobi-Bellman equation of our control problem is a second order degenerate elliptic integro-differential variational inequality of the form

$$F(x, v, v_b, v_s, v_y, v_{ss}) := \max\Big(U(y) - \delta v + \mathcal{A}v, -v_b + \beta v_y, -(1+\lambda)v_b + v_s, (1-\mu)v_b - v_s \Big) = 0. \tag{7}$$

The point is now that value function (3) as well as the optimal control Π^* (whose existence is guaranteed by Thm. 4.5) can be found by (numerically) solving the fully nonlinear partial differential equation in (7).

The main result of our previous work [4] was a characterization of the value function (3) as the unique constrained viscosity solution [6] of the Hamilton-Jacobi-Bellman equation, which indeed constitutes a starting point for computing (numerically) the optimal value (3) as well as the optimal control Π^*.

Since we have only been able to show that the value function is continuous (see [4]), we cannot interpret the value function as a solution of (7) in the usual classical sense, but we have to resort to a weaker notion of solution that does not require differentiability of candidate solutions. The proper notion of weak solutions turns out to be that of constrained viscosity solutions as described in, e.g., Crandall, Ishii, and Lions [6].

We recall that the value function V is a *constrained viscosity solution* of (7) if it is simultaneously a viscosity subsolution in $\overline{\mathcal{D}}$ and a viscosity supersolution in $\mathcal{D}$. For example, the value function V is a *viscosity subsolution* of (7) in $\overline{\mathcal{D}}$ if $\forall \phi \in C^2(\overline{\mathcal{D}})$ (growing at most linearly as $x \to \infty$) we have:

$$\begin{cases} \text{for each } x \in \overline{\mathcal{D}} \text{ s. t. } V \leq \phi \text{ and } (V - \phi)(x) = 0, \\ F(x, \phi, \phi_b, \phi_s, \phi_y, \phi(x)) \geq 0. \end{cases} \tag{8}$$

A *viscosity supersolution* is defined similarly, see [4] for details.

The purpose of this section is to give an alternative proof of the viscosity subsolution property of the value function, which is simpler than the proof in [4]. The proof below exploits that we have Thm. 4.5 at our disposal. We refer to [4] for results concerning continuity of the value function as well as uniqueness of the viscosity solution characterization.

Theorem 5.2. *The value function* (3) *is a constrained viscosity solution of* (7).

Proof. The proof of the viscosity supersolution property goes as before [4]. We therefore concentrate on the viscosity subsolution property.

Let ϕ be as in (8). Arguing by contradiction, we suppose that the subsolution inequality (8) is violated. Then, by continuity, there is a nonempty open ball $\mathcal{N}$ centered at x and $\varepsilon > 0$ such that $V \leq \phi - \varepsilon$ on $\partial\mathcal{N} \cap \mathcal{D}$ and in $\overline{\mathcal{N} \cap \mathcal{D}}$ we have

$$\beta\phi_y - \phi_b \leq 0, \quad -(1+\lambda)\phi_b + \phi_s \leq 0, \quad (1-\mu)\phi_b - \phi_s \leq 0, \tag{9}$$

as well as $U(\cdot) - \delta\phi + \mathcal{A}\phi \leq -\varepsilon\delta$.

From Thm. 4.5, there exists an optimal investment-consumption strategy $\Pi^*(t) = (L^*(t), M^*(t), C^*(t)) \in \mathcal{A}_x$. Let $X^*(t) = (B^*(t), S^*(t), Y^*(t))$ denote the corresponding optimal trajectory with $X^*(0) = x$. In Lemma 5.3 below, it is shown that $X^*(t)$ has no control-jumps P - a.s. at x. Hence P - a.s., we have

$$\tau = \inf\{t \in [0,\infty) : X^*(t) \notin \overline{\mathcal{N} \cap \mathcal{D}}\} > 0.$$

Let us introduce the short-hand notation $\hat{X}^*(t)$ for the vector

$$\Big(B(t-) - \Delta C^*(t) - (1+\lambda)\Delta L^*(t) + (1+\mu)\Delta M^*(t),$$
$$S(t-) + \Delta L^*(t) - \Delta M^*(t), \, Y(t-) + \beta\Delta C^*(t)\Big),$$

and let $\Delta^{\Pi^*}\phi(t) := \phi(\hat{X}(t)) - \phi(X(t-))$. Note that by the dynamic programming principle (6), we can without loss of generality assume that $\hat{X}^*(t) \in \overline{\mathcal{N} \cap \mathcal{D}}$. Let $L^{*,c}$, $M^{*,c}$, and $C^{*,c}$ denote the continuous parts of L^*, M^*, and C^*, respectively.

Using Itô's formula for semimartingales together with the inequalities stated above (see 9), we get

$$\begin{aligned}
V(x) &= \mathbb{E}\Big[\int_0^\tau e^{-\delta t} U(Y^*(t))\,dt + e^{-\delta\tau} V(X^*(\tau))\Big] \\
&\leq \mathbb{E}\Big[\int_0^\tau e^{-\delta t} U(Y^*(t))\,dt + e^{-\delta}\phi(X^*(\tau))\Big] \\
&\leq \mathbb{E}\Big[\phi(x) + \int_0^\tau e^{-\delta t}\Big(U(Y^*(t)) - \delta\phi(X^*(t)) + \mathcal{A}\phi(X^*(t))\Big)\,dt\Big] \\
&\quad + \mathbb{E}\Big[\int_0^\tau e^{-\delta t}\Big(-\phi_b + \beta\phi_y\Big)\,dC^{*,c}(t)\Big] \\
&\quad + \mathbb{E}\Big[\int_0^\tau e^{-\delta t}\Big(-(1+\lambda)\phi_b + \phi_s\Big)\,dL^{*,c}(t)\Big] \\
&\quad + \mathbb{E}\Big[\int_0^\tau e^{-\delta t}\Big((1-\mu)\phi_b - \phi_s\Big)\,dM^{*,c}(t)\Big] \\
&\quad + \mathbb{E}\Big[\sum_{t \in [0,1] \cap [0,\tau]} e^{-\delta t}\Delta^{\Pi^*}\phi(t)\Big] \leq \phi(x) - \varepsilon\mathbb{E}\big[e^{-\delta\tau}\big] < \phi(x),
\end{aligned}$$

which is a contradiction since $(V - \phi)(x) = 0$. □

Lem. 5.3 below, which was used in the proof of Thm. 5.2, is similar to Lem. 3.5 in Davis, Panas, and Zariphopoulou [8].

Lemma 5.3. *Let $A = A(\omega)$ denote the event that the optimal trajectory $X^*(t)$ starting at $x = (b, s, y)$ has an initial control-jump of size $(\varepsilon_L, \varepsilon_M, \varepsilon_C) > 0$. Suppose that the inequalities in* (9) *hold. Then $P(A) = 0$.*

Proof. Notice that the state (after the control-jump) is

$$\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C) := \Big(b - (1+\lambda)\varepsilon_L + (1-\mu)\varepsilon_M - \varepsilon_C,\ s + \varepsilon_L - \varepsilon_M,\ y + \beta\varepsilon_C\Big).$$

By the dynamic programming principle (6), we can without loss of generality assume that $\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C) \in \overline{\mathcal{N} \cap \mathcal{D}}$. Again by (6), we have

$$V(x) = \mathbb{E}\big[V\big(\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C)\big)\big] = \int_{A(\omega)} V\big(\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C)\big)\,dP + \int_{\Omega - A(\omega)} V(x)\,dP.$$

From this equality it follows that

$$\int_{A(\omega)} \Big(V\big(\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C)\big) - V(x)\Big)\,dP = 0,$$

and, since $V \le \phi$ and $(V - \phi)(x) = 0$ (recall that ϕ comes from (8)),

$$\int_{A(\omega)} \Big(\phi\big(\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C)\big) - \phi(x)\Big)\,dP \ge 0. \tag{10}$$

From (9), we get

$$\phi\big(\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C)\big) \le \phi\big(\hat{x}(\varepsilon, 0, 0)\big), \qquad \forall \varepsilon \le \varepsilon_L, \tag{11}$$

$$\phi\big(\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C)\big) \le \phi\big(\hat{x}(0, \varepsilon, 0)\big), \qquad \forall \varepsilon \le \varepsilon_M, \tag{12}$$

$$\phi\big(\hat{x}(\varepsilon_L, \varepsilon_M, \varepsilon_C)\big) \le \phi\big(\hat{x}(0, 0, \varepsilon)\big), \qquad \forall \varepsilon \le \varepsilon_C. \tag{13}$$

Suppose $\varepsilon_L > 0$. We then claim that

$$\big[-(1+\lambda)\phi_b(x) + \phi_s(x)\big]P(A) \ge 0. \tag{14}$$

From (10) and (11) it follows that

$$\int_{A(\omega)} \Big(\phi\big(\hat{x}(\varepsilon, 0, 0)\big) - \phi(x)\Big)\,dP \ge 0, \qquad \forall \varepsilon \le \varepsilon_L,$$

and therfore by Fatou's lemma

$$\int_{A(\omega)} \limsup_{\varepsilon \to 0} \Big[\frac{\phi\big(b - (1+\lambda)\varepsilon,\ s + \varepsilon,\ y\big) - \phi(b, s, y)}{\varepsilon}\Big]\,dP \ge 0.$$

Hence, (14) follows. Similarly, if $\varepsilon_M > 0$, we can use (10) and (12) to prove

$$\big[(1-\mu)\phi_b(x) - \phi_s(x)\big]P(A) \ge 0. \tag{15}$$

Finally, if $\varepsilon_C > 0$, we can use (10) and (13) to prove

$$\big[-\phi_b(x) + \beta\phi_y(x)\big]P(A) \ge 0. \tag{16}$$

Summing up, if at least one of the jump-sizes $\varepsilon_L, \varepsilon_M, \varepsilon_C$ is greater than zero, then we can conclude (from (9), (14), (15), (16)) that $P(A) = 0$. $\square$

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