THE CLASS TYPE G DISTRIBUTIONS ON $\mathbb{R}^d$ AND RELATED SUBCLASSES OF INFINITELY DIVISIBLE DISTRIBUTIONS

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Abstract. Classes of infinitely divisible distributions obtained by iteration of Gaussian randomization of Lévy measures are introduced and studied. Their relation to Urbanik–Sato nested classes of selfdecomposable distributions is also established.

1. Introduction

In our previous paper [MR00], we studied the class of type G distributions on $\mathbb{R}^d$ defined in the following way. A symmetric infinitely divisible probability distribution μ on $\mathbb{R}^d$ is of type G if its Lévy measure ν is of the form

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)], \qquad A \in \mathcal{B}_0(\mathbb{R}^d), \tag{1.1}$$

where ν_0 is a Borel measure on $\mathbb{R}^d \setminus \{0\}$, Z is the standard normal random variable, and $\mathcal{B}_0(\mathbb{R}^d)$ is the class of all Borel sets A in $\mathbb{R}^d$ such that $A \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$. Such kind of distributions combine Gaussian and Poissonian structures in a nontrivial way (see Section 5 in [MR00]). Denote by $TG(\mathbb{R}^d)$ the class of type G distributions on $\mathbb{R}^d$.

A typical representative of the class $TG(\mathbb{R}^d)$ is a symmetric stable distribution. In this paper we will use the following convention. Given a class of measures H on $\mathbb{R}^d$, we will denote by $\widetilde{H}$ the subset of H consisting of symmetric measures. Denote by $S(\mathbb{R}^d)$ and $I(\mathbb{R}^d)$ the classes of stable and infinitely divisible

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distributions on $\mathbb{R}^d$, respectively. Therefore, we have $\widetilde{S}(\mathbb{R}^d) \subset TG(\mathbb{R}^d) \subset \widetilde{I}(\mathbb{R}^d)$. Our goal is to introduce and investigate the nested classes $TG_m(\mathbb{R}^d)$, $m \geq 1$, between $TG_0(\mathbb{R}^d) := TG(\mathbb{R}^d)$ and $\widetilde{S}(\mathbb{R}^d)$, using the procedure somewhat analogous to Urbanik-Sato construction of subclasses of selfdecomposable distributions.

In Section 2, we define the classes $TG_m(\mathbb{R}^d)$, $m \geq 1$, and show that they form a strictly descending sequence. In Section 3, we compare our nested subclasses of $TG_0(\mathbb{R}^d)$ and those of the class $L_0(\mathbb{R}^d)$ of selfdecomposable distributions introduced and studied by Urbanik [U73] and Sato [S80]. A necessary and sufficient condition for a type G distribution on $\mathbb{R}^1$ to be selfdecomposable was given in [R91]. We generalize this result to $\mathbb{R}^d$ and give an answer to the converse problem: When is a symmetric selfdecomposable distribution of type G? We also study related problems.

Every distribution $\mu \in TG_m(\mathbb{R}^d)$ has its predecessor $\mu_0 \in TG_{m-1}(\mathbb{R}^d)$, as defined in Section 2. In Section 4, we study the relationship between μ and μ_0 along the following lines : If μ belongs to a certain class of distributions, then does μ_0 belong to the same class? The answers are obtained for some important classes in Theorem 4.1. Section 5 contains some examples and Section 6 discusses open problems.

We conclude the Introduction by stating a basic characterization theorem for type G distributions, which has been proved in [MR00], and will also be needed later in this paper.

Theorem A ([MR00]). *A symmetric probability measure μ on $\mathbb{R}^d$ is of type G if and only if it is infinitely divisible and its Lévy measure ν is either zero or represented as*

$$\nu(EB) = \int_B \lambda(dx) \int_E g_x(r^2)dr \qquad \text{for} \quad E \in \mathcal{B}(\mathbb{R}_+),\ B \in \mathcal{B}(S),$$

where λ is a probability measure on S and $g_x(r)$ is a jointly measurable function which, for any fixed x, is completely monotone on $(0, \infty)$ and satisfies

$$\int_0^\infty (1 \wedge r^2) g_x(r^2)\, dr = c \in (0, \infty)$$

with c independent of x. This representation is unique in the sense that, if $\nu \neq 0$ and two pairs (λ, g_x) and $(\widetilde{\lambda}, \widetilde{g}_x)$ both satisfy the above conditions, then $\lambda = \widetilde{\lambda}$ and

$g_x = \widetilde{g}_x$ *for* λ*-a.e.* x*. Moreover,* λ *is a symmetric probability measure and* $g_x = g_{-x}$ λ*-a.e.*

2. Subclasses of the class of type G distributions

In the following, if μ is infinitely divisible, we denote its Lévy measure by $\nu(\mu)$.

We first rewrite the definition of type G distribution. In the definition (1.1), ν_0 is a Borel measure. However, by Proposition 2.2 (i)–(ii) in [MR00], ν_0 in (1.1) is also a Lévy measure and can always be taken symmetric. For any $\mu_0 \in \widetilde{I}(\mathbb{R}^d)$, define $K(\mu_0)$ as the infinitely divisible distribution μ having the same Gaussian component as μ_0 and Lévy measure ν given by (1.1) with $\nu_0 = \nu_0(\mu_0)$. The symmetric distribution μ_0 will be called the *predecessor* of μ (relative to the operation K). The predecessor is uniquely defined. Indeed, suppose that μ has two predecessors μ_1 and μ_2. Then ν satisfies (1.1) with $\nu_0 = \nu_1(\mu_1)$ and $\nu_0 = \nu_2(\mu_2)$. By Proposition 2.2 (iii) in [MR00] $\nu_1 = \nu_2$, and since μ_1 and μ_2 have the same Gaussian part, $\mu_1 = \mu_2$. We have just shown that K is one-to-one. If we write

$$K(H) = \{K(\mu_0) : \ \mu_0 \in H\}, \quad H \subset \widetilde{I}(\mathbb{R}^d),$$

then

$$TG(\mathbb{R}^d) = K(\widetilde{I}(\mathbb{R}^d)).$$

Put $TG_{-1}(\mathbb{R}^d) = \widetilde{I}(\mathbb{R}^d)$ and $TG_0(\mathbb{R}^d) = TG(\mathbb{R}^d)$. Define for $1 \le m < \infty$,

$$TG_m(\mathbb{R}^d) = K(TG_{m-1}(\mathbb{R}^d)),$$

and

$$TG_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} TG_m(\mathbb{R}^d).$$

Theorem 2.1. $\widetilde{I}(\mathbb{R}^d) \supset TG_0(\mathbb{R}^d) \supset TG_1(\mathbb{R}^d) \supset \cdots \supset TG_m(\mathbb{R}^d) \supset TG_{m+1}(\mathbb{R}^d) \supset \cdots \supset TG_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d)$.

Proof. By the definition,

$$TG_{-1}(\mathbb{R}^d) \supset TG_0(\mathbb{R}^d).$$

Suppose that $TG_{m-1}(\mathbb{R}^d) \supset TG_m(\mathbb{R}^d)$ for some $0 \le m < \infty$. If $\mu \in TG_{m+1}(\mathbb{R}^d)$, then $\nu(\mu)(A) = \mathbf{E}[\nu_0(Z^{-1}A)]$, where ν_0 is the Lévy measure of the predecessor

$\mu_0 \in TG_m(\mathbb{R}^d)$. By the induction hypothesis, we have that $\mu_0 \in TG_{m-1}(\mathbb{R}^d)$. Hence $\mu \in TG_m(\mathbb{R}^d)$, concluding

$$TG_{m+1}(\mathbb{R}^d) \subset TG_m(\mathbb{R}^d).$$

The assertion $TG_m(\mathbb{R}^d) \supset TG_\infty(\mathbb{R}^d)$ is trivial from its definition.

We next show that $TG_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d)$. If $\mu_0 \in \widetilde{S}(\mathbb{R}^d)$, then $\nu(A) = E[\nu_0(Z^{-1}A)]$ is the Lévy measure of a symmetric stable distribution, where ν_0 is the Lévy measure of μ_0. Thus $K(\widetilde{S}(\mathbb{R}^d)) \subset \widetilde{S}(\mathbb{R}^d)$. Conversely, if $\mu \in \widetilde{S}(\mathbb{R}^d)$, then

$$\nu(\mu)(A) = \mathbf{E}[\nu_0(Z^{-1}A)],$$

where ν_0 is also the Lévy measure of a distribution in $\widetilde{S}(\mathbb{R}^d)$. For, since the Lévy measure of $\mu \in \widetilde{S}(\mathbb{R}^d)$ satisfies the condition $a^\alpha \nu(\mu)(A) = \nu(\mu)(a^{-1}A)$, for every $a > 0$ and $A \in \mathcal{B}_0(\mathbb{R}^d)$, where $\alpha \in (0,2]$ is the index of stability, (1.1) holds with $\nu_0 = (\mathbf{E}[|Z|^\alpha])^{-1}\nu$. Hence $\widetilde{S}(\mathbb{R}^d) \subset K(\widetilde{S}(\mathbb{R}^d))$ and thus $K(\widetilde{S}(\mathbb{R}^d)) = \widetilde{S}(\mathbb{R}^d)$, namely, $\widetilde{S}(\mathbb{R}^d)$ is invariant under the operation K. We thus have, for each $m \geq 0$,

$$\widetilde{S}(\mathbb{R}^d) = K^m(\widetilde{S}(\mathbb{R}^d)) \subset K^m(\widetilde{I}(\mathbb{R}^d)) = TG_m(\mathbb{R}^d),$$

where K^m is the mth iteration of K. Thus $\widetilde{S}(\mathbb{R}^d) \subset \bigcap_{m \geq 0} TG_m(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d)$. This completes the proof. $\square$

It might be asked whether the inclusions in Theorem 2.1 are strict or not. The answer is the following.

Theorem 2.2. *The inclusions in Theorem 2.1 are all strict, namely*

$$\begin{aligned}\widetilde{I}(\mathbb{R}^d) \supsetneqq TG_0(\mathbb{R}^d) \supsetneqq TG_1(\mathbb{R}^d) \supsetneqq \cdots \supsetneqq TG_m(\mathbb{R}^d) \supsetneqq TG_{m+1}(\mathbb{R}^d)& \\ \supsetneqq \cdots \supsetneqq TG_\infty(\mathbb{R}^d) \supsetneqq \widetilde{S}(\mathbb{R}^d).&\end{aligned}$$

Proof. First note that $TG_{-1}(\mathbb{R}^d) \supsetneqq TG_0(\mathbb{R}^d)$, since the existence of non-type G infinitely divisible distribution is assured by Theorem A.

We next show that if for some $m \geq 0$,

$$TG_{m-1}(\mathbb{R}^d) \supsetneqq TG_m(\mathbb{R}^d), \tag{2.1}$$

then

$$TG_m(\mathbb{R}^d) \supsetneqq TG_{m+1}(\mathbb{R}^d).$$

If (2.1) is true, then there exists a μ_0 such that $\mu_0 \in TG_{m-1}(\mathbb{R}^d)$ but $\mu_0 \notin TG_m(\mathbb{R}^d)$. Let $\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)]$, where $\nu_0 = \nu_0(\mu_0)$. Then the infinitely divisible distribution μ with the Lévy measure ν is in $TG_m(\mathbb{R}^d)$. However $\mu \notin TG_{m-1}(\mathbb{R}^d)$. Because if $\mu \in TG_{m-1}(\mathbb{R}^d)$, then the corresponding uniquely determined μ_0 must be in $TG_m(\mathbb{R}^d)$, which is impossible. We thus conclude that

$$TG_{m-1}(\mathbb{R}^d) \supsetneqq TG_m(\mathbb{R}^d), \quad \forall m \geq 0. \tag{2.2}$$

We next show that

$$TG_m(\mathbb{R}^d) \supsetneqq TG_\infty(\mathbb{R}^d), \quad \forall m \geq 0.$$

If there exists an m_0 such that

$$TG_{m_0}(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d),$$

then

$$TG_{m_0}(\mathbb{R}^d) = TG_{m_0+1}(\mathbb{R}^d) = \cdots = TG_\infty(\mathbb{R}^d),$$

which contradicts (2.2).

Finally the fact that $TG_\infty(\mathbb{R}^d) \supsetneqq \widetilde{S}(\mathbb{R}^d)$ follows from Corollary 3.1 in Section 3, and so the rest of the proof is postponed to the end of Section 3. $\square$

The class $TG_\infty(\mathbb{R}^d)$ has the following special property.

Theorem 2.3. *$TG_\infty(\mathbb{R}^d)$ is invariant under the operation K and the largest class among such classes.*

Proof. By Theorem 2.1,

$$TG_m(\mathbb{R}^d) \supset TG_{m+1}(\mathbb{R}^d) = K(TG_m(\mathbb{R}^d)),$$

and hence

$$\bigcap_{m\geq 0} TG_m(\mathbb{R}^d) \supset \bigcap_{m\geq 0} K(TG_m(\mathbb{R}^d)) \supset K\left(\bigcap_{m\geq 0} TG_m(\mathbb{R}^d)\right).$$

Thus

$$TG_\infty(\mathbb{R}^d) \supset K(TG_\infty(\mathbb{R}^d)).$$

Let us show the converse inclusion. Let $\mu \in TG_\infty(\mathbb{R}^d)$. Then for any $m \geq 0$, $\mu \in TG_m(\mathbb{R}^d)$. Hence μ has the predecessor μ_0 in every class $TG_{m-1}(\mathbb{R}^d)$. Since the predecessor is uniquely defined,

$$\mu_0 \in \bigcap_{m \geq 0} TG_m(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d),$$

and hence

$$\mu \in K(TG_\infty(\mathbb{R}^d)).$$

We thus conclude that

$$K(TG_\infty(\mathbb{R}^d)) = TG_\infty(\mathbb{R}^d).$$

We next show that $TG_\infty(\mathbb{R}^d)$ is the largest class among such classes. Suppose that $H(\subset \widetilde{I}(\mathbb{R}^d))$ satisfies that $K(H) = H$. As before, for each $m \geq 0$,

$$H = K^m(H) \subset K^m(\widetilde{I}(\mathbb{R}^d)) = TG_m(\mathbb{R}^d),$$

and thus

$$H \subset \bigcap_{m \geq 0} TG_m(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d).$$

This completes the proof. $\square$

In one dimensional case ($d = 1$), a random variable X with distribution μ in $TG_0(\mathbb{R}^1)$ can be characterized by

$$X \stackrel{d}{=} V^{1/2} Z,$$

where V is some nonnegative infinitely divisible random variable independent of Z and $\stackrel{d}{=}$ means equivalence in law. Then a natural question is how we can characterize X with μ in $TG_m(\mathbb{R}^1), m = 1, 2, \ldots$, or what type of restriction on V assures that μ belongs to $TG_m(\mathbb{R}^1)$.

To answer this question, we need an observation found in [MR00]. For any given nonnegative infinitely divisible random variable V, there exists a Lévy process $\{V_0(t)\}$ such that its quadratic variation $[V_0, V_0](t)$ satisfies $[V_0, V_0](1) \stackrel{d}{=} V$. Then the Lévy measure ν_0 in (1.1) is the Lévy measure of $V_0(1)$. We call $\{V_0(t)\}$ the Lévy process associated with V. We thus have the following equivalence from the definition of $TG_m(\mathbb{R}^1)$.

Theorem 2.4. *Let $m = 1, 2, \ldots$. Then the following are equivalent.*

(i) $\mu \in TG_m(\mathbb{R}^1)$.

(ii) Let X be a random variable with distribution μ. Then

$$X \stackrel{d}{=} V^{1/2} Z,$$

where Z is the standard normal random variable and V is some nonnegative infinitely divisible random variable independent of Z, and the distribution μ_0 of $V_0(1)$, $\{V_0(t)\}$ being the Lévy process associated with V, belongs to $TG_{m-1}(\mathbb{R}^1)$.

3. The Urbanik-Sato nested subclasses of symmetric selfdecomposable distributions

Urbanik [U73] and Sato [S80] introduced and studied the nested classes $L_m(\mathbb{R}^d)$, $m = 0, 1, 2, \ldots, \infty$, between $I(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$, which are defined in the following way.

In general, for $H \subset I(\mathbb{R}^d)$, define

$$Q(H) = \{\mu \in I(\mathbb{R}^d) : \text{for any } a \in (0,1), \text{there exists } \rho_a \in H$$
$$\text{such that } \widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\rho}_a(\theta),\ \forall \theta \in \mathbb{R}^d\},$$

where $\widehat{\mu}$ is the characteristic function of μ.

Then, $L_0(\mathbb{R}^d)$ is defined as

$$L_0(\mathbb{R}^d) = Q(I(\mathbb{R}^d)),$$

and $L_m(\mathbb{R}^d), m = 1, 2, \ldots$, are defined inductively as

$$L_m(\mathbb{R}^d) = Q(L_{m-1}(\mathbb{R}^d))$$

and

$$L_\infty(\mathbb{R}^d) = \bigcap_{m \geq 0} L_m(\mathbb{R}^d).$$

Then it was shown that

$$I(\mathbb{R}^d) \supset L_0(\mathbb{R}^d) \supset L_1(\mathbb{R}^d) \supset \cdots \supset L_\infty(\mathbb{R}^d) \supset S(\mathbb{R}^d).$$

Distributions in $L_0(\mathbb{R}^d)$ are called selfdecomposable. Throughout this paper, we are only concerned with symmetric distributions. Therefore we will consider classes $\widetilde{L}_m(\mathbb{R}^d)$. Now we have two sequences of nested classes between $\widetilde{I}(\mathbb{R}^d)$ and $\widetilde{S}(\mathbb{R}^d)$.

$$\text{(i)}\quad \widetilde{I}(\mathbb{R}^d) \supset TG_0(\mathbb{R}^d) \supset TG_1(\mathbb{R}^d) \supset \cdots \supset TG_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d)$$

and

$$\text{(ii)}\quad \widetilde{I}(\mathbb{R}^d) \supset \widetilde{L}_0(\mathbb{R}^d) \supset \widetilde{L}_1(\mathbb{R}^d) \supset \cdots \supset \widetilde{L}_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d).$$

Then a natural question is to compare two sequences. The following is due to Sato [S80].

Theorem B ([S80]). *A probability measure $\mu \in I(\mathbb{R}^d)$ is selfdecomposable, namely in $L_0(\mathbb{R}^d)$ if and only if its Lévy measure ν is either zero or represented as*

$$\nu(EB) = \int_B \lambda(dx) \int_E \frac{k_x(r)}{r} dr \qquad \textit{for} \quad E \in \mathcal{B}(\mathbb{R}_+),\ B \in \mathcal{B}(S)$$

where λ is a probability measure on S and $k_x(r)$ is, for any fixed x, a nonnegative nonincreasing right-continuous function of r satisfying

$$\int_0^\infty (1 \wedge r^2) \frac{k_x(r)}{r}\, dr = c \in (0, \infty)$$

with c independent of x, and for any r, $k_x(r)$ is a measurable function of x. This representation is unique in the sense that, if $\nu \neq 0$ and two pairs (λ, k_x) and $(\widetilde{\lambda}, \widetilde{k}_x)$ both satisfy the above conditions, then $\lambda = \widetilde{\lambda}$ and $k_x = \widetilde{k}_x$ for λ-a.e. x.

A question when a given type G distribution on $\mathbb{R}^1$ is selfdecomposable was answered in [R91], namely, a type G distribution is selfdecomposable if and only if $x^{1/2} g_x(r)$ is nonincreasing with respect to r on $(0, \infty)$. The proof in [R91] did not use Theorem B, but once we have Theorems A and B, we can relate selfdecomposable and type G distributions in $\mathbb{R}^d$ using spectral forms of their Lévy measures (which are unique).

Theorem 3.1. *(i) Let $\mu \in TG_0(\mathbb{R}^d)$. Then $\mu \in L_0(\mathbb{R}^d)$ if and only if for λ-a.e. x $r^{1/2} g_x(r)$ is nonincreasing with respect to r on $(0, \infty)$.*

(ii) Let $\mu \in \widetilde{L}_0(\mathbb{R}^d)$. Then $\mu \in TG_0(\mathbb{R}^d)$ if and only if for λ-a.e. x $k_x(r^{1/2})/r^{1/2}$ is complete monotone.

Proof. It follows from Theorems A and B that

$$rg_x(r^2) = k_x(r) \tag{3.1}$$

for λ-a.e. x. The theorem follows from (3.1). $\square$

Sato [S80] also gave a necessary and sufficient condition for $\mu \in L_m(\mathbb{R}^d)$, $m = 1, 2, ..., \infty$. Define $h_x(s) = k_x(e^{-s})$, and call it the h-function of $\mu \in L_0(\mathbb{R}^d)$. For $\delta > 0$, let Δ_δ be the difference operator, $\Delta_\delta f(s) = f(s+\delta) - f(s)$, and Δ_δ^n be its nth iteration. We say that a function $f(s)$ is monotone of order n if

$$\Delta_\delta^j f(s) \geq 0 \quad \text{for } \delta > 0, s \in \mathbb{R}^1, \tag{3.2}$$

for any $j = 0, 1, ..., n$. When (3.2) holds for all integers j, f is called absolutely monotone. Then one of results by Sato [S80] is the following.

Theorem C ([S80]). *Let $m = 0, 1, 2, ..., \infty$. A probability measure μ belongs to $L_m(\mathbb{R}^d)$ if and only if $\mu \in L_0(\mathbb{R}^d)$ and h-function $h_x(s)$ of μ is monotone of order $m + 1$ for λ-a.e. x, where λ is the spherical component of the Lévy measure of μ, and when $m = \infty$, being monotone of order $m+1$ is understood as being absolutely monotone.*

The next theorem is a direct consequence of Theorem C and the relation (3.1).

Theorem 3.2. *Let $\mu \in TG_0(\mathbb{R}^d)$, and $m = 0, 1, 2, ..., \infty$. Then $\mu \in L_m(\mathbb{R}^d)$ if and only if*

$$h_x(s) = e^{-s} g_x(e^{-2s})$$

is monotone of order $m + 1$ (absolutely monotone when $m = \infty$) for λ-a.e. x.

In [MR00], we have shown that $TG_0(\mathbb{R}^d)$ is closed under convolution and weak convergence. By exactly the same argument, we can show the following.

Theorem 3.3. *The classes $TG_m(\mathbb{R}^d)$, $m = 1, 2, ..., \infty$, are closed under convolution and weak convergence.*

Corollary 3.1. $TG_\infty(\mathbb{R}^d) \supset \widetilde{L}_\infty(\mathbb{R}^d)$.

Proof. It is known ([S80]) that $L_\infty(\mathbb{R}^d)$ is the smallest class containing the class $S(\mathbb{R}^d)$, closed under convolution and weak convergence, and thus $\widetilde{L}_\infty(\mathbb{R}^d)$ is the smallest class containing the class $\widetilde{S}(\mathbb{R}^d)$, closed under convolution and weak convergence. This fact combined with Theorem 3.3 for $m = \infty$ yields the conclusion. □

A consequence of Corollary 3.1 is that convolutions of symmetric stable distributions of different indices are of type G. This fact is pointed out in [R91] for the case $d = 1$.

Proof of Theorem 2.2 (continued). As stated above in the proof of Corollary 3.1, we know that $\widetilde{L}_\infty(\mathbb{R}^d) \underset{\neq}{\supseteq} \widetilde{S}(\mathbb{R}^d)$, because, for instance, convolutions of symmetric stable distributions of different indices are in $\widetilde{L}(\mathbb{R}^d)$ but not in $\widetilde{S}(\mathbb{R}^d)$. Thus by Corollary 3.1,

$$TG_\infty(\mathbb{R}^d) \supset \widetilde{L}_\infty(\mathbb{R}^d) \underset{\neq}{\supseteq} \widetilde{S}(\mathbb{R}^d).$$

This completes the proof of Theorem 2.2. □

4. Some invariant properties of type G distributions

The first two statements, (i) and (ii) of Theorem 4.1, give examples of invariant properties under the operation K. (iii) and (iv) show that selfdecomposability of $K(\mu_0)$ is inherited from its predecessor μ_0 but is not a K-invariant property (see Section 2 for the definition of K).

Theorem 4.1. *Suppose that $\mu \in TG_m(\mathbb{R}^d)$ and let $\mu_0 \in TG_{m-1}(\mathbb{R}^d)$ be its predecessor, $m \geq 0$. Then the following holds.*
(i) μ is operator stable if and only if μ_0 is operator stable.
(ii) μ is $\mathbb{R}^d$-valued semi-stable if and only if μ_0 is $\mathbb{R}^d$-valued semi-stable.
(iii) If μ_0 is selfdecomposable, then so is μ.
(iv) There is a type G μ such that μ is selfdecomposable, but μ_0 is not selfdecomposable.

Proof. (i) The "if" part. If μ_0 is operator stable with some exponent matrix M, then its Lévy measure ν_0 satisfies that for any $a > 0$

$$a\nu_0(A) = \nu_0(b^{-M}A), \quad A \in \mathcal{B}_0(\mathbb{R}^d), \tag{4.1}$$

where $t^M = \sum_{k=0}^{\infty} \frac{1}{k!}(\log t)^k M^k$, for $t > 0$ and a matrix M. Then we have

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)] = \mathbf{E}[a^{-1}\nu_0(Z^{-1}a^{-M}A)] = a^{-1}\nu(a^{-M}A),$$

concluding that μ is operator stable.

The "only if" part. If μ is operator stable with some exponent M, then its Lévy measure ν satisfies the relation in (4.1) for ν instead of ν_0. Thus we have

$$\mathbf{E}[a\nu_0(Z^{-1}A)] = \mathbf{E}[\nu_0(Z^{-1}a^{-M}A)],$$

and by Proposition 2.3 in [MR00], we obtain

$$a\nu_0(\cdot) = \nu_0(a^{-M}\cdot),$$

concluding that μ_0 is operator stable.

(ii) The "if" part. If μ_0 is $\mathbb{R}^d$-valued semi-stable, then for some $r \in (0,1)$ and $\alpha \in (0,2]$,

$$r\nu_0(A) = \nu_0(r^{-1/\alpha}A), \quad A \in \mathcal{B}_0(\mathbb{R}^d). \tag{4.2}$$

Then obviously, ν satisfies (4.2) for the same r and α, which assures the semi-stability of μ. The "if" part can be shown as in the second half part of the proof of (i).

(iii) Since μ_0 is selfdecomposable, we have for each $a \in (0,1)$,

$$\nu_0(A) = \nu_0(aA) + \nu_0^a(A),$$

where ν_0^a is a Lévy measure. Thus the Lévy measure ν of μ satisfies

$$\nu(A) = \nu(aA) + \nu^a(A),$$

where ν^a is another Lévy measure. This implies the selfdecomposability of μ.

(iv) We use the same idea for Theorem 4.1 in [MR00]. Let $D_1 = \{x \in \mathbb{R}^d : 1 < |x| < 2\}$ and $D_2 = \{x \in \mathbb{R}^d : 0 < |x| < 1\}$. Let

$$\rho_0(A) = \lambda_d(A \cap D_1) - \varepsilon \lambda_d(A \cap D_2), \quad 0 < \varepsilon < 1,$$

and

$$\rho(A) = \mathbf{E}[\rho_0(Z^{-1}A)], \tag{4.3}$$

where λ_d is the Lebesgue measure In $\mathbb{R}^d$. Then we have shown in the proof of Theorem 4.1 in [MR00] that ρ_0 is not a measure, but ρ is a measure. Furthermore, these two ρ_0 and ρ satisfy conditions in (2.1) in Proposition 2.1 of [S98], and thus we can define

$$\nu_0(A) = \int_{\mathbb{R}^d} \rho_0(dx) \int_0^\infty \mathbf{1}_A(e^{-t}x)dt \tag{4.4}$$

and

$$\nu(A) = \int_{\mathbb{R}^d} \rho(dx) \int_0^\infty \mathbf{1}_A(e^{-t}x)dt. \tag{4.5}$$

By a theorem due to Urbanik [U69], (4.5) is the Lévy measure of some selfdecomposable distribution, because ρ is a measure. On the other hand, Sato [S98] showed that ν_0 is a Lévy measure, but the distribution whose Lévy measure is ν_0 in (4.4) is not selfdecomposable, (Proposition 2.2 of [S98]). It follows from (4.3) that

$$\begin{aligned}
\nu(A) &= \int_{\mathbb{R}^d} \mathbf{E}[\rho_0(Z^{-1}dx)] \int_0^\infty \mathbf{1}_A(e^{-t}x)dt \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} \rho_0(dx) \int_0^\infty \mathbf{1}_{Z^{-1}A}(e^{-t}x)dt\right] \\
&= \mathbf{E}[\nu_0(Z^{-1}A)].
\end{aligned}$$

Thus the infinitely divisible probability measure, whose Lévy measure is ν in (4.5), is of type G and satisfies our requirements in the statement (iv). This completes the proof of (iv). $\square$

Related to Theorem 4.1 (iv), we want to know under what conditions in addition to the selfdecomposability of μ, μ_0 is selfdecomposable. To answer this question, we first prove the following.

Theorem 4.2. *For any $H \subset \widetilde{I}(\mathbb{R}^d)$,*

$$K(Q(H)) = Q(K(H)).$$

Proof. We first show that $K(Q(H)) \subset Q(K(H))$. Suppose $\mu \in K(Q(H))$. Then its Lévy measure ν is represented as in (1.1), and μ_0 whose Lévy measure is ν_0 in (1.1) satisfies that for each $a \in (0,1)$, there exists $\rho^a \in H$ such that $\widehat{\mu}_0(\theta) = \widehat{\mu}_0(a\theta)\widehat{\rho_0^a}(\theta)$. Thus the respective Lévy measures ν_0 and ν_0^a of μ_0 and ρ_0^a satisfy

$$\nu_0(A) = \nu_0(aA) + \nu_0^a(A).$$

Hence we have

$$\nu(A) = \mathbf{E}[\nu_0(aZ^{-1}A)] + \mathbf{E}[\nu_0^a(Z^{-1}A)] = \nu(aA) + \xi^a(A),$$

implying that

$$\widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\eta^a}(\theta),$$

where $\eta^a \in I(\mathbb{R}^d)$ is the probability distribution with Lévy measure ξ^a and $\eta^a \in K(H)$. This concludes that $\mu \in Q(K(H))$.

We next show that $Q(K(H)) \subset K(Q(H))$. Suppose $\mu \in Q(K(H))$. Then for any $a \in (0,1)$ there exists $\rho_a \in K(H)$ such that

$$\widehat{\mu}(\theta) = \widehat{\mu}(a\theta) = \widehat{\rho}_a(\theta),$$

If $\rho_a \in K(H)$, then its Lévy measure ν^a is represented as

$$\nu^a(A) = \mathbf{E}[\nu_0^a(Z^{-1}A)] \tag{4.6}$$

for some Lévy measure ν_0^a, depending on a, whose corresponding infinitely divisible distribution belongs to the class H. However, (4.6) is equivalent to that

$$\mathbf{E}[\nu_0^a(Z^{-1}A)] = \nu(A) - \nu(aA) = \nu((1-a)A)$$

for any $a \in (0,1)$ and any $A \in \mathcal{B}_0(\mathbb{R}^d)$. This means that ν has the same property as in (1.1). By the uniqueness of ν_0 in (1.1), the infinitely divisible distribution μ_0 with Lévy measure ν_0 belongs to the same class as ρ_a, namely, $\mu_0 \in K(H)$.

We thus conclude that $\mu \in Q(K(H))$, and the proof of Theorem 4.2 is completed. $\square$

Theorem 4.3. *If μ is selfdecomposable, namely if for any $a \in (0,1)$, there exists $\rho_a \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\rho}_a(\theta)$, and further if ρ_a is of type G, then μ_0, the predecessor of μ, is selfdecomposable.*

Proof. Applying Theorem 4.2 to the case $H = I(\mathbb{R}^d)$, we have

$$K(L_0(\mathbb{R}^d)) = Q(TG_0(\mathbb{R}^d)).$$

Therefore, the following two statements are equivalent:
(i) μ is selfdecomposable such that for any $a \in (0,1)$, $\widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\rho}_a(\theta)$, where ρ_a is of type G.
(ii) μ is of type and its predecessor μ_0 is selfdecomposable.
This equivalence concludes the statement of the theorem. □

5. Some examples

Here we give simple examples of $\mu \in TG_m(\mathbb{R}^1), m = 0, 1$. We start with a lemma.

Lemma 5.1. *Let Z be the standard normal random variable and Y be a positive random variable independent of Z. Then $|Z|^p Y$ is infinitely divisible for any $p \geq 2$.*

Proof. Let f be the density of $|Z|^p$. Observe that

$$f(x) = \frac{2}{p\sqrt{2\pi}} x^{-1+1/p} \exp\{-x^{2/p}/2\}$$

is a completely monotone function (see, e.g., E 55.1, page 424 in [S99]). By Bernstein's theorem

$$f(x) = \int_{(0,\infty)} e^{-xu} \pi(du), \ \ x > 0$$

for some measure π. Denoting by G the distribution of Y, we have for every $a > 0$,

$$\begin{aligned} P\{|Z|^p Y \leq a\} &= \int_0^\infty \int_0^\infty \int_0^\infty \mathbf{1}(xy \leq a) e^{-xu} \pi(du) dx G(dy) \\ &= \int_0^\infty \int_0^\infty u^{-1}(1 - e^{-au/y}) \pi(du) G(dy). \end{aligned}$$

Hence $|Z|^p Y$ has completely monotone density and as such is infinitely divisible by a theorem of Goldie (see Theorem 51.6 in [S99]). □

Example 5.1. *If $Z_1, ..., Z_n$ are i.i.d. standard normal random variables, then $Z_1 \cdots Z_n$ is of type G.*

Proof. $Z_1 \cdots Z_n \stackrel{d}{=} Z_1|Z_2 \cdots Z_n|$ and $|Z_2 \cdots Z_n|^2$ is infinitely divisible by the above lemma. □

We are now going to show that the distribution of Z_1Z_2 belongs to $TG_1(\mathbb{R}^1)$.

Example 5.2. *Let Z_1 and Z_2 be independent standard normal random variables. Then the distribution of Z_1Z_2 is in $TG_1(\mathbb{R}^1)$.*

Proof. Since

$$X := Z_1Z_2 \stackrel{d}{=} Z_1(|Z_2|^2)^{1/2},$$

V in (1.1) is $|Z_2|^2$ in this case. By Theorem 2.4, it is enough to show that the distribution of $V_0(1)$ associated with $V = |Z_2|^2$ is of type G. Note that $|Z_1|^2$ is χ^2-distribution with freedom 1, thus is nonnegative infinitely divisible, and its Lévy measure is of the form

$$\nu([x, \infty)) = \int_x^\infty \frac{e^{-u/2}}{u} du.$$

Then ν_0 satisfies for $x > 0$

$$\begin{aligned}\nu_0([x, \infty)) &= \frac{1}{2}\nu([x^{1/2}, \infty)) = \frac{1}{2}\int_{x^{1/2}}^\infty \frac{e^{-u/2}}{u} du \\ &= \frac{1}{4}\int_x^\infty \frac{e^{-v^{1/2}/2}}{v} dv = \int_x^\infty g(v^2) dv.\end{aligned}$$

By a characterization for type G distributions (see Theorem 1 of [R91], also see Theorem 2.5 of [MR00]), it is enough to check that

$$g(x) = x^{-1/2} e^{-x^{1/4}/2}$$

is completely monotone. However, this is true, (see again e.g., E 55.1, page 424 in [S99]). The proof is completed. □

6. Further problems

We conclude the paper by stating some further problems which naturally arise form the observations in this paper.

Problem 1 : In Theorem A, we gave a necessary and sufficient condition for that $\mu \in TG_0(\mathbb{R}^d)$. Namely, $\mu \in TG_0(\mathbb{R}^d)$ if and only if the radial component of its Lévy measure has a density involving a completely monotone function $g_x(\cdot)$. What additional conditions on $g_x(\cdot)$ assure that $\mu \in TG_m(\mathbb{R}^d)$?

Problem 2 : Related to Corollary 3.1, we conjecture that $TG_\infty(\mathbb{R}^d) = \widetilde{L}_\infty(\mathbb{R}^d)$, namely $TG_\infty(\mathbb{R}^d)$ is also the *smallest* class containing the class $\widetilde{S}(\mathbb{R}^d)$ of all symmetric stable distributions, closed under convolution and weak convergence.

Problem 3 : In Examples 5.1 and 5.2, we have shown that the distribution of the product $Z_1 \cdots Z_n$ is of type G and furthermore the distribution of $Z_1 Z_2$ belongs to $TG_1(\mathbb{R}^d)$. Can one say more about the distribution of $Z_1 \cdots Z_n$?

Problem 4 : Type G distributions are continuous but are they absolutely continuous? If not, what is the smallest $m(\geq 1)$ such that any $\mu \in TG_m(\mathbb{R}^d)$ is absolutely continuous?

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