SUBORDINATION AND SELFDECOMPOSABILITY

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Abstract. Two facts are established concerning subordination and selfdecomposability. 1. Any subordinated process arising from a Brownian motion with drift and a selfdecomposable subordinator is selfdecomposable. 2. Selfdecomposable distributions of type G are not necessarily of type G_L. Consequences of the first fact on smoothness of the distributions are discussed.
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1. Introduction and results

A distribution μ on $\mathbb{R}$ is called selfdecomposable if, for every $b > 1$, there exists a distribution μ_b on $\mathbb{R}$ such that their characteristic functions $\widehat{\mu}(z)$, $\widehat{\mu}_b(z)$ satisfy

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z), \qquad z \in \mathbb{R}. \tag{1.1}$$

The class L of selfdecomposable distributions is a subclass of the class of infinitely divisible distributions. Its importance in the theory of Lévy processes, processes of Ornstein–Uhlenbeck type, and selfsimilar additive processes and in applications is now getting greater. See Barndorff-Nielsen and Shephard (2000) and Sato (1999). For other aspects of selfdecomposability, see Bondesson (1992) and Jurek and Mason (1993). A Lévy process whose distribution at each t is selfdecomposable is called a selfdecomposable process. A random variable with a selfdecomposable distribution is called a selfdecomposable random variable.

It is an interesting problem to see whether selfdecomposability is preserved under various transformations. In the case of a transformation which preserves selfdecomposability, it is also interesting how big the image of the class L is. Similar problems are considered on subclasses of the class L such as the classes L_m. See, for example,

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Sato and Yamazato (1983) for the transformation from background driving Lévy processes to processes of Ornstein–Uhlenbeck type. In this paper we study two problems of this sort.

A relation of subordination and selfdecomposability was found by Halgreen (1979) and Ismail and Kelker (1979). They noticed essentially the following fact. Let $\{X_t\colon t \geq 0\}$ and $\{Z_t\colon t \geq 0\}$ be the Brownian motion on $\mathbb{R}$ and a selfdecomposable subordinator, respectively. Let $\{Y_t\colon t \geq 0\}$ be the subordinated process arising from them. By this we mean that

$$\{X_t\} \text{ and } \{Z_t\} \text{ are independent} \tag{1.2}$$

and that

$$Y_t = X_{Z_t}, \qquad t \geq 0. \tag{1.3}$$

Then $\{Y_t\}$ is selfdecomposable. The process $\{Y_t\}$ is a Lévy process and the procedure above to get $\{Y_t\}$ is Bochner's subordination. This fact gives the selfdecomposability of normal inverse Gaussian distributions. Halgreen (1979) asked a question whether the same conclusion remains true if the Brownian motion is replaced by a Brownian motion with drift. He gave an affirmative answer under the restriction that the distribution of Z_1 is of Bondesson class, sometimes called *GGC* (generalized gamma convolutions) (see Sato (1999) p. 389 for definition). The same result was obtained also by Shanbhag and Sreehari (1979). In this way the two papers established selfdecomposability of generalized hyperbolic distributions.

One of the results in this paper is the following affirmative answer to Halgreen's question.

Theorem 1.1. *Let $\{X_t\colon t \geq 0\}$ be a Brownian motion with drift on $\mathbb{R}$ and let $\{Z_t\colon t \geq 0\}$ be a selfdecomposable subordinator. Then the subordinated process $\{Y_t\colon t \geq 0\}$ arising from them is selfdecomposable.*

Using the terminology of Barndorff-Nielsen and Shephard (2000), we can express Theorem 1.1 in this way: normal variance-mean mixtures using selfdecomposable mixing distributions are selfdecomposable. A related paper is Barndorff-Nielsen and Halgreen (1977).

The Brownian motion is strictly stable with index 2. Brownian motions with drift are stable with index 2, but not strictly stable. It is known that if the subordinand $\{X_t\}$ is strictly stable and the subordinator $\{Z_t\}$ is selfdecomposable, then the subordinated process $\{Y_t\}$ is selfdecomposable. This is another extension of the result of Halgreen and Ismail and Kelker and a multivariate generalization of this fact is given in Theorem 6.1 of Barndorff-Nielsen, Pedersen, and Sato (2000). In Section 4 we give a simple proof of this fact. We do not know whether Theorem 1.1 can be generalized to the case where the subordinand $\{X_t\}$ is a stable process which is not strictly stable.

A random variable Y is called of type G if there are a standard Gaussian random variable X and a nonnegative infinitely divisible random variable Z such that

$$X \text{ and } Z \text{ are independent} \tag{1.4}$$

and

$$Y \stackrel{\mathrm{d}}{=} Z^{1/2} X; \tag{1.5}$$

see Rosinski (1991) for an account of this class. The distribution of Y is also called of type G. In other words it is a normal variance mixture using an infinitely divisible mixing distribution. It is easy to see that a distribution μ on $\mathbb{R}$ is of type G if and only if μ is the distribution at time 1 of the subordinated process $\{Y_t\}$ arising from the Brownian motion $\{X_t\}$ and a subordinator $\{Z_t\}$. Thus the result of Halgreen and Ismail and Kelker is paraphrased as follows: a random variable Y of type G is selfdecomposable if the Z in the definition of type G is selfdecomposable, that is, if Y is of class G_L in the terminology of Jian (2000). The second result of this paper is that the converse assertion is not true.

Theorem 1.2. *There is a selfdecomposable random variable Y of type G for which one cannot find a nonnegative selfdecomposable Z and a standard Gaussian X satisfying* (1.4) *and* (1.5).

In other words, a selfdecomposable random variable of type G is not necessarily of type G_L. This disproves a conjecture of Jian (2000), p. 40.

Proofs of Theorems 1.1 and 1.2 are given in Sections 2 and 3.

2. Proof of Theorem 1.1

The generating triplet (A, ν, γ) of an infinitely divisible distribution μ on $\mathbb{R}$ is, by definition, a triplet of a nonnegative number A, a σ-finite measure ν satisfying $\nu(\{0\}) = 0$ and $\int (1 \wedge x^2)\nu(dx) < \infty$, and a real number γ such that

$$\widehat{\mu}(z) = \exp\left[-\frac{A}{2}z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx1_{[-1,1]}(x))\nu(dx) + i\gamma z\right]. \tag{2.1}$$

Here A and ν are called the Gaussian variance and the Lévy measure of μ, respectively. A Lévy process $\{X_t \colon t \geq 0\}$ is said to have the generating triplet (A, ν, γ) if the distribution of X_1 has the generating triplet (A, ν, γ). $\{X_t\}$ is a subordinator if and only if $A = 0$, $\nu((-\infty, 0)) = 0$, $\int_{(0,1]} x\nu(dx) < \infty$, and $\gamma \geq \int_{(0,1]} x\nu(dx)$, that is,

$$\widehat{\mu}(z) = \exp\left[\int_{(0,\infty)} (e^{izx} - 1)\nu(dx) + i\gamma_0 z\right] \tag{2.2}$$

with $\gamma_0 = \gamma - \int_{(0,1]} x\nu(dx) \geq 0$, which is called the drift of the subordinator. See Sato (1999) for detailed exposition.

Let $\{X_t\}$ be a Lévy process on $\mathbb{R}$ with generating triplet (A, ν, γ) and $\{Z_t\}$ be a subordinator with Lévy measure ρ and drift β_0. Let $\{Y_t\}$ be the subordinated Lévy process arising from them. Then the generating triplet $(A^\sharp, \nu^\sharp, \gamma^\sharp)$ of $\{Y_t\}$ is given as follows:

$$A^\sharp = \beta_0 A, \tag{2.3}$$

$$\nu^\sharp(B) = \beta_0\nu(B) + \int_{(0,\infty)} \mu^s(B)\rho(ds), \qquad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}), \tag{2.4}$$

$$\gamma^\sharp = \beta_0\gamma + \int_{(0,\infty)} \rho(ds) \int_{[-1,1]} x\mu^s(dx), \tag{2.5}$$

where μ^s is the distribution of X_s and $\mathcal{B}(\mathbb{R} \setminus \{0\})$ is the class of Borel subsets of $\mathbb{R} \setminus \{0\}$. See Theorem 30.1 of Sato (1999).

Lemma 2.1. *Fix a selfdecomposable process $\{X_t\}$. Suppose that, if $\{Z_t\}$ is a selfdecomposable subordinator satisfying $\beta_0 = 0$ and $\rho(ds) = s^{-1}1_{(0,a]}(s)ds$ for some $a \in (0, \infty)$, then the subordinated process $\{Y_t\}$ is selfdecomposable. Then, for any choice of a selfdecomposable subordinator $\{Z_t\}$, the subordinated process $\{Y_t\}$ is selfdecomposable.*

Proof. An infinitely divisible distribution is selfdecomposable if and only if its Lévy measure is of the form $|x|^{-1}k(x)dx$ with $k(x)$ increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$ (we are using the words increase and decrease in the weak sense). See Corollary 15.11 of Sato (1999). Let $\{Z_t\}$ be a selfdecomposable subordinator with Lévy measure ρ and drift β_0. Then $\rho(ds)=s^{-1}k(s)1_{(0,\infty)}(s)ds$ with $k(s)$ decreasing on $(0,\infty)$. The function $k(s)$ is the limit of an increasing sequence of functions $k_n(s)$, $n=1,2,\ldots$, of the form

$$k_n(s)=\sum_{j=1}^{m_n} b_{n,j}1_{(0,a_{n,j}]}(s),$$

where m_n is a positive integer, $0<a_{n,1}<\cdots<a_{n,m_n}$, and $b_{n,j}>0$ for $j=1,\ldots,m_n$. Let $\{Z_t^{(n)}\}$ be the subordinator with Lévy measure $s^{-1}k_n(s)1_{(0,\infty)}(s)ds$ and drift β_0. Recall that convolutions of selfdecomposable distributions are selfdecomposable. By virtue of (2.4), the assumption implies that the subordinated process $\{Y_t^{(n)}\}$ arising from $\{X_t\}$ and $\{Z_t^{(n)}\}$ is selfdecomposable. Since the limit of a sequence of selfdecomposable distributions is selfdecomposable, it follows that $\{Y_t\}$ is selfdecomposable. $\square$

Proof of Theorem 1.1. Let $\{X_t\}$, $\{Z_t\}$, and $\{Y_t\}$ be the processes in the theorem. Let (A,ν,γ), ρ, β_0, and $(A^\sharp,\nu^\sharp,\gamma^\sharp)$ be as above. Then $A=\frac{1}{2}$, $\nu=0$, and $\gamma\neq 0$. By Lemma 2.1 it is enough to prove the theorem under the assumption that $\rho(ds)=s^{-1}1_{(0,a]}(s)ds$ for some $a\in(0,\infty)$ and $\beta_0=0$. Thus, by (2.4), we have $\nu^\sharp(dx)=|x|^{-1}k^\sharp(x)dx$ with

$$k^\sharp(x)=|x|\int_0^a \frac{1}{\sqrt{2\pi s}}e^{-(x-\gamma s)^2/(2s)}\frac{ds}{s}. \tag{2.6}$$

All we have to show is that $k^\sharp(x)$ is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. The discussion for $x<0$ is reduced to that for $x>0$ by changing γ to $-\gamma$. So we only consider $x>0$. By change of variables $x^2/s=u$ we get

$$\begin{aligned}\sqrt{2\pi}k^\sharp(x)&=x\int_{x^2/a}^\infty e^{-(u^{1/2}-\gamma x u^{-1/2})^2/2}(x^2/u)^{-3/2}(x^2/u^2)du\\&=\int_{x^2/a}^\infty e^{-(u^{1/2}-\gamma x u^{-1/2})^2/2}u^{-1/2}du.\end{aligned} \tag{2.7}$$

Hence, if $\gamma<0$, then $k^\sharp(x)$ is decreasing in x on $(0,\infty)$.

Henceforth we assume that $\gamma > 0$. Write $b = a^{-1/2}$. We have, from (2.7),

$$\sqrt{2\pi}\frac{dk^\sharp(x)}{dx} = -e^{-(bx-\gamma/b)^2/2}(bx)^{-1}2b^2x + \int_{b^2x^2}^{\infty} e^{-(u^{1/2}-\gamma xu^{-1/2})^2/2}(u^{1/2}-\gamma xu^{-1/2})(\gamma u^{-1/2})u^{-1/2}du = \int_{b^2x^2}^{\infty} e^{-(u^{1/2}-\gamma xu^{-1/2})^2/2} f(u)du, \tag{2.8}$$

where

$$f(u) = -b(1-\gamma^2x^2u^{-2}) + \gamma(1-\gamma xu^{-1})u^{-1/2}.$$

If $x \geq \gamma b^{-2}$, then, rewriting $f(u)$ as

$$f(u) = -b + \gamma^2bx^2u^{-2} + \gamma u^{-1/2} - \gamma^2xu^{-3/2} = -b(1-\gamma b^{-1}u^{-1/2}) - \gamma^2xu^{-3/2}(1-bxu^{-1/2})$$

and noticing that $1-\gamma b^{-1}u^{-1/2} > 1-\gamma b^{-2}x^{-1} \geq 0$ and $1-bxu^{-1/2} > 0$ for $u > b^2x^2$, we get $dk^\sharp(x)/dx < 0$ from (2.8). If $0 < x < \gamma b^{-2}$, then $b^2x^2 < \gamma x$ and we get

$$\sqrt{2\pi}\frac{dk^\sharp(x)}{dx} = I_1 + I_2,$$

where

$$I_1 = \int_{b^2x^2}^{\gamma x} e^{-(u^{1/2}-\gamma xu^{-1/2})^2/2} f(u)du, \quad I_2 = \int_{\gamma x}^{\infty} e^{-(u^{1/2}-\gamma xu^{-1/2})^2/2} f(u)du.$$

The integral I_2 is, by change of variables $\gamma^2x^2u^{-1} = v$, written as

$$I_2 = \int_0^{\gamma x} e^{-(\gamma xv^{-1/2}-v^{1/2})^2/2}[-b(1-\gamma^{-2}x^{-2}v^2) + \gamma(1-\gamma^{-1}x^{-1}v)\gamma^{-1}x^{-1}v^{1/2}]\gamma^2x^2v^{-2}dv = -\int_0^{\gamma x} e^{-(v^{1/2}-\gamma xv^{-1/2})^2/2} f(v)dv.$$

Thus we get

$$I_1 + I_2 = -\int_0^{b^2x^2} e^{-(u^{1/2}-\gamma xu^{-1/2})^2/2} f(u)du = -\int_0^{b^2x^2} e^{-(u^{1/2}-\gamma xu^{-1/2})^2/2}(\gamma xu^{-1}-1)(b(\gamma xu^{-1}+1)-\gamma u^{-1/2})du.$$

The function $\gamma xu^{-1/2} + u^{1/2}$ is strictly decreasing for $0 < u < \gamma x$. Since $b^2x^2 < \gamma x$, we have, for $0 < u < b^2x^2$,

$$\gamma xu^{-1/2} + u^{1/2} > \gamma b^{-1} + bx \geq \gamma b^{-1}.$$

Hence $I_1 + I_2 < 0$ for $0 < x < \gamma b^{-2}$, which finishes the proof that $dk^\sharp(x)/dx < 0$ for $x > 0$. $\square$

3. Proof of Theorem 1.2

We prepare two simple lemmas.

Lemma 3.1. *Suppose that Y satisfies* (1.4) *and* (1.5) *with a standard Gaussian random variable X and a nonnegative random variable Z. Then the distribution of Z is determined by the distribution of Y.*

Proof. For any real u we have

$$E[e^{iuY}] = E[e^{iuZ^{1/2}X}] = E[e^{-u^2 Z/2}] = \int_{[0,\infty)} e^{-u^2 s/2} P_Z(ds),$$

where P_Z is the distribution of Z. Thus the characteristic function of the distribution P_Y of Y determines the Laplace transform of P_Z. Hence P_Z is determined by P_Y. $\square$

Lemma 3.2. *Let $a > b > 0$ and let*

$$f(s) = \begin{cases} 1, & 0 < s < a, \\ -1, & a \le s < a+b, \\ 1, & a+b \le s. \end{cases}$$

Then

$$\int_0^\infty e^{-xs} f(s)ds > 0 \quad \textit{for all } x > 0.$$

Proof. We have

$$\int_0^\infty e^{-xs} f(s)ds = \int_0^a e^{-xs}ds - \int_a^{a+b} e^{-xs}ds + \int_{a+b}^\infty e^{-xs}ds = \frac{1}{x} g(x)$$

for $x > 0$, where

$$g(x) = 1 - 2e^{-ax} + 2e^{-(a+b)x}.$$

We have $g(0+) = 1$, $g(\infty) = 1$, and $g'(x) = 2e^{-ax}(a - (a+b)e^{-bx})$. Hence $g(x)$ takes its minimum at $x = x_0 = \frac{1}{b} \log \frac{a+b}{a}$. The minimum value is expressed as

$$g(x_0) = 1 - \frac{2b}{a+b} \left(1 + \frac{b}{a}\right)^{-a/b} > 0,$$

since $a > b > 0$. This proves the assertion. $\square$

Proof of Theorem 1.2. By Lemma 3.1 it is enough to construct a selfdecomposable subordinated process $\{Y_t\}$ arising from the Brownian motion $\{X_t\}$ and a non-selfdecomposable subordinator $\{Z_t\}$. Assume, for a moment, that such $\{Y_t\}$,

$\{X_t\}$, and $\{Z_t\}$ are found. Then, as in Section 2, the generating triplet $(A^\sharp, \nu^\sharp, \gamma^\sharp)$ of $\{Y_t\}$ is expressed by the Lévy measure ρ and the drift β_0 of $\{Z_t\}$ as

$$A^\sharp = \beta_0, \tag{3.1}$$

$$\nu^\sharp(B) = \int_{(0,\infty)} \rho(ds) \int_B \frac{1}{\sqrt{2\pi s}} e^{-x^2/(2s)} dx. \tag{3.2}$$

By the selfdecomposability the Lévy measure $\nu^\sharp$ is of the form $\nu^\sharp(dx) = |x|^{-1}k^\sharp(x)dx$ with $k^\sharp(x)$ being decreasing on $(0,\infty)$. We have $k^\sharp(-x) = k^\sharp(x)$. Assume further that $\rho(ds) = s^{-1}k(s)ds$ on $(0,\infty)$ with a nonnegative measurable function $k(s)$. Then

$$\begin{aligned} k^\sharp(x) &= x \int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-x^2/(2s)} \frac{k(s)}{s} ds \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi r}} e^{-1/(2r)} \frac{k(x^2 r)}{r} dr \end{aligned} \tag{3.3}$$

for $x > 0$. We have

$$\frac{dk^\sharp(x)}{dx} = 2x \int_0^\infty \frac{1}{\sqrt{2\pi r}} e^{-1/(2r)} k'(x^2 r) dr \tag{3.4}$$

for $x > 0$, supposing that k is differentiable except at a finite number of points and the order of integration in r and differentiation in x is interchangeable. By change of variables $x^2 r = 1/u$ we get

$$\frac{dk^\sharp(x)}{dx} = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x^2 u/2} u^{-3/2} k'(u^{-1}) du. \tag{3.5}$$

Now we fix $a > b > 0$ and want to choose a function $k(s)$ such that

$$u^{-3/2} k'(u^{-1}) = -f(u),$$

where f is the function in Lemma 3.2. That is,

$$k'(u^{-1}) = \begin{cases} -u^{3/2}, & u \in (0,a) \cup (a+b,\infty), \\ u^{3/2}, & u \in (a, a+b). \end{cases}$$

This is equivalent to

$$k'(s) = \begin{cases} -s^{-3/2}, & s \in (0, \frac{1}{a+b}) \cup (\frac{1}{a}, \infty), \\ s^{-3/2}, & s \in (\frac{1}{a+b}, \frac{1}{a}). \end{cases} \tag{3.6}$$

Take $k(s)$ such that

$$k(s) = \begin{cases} 2s^{-1/2}, & s \in (0, \frac{1}{a+b}) \cup [\frac{1}{a}, \infty), \\ -2s^{-1/2} + c, & s \in [\frac{1}{a+b}, \frac{1}{a}), \end{cases} \tag{3.7}$$

where c is chosen to be $-2(a+b)^{1/2}+c \geq 0$. Then $k(s) \geq 0$ on $(0,\infty)$, $\int_0^1 k(s)ds < \infty$, $\int_1^\infty s^{-1}k(s)ds < \infty$, and (3.6) is satisfied. Define $k^\sharp(x)$ for $x > 0$ by (3.3) and define $k^\sharp(-x) = k^\sharp(x)$. Then the condition on the interchangeability of the order of integration and differentiation is satisfied. By Lemma 3.2 and by (3.5), $dk^\sharp(x)/dx$ is negative for $x > 0$. Hence $\nu^\sharp(dx) = |x|^{-1}k^\sharp(x)dx$ and $\rho(ds) = s^{-1}k(s)ds$ are the Lévy measures that we wanted to construct. Indeed, the Lévy process with Lévy measure $\nu^\sharp$ is selfdecomposable since $k^\sharp(x)$ is decreasing on $(0,\infty)$ and increasing on $(-\infty,0)$; the subordinator with Lévy measure ρ is not selfdecomposable since $k(s)$ is strictly increasing on $[\frac{1}{a+b}, \frac{1}{a})$. □

4. Remarks

Let $\{X_t\}$, $\{Z_t\}$, and $\{Y_t\}$ be the processes in Theorem 1.1. Let γ, ρ, β_0, and $(A^\sharp, \nu^\sharp, \gamma^\sharp)$ be as in the proof of the theorem. Let P_{Z_t} and P_{Y_t} be the distributions of Z_t and Y_t, respectively. Then, the two functions $k(s)$ and $k^\sharp(x)$ which express $\rho(ds) = s^{-1}k(s)ds$ and $\nu^\sharp(dx) = |x|^{-1}k^\sharp(x)dx$ are connected as follows:

$$\begin{aligned} k^\sharp(x) &= |x| \int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-(x-\gamma s)^2/(2s)} \frac{k(s)}{s} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(u^{1/2}-\gamma x u^{-1/2})^2/2} k(x^2u^{-1}) u^{-1/2} du \\ &= \frac{e^{\gamma x}}{\sqrt{2\pi}} \int_0^\infty e^{-(u/2)-\gamma^2x^2/(2u)} k(x^2u^{-1}) u^{-1/2} du \end{aligned} \tag{4.1}$$

for $x \in \mathbb{R} \setminus \{0\}$. It follows that

$$k^\sharp(0+) = k^\sharp(0-) = \frac{k(0+)}{\sqrt{2\pi}} \int_0^\infty e^{-u/2} u^{-1/2} du = k(0+), \tag{4.2}$$

since $\int_0^\infty e^{-u/2}u^{-1/2}du = \sqrt{2}\Gamma(1/2) = \sqrt{2\pi}$. Notice that the result (4.2) does not depend on γ. The same result holds also when $\{X_t\}$ is the Brownian motion ($\gamma = 0$). ¿From the selfdecomposability proved in Theorem 1.1 and from (4.2) follow many properties of P_{Y_t}. First, P_{Y_t} is unimodal by Yamazato's theorem (1978). Second, the smoothness of P_{Y_t} is expressed by $k^\sharp(0+) + k^\sharp(0-)$ as in Sato and Yamazato (1978). Some of the results are given below; see also Sections 28 and 53 of Sato (1999). Let $c_Z = k(0+)$ and $c_Y = k^\sharp(0+) + k^\sharp(0-)$. Then

$$c_Y = 2c_Z \tag{4.3}$$

by (4.2). If c_Z and c_Y are infinite or if $\beta_0 > 0$, then P_{Z_t} and P_{Y_t} have C^∞ densities on $\mathbb{R}$ for every $t > 0$. Assume that $k(0+) < \infty$ and $\beta_0 = 0$. Thus, by (2.5), $\{Y_t\}$ has drift 0. Define $N_Z(t)$ and $N_Y(t)$ as the integers satisfying $N_Z(t) < tc_Z \le N_Z(t) + 1$ and $N_Y(t) < tc_Y \le N_Y(t) + 1$. Then, for $t > 0$, P_{Y_t} (resp. P_{Z_t}) has $C^{N_Y(t)}$ (resp. $C^{N_Z(t)}$) density on $\mathbb{R}$ but does not have $C^{N_Y(t)+1}$ (resp. $C^{N_Z(t)+1}$) density on $\mathbb{R}$. In this sense, the distribution of $\{Y_t\}$ is twice as smooth as the distribution of $\{Z_t\}$. For $t \le 1/c_Y$, P_{Y_t} has mode 0 and the density $f_Y(t,x)$ of P_{Y_t} satisfies $f_Y(t, 0+) = \infty$ and $f_Y(t, 0-) = \infty$. On the other hand, for $t < 1/c_Z$, P_{Z_t} has mode 0 and $f_Z(t, 0+) = \infty$; for $t = 1/c_Z$, P_{Z_t} has mode 0 and, in order that $f_Z(1/c_Z, 0+) = \infty$, it is necessary and sufficient that $\int_0^1 (c_Z - k(s))s^{-1}ds = \infty$.

In Theorem 1.1 we have treated a Bownian motion with drift, which is not strictly stable. In the case of strictly stable subordinands, more results are known. The following fact is a special case of Theorem 6.1 of Barndorff-Nielsen, Pedersen, and Sato (2000). Its simple proof is presented here, as the proof of the general theorem is complicated. It suggests the nature of the first problem treated in this paper.

Proposition 4.1. *A subordinated process on $\mathbb{R}$ arising from a strictly stable subordinand and a selfdecomposable subordinator is selfdecomposable.*

Proof. As before denote by $\{X_t\}$, $\{Z_t\}$, and $\{Y_t\}$ the subordinand, the subordinator, and the subordinated. If $\{X_t\}$ is a linear deterministic motion, the assertion is trivial. So we assume that $\{X_t\}$ is strictly stable with index $\alpha \in (0, 2]$ and not a linear deterministic motion. We have

$$\{X_{at} \colon t \ge 0\} \stackrel{\mathrm{d}}{=} \{a^{1/\alpha} X_t \colon t \ge 0\}$$

for all $a > 0$. The distribution of X_t for all $t > 0$ has a density $p_t(x)$ satisfying

$$p_t(x) = t^{-1/\alpha} p_1(t^{-1/\alpha} x). \tag{4.4}$$

Using the same notation for the generating triplets as in Section 2, we have

$$\nu^\sharp(B) = \beta_0 \nu(B) + \int_B dx \int_0^\infty p_s(x) s^{-1} k_Z(s) ds, \qquad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

where $\rho(ds) = s^{-1}k_Z(s)ds$ with $k_Z(s)$ being decreasing on $(0,\infty)$. Since stable processes are selfdecomposable, we have $\nu(dx) = |x|^{-1}k_X(x)dx$ with $k_X(x)$ being increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$. Thus $\nu^\sharp$ is written as $\nu^\sharp(dx) = |x|^{-1}k^\sharp(x)dx$

with

$$k^\sharp(x) = \beta_0 k_X(x) + |x| \int_0^\infty p_s(x) s^{-1} k_Z(s) ds. \tag{4.5}$$

Combining (4.4) and (4.5) and using a variable $r = s^{-1/\alpha}|x|$, we obtain

$$\begin{aligned}
k^\sharp(x) &= \beta_0 k_X(x) + |x| \int_0^\infty p_1(s^{-1/\alpha} x) s^{-1-1/\alpha} k_Z(s) ds \\
&= \beta_0 k_X(x) + |x| \int_0^\infty p_1(r \operatorname{sgn} x)(|x|^\alpha r^{-\alpha})^{-(\alpha+1)/\alpha} k_Z(|x|^\alpha r^{-\alpha}) \alpha |x|^\alpha r^{-\alpha-1} dr \\
&= \beta_0 k_X(x) + \alpha \int_0^\infty p_1(r \operatorname{sgn} x) k_Z(|x|^\alpha r^{-\alpha}) dr
\end{aligned}$$

for $x \neq 0$. Therefore the monotonicity of $k_X(x)$ and $k_Z(s)$ leads to the monotonicity of $k^\sharp(x)$; $k^\sharp(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. This shows that $\{Y_t\}$ is selfdecomposable. $\square$

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