Quantum Stochastic Approach to the Description of Quantum Measurements

Elena R. Loubenets*

*Moscow State Institute of Electronics and Mathematics,
Technical University*

In the present paper we consider the problem of description of a generalized quantum measurement with outcomes in a measurable space. Analyzing the concepts of operational approach in quantum measurement theory, we introduce the notion of a *quantum stochastic representation of an instrument*. We show that the description of a generalized quantum measurement can be considered in the frame of a new general approach based on the notion of a family of *quantum stochastic evolution operators.* Such approach gives not only the complete *statistical* description of any quantum measurement but the complete description in a Hilbert space of the *stochastic* behaviour of a quantum system under a measurement.
In the frame of the proposed approach, which we call *quantum stochastic*, all possible schemes of measurements upon a quantum system can be considered.
In the case of repeated or continuous in time measurements the quantum stochastic approach allows to define in the most general case the notion of a family of posterior pure state trajectories (quantum trajectories in discrete or continuous time) in a Hilbert space of a quantum system and to give their probabilistic treatment.

Keywords: quantum measurement theory, quantum stochastic representations of an instrument, quantum stochastic evolution operators.

1.Introduction

The behaviour of an isolated quantum system, which is not observed, is quantum deterministic since it is described by the Schrödinger equation, whose solutions are reversible in time.
Under a measurement the behaviour of a quantum system becomes irreversible in time and stochastic. Not only is the outcome of a measured quantum quantity random, being defined with some probability distribution, but the state of the quantum system under measurement becomes random as well.
In this paper we present the g*eneral approach* to the desciption of quantum measurements based on the introduction of the physically important mathematical notion of a family of *quantum*

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stochastic evolution operators. Every operator of this family defines, under a generalized measurement, *a posterior pure state outcome* of a quantum system in a Hilbert space.
The quantum stochastic approach (QSA) gives not only the *complete statistical description* of any quantum measurement, which implies both the knowledge of the probability distribution of different outcomes of the measurement and a *statistical* description of the state change of the quantum system under the measurement, but the QSA gives also the *complete stochastic description* of the random behaviour of a quantum system under the measurement, in the sense of specifying the *probabilistic transition law governing the change from the initail state to a final one*.
We generalize as far as possible our results presented in [16-18], where the notion of a quantum stochastic operator was first defined for the description of conditional evolution of continuously observed quantum systems in the general case of non-demolition measurements.
In section 2 we review the main concepts of quantum measurement theory.
In section 3 we present the main ideas of the quantum stochastic approach to the description of quantum measurements.
In section 4 we give the semiclassical interpretation of our results.

2. The main approaches to the description of quantum measurements

Let us first review the main approaches to the description of quantum measurements available up to the moment. Under a quantum measurement we mean such physical experiment upon a system, which, resulting in the observation of a value of a quantum system variable, may cause only a change in a quantum system state, but not the quantum system's destruction.

2.1 Von Neumann approach

Let H_S be a complex separable Hilbert space of a quantum system. According to the von Neumann approach [1] only self-adjoint operators on H_S are allowed to represent *real-valued variables* of a quantum system, which can be measured. The probability distribution of different outcomes of a measurement on a quantum observable is described by the spectral projection-valued measure $\hat{P}(\cdot)$ on $(R, B(R))$ corresponding, due to the spectral theorem, to the self-adjoint operator representing this observable.
In the case of discrete spectrum of a measured quantum observable the famous von Neumann reduction postulate [1] prescribes the well-known "jump" of a state of a quantum system under a measurement. In the case of continuous spectrum the description of a state change of a quantum system under a measurement is not formalized.
The simultaneous measurement of n quantum observables is allowed if and only if the corresponding self-adjoint operators and, consequently, spectral projection-valued measures, commute. Such measurement is described by the spectral projection-valued measure

(1) $$\hat{P}(E_1 \times E_2 \times ... \times E_n) = \hat{P}(E_1)\hat{P}(E_2) \cdot ..... \cdot \hat{P}(E_n)$$

on $(R^n, B(R^n))$ common for all n commuting self-adjoint operators.

The generalizations of von Neumann's approach, to be discussed in the sequel, are caused by the fact that the approach does not describe all measurements possible upon a quantum system, and it does not describe a state reduction of a quantum system in the general case where the spectrum of a measured quantum observable may be continuous or complicated.

2.2 The description of a generalized quantum measurement

In the further developments of quantum measurement theory [2-7] the mathematical notion of a probability operator-valued (POV) measure is used for the description of a probability distribution on a space of outcomes in the case of any measurement possible upon a quantum system.

Let Ω be a set of outcomes of the most general nature possible under a quantum measurement and F be a $\sigma-$algebra of subsets of Ω. Let $L(H_s)$ be the space of all bounded linear operators on H_S.

A POV measure $\hat{M}(\cdot): F \to L(H_S)$ of a quantum measurement with outcomes in the measurable space (Ω,F) is a positive operator-valued measure on (Ω,F), satisfying the condition $\hat{M}(\Omega)=\hat{I}$.

Given a POV measure, a scalar probability measure $\mu_\rho(\cdot)$ on (Ω,F), describing the probability distribution of possible outcomes of a measurement upon the quantum system, being at the instant before the measurement in the state $\hat{\rho}_S$, is given by

(2) $$\mu_\rho(E) = \mathrm{tr}[\hat{\rho}_S\,\hat{M}(E)], \quad \forall E \in F.$$

In contrast to a spectral projection-valued measure, which is one-to-one defined by a self-adjoint operator, different possible measurements on even the same quantum observable represented by a self-adjoint operator $\hat{B}$ are described by different POV measures $\hat{M}_B(\cdot)$ on $(R, B(R))$ and induce different integral representations for

(3) $$\hat{B} = \int_{-\infty}^{\infty} \lambda\, \hat{M}_{\hat{B}}(d\lambda).$$

A POV measure is sometimes called a generalized observable [3] or semiobservable [6] of a quantum system. A spectral projection-valued measure $\hat{P}(\cdot)$ on $(R, B(R))$ (and the corresponding self-adjoint operator, for short) is called a von Neumann observable.

2.3 Operational approach

The notion of a POV measure does not, however, describe in any way a state change of a quantum system under a measurement. Thus, with respect to a quantum system it does not give the *complete statistical description of a measurement*.

We recall that in the case of discrete spectrum of a measured quantum quantity the von Neumann approach gives the complete statistical description of a measurement describing both *a probability distribution of different outcomes of a measurement and a state change of a quantum system under a measurement*.

The complete *statistical* description of any quantum measurement is given by the mathematical notion of an instrument [2-6] or an operation-valued measure $\hat{T}(\cdot)$ on (Ω, F), satisfying the condition $\hat{T}(\Omega)[\hat{I}] = \hat{I}$.

Given the instrument of a measurement, the POV measure of that measurement. is defined as

(4a) $$\hat{M}(E) = \hat{T}(E)[\hat{I}], \forall E \in F.$$

The scalar probability measure on (Ω,F) defining a probability distribution of possible outcomes under a measurement upon a quantum system being before the measurement in the state $\hat{\rho}_S$ is

(4b) $$\mu_\rho(E) = \mathrm{tr}[\hat{\rho}_S \hat{T}(E)[\hat{I}]].$$

The conditional expectation of any von Neumann observable $\hat{Z}$ at the instant immediately after the measurement, under the condition that the observed outcome belongs to the subset E, is given by

(5a) $$\mathrm{Ex}\{\hat{Z} \mid E\} = \frac{\mathrm{tr}[\hat{\rho}_S \hat{T}(E)[\hat{Z}]]}{\mu_\rho(E)},$$

and the quantum mean value is

(5b) $$<\hat{Z}> \equiv \mathrm{Ex}\{\hat{Z} \mid \Omega\} = \mathrm{tr}[\hat{\rho}_S \hat{T}(\Omega)[\hat{Z}]].$$

The knowledge of an instrument gives the statistical description of any state change of a quantum system caused by a measurement. The posterior (conditional) statistical operator of a quantum system $\hat{\rho}(E)$, conditioned by the outcome being in E, is defined by the relation

(6a) $$\mathrm{Ex}\{\hat{Z} \mid E\} = \frac{\mathrm{tr}[\hat{\rho}_S \hat{T}(E)[\hat{Z}]]}{\mu_\rho(E)} = \mathrm{tr}[\hat{\rho}(E)\hat{Z}].$$

The unconditional (a *priori*) state $\hat{\rho}(\Omega)$ of a quantum system defines the quantum mean value

(6b) $$<\hat{Z}> = \mathrm{tr}[\hat{\rho}(\Omega)\hat{Z}]$$

of a von Neumann observable $\hat{Z}$ at the instant after a measurement if the results of a measurement are ignored.

Any conditional state change of a quantum system can be completely described in a Hilbert space H_S by a family of normalized statistical operators $\{\hat{\rho}_N(\omega), \omega \in \Omega\}$ called usually a family of posterior states [7-9]. For any instrument and a premeasurement state $\hat{\rho}_S$ of a quantum system the family $\{\hat{\rho}_N(\omega), \omega \in \Omega\}$ always exists and is defined uniquely $\mu_\rho(\cdot)$ - almost surely by

(7a) $$\mathrm{tr}[\hat{\rho}_S\, \hat{T}(E)[\hat{A}]] = \int_{\omega \in E} \mathrm{tr}[\hat{\rho}_N(\omega)\hat{A}]\, \mu_\rho(d\omega),$$

$$\forall \hat{A} \in L(H_S),\ \forall E \in F.$$

From (5) and (7a) it follows that the family $\{\hat{\rho}_N(\omega), \omega \in \Omega\}$ determines the conditional expectation

(7b) $$\mathrm{Ex}\{\hat{Z} \mid E\} = \frac{\int_{\omega \in \mathrm{E}} \mathrm{tr}[\hat{\rho}_N(\omega)\hat{Z}]\mu_\rho(d\omega)}{\mu_\rho(E)},$$

as well as the quantum mean value

(7c) $$<\hat{Z}> = \int_{\Omega} \mathrm{tr}[\hat{\rho}_N(\omega)\hat{Z}]\, \mu_\rho(d\omega)$$

of any von Neumann observable $\hat{Z}$ at the instant immediately after a measurement.

The normalized posterior state $\hat{\rho}(E)$ of a quantum system, conditioned by the outcome $\omega \in E$, is presented through the family of normalized posterior statistical operators $\{\hat{\rho}_N(\omega), \omega \in \Omega\}$ as

(8) $$\hat{\rho}(E) = \frac{\int_{\omega \in E} \hat{\rho}_N(\omega)\mu_\rho(d\omega)}{\mu_\rho(E)}.$$

There is a one- to-one correspondence between a POV measure and a family of posterior statistical operators on the one side and an instrument on the other side [7-9]. Knowing a POV measure and a family of posterior statistical operators one can reconstruct the instrument.

2.4 Von Neumann measurement

We would like to reformulate now what we mean by a *von Neumann measurement.*

Definition.

A von Neumann measurement is a measurement upon a von Neumann observable with discrete spectrum, which is described :

a) by the spectral projection-valued measure corresponding to this observable;

b) by the state reduction, defined by von Neumann reduction postulate.

Thus, by definition, a POV measure and a family of posterior statistical operators of a von Neumann measurement are given on $(R, B(R))$ by

(9a) $$\hat{M}(E) \equiv \hat{P}(E) = \sum_{\lambda_j \in E} \hat{P}_j,$$

(9b) $$\hat{\rho}_N(\{\lambda_j\}) = \frac{\hat{P}_j \hat{\rho}_S \hat{P}_j}{\mathrm{tr}[\hat{\rho}_S \hat{P}_j]}.$$

From (9) it follows that the *unique* instrument corresponding to a von Neumann measurement has the form

(10) $$\hat{T}(E)[\hat{A}] = \sum_{\lambda_j \in E} \hat{P}_j \hat{A} \hat{P}_j, \ \forall E \in B(R), \ \forall \hat{A} \in L(H_S).$$

In [1, p.442] von Neumann showed that the state reduction (9b), first *postulated* by him in his projection postulate, can be derived in the scheme of an indirect measurement.
Consider a possible indirect quantum measurement of a von Neumann observable

(11) $$\hat{B} = \sum_j \lambda_j \hat{P}_j$$

presented in [6]. Let $\{\psi_{jk}\}$ be the complete orthonormal set of eigenvectors of the observable (11)

(12) $$\hat{B}\psi_{jk} = \lambda_j \psi_{jk}, \quad \hat{P}_j = \sum_k |\psi_{jk}><\psi_{jk}|.$$

Let K be another separable Hilbert space, $\{\eta_i\}$ and η be, respectively, a set of complete orthonormal vectors and an unit vector in K. Let $\hat{U}$ be a unitary operator on $H_S \otimes K$ satisfying the relation

(13) $$\hat{U}(\psi_{jk} \otimes \eta) = \psi_{jk} \otimes \eta_j \ .$$

The measurement upon the observable $\sum_i \lambda_i |\eta_i><\eta_i|$ on the Hilbert space K, described on the Hilbert space $H_S \otimes K$ of the extended system by a projection-valued measure

(14) $$\hat{U}^+(\hat{I} \otimes \sum_{\lambda_i \in E} |\eta_i><\eta_i|)\hat{U}$$

gives the indirect measurement of the observable (11) with the instrument

(15a) $$\hat{T}(E)[\hat{A}] = <\eta, \hat{U}^+(\hat{A} \otimes (\sum_{\lambda_i \in E} |\eta_i><\eta_i|))\hat{U}\eta>_K = \sum_{\lambda_j \in E} \hat{P}_j \hat{A} \hat{P}_j,$$
$$\forall E \in B(R), \ \forall \hat{A} \in L(H_S),$$

on $(R, B(R))$. In (15a) for any operator $\hat{Q} \in L(H_S \otimes K)$ we have used the notation

(15b) $$<\eta, \hat{Q}\eta>_K, \forall \eta \in K$$

in the sense of the extension by linearity of the relation

(15c) $$<\eta, (\hat{X} \otimes \hat{Y})\eta>_K := (<\eta, \hat{Y}\eta>_K)\hat{X},$$
$$\forall \eta \in K, \ \forall \hat{X} \in L(H_S), \ \forall \hat{Y} \in L(K).$$

Thus, the considered measurement, described by the instrument (15a), is von Neumann. The state reduction under this indirect measurement is given by (9b).
We would like to emphasize that this statement is valid for any pair - a set $\{\eta_i\}$ of orthonormal vectors and an unit vector η in K. Thus, the concept of a *von Neumann measurement* described by the special kind of an instrument (10), corresponds to *different indirect measurements* described by (13).

Let us construct now one more example of an indirect measurement upon the observable (11) of a quantum system, where the POV measure will be again the spectral projection-valued measure (9a), corresponding to the observable (11), but the state reduction will be quite different from (9b).
Suppose, for simplicity, that every eigenvalue λ_j of the observable (11) has multiplicity $k = 1$

and ψ_j is a corresponding eigenvector. Let $\{\phi_m\}$ be another complete set of orthonormal vectors in H_S. Let again K be some separable Hilbert space, $\{\eta_i\}$ and η be, respectively, a complete set of orthonormal vectors and an unit vector in K. Let $\hat{U}_1$ be a unitary operator on $H_S \otimes K$, satisfying the relation

(16) $$\hat{U}_1(\psi_j \otimes \eta) = \phi_j \otimes \eta_j \ .$$

The measurement upon the observable $\sum_i \lambda_i \,|\eta_i><\eta_i\,|$ on the Hilbert space K, described in the Hilbert space $H_S \otimes K$ of the extended system by a projection-valued measure

(17) $$\hat{U}_1^+(\hat{I} \otimes \sum_{\lambda_i \in E} |\,\eta_i><\eta_i\,|)\hat{U}_1,$$

gives the indirect measurement of the observable (11) of the quantum system with the instrument

(18) $$\hat{T}(E)[\hat{A}] = <\eta,(\hat{U}^+(\hat{A} \otimes \sum_{\lambda_i \in E} |\,\eta_i><\eta_i\,|)\hat{U})\eta>_K = \sum_{\lambda_j \in E} \hat{V}_j^+ \hat{A} \hat{V}_j,$$

where

(18) $$\hat{V}_j = |\,\phi_j><\psi_j\,| \ .$$

For such indirect measurement the POV measure

(20) $$\hat{M}(E) \equiv \hat{P}(E) = \sum_{\lambda_j \in E} \hat{P}_j$$

is the spectral projection-valued measure of the observable (11), but the family of posterior statistical operators consists of

(21) $$\hat{\rho}_N(\{\lambda_j\}) = \frac{\hat{V}_j \hat{\rho}_S \hat{V}_j^+}{\mathrm{tr}[\hat{\rho}_S \hat{V}_j^+ \hat{V}_j]} = |\,\phi_j><\phi_j\,|.$$

We see that although the considered indirect measurement is described by the *spectral projection-valued measure* corresponding to the observable (11), the *state reduction* given by (21) is *quite different* to the *von Neumann one* (9b).
Since the type of the state reduction (9b) in a von Neumann measurement just corresponds to the *von Neumann repeatability hypothesis,* we can conclude from the above example that even in the case of discrete character of a measured quantum quantity *the repeatability hypothesis is valid only for a special kind of quantum measurements described by (9) and called von Neumann.*

Thus, *even in the case of discrete outcomes a measurement* upon a quantum system *described by a projection-valued measure is not necessarily von Neumann.*

2.5 Statistical realizations of an instrument

As well as in the von Neumann approach as in the operational approach the notion of a projection-valued measure on (Ω, F) plays a fundamental role.
Introduce the following notation. Let $\hat{\sigma}$ be a statistical operator on a separable Hilbert space K and $\hat{Q}$ be an operator belonging to $L(H_S \otimes K)$. There exists [6] a uniquely determined completely positive linear map $E_\sigma : L(H_S \otimes K) \to L(\hat{H}_S)$ such that the relation

(22a) $$tr[\hat{\rho}\, E_\sigma[\hat{Q}]] = tr[(\hat{\rho} \otimes \hat{\sigma})\hat{Q}]$$

is valid for any statistical operator $\hat{\rho}$ on H_S.
In [6] it was shown that for *any instrument* on a Borel space (Ω, F) there exist a Hilbert space K, a statistical operator $\hat{\sigma}$ on K, an unitary operator $\hat{U}$ and a projection-valued measure $\hat{I} \otimes \hat{P}(\cdot)$ on $H_S \otimes K$, such that an instrument can be presented in the form:

(22b)
$$\hat{T}(E)[\hat{A}] = E_{\sigma}[\hat{U}^{+}(\hat{A}\otimes\hat{P}(E))\hat{U}],$$
$$\forall E\in F, \quad \forall\hat{A}\in L(H_S),$$

A 4-tuple

(22c)
$$(K,\hat{\sigma},\hat{P}(\cdot),\hat{U})$$

is called a *measuring process* of the corresponding generalized observable (a POV measure) or a *statistical realization* of an instrument. For a given instrument a statistical realization always exists but may not be unique.

If in (22c) a Hilbert space K is separable then the corresponding statistical realization is called *separable.*

In quantum theory a Hilbert space H_S of a system is always separable, while the value space is mostly a *standard* Borel space (that is a Borel space which is Borel isomorphic to a complete separable metric space).

If (Ω,F_B) *is a standard Borel space and a Hilbert space* H_S *of a quantum system is separable, then there exists a separable statistical realization of an instrument* (Ω,F_B) [6].

In any real physical situation the measurement is always performed by the measurement of some observable of a quantum apparatus (often called a reservoir). The dynamical measurement model and the scheme of a measurement are given in the frame of the Hamiltonian formalism, that is in terms of interaction between a quantum system and a reservoir. That is why the notion of a statistical realization corresponds to a clear physical interpretation. However, for measurements with outcomes in a complicated value space (as, for example, in the case of repeated measurements) there may not be identity between the dynamical measurement model plus the scheme of measurement on the one side and the statistical realization of the corresponding instrument on the other side.

3. Quantum stochastic approach

From the above review of the main general approaches to the description of quantum measurements we see that the mathematical notion of an instrument while being very important for the formalization of the complete statistical description of any quantum measurement does not give the description in a Hilbert space H_S of the stochastic behaviour of a quantum system under a measurement**.**

However, the description of stochastic, irreversible in time behaviour of a quantum system under consecutive measurements is very important, in particular, in the case of continuous in time measurement of an open system, where the evolution of the continuously observed open system differs from that described by reversible in time solutions of the Schrödinger equation.

We would like now to introduce the more detailed general approach to the description of quantum measurements based on introduction of a family of operator -valued functions on Ω *(quantum stochastic evolution operators) and giving not only the complete statistical description of a measurement but the complete description of the stochastic behaviour of an observed quantum system in a Hilbert space* H_S *as well.*

3.1 Quantum stochastic representations of an instrument

Let

(23)
$$\hat{T}(E)[\hat{A}], \quad \forall E\in F_B, \ \forall\hat{A}\in L(H_S).$$

be an instrument on a standard Borel space (Ω,F_B) with values in $L(H_S)$, where the Hilbert space H_S of the quantum system is separable and infinite–dimensional.

Let $\gamma=\{K,\hat{\sigma},\hat{P}(\cdot),\hat{U}\}$ be a separable statistical realization of the instrument (23).

<u>Definition.</u>

We shall say that a separable statistical realization $\gamma' = \{K', \hat{\sigma}', \hat{P}'(\cdot), \hat{U}'\}$ *of an instrument (23) is unitarily equivalent to* γ *if there exists an isometry* $\hat{W} : K \to K'$ *under which*

(24a) $$\hat{\sigma}' = \hat{W}^{-1}\hat{\sigma}\,\hat{W},\ \hat{P}'(\cdot) = \hat{W}^{-1}\hat{P}(\cdot)\hat{W},\ \hat{U}' = (\hat{I}\otimes\hat{W}^{-1})\hat{U}(\hat{I}\otimes\hat{W}).$$

From (22b) it follows that $\gamma_\xi = \{K, \hat{\sigma}, \hat{P}(\cdot), e^{i\xi}\hat{U}\}$ with $\forall\xi \in R$ is also a separable statistical realization of the instrument (23). We shall say that a statistical realization γ_ξ is phase-equivalent to the statistical realization γ. Let $G(\gamma_\xi)$ be the set of all separable statistical realizations of an instrument (23) unitarily equivalent to a statistical realization γ_ξ.
Introduce $G_\gamma = \{G(\gamma_\xi), \xi \in R\}$ - the class of all separable statistical realizations of the instrument (23) unitarily and phase equivalent to the statistical realization γ. The class G_γ includes, in particular, all unitarily and phase equivalent separable statistical realizations on one and the same Hilbert space K'. All separable Hilbert spaces K' inside a class G_γ are isomorphic to each other and have the same dimension D_γ.
Let H_R denote some *separable Hilbert space* corresponding to a class G_γ and let

(24b) $$\{H_R, \hat{\rho}_R, \hat{P}_R(\cdot), \hat{U}_R\}$$

be any separable statistical realization on the Hilbert space H_R from the class G_γ.
Consider on (Ω, F_B) the family of positive scalar Borel measures $\{\mu_\varphi(\cdot) = <\varphi, \hat{P}_R(\cdot)\varphi>, \forall\varphi \in H_R\}$, induced by a projection-valued measure $\hat{P}_R(\cdot)$. For any projection-valued measure in a separable Hilbert space H_R there exists [10] $\tilde{\varphi} \in H_R$ such that with respect to a subset $E \in F_B$ the equations $\mu_{\tilde{\varphi}}(E) = 0$ and $\hat{P}_R(E) = \hat{0}$ are equivalent. The element $\tilde{\varphi} \in H_R$ is said to be an element of maximum type [10] for a projection-valued measure $\hat{P}_R(\cdot)$. Denote by $[\mu_{\tilde{\varphi}}]$ the type of the scalar measure $\mu_{\tilde{\varphi}}(\cdot)$ (i.e. $[\mu_{\tilde{\varphi}}]$ is the class of positive scalar measures equivalent to $\mu_{\tilde{\varphi}}(\cdot)$).

<u>Definition</u>

The spectral type $[\hat{P}_R(\cdot)]$ *of a projection-valued measure* $\hat{P}_R(\cdot)$ *on* (Ω, F_B) *is defined to be equal to the type* $[\mu_{\tilde{\varphi}}]$ *of the positive scalar Borel measure* $\mu_{\tilde{\varphi}}(\cdot) = <\tilde{\varphi}, \hat{P}_R(\cdot)\tilde{\varphi}>$, *induced by an element of the maximum type* [10].

Let $\nu(\cdot)$ be a positive scalar Borel measure on (Ω, F_B) of the type $[\nu(\cdot)] = [\hat{P}_R(\cdot)]$. For any $\phi \in H_R$ introduce the subset

(25a) $$\Omega(\phi) = \{\omega \mid \omega \in \Omega, \frac{d\mu_\phi}{d\nu}(\omega) > 0\},$$

which is defined ν - almost surely and does not depend on the choice of the scalar measure $\nu(\cdot)$ on (Ω, F_B) out of the class of equivalent scalar measures of the type $[\hat{P}_R(\cdot)]$.
The following statements are valid [10]:

(25b) $$\Omega(\phi) \subset \Omega(\varphi) \Leftrightarrow [\mu_\phi] \prec [\mu_\varphi];$$

(25c) $$\Omega(\phi) = \Omega(\varphi) \Leftrightarrow [\mu_\phi] = [\mu_\varphi];$$

(25d) $$\hat{P}_R(\Omega(\phi))\phi = \phi, \quad \forall\phi \in H_R;$$

(25e) $$\Omega(\eta_1) \cap \Omega(\eta_2) = \varnothing \Rightarrow\ <\eta_1, \hat{P}(\cdot)\eta_2> = 0;$$

(25f) $$<\eta_1, \hat{P}(\cdot)\eta_2> = 0 \Rightarrow \Omega(\eta_1 + \eta_2) = \Omega(\eta_1) \cup \Omega(\eta_2);$$

If

(25g) $$[\mu_\phi] = [\hat{P}_R(\cdot)],\ \eta = \hat{P}_R(E)\phi,\ E \in F_B,$$

then

(25h) $$\Omega(\eta) = E\,.$$

For any projection-valued measure $\hat{P}_R(\cdot)$ on (Ω, F_B) there exists [10] a family of elements $\{\eta_j,\ \eta_j \in H_R,\ j = 1,...,m;\ 1 \le m \le \infty\}$, satisfying $< \eta_l, \hat{P}(\cdot)\eta_k >= 0,\ \forall l,k$, such that

(26a) $$H_R = \sum_j \oplus H_{\eta_j},$$

$$H_{\eta_j} = \{\hat{Z}_\psi \eta_j \mid \psi \in S(\Omega, \hat{P}_R), \eta_j \in D_\psi\}$$

and

(26b) $$[\hat{P}_R] = [\mu_{\eta_1}] \succ [\mu_{\eta_2}] \succ ....$$

In (26a) $S(\Omega, \hat{P}_R)$ is a class of $\hat{P}_R$ - measurable, $\hat{P}_R$ - almost surely finite functions: $\Omega \to C$; $\hat{Z}_\psi$ is the operator defined by the relation

(26c) $$\hat{Z}_\psi = \int_\Omega \psi(\omega)\hat{P}_R(d\omega)$$

with the domain $D_\psi = \{ f \in H_R \mid \int_\Omega |\psi(\omega)|^2\ \mu_f(d\omega) < \infty\}$.

If $m > 1$, then the decomposition (26a) of a Hilberts space H_R is not unique.
From (25) and (26b) it follows that

(27a) $$\Omega = \Omega(\eta_1) \supset \Omega(\eta_2) \supset ....$$

Introduce the sets $\Omega_k,\ k = 1,...,m$ by the relations

(27b) $$\Omega_k = \Omega(\eta_k) \setminus \Omega(\eta_{k+1}), \quad k < m,$$

$$\Omega_m = \bigcap_{k=1,....,m} \Omega(\eta_k)\,.$$

<u>*Definition*</u>

The $\hat{P}_R$ - measurable function $N_{P_R} : \Omega \to N$ defined by the relation

(28a) $$N_{P_R}(\omega) = k, \quad for \quad \omega \in \Omega_k, \quad k = 1,...,m$$

$\hat{P}_R$ - almost surely is called a multiplicity function of a projection- valued measure $\hat{P}_R(\cdot)$ on (Ω, F_B) [10].

Since the type $[\hat{P}_R(\cdot)]$ and the multiplicity function $N_{P_R}(\omega)$ of a projection-valued measure $\hat{P}_R(\cdot)$ are *unitary invariants* [10], they are *invariants of the class* G_γ, that is the same for all projection-valued measures from the class G_γ.

Let $H(\nu, N)$ be a direct integral [10-12] of separable Hilbert spaces $H(\omega)$ on (Ω, F_B), induced by a positive scalar Borel measure $\nu(\cdot)$ of the type $[\nu(\cdot)] = [\hat{P}_R(\cdot)]$) and by the dimension function $N(\omega) = \dim H(\omega)$, being equal to the multiplicity function $N_{P_R}(\omega)$ of the projection-valued measure $\hat{P}_R(\cdot)$ on (Ω, F_B):

(28b) $$H(\nu, N) = \int_\Omega \oplus H(\omega)\nu(d\omega)\,.$$

Then there exists [10-12] an isometry $\hat{R} :\ H_R \to H(\nu, N)$ such that

(28c) $$\hat{P}_R(E) = \int_{\omega\in E} \hat{R}^{-1}(\sum_{n=1}^{N(\omega)} \hat{p}_n(\omega))\hat{R}\nu(d\omega),$$

where $\hat{p}_n(\omega),\ n = 1,...,N(\omega),$ are one-dimensional mutually ortogonal projection-valued densities on $H(\nu,N)$

(28d) $$\hat{p}_n(\omega_1)\,\hat{p}_m(\omega_2) = \delta(\omega_1 - \omega_2)\delta_{nm}\hat{p}_n(\omega_1)$$

In (28d) the delta function is with respect to the measure $\nu(\cdot)$.

Due to the definition of a statistical realization (22), the instrument (23) is given through the functional elements of the considered statistical realisation (24b) as

(29) $$\begin{aligned} &\hat{T}\,(E)[\hat{A}] = E_{\rho_R}[\hat{U}_R^+(\hat{A}\otimes\hat{P}_R(E))\hat{U}_R],\\ &\forall E\in F,\quad \forall\hat{A}\in L(H_S). \end{aligned}$$

The statistical operator $\hat{\rho}_R$ is uniquely represented in the form

(30) $$\begin{aligned} &\hat{\rho}_R = \sum_{i=1}^{N_\gamma}\alpha_i\,|\varphi_i\rangle\langle\varphi_i|,\qquad \langle\varphi_i|\varphi_i\rangle_{H_R} = \delta_{ij},\\ &\alpha_i \geq 0,\quad \sum_i \alpha_i = 1. \end{aligned}$$

The positive integer N_γ (it may be infinite) and the sequence of non negative numbers $\alpha_\gamma = (\alpha_1,\alpha_2,...)$ are *invariants* of the class G_γ.

Thus, $D_\gamma, [\hat{P}_R], N_{P_R}, N_\gamma, \alpha_\gamma$ *present invariants of the class* G_γ.

Introduce also a family $\tilde{G}_\gamma$ of all separable statistical realizations of an instrument from classes G_γ different only by an invariant α_γ.

Denote by $\hat{q}_n(\omega) = \hat{R}^{-1}\hat{p}_n(\omega)\hat{R}$ one-dimensional mutually ortogonal projection-valued densities on H_R:

(31) $$\hat{q}_n(\omega_1)\,\hat{q}_m(\omega_2) = \delta(\omega_1 - \omega_2)\delta_{nm}\hat{q}_n(\omega_1)$$

and introduce the following notation for scalar products

(32a) $$q_{ji}^{(n)}(\omega) = \langle\varphi_j, \hat{q}_n(\omega)\varphi_i\rangle$$

and scalar measures on (Ω, F_B)

(32b) $$\begin{aligned} &\nu_{in}(d\omega) = q_{in}(\omega)\,\nu(d\omega),\\ &q_{in}(\omega) \equiv q_{ii}^{(n)}(\omega) \geq 0,\quad \forall\omega\in\Omega. \end{aligned}$$

The following relation is valid for the scalar products introduced in (32a)

(32c) $$\int_\Omega (\sum_{n=1}^{N(\omega)} q_{ji}^{(n)}(\omega))\nu(d\omega) = \delta_{ji},$$

Consequently,

(32d) $$\nu_i(d\omega) = \sum_{n=1}^{N(\omega)} \alpha_i \nu_{in}(d\omega)$$

is a probability measure on (Ω, F_B)

<u>*Definition*</u>

For any i, n *define the operator-valued* ν_{in} *- measurable function* $\hat{V}_{in}(\omega): \Omega \to L(H_S)$ *by the relation*

(33) $$\begin{aligned} &(\hat{V}_{in}(\omega)\otimes\hat{q}_n(\omega))(\psi\otimes\varphi_i) = (\hat{I}\otimes\hat{q}_n(\omega))\hat{U}_R(\psi\otimes\varphi_i),\\ &\forall\psi\in H_S, \end{aligned}$$

ν_{in} *- almost surely.*

It is easy to show that for different separable statistical realizations of an instrument from the class $\widetilde{G}_\gamma$ the operator-valued functions, defined by (33), with the definite indexes i,n may differ from each other ν_{in} - almost surely only as $\hat{V}_{in}^{'}(\omega) = e^{i\xi}\hat{V}_{in}(\omega)$, that is they are identical up to the phase-equivalence. If this relation is valid between all the operator-valid functions from families $\{\hat{V}_{in}'(\omega)\}$ and $\{\hat{V}_{in}(\omega)\}$, then we shall say that the corresponding families are identical ν_{in} - almost surely up to the phase-equivalence.
From the definition (33) the following resolutions follow

(34a) $$\hat{U}_R(\psi \otimes \varphi_i) = \int_\Omega (\sum_{n=1}^{N(\omega)} \hat{V}_{in}(\omega) \otimes \hat{q}_n(\omega))\nu(d\omega)(\psi \otimes \varphi_i),$$
$$\forall \psi \in H_S,$$

(34b) $$<\varphi_j, \hat{U}_R \varphi_i>_{H_R} = \int_\Omega (\sum_{n=1}^{N(\omega)} \hat{V}_{in}(\omega) q_{ji}^{(n)}(\omega))\nu(d\omega),$$

(34c) $$E_{\rho_R}[\hat{U}_R] = \int_\Omega (\sum_{i,n} \alpha_i q_{in}(\omega)\hat{V}_{in}(\omega))\nu(d\omega),$$

(34d) $$\int_\Omega (\sum_{n=1}^{N(\omega)} \hat{V}_{jn}^{+}(\omega)\hat{V}_{in}(\omega) q_{ji}^{(n)}(\omega))\nu(d\omega) = \delta_{ji}\hat{I}, \quad \forall j,i\ .$$

It is easy to show that the scalar measures $\nu_{in}(\cdot)$, as well as the scalar probability measures $\nu_i(\cdot)$ are *invariants* of the family $\widetilde{G}_\gamma$. Due to (30), (32) and (34)

(35) $$tr[\hat{\rho}_R \hat{P}_R(E)] = \\ = \int_{\omega \in E} \sum_{i,n} \alpha_i \nu_{in}(d\omega) = \nu_0(E),\ \forall E \in F_B$$

and defines a scalar probability measure $\nu_0(\cdot)$ on (Ω, F_B), which is an *invariant* of the class G_γ. Considering all mentioned above the following statement is valid.

Proposition 1
Let (Ω, F_B) be a standard Borel space, H_S - a complex separable Hilbert space of a quantum system. The family of complex measures

(36) $$\Lambda = \{q_{ji}^{(n)}(\omega)\nu(d\omega) \mid \omega \in \Omega,\ i,\ j = 1,2,..,N_\gamma;\ n = 1,2,...,N(\omega); \int_\Omega (\sum_{n=1}^{N(\omega)} q_{ji}^{(n)}(\omega))\nu(d\omega) = \delta_{ji}\},$$

defined by (32) and corresponding to a separable statistical realization of a class $\widetilde{G}_\gamma$ does not depend on the chosen statistical realization.
The family of operator-valued ν_{in} - measurable functions: $\Omega \to L(H_S)$

(37) $$V = \{\hat{V}_{in}(\omega), \omega \in \Omega, i = 1,2,..,N_\gamma;\ n = 1,2,...N(\omega); \int_\Omega (\sum_{n=1}^{N(\omega)} \hat{V}_{jn}^{+}(\omega)\hat{V}_{in}(\omega) q_{ji}^{(n)}(\omega))\nu(d\omega) = \delta_{ji}\hat{I}\},$$

defined by (33) ν_{in} - almost surely and corresponding to a separable statistical realization of a class $\widetilde{G}_\gamma$ is independent of the chosen statistical realization up to the phase-equivalence.
The families (36), (37) are functional invariants of a class $\widetilde{G}_\gamma$, and they do not depend on the choice of the scalar measure $\nu(\cdot)$ on (Ω, F_B) out of the class of equivalent scalar measures of the type $[\hat{P}_R(\cdot)]$, which is an invariant of the class $\widetilde{G}_\gamma$.

We recall that $\{D_\gamma, [P_R], N_{P_R}, N_\gamma\}$ are *invariants of* $\widetilde{G}_\gamma$.

Using (29), (32),(33), (34), we get the following *representation* of an instrument (23) *valid for all separable statistical realizations from a class* G_γ

(38)
$$\hat{T}(E)[\hat{A}] = \int_{\omega\in E} \Big(\sum_{i,n} \alpha_i q_{in}(\omega)\, \hat{V}_{in}^{+}(\omega)\hat{A}\, \hat{V}_{in}(\omega)\Big)\, \nu(d\omega),$$
$$\forall E \in F_B, \quad \forall \hat{A} \in L(H_S).$$

The corresponding representation of the POV measure is

(39)
$$\hat{M}(E) = \int_{\omega\in E} \Big(\sum_{i,n} \alpha_i q_{in}(\omega) \hat{V}_{in}^{+}(\omega)\; \hat{V}_{in}(\omega)\Big)\, \nu(d\omega), \quad \forall E \in F_B.$$

We shall call a representation (38) of an instrument *quantum stochastic.*
The following theorem is valid.

<u>*Theorem 1*</u>
Let (Ω, F_B) *be a standard Borel space and* H_S *- a complex separable Hilbert space of a quantum system. For any instrument* $\hat{T}(E)[\hat{A}\,],\ \forall E \in F_B, \forall \hat{A} \in L(H_S)$ *on* (Ω, F_B) *with values in* $L(H_S)$ *there exists a quantum stochastic representation .*

The form (38) of a quantum stochastic representation of an instrument is in perfect agreement with the general Stinespring theorem [13] on completely positive maps.
In the general case the problem of existence of a representation of an instrument similar to (38) was considered in [14]. The form of the representation obtained in [14] can be derived from (38) and corresponds to a special case.
In the special case of description of continuous in time nondemolition observation upon an open system the representation of an instrument in a form (38) was considered in [15-18].

<u>*Definition*</u>
We shall say that we have different quantum stochastic representations of the same instrument if in (38) the pairs (V, Λ) *are different.*
It is easy to show that for every quantum stochastic representation of a *given* instrument the *sequence* α_γ is *unique*, and consequently, for the given instrument, we have $\tilde{G}_\gamma = G_\gamma$.

<u>*Proposition 2*</u>
There is a one-to-one correspondence between the set $\{G_\gamma\}$ *of different classes* G_γ *of separable statistical realizations of an instrument and the set* $\{(V, \Lambda)\}$ *of different pairs* (V, Λ), *corresponding to different quantum stochastic representations of this instrument.*

From (35) and (39) it follows that *for the definite quantum stochastic representation* the scalar probability measure (4b) defining a probability distribution of outcomes under a measurement, is given by

(40a)
$$\mu_\rho(E) = \int_{\omega\in E} \mathrm{tr}[\hat{\rho}(\omega)]\nu_0(d\omega),$$

where the scalar probability measure $\nu_0(\cdot)$ is given by (35) and

(40b)
$$\hat{\rho}(\omega) = \sum_{i,n} \alpha_i \theta_{in}(\omega) \hat{V}_{in}(\omega) \hat{\rho}_S \hat{V}_{in}^{+}(\omega),$$

(40c)
$$\theta_{in}(\omega) = \frac{q_{in}(\omega)}{\sum_{i,n} \alpha_i q_{in}(\omega)}.$$

Introduce the notation

(41a) $$\hat{\rho}_{in}(\omega) = \hat{V}_{in}(\omega)\hat{\rho}_S \hat{V}_{in}^{+}(\omega).$$

Then from (6), (38) and (40) we have that the family of normalized posterior statistical operators $\{\hat{\rho}_N(\omega), \omega \in \Omega\}$, defined by (7a), is given by

(41b) $$\hat{\rho}_N(\omega) = \frac{\hat{\rho}(\omega)}{\mathrm{tr}[\hat{\rho}(\omega)]},$$

(41c) $$\hat{\rho}(\omega) = \sum_{i,n} \alpha_i \theta_{in}(\omega)\hat{\rho}_{in}(\omega).$$

Thus, $\hat{\rho}_N(\omega)$ is a sum of normalized statistical operators

(41d) $$\hat{\rho}_{in}^{(N)}(\omega) = \frac{\hat{\rho}_{in}(\omega)}{tr[\hat{\rho}_{in}(\omega)]}$$

with statistical weights

(42) $$\beta_{in}(\omega) = \frac{\alpha_i q_{in}(\omega)\mathrm{tr}[\hat{\rho}_{in}(\omega)]}{\sum_{j,l} \alpha_j q_{jl}(\omega)\mathrm{tr}[\hat{\rho}_{jl}(\omega)]}.$$

The a priori (unconditional) statistical operator of a quantum system at the instant after a measurement can be represented as

(43a) $$\hat{\rho}(\Omega) = \int_{\Omega} \sum_{i,n} \alpha_i \hat{\rho}_{in}(\omega) \nu_{in}(d\omega) = \\ = \int_{\Omega} \sum_{i,n} \alpha_i \hat{\rho}_{in}^{(N)}(\omega) \mu_{in}(d\omega)$$

with a scalar measure

(43b) $$\mu_{in}(d\omega) = \mathrm{tr}[\hat{\rho}_{in}(\omega)]\nu_{in}(d\omega).$$

We can rewrite (40a) in the form

(43c) $$\mu_{\rho}(d\omega) = \sum_{i,n} \alpha_i \mu_{in}(d\omega).$$

In the physical literature on quantum measurements , in the special case when $\Omega = R$ and the spectrum of a measured quantum quantity is discrete, the formulae for a posterior statistical operator and a POV measure, similar in some sense to (39) and (40b), were considered in [19, 20].

3.2 Quantum stochastic measurement model

In section 2.3 it was pointed out that an instrument gives the complete statistical description of a quantum measurement. Consider the definite quantum stochastic representation (38) of an instrument and try to understand what is different under different quantum stochastic representations of the same instrument.

Suppose for simplicity that the state of a quantum system at the instant before a measurement is pure that is $\hat{\rho}_S = |\psi_0 \rangle\langle \psi_0|$. In this case

(44a) $$\hat{\rho}_{in}(\omega) = \hat{V}_{in}(\omega)|\psi_0 \rangle\langle \psi_0| V_{in}^{+}(\omega)$$

of (41a) represent *pure states* and the normalized posterior statistical operator (41b) can be presented in the form

(44b) $$\hat{\rho}_N(\omega) = \sum_{i,n} \beta_{in}(\omega)|\Psi_{in}(\omega)\rangle\langle \Psi_{in}(\omega)|,$$

(44c) $$\beta_{in}(\omega) = \frac{\alpha_i q_{in}(\omega)\|\hat{V}_{in}(\omega)\psi_0\|^2}{\sum_{j,k} \alpha_j q_{jk}(\omega)\|\hat{V}_{jk}(\omega)\psi_0\|^2}$$

with the notation

(45a) $$\Psi_{in}(\omega)=\frac{\hat{V}_{in}(\omega)\psi_0}{\|\hat{V}_{in}(\omega)\psi_0\|_{H_S}}$$

for a posterior pure state defined by the operator $\hat{V}_{in}(\omega)$.
The following orthonormalization relation is valid for the set of posterior pure states:

(45b) $$\int_{\Omega}\sum_{n=1}^{N(\omega)}<\hat{V}_{jn}(\omega)\psi_0,\hat{V}_{in}(\omega)\psi_0>_{H_S}\ q_{ji}^{(n)}(\omega)\nu(d\omega)=\delta_{ji}\|\psi_0\|^2_{H_S},\quad \forall j,i\ .$$

From (44), (45) it follows that for different quantum stochastic representations of the same instrument the corresponding families of posterior pure states

(45c) $$\{\Psi_{in}(\omega),\omega\in\Omega,\ i=1,...,N_\gamma;n=1,...,N(\omega)\}$$

and their statistical weights $\beta_{in}(\omega)$ in the normalized posterior statistical operator $\hat{\rho}_N(\omega)$ (which is the same under different quantum stochastic representations of the same instrument) are different.
For all quantum stochastic representations of the instrument with $i=1,n=1$, for $\forall\omega\in\Omega$, $\hat{\rho}_N(\omega)$ is a pure state.
The normalized posterior statistical operator $\hat{\rho}(E)$, conditioned by the outcome $\omega\in E$, is defined by the set $\{\Psi_{in}(\omega)\}$ of posterior pure states as

(46a) $$\hat{\rho}(E)=\frac{\int_{\omega\in E}\sum_{i,n}\alpha_i\,|\Psi_{in}(\omega)><\Psi_{in}(\omega)|\,\mu_{in}(d\omega)}{\int_{\omega\in E}\sum_{i,n}\alpha_i\mu_{in}(d\omega)}$$

with a scalar measure

(46b) $$\mu_{in}(d\omega)=\|\hat{V}_{in}(\omega)\psi_0\|^2\ \nu_{in}(d\omega),$$

defining a probability distribution in the "i,n" transition channel of a quantum stochastic representation. The scalar probability measure (43) of the whole measurement is given by

(46c) $$\mu_\rho(d\omega)=\sum_{i,n}\alpha_i\mu_{in}(d\omega)$$

We can interpret then the scalar measure $\nu_{in}(d\omega)$ in (46b) as one describing the input probability distribution of different outcomes in the "i,n" transition channel of a given quantum stochastic representation and the scalar measure $\mu_{in}(d\omega)$ as one describing an output probability distribution of different outcomes on (Ω,F_B) in every "i,n" transition channel.
The scalar probability measures

(47a) $$\nu_i(E)=\int_{\omega\in E}\sum_{n=1}^{N(\omega)}\nu_{in}(d\omega)=\int_{\omega\in E}(\sum_{n=1}^{N(\omega)}q_{in}(\omega))\nu(d\omega),$$

(47b) $$\mu_i(E)=\int_{\omega\in E}\sum_{n=1}^{N(\omega)}\mu_{in}(d\omega)$$

describe, respectively, input and output probability distributions in the "i" channel of a quantum stochastic representation. The numbers $\{\alpha_i,\ \alpha_i\geq 0,\ \sum_i\alpha_i=1\}$ describe weights of different "i" channels.
From (46a) it follows that $\Psi_{in}(\omega)$ can be interpreted as a random posterior pure state *outcome* in a Hilbert space H_S of a quantum system, conditioned by the observed value $\omega\in d\omega$ and the "i,n"

random transition channel. For the definite ω the statistical weights of different posterior pure state outcomes are defined by $\beta_{in}(\omega)$.

Thus, the random operators $\hat{V}_{in}(\omega)$ describe in a Hilbert space H_S the posterior behaviour of a quantum system conditioned by the observed outcome $\omega \in d\omega$ in the $"i,n"$ *random transition channel.*

Analysing the definition of a class G_γ of unitarily equivalent separable statistical realizations of an instrument given in the previous section, we conclude that *different quantum stochastic representations* of the same instrument can be *identified* with *different quantum measurements. Although the statistical description of these quantum measurements (the POV measure and the family of normalized posterior statistical operators) is the same, the stochastic behaviour of a quantum system in a Hilbert space under these measurements may be different.*
Physically the notion of different transition channels under a measurement (a quantum stochastic representation) corresponds under the same observed outcome of a measured quantum variable to different random quantum transitions of a quantum system (reservoir) modelling the measuring device under a measurement.

The following statements follow from our *identification of a quantum stochastic representation with a concrete quantum measurement.*

Proposition 3

For any quantum measurement with outcomes in a standard Borel space (Ω, F_B) *of a quantum system being at the instant before the measurement in a state* $\hat{\rho}_S$*, there exist:*
the unique family of complex measures

(48) $$\Lambda = \{q_{ji}^{(n)}(\omega)\nu(d\omega) \,|\, \omega \in \Omega;\ i,j = 1,2,..,N_\gamma;\ n = 1,2,...,N(\omega); \int_\Omega (\sum_{n=1}^{N(\omega)} q_{ji}^{(n)}(\omega))\nu(d\omega) = \delta_{ji}\};$$

the unique (up to the phase- equivalence) family of operator-valued ν_{in} *- measurable functions:*
$\Omega \to L(H_S)$

(49) $$V = \{\hat{V}_{in}(\omega), \omega \in \Omega, i = 1,2,..,N_\gamma; n = 1,2,...N(\omega); \int_\Omega (\sum_{n=1}^{N(\omega)} \hat{V}_{jn}^{+}(\omega)\hat{V}_{in}(\omega)q_{ji}^{(n)}(\omega))\nu\ (d\omega) = \delta_{ji}\hat{I}\}$$

defined ν_{in} *- almost surely, where* $\nu_{in}(d\omega) = q_{ii}^{(n)}(\omega)\nu(d\omega)$*;*
the unique sequence of numbers

(50) $$\alpha = (\alpha_1, \alpha_2, ...) \text{ with } \alpha_i \geq 0, \quad \sum_i \alpha_i = 1,$$

such that the complete statistical description (a POV measure and a family of normalized posterior statistical operators) of a measurement and the complete stochastic description of the behaviour of a quantum system under a measurement (a family of posterior pure states and their probability distribution) are given by:
1)*The POV measure*

(51a) $$\hat{M}(E) = \int_{\omega \in E} \sum_{i,n} \alpha_i \hat{V}_{in}^{+}(\omega)\ \hat{V}_{in}(\omega)\, \nu_{in}(d\omega), \quad \forall E \in F_B,$$

(51b) $$\nu_{in}(d\omega) = q_{in}(\omega)\nu(d\omega),$$
$$q_{in}(\omega) = q_{ii}^{(n)}(\omega);$$

with operator-valued density

(51c) $$\hat{m}(\omega) = \sum_{i,n} \alpha_i \theta_{in}(\omega)\hat{V}_{in}^{+}(\omega)\hat{V}_{in}(\omega), \qquad \theta_{in}(\omega) = \frac{q_{in}(\omega)}{\sum_{i,n} \alpha_i q_{in}(\omega)}$$

with respect to the scalar probability measure

(51d) $$\nu_0(d\omega) = \sum_{i,n} \alpha_i \nu_{in}(d\omega).$$

2) *The family* $\{\hat{\rho}(\omega), \omega \in \Omega\}$ *of unnormalized posterior statistical operators*

(52a) $$\hat{\rho}(\omega) = \sum_{i,n} \alpha_i \theta_{in}(\omega) \hat{\rho}_{in}(\omega),$$

(52b) $$\hat{\rho}_{in}(\omega) = \hat{V}_{in}(\omega) \hat{\rho}_S \hat{V}_{in}^+(\omega).$$

3) *The family of random operators (49) describing the stochastic behaviour of the quantum system under a measurement. Every operator* $\hat{V}_{in}(\omega)$ *defines in the Hilbert space* H_S *a posterior pure state outcome of the observed quantum system conditioned by the observed result* $\omega \in d\omega$ *and the* "i,n" *random transition channel. For any* $\psi_0 \in H_S$ *the following orthonormalization relation for a family of posterior pure states is valid:*

(53a) $$\int_\Omega \sum_{n=1}^{N(\omega)} < \hat{V}_{jn}(\omega)\psi_0, \hat{V}_{in}(\omega)\psi_0 >_{H_S} q_{ji}^{(n)}(\omega) \nu(d\omega) = \delta_{ji} \|\psi_0\|_{H_S}^2, \quad \forall j,i\,.$$

The probability distribution of different outcomes in a random "i,n" *transition channel is given by*

(53b) $$\mu_{in}(d\omega\cdot) = \mathrm{tr}[\rho_{in}(\omega)] \nu_{in}(d\omega).$$

For the definite observed value ω *the statistical weights of different random transition channels are*

(53c) $$\beta_{in}(\omega) = \frac{\alpha_i q_{in}(\omega) \mathrm{tr}[\hat{\rho}_{in}(\omega)]}{\sum_{j,k} \alpha_j q_{jk}(\omega) \mathrm{tr}[\hat{\rho}_{jk}(\omega)]}\,.$$

4) The scalar probability measure of the whole measurement is given by the expression

(54) $$\mu_\rho(E) = \int_{\omega \in E} \mathrm{tr}[\hat{\rho}(\omega)] \nu_0(d\omega) = \int_{\omega \in E} \sum_{i,n} \alpha_i \mu_{in}(d\omega).$$

through the scalar probability measure (51d) and the scalar measures (53b) in different random "i,n" *transition channels.*

We shall call $\hat{V}_{in}(\omega)$ a *quantum stochastic evolution operator* and $\nu_0(\cdot)$, $\mu_\rho(\cdot)$ input and output scalar probability measures, respectively.
The derivation of the families (48) and (49) in any concrete quantum measurement is based on their definitions (32), (33) .

<u>*Proposition 4*</u>
For any triple $\{V, \Lambda, \alpha_\gamma\}$, *defined in (48) - (50), and a projection-valued measure* $\hat{P}(\cdot)$ *on* (Ω, F_B), *consistent with this triple, there exists a measurement upon a quantum system, described by formulae (51) - (54).*

In the proposition 4 a projection-valued measure $\hat{P}(\cdot)$ is said to be *consistent* with the families (48), (49) *if a type* $[\hat{P}(\cdot)]$ *and a multiplicity function* N_P *of* $\hat{P}(\cdot)$ *are the same as the similar invariants in* (48), (49).

<u>4. Semiclassical stochastic model of a quantum measurement</u>

In quantum theory there was always a wish to combine the classical description of a measuring

apparatus for an observer with the quantum description of an observed system.
The results we derived in the previous sections allow us to introduce such kind of interpretation of the description of a quantum measurement.

Definition
We shall say that a family of scalar measures

(55a) $$\Lambda = \{q_{in}(\omega)\nu(d\omega) \mid i = 1,2,..., N_\gamma;\ n = 1,2,..., N(\omega); \int_\Omega (\sum_{n=1}^{N(\omega)} q_{in}(\omega))\nu\ (d\omega) = 1\}$$

on a measurable space (Ω, F) *describes a classical premeasurement state* Λ *of a quantum apparatus, if for any measurement* $"\alpha"$*:*

(55b) $$\alpha = (\alpha_1, \alpha_2, ...) \text{ with } \alpha_i \geq 0, \quad \sum_i \alpha_i = 1,$$

performed by a "free" apparatus in a state Λ, *an input scalar probability measure of* $"\alpha"$ *measurement is given by*

(55c) $$\nu_0^{(\alpha)}(d\omega) = \sum_{i,n} \alpha_i\, q_{in}(\omega)\nu(d\omega).$$

Different $"\alpha"$ correspond to *different preparations of the quantum state of a measuring apparatus.*

From this definition it follows that :
For all measurements $"\alpha"$ *performed by a measuring device being in a classical premeasurement state* Λ *there exists the unique family of quantum stochastic operators (49) such that for any premeasurement quantum state* $\hat{\rho}_S$ *of a quantum system the POV measure and the family of posterior statistical operators are defined by formulae (51), (52).*

5. Concluding remarks

In the present paper we review the main approaches to the description of quantum measurements.
For a generalized quantum measurement we introduce the notion of *a quantum stochastic representation of an instrument* and prove that every quantum stochastic representation is induced uniquely by the definite class of unitarily and phase equivalent statistical realizations of this instrument.
We define the notion of a family of *quantum stochastic operators, describing under a measurement the conditional evolution of a quantum system in a Hilbert space*. In the general case of continuous in time nondemolition measurement of an open system, where a space of outcomes is a space of trajectories, our general definition (33) of a quantum stochastic evolution operator coincides with the definition of a quantum stochastic evolution operator introduced in [16-18] (formulae (61), (22) and (11), respectively).
We show that a quantum measurement can be wholly described in the frame of a new *general* approach based on the introduced notion of a family of quantum stochastic evolution operators.
The proposed approach allows to give:

1) the complete statistical description of any quantum measurement;
2) the complete description in a Hilbert space of the stochastic behaviour of a quantum system under a measurement;
3) to give the semiclassical interpretation of the description of a quantum measurement;
4) to formalize the consideration of all possible cases of quantum measurements including measurements continuous in time.

In a sequel to this paper we will consider in detail the further application of the proposed general approach to the description of different concrete types of measurements.
We shall, in particular, present the further development of our results given in [16-18] on the description (*without assuming any Markov property*) of measurements continuous in time. For this

particular case the *quantum stochastic approach allows to define in the most general case the notion of posterior pure state trajectories (quantum trajectories) in a Hilbert space of a quantum system, to give their probabilistic treatment and in the frame of Hamiltonian formalism to derive* the *general integral equation for a posterior pure state trajectory.*

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Moscow State Institute of Electronics and Mathematics,
Technical University,
109028, Moscow, Russia

erl@erl.msk.ru
pm@miem.edu.ru