On Criteria for the Uniform Integrability of Brownian Stochastic Exponentials

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Abstract. This paper deals with various sufficient (as well as necessary and sufficient) conditions for the uniform integrability of the exponential martingales of the form

$$Z_t = \exp\Big\{B_{t\wedge\tau} - \frac{1}{2}\,t\wedge\tau\Big\}, \quad t \geq 0,$$

where B is a Brownian motion and τ is a stopping time. We give an overview of the known results and present some new criteria (Theorems 3.2, 4.1).

As an auxiliary lemma, we prove the following statement that is interesting in itself: for any function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, the upper limit $\limsup_{t\uparrow\infty}(B_t - \varphi(t))$ either equals $+\infty$ a.s. or equals $-\infty$ a.s. This provides a simple criterion for distinguishing lower and upper functions of a Brownian motion.

Key words and phrases. Exponential martingales, Novikov's condition, Kazamaki's condition, consistent probability measures, upper and lower functions of a Brownian motion.

1 Introduction and Known Results

1. Let $B = (B_t)_{t\geq 0}$ be a Brownian motion on some filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\big)$ and $\tau = \tau(\omega)$ be a $(\mathcal{F}_t)$-stopping time taking values in $[0, \infty]$.

Set $M_t = B_{t\wedge\tau}$. The process $M = (M_t)_{t\geq 0}$ is a continuous square-integrable martingale with the (predictable) quadratic variation $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$ given by $\langle M\rangle_t = t\wedge\tau$. The process $Z = (Z_t)_{t\geq 0}$ defined as

$$Z_t = \exp\Big\{M_t - \frac{1}{2}\,\langle M\rangle_t\Big\}$$

is called the *Doléans exponential* (or the *stochastic exponential*) of M. These processes arise naturally in many aspects of the stochastic analysis as well as in its applications (stochastic optimal control, nonlinear filtering, stochastic mathematical finance, etc.). The important problem is to find out whether the process Z is *uniformly integrable.*

The uniform integrability of Z is equivalent to the condition $\mathsf{E} Z_\infty = 1$, or in other words,

$$\mathsf{E}\exp\Big\{B_\tau - \frac{1}{2}\tau\Big\} = 1. \tag{1.1}$$

Here, the expression $\exp\{B_\tau - \tau/2\}$ is taken to be equal to zero on the set $\{\omega : \tau(\omega) = \infty\}$. This convention is natural in view of the limit relation

$$\lim_{t\to\infty} \exp\Big\{B_t - \frac{1}{2}t\Big\} = \lim_{t\to\infty} \exp\Big\{t\Big(\frac{B_t}{t} - \frac{1}{2}\Big)\Big\} = 0 \quad \text{a.s.}$$

which follows from the strong law of large numbers for a Brownian motion: $\lim_{t\to\infty} B_t/t = 0$ a.s.

2. There are many papers dealing with the sufficient conditions that should be imposed on τ in order to guarantee (1.1).

For a uniformly bounded τ (i.e. $\tau(\omega) \le c$), property (1.1) is a consequence of Doob's optional stopping theorem (see, for example, [13; Ch. II, (3.2)]). I.I. Gikhman and A.V. Skorokhod [3] proved that the condition

$$\exists \varepsilon > 0 : \ \mathsf{E}\exp\{(1+\varepsilon)\tau\} < \infty$$

guarantees (1.1). R.S. Liptser and A.N. Shiryaev [10] showed that a weaker assumption

$$\exists \varepsilon > 0 : \ \mathsf{E}\exp\Big\{\Big(\frac{1}{2} + \varepsilon\Big)\tau\Big\} < \infty$$

is sufficient for (1.1). A.A. Novikov [11] proved that one can set $\varepsilon = 0$ in the above condition, i.e. that the assumption

$$\mathsf{E}\exp\Big\{\frac{1}{2}\tau\Big\} < \infty \tag{1.2}$$

implies (1.1) while no condition of the form

$$\mathsf{E}\exp\Big\{\Big(\frac{1}{2} - \varepsilon\Big)\tau\Big\} < \infty \tag{1.3}$$

with $\varepsilon > 0$ is sufficient for (1.1).

Let us now consider the following example:

$$\tau = \inf\{t \ge 0 : B_t = 1\}. \tag{1.4}$$

We have $\mathsf{E}\sqrt{\tau} = \infty$, and consequently, $\mathsf{E}\exp\{\tau/2\} = \infty$. On the other hand, it is well known (see, for example, [16; p. 248]) that (1.1) holds for this stopping time τ. Thus, condition (1.2) is not necessary for (1.1).

It is of interest to mention in this connection *Kazamaki's condition* (see [7]):

$$\sup_{t\ge 0} \mathsf{E}\exp\Big\{\frac{1}{2}B_{t\wedge\tau}\Big\} < \infty. \tag{1.5}$$

This condition is sufficient for (1.1) and is weaker than (1.2) in view of the inequality

$$\begin{aligned}
&\mathsf{E}\exp\Big\{\frac{1}{2}B_{t\wedge\tau}\Big\} = \mathsf{E}\exp\Big\{\frac{1}{2}B_{t\wedge\tau} - \frac{1}{4}t\wedge\tau\Big\}\exp\Big\{\frac{1}{4}t\wedge\tau\Big\} \\
&\le \Big(\mathsf{E}\exp\Big\{B_{t\wedge\tau} - \frac{1}{2}t\wedge\tau\Big\}\Big)^{1/2}\Big(\mathsf{E}\exp\Big\{\frac{1}{2}t\wedge\tau\Big\}\Big)^{1/2} \\
&= \Big(\mathsf{E}\exp\Big\{\frac{1}{2}t\wedge\tau\Big\}\Big)^{1/2} \le \Big(\mathsf{E}\exp\Big\{\frac{1}{2}\tau\Big\}\Big)^{1/2}.
\end{aligned}$$

Note that (1.5) holds for the stopping time given by (1.4) while (1.2) is violated for this stopping time. In other words, Kazamaki's condition (1.5) is strictly weaker than Novikov's condition (1.2).

As already pointed out, condition (1.3) is insufficient for (1.1). It is interesting to note, however, that any of the conditions

$$\lim_{\varepsilon\downarrow 0}\Bigl(\mathsf{E}\exp\Bigl\{\frac{1-\varepsilon}{2}\,\tau\Bigr\}\Bigr)^{\varepsilon}=1,$$
$$\lim_{\varepsilon\downarrow 0}\Bigl(\sup_{t\ge 0}\mathsf{E}\exp\Bigl\{\frac{1-\varepsilon}{2}\,B_{t\wedge\tau}\Bigr\}\Bigr)^{\varepsilon}=1$$

implies (1.1) (see [9; p. 160], [18; Théorème 1]).

Condition (1.2) can also be weakened by another way. If there exists a *lower function* φ of a Brownian motion (the definition of a lower function is given in Section 2) such that

$$\mathsf{E}\exp\Bigl\{\frac{1}{2}\,\tau-\varphi(\tau)\Bigr\}<\infty \tag{1.6}$$

(here, τ is supposed to be a.s. finite), then (1.1) is satisfied. This was proved in [12] and [9; p. 159] with some additional monotonicity assumptions made on the lower function φ.

Let us also mention the paper [8] that deals with the conditions similar to (1.6) as well as with the weakening of Kazamaki's condition.

3. One of the aims of this paper is to give some new sufficient conditions for (1.1) involving the lower functions (see Section 4).

In Section 2 we prove a lemma related to the lower functions. This lemma is used in the subsequent proofs. Besides, it is in itself noteworthy.

In Section 3 we present a simple proof of a *necessary and sufficient* condition for (1.1). This condition has a particularly simple formulation for the stopping times of the form $\tau=\inf\{t\ge 0: B_t\ge f(t)\}$, where $f:\mathbb{R}_+\to\mathbb{R}$ is a continuous function.

Section 5 contains several (counter-)examples.

In Section 6 we show that, for any continuous local martingale M, the problem of the uniform integrability of its stochastic exponential can be reduced to (1.1).

2 Upper and Lower Functions of Brownian Motion

Let $(B_t)_{t\ge 0}$ be a standard linear Brownian motion started at zero and let $\varphi:\mathbb{R}_+\to\mathbb{R}$ be a continuous function. The set

$$A=\bigl\{\omega:\exists t=t(\omega)>0:\forall s\ge t,\ B_s(\omega)<\varphi(s)\bigr\}$$

belongs to the tail σ-field $\mathcal{X}=\bigcap_{t>0}\sigma(B_s;\ s\ge t)$. The σ-field $\mathcal{X}$ is trivial (this follows from Blumenthal's zero-one law combined with the time-inversion property of a Brownian motion). Hence, $\mathsf{P}(A)$ equals 0 or 1. We will now cite the classical definition of the lower and upper functions (see, for example, [5; §1.8]).

Definition 2.1. If $\mathsf{P}(A)=0$, then φ is called a *lower function* of a Brownian motion. If $\mathsf{P}(A)=1$, then φ is called an *upper function* of a Brownian motion.

`Remark`. It can be proved that, for *any* function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, the set A is measurable and belongs to the σ-field $\mathcal{X}$. Thus, any function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is either a lower function or an upper function of a Brownian motion. □

Lemma 2.2. *For any (continuous) function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, one has*

$$\limsup_{t\to\infty}(B_t - \varphi(t)) \overset{\text{a.s.}}{=} \begin{cases} +\infty & \text{if } \varphi \text{ is a lower function,} \\ -\infty & \text{if } \varphi \text{ is an upper function.} \end{cases}$$

`Proof`. The random variable $\limsup_{t\to\infty}(B_t - \varphi(t))$ is measurable with respect to the tail σ-field $\mathcal{X}$. As $\mathcal{X}$ is trivial, there exists a constant $\alpha \in [-\infty, \infty]$ such that

$$\limsup_{t\to\infty}(B_t - \varphi(t)) = \alpha \quad \text{a.s.}$$

Suppose that $\alpha \in (-\infty, \infty)$. Set

$$\mathsf{Q} = \mathrm{Law}(B_t;\, t \geq 0), \qquad \widetilde{\mathsf{Q}} = \mathrm{Law}(B_t + t \wedge 1;\, t \geq 0).$$

Let X denote the coordinate process on $C(\mathbb{R}_+, \mathbb{R})$ (i.e. $X_t : C(\mathbb{R}_+, \mathbb{R}) \ni x \mapsto x(t)$). Then

$$\limsup_{t\to\infty}(X_t - \varphi(t)) = \alpha \quad \mathsf{Q}\text{-a.s.}$$
$$\limsup_{t\to\infty}(X_t - \varphi(t)) = \alpha + 1 \quad \widetilde{\mathsf{Q}}\text{-a.s.}$$

On the other hand, the general theory of the change of measure (see [6; Ch. IV, (4.23)]) guarantees that $\widetilde{\mathsf{Q}} \sim \mathsf{Q}$. Consequently, $\alpha = \alpha + 1$. The obtained contradiction shows that α equals either $+\infty$ or $-\infty$. Obviously, in the former case φ is a lower function while in the latter case φ is an upper function. □

`Remark`. Sometimes one defines “lower” and “upper” (strictly positive) functions of a Brownian motion using the expression

$$\limsup_{t\to\infty} \frac{B_t}{\varphi(t)}$$

(note that this random variable is a.s. equal to a constant $\alpha \in [0, \infty]$). However, this approach does not allow the lower functions and the upper functions to be distinguished completely for the following reason. If $\alpha > 1$, then φ is a lower function (in the sense of Definition 2.1); if $\alpha < 1$, then φ is an upper function. But there exist lower functions as well as upper functions φ for which $\alpha = 1$.

In order to prove the last assertion, take $\varphi(t) = \sqrt{2t \ln\ln t}$. The Kolmogorov-Petrovsky criterion (see [5; §1.8]) shows that $\varphi(t)$ is a lower function. For any $\varepsilon > 0$, the function $(1+\varepsilon)\varphi(t)$ is an upper function. Thus, there exists an increasing sequence $(t_n)_{n=1}^\infty$ of real numbers such that, for any $n \in \mathbb{N}$,

$$\mathsf{P}\Big\{\forall s \geq t_n,\ B_s < \Big(1 + \frac{1}{n}\Big)\varphi(s)\Big\} \geq 1 - \frac{1}{2^n}.$$

Thanks to the Borel-Cantelli lemma, the function

$$\psi(t) = \Big(1 + \frac{1}{n}\Big)\varphi(t) \quad \text{if } t \in [t_n, t_{n+1})$$

is an upper function. Furthermore, by the law of the iterated logarithm, we have

$$\limsup_{t\to\infty} \frac{B_t}{\varphi(t)} = \limsup_{t\to\infty} \frac{B_t}{\psi(t)} = 1 \quad \text{a.s.}$$

Obviously, one can construct a *continuous* upper function ψ with the same property. □

3 Criteria for the Uniform Integrability of Exponential Martingales

When considering the sufficient conditions for (1.1), it is useful to introduce the following classes of stopping times:

(A) $\mathfrak{M}^A$ is the class of all $(\mathcal{F}_t)$-stopping times;

(B) $\mathfrak{M}^B$ is the class of $(\mathcal{F}_t^B)$-stopping times, where $\mathcal{F}_t^B = \bigcap_{\varepsilon>0} \sigma(B_s;\ s \le t+\varepsilon)$;

(C) $\mathfrak{M}^C$ is the class of the stopping times that have the form $\tau = \inf\{t \ge 0 : B_t \ge f(t)\}$ for some continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) > 0$.

1. Let $(B_t)_{t\ge0}$ be a Brownian motion on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathsf{P}\big)$ and τ be a $(\mathcal{F}_t)$-stopping time, i.e. $\tau \in \mathfrak{M}^A$. Let us consider the space $C(\mathbb{R}_+, \mathbb{R}^2)$ of continuous functions $x = (x^1(t), x^2(t))_{t\ge0}$. This space is endowed with the Borel σ-field $\mathcal{C}^{(2)}$. Let $X = (X_t^1, X_t^2)_{t\ge0}$ denote the coordinate process on $C(\mathbb{R}_+, \mathbb{R}^2)$ (i.e. $X_t^1(x) = x^1(t)$, $X_t^2(x) = x^2(t)$) and $\mathcal{C}_t^{(2)} = \bigcap_{\varepsilon>0} \sigma(X_s;\ s \le t+\varepsilon)$ denote the canonical filtration. Set

$$\mathsf{Q} = \operatorname{Law}\big(B_t, (t-\tau)^+;\ t \ge 0\big) \tag{3.1}$$

(thus, Q is a measure on $\mathcal{C}^{(2)}$). Obviously, (1.1) is equivalent to:

$$\mathsf{E}_{\mathsf{Q}} \exp\Big\{X_T^1 - \frac{1}{2}T\Big\} = 1, \tag{3.2}$$

where

$$T = \inf\{t \ge 0 : X_t^2 > 0\}. \tag{3.3}$$

Set $\mathsf{Q}_t = \mathsf{Q}|\mathcal{C}_t^{(2)}$ and consider the measures $(\widetilde{\mathsf{Q}}_t)_{t\ge0}$ defined by

$$\frac{d\widetilde{\mathsf{Q}}_t}{d\mathsf{Q}_t} = \exp\Big\{X_t^1 - \frac{1}{2}t\Big\}. \tag{3.4}$$

Then the measures $(\widetilde{\mathsf{Q}}_t)$ are consistent in the sense that $\widetilde{\mathsf{Q}}_t|\mathcal{C}_s^{(2)} = \widetilde{\mathsf{Q}}_s$ for $s \le t$.

Proposition 3.1. *Let $d \in \mathbb{N}$ and $(\mathbb{P}_t)_{t\ge0}$ be a family of consistent probability measures on $\big(\mathcal{C}_t^{(d)}\big)$, where $\big(\mathcal{C}_t^{(d)}\big)$ is the canonical filtration on the space $C(\mathbb{R}_+, \mathbb{R}^d)$. Then there exists a unique measure $\mathbb{P}$ on the Borel σ-field $\mathcal{C}^{(d)}$ such that $\mathbb{P}|\mathcal{C}_t^{(d)} = \mathbb{P}_t$ for any $t \ge 0$.*

For the proof, see [17; (1.3.5)].

`Remark`. Suppose that Ω is an arbitrary probability space endowed with a filtration $(\mathcal{G}_t)_{t\ge0}$ and $(\mathbb{P}_t)_{t\ge0}$ is a family of consistent probability measures on $(\mathcal{G}_t)$. Then it may happen that the family $(\mathbb{P}_t)$ can not be extended to a measure $\mathbb{P}$ on $\bigvee_{t\ge0} \mathcal{G}_t$ (see [2], [14; Ch. II, §3] for the corresponding examples). □

The following theorem provides a necessary and sufficient condition for (3.2) (and hence, for (1.1)).

Theorem 3.2. *Suppose that $\tau \in \mathfrak{M}^A$. Let $\widetilde{\mathsf{Q}}$ be the measure such that $\widetilde{\mathsf{Q}}|\mathcal{C}_t^{(2)} = \widetilde{\mathsf{Q}}_t$, where $\widetilde{\mathsf{Q}}_t$ is given by (3.4). Then (3.2) is satisfied if and only if $\widetilde{\mathsf{Q}}\{T < \infty\} = 1$.*

Proof. Set $\mathsf{Q}_T = \mathsf{Q}|\mathcal{C}_T^{(2)}$, $\widetilde{\mathsf{Q}}_T = \widetilde{\mathsf{Q}}|\mathcal{C}_T^{(2)}$. By [6; Ch. III, (3.4)], we have

$$\frac{d\widetilde{\mathsf{Q}}_T}{d\mathsf{Q}_T} = \exp\Big\{X_T^1 - \frac{1}{2}T\Big\}$$

on the set $\{T < \infty\}$. Therefore,

$$\widetilde{\mathsf{Q}}\{T < \infty\} = \mathsf{E}_{\mathsf{Q}}\Big[I(T < \infty)\exp\Big\{X_T^1 - \frac{1}{2}T\Big\}\Big] = \mathsf{E}_{\mathsf{Q}}\exp\Big\{X_T^1 - \frac{1}{2}T\Big\} \tag{3.5}$$

(we used the fact that $\exp\{X_T^1 - T/2\}$ is taken to be equal to zero on the set $\{T = \infty\}$). □

2. Let us now consider the case where τ is a $(\mathcal{F}_t^B)$-stopping time, i.e. $\tau \in \mathfrak{M}^B$. Here, $\mathcal{F}_t^B = \bigcap_{\varepsilon > 0}\sigma(B_s;\, s \le t + \varepsilon)$. Then there exists a map $T : C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ such that $\tau = T(B)$ and T is a $(\mathcal{C}_t)$-stopping time (here, $(\mathcal{C}_t)$ denotes the canonical filtration on $C(\mathbb{R}_+, \mathbb{R})$).

Theorem 3.3. *Suppose that* $\tau \in \mathfrak{M}^B$. *Let* $\widetilde{\mathsf{Q}}$ *be the distribution of a Brownian motion with the unit drift, i.e.* $\widetilde{\mathsf{Q}} = \mathrm{Law}(B_t + t;\, t \ge 0)$. *Then* (1.1) *is satisfied if and only if* $\widetilde{\mathsf{Q}}\{T < \infty\} = 1$.

Proof. Condition (1.1) is equivalent to the property

$$\mathsf{E}_{\mathsf{Q}}\exp\Big\{X_T - \frac{1}{2}T\Big\} = 1,$$

where X is the coordinate process on $C(\mathbb{R}_+, \mathbb{R})$ and Q is the Wiener measure. By Girsanov's theorem combined with [6; Ch. III, (3.4)], we have

$$\frac{d\widetilde{\mathsf{Q}}_T}{d\mathsf{Q}_T} = \exp\Big\{X_T - \frac{1}{2}T\Big\}$$

on the set $\{T < \infty\}$. The equality similar to (3.5) completes the proof. □

3. Let us now suppose that $\tau = \inf\{t \ge 0 : B_t \ge f(t)\}$, where $f : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function with $f(0) > 0$, i.e. $\tau \in \mathfrak{M}^C$. The following result was obtained in [15]. We give here another proof.

Theorem 3.4. *Suppose that* $\tau \in \mathfrak{M}^C$. *Then condition* (1.1) *is satisfied if and only if the function* $f(t) - t$ *is a lower function of a Brownian motion.*

Proof. Let Q and $\widetilde{\mathsf{Q}}$ be the same as in Theorem 3.3. Set

$$T = \inf\{t \ge 0 : X_t \ge f(t)\},$$

where X is the coordinate process on $C(\mathbb{R}_+, \mathbb{R})$. By Theorem 3.3, condition (1.1) is equivalent to the equality $\widetilde{\mathsf{Q}}\{T < \infty\} = 1$. This equality is, in turn, equivalent to $\mathsf{Q}\{\widetilde{T} < \infty\} = 1$, where

$$\widetilde{T} = \inf\{t \ge 0 : X_t \ge f(t) - t\}.$$

If $f(t) - t$ is a lower function, then, obviously, $\mathsf{Q}\{\widetilde{T} < \infty\} = 1$.

Suppose now that $f(t) - t$ is an upper function. Then there exists $t_0 \ge 0$ such that

$$\mathsf{Q}\{\forall s \ge t_0,\, X_s < f(s) - s\} > 0.$$

Consequently, there exists $a \in \mathbb{R}$ such that

$$\mathsf{Q}\{\forall s \geq t_0,\, a + X_s - X_{t_0} < f(s) - s\} > 0. \tag{3.6}$$

Obviously, we have

$$\mathsf{Q}\big(\{\forall s \leq t_0,\, X_s < f(s) - s\} \cap \{X_{t_0} < a\}\big) > 0. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\mathsf{Q}\{\forall s \geq 0,\, X_s < f(s) - s\} > 0.$$

This means that $\mathsf{Q}\{\widetilde{T} < \infty\} < 1$. □

Remark. Take $\tau = \inf\{t \geq 0 : B_t \geq 1 + t\}$. By Theorem 3.4, (1.1) is then satisfied. On the other hand, τ is infinite with positive P-probability. Thus, the integral conditions like (1.2) are far from being necessary for (1.1). □

Let us now consider the following example related to Theorem 3.4.

Example 3.5. Let $a \in (0, \infty)$, $b \in (-\infty, \infty)$ and

$$\tau_{a,b} = \inf\{t \geq 0 : B_t \geq a + bt\}.$$

It is well known (see [1; (4.32)] or [16; p. 759]) that

$$\mathsf{P}\{\tau_{a,b} \leq t\} = 1 - \Phi\Big(\frac{a+bt}{\sqrt{t}}\Big) + e^{-2ab}\Phi\Big(\frac{bt-a}{\sqrt{t}}\Big).$$

Hence, the density

$$p_{a,b}(t) = \frac{\partial \mathsf{P}\{\tau_{a,b} \leq t\}}{\partial t}$$

is given by

$$p_{a,b}(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\Big\{-\frac{(a+bt)^2}{2t}\Big\}. \tag{3.8}$$

Therefore,

$$\begin{aligned}
\mathsf{E}\exp\Big\{B_{\tau_{a,b}} - \frac{1}{2}\,\tau_{a,b}\Big\} &= \mathsf{E}\exp\Big\{a + b\tau_{a,b} - \frac{1}{2}\,\tau_{a,b}\Big\} I(\tau_{a,b} < \infty) \\
&= \frac{ae^{a(1-b)}}{\sqrt{2\pi}} \int_0^\infty \frac{1}{t^{3/2}} \exp\Big\{bt - \frac{1}{2}\,t - \frac{a^2}{2t} - \frac{b^2 t}{2}\Big\} dt \\
&= \exp\big\{a(1-b) - a|b-1|\big\} = \begin{cases} 1 & \text{if } b \leq 1, \\ e^{-2a(b-1)} & \text{if } b > 1. \end{cases}
\end{aligned}$$

In order to calculate the integral, we used the change of variables $u = t^{-1/2}$ and the equality

$$\int_0^\infty \exp\Big\{-\alpha u^2 - \frac{\beta}{u^2}\Big\} du = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-2\sqrt{\alpha\beta}}, \quad \alpha > 0, \beta \geq 0$$

(see [4; (3.325)]). Thus,

$$\mathsf{E}\exp\Big\{B_{\tau_{a,b}} - \frac{1}{2}\,\tau_{a,b}\Big\} = 1 \ \text{ if } b \leq 1$$

and

$$\mathsf{E}\exp\Big\{B_{\tau_{a,b}} - \frac{1}{2}\,\tau_{a,b}\Big\} < 1 \ \text{ if } b > 1.$$

Moreover,

$$\mathsf{P}\{\tau_{a,b} < \infty\} = \int_0^\infty p_{a,b}(t)\,dt = \begin{cases} 1 & \text{if } b \le 0, \\ e^{-2ab} & \text{if } b > 0. \end{cases}$$

It is reasonable to ask the question: for which functions ψ is (1.1) satisfied with the following stopping time

$$\tau = \inf\big\{t \ge 0 : B_t \ge a + t + \psi(t)\big\}.$$

Theorem 3.4 shows that (1.1) is satisfied with such τ if and only if ψ is a lower function of a Brownian motion. □

4 Some New Conditions for the Uniform Integrability

The following new criteria improve the results of [8] as well as some conditions of [9] and [12]. In particular, our results improve Novikov's criterion (1.2) and Kazamaki's criterion (1.5).

Theorem 4.1. *Let φ be a lower function of a Brownian motion. Then any of the conditions*

$$\limsup_{t\to\infty} \mathsf{E}\exp\Big\{\frac{1}{2}\,t\wedge\tau - \varphi(t\wedge\tau)\Big\} < \infty, \tag{4.1}$$

$$\limsup_{t\to\infty} \mathsf{E}\exp\Big\{\frac{1}{2}\,B_{t\wedge\tau} - \frac{1}{2}\,\varphi(t\wedge\tau)\Big\} < \infty \tag{4.2}$$

is sufficient for (1.1).

Proof. We will give the proof only for condition (4.2) as (4.1) is treated similarly (see also [8]). Let Q, $\widetilde{\mathsf{Q}}$ and T be the same as in (3.1), (3.3), (3.4). For any $t \ge 0$, we have

$$\begin{aligned}
&\mathsf{E}_{\mathsf{Q}}\exp\Big\{\frac{1}{2}\,X^1_{t\wedge T} - \frac{1}{2}\,\varphi(t\wedge T)\Big\} \\
&= \mathsf{E}_{\widetilde{\mathsf{Q}}}\exp\Big\{\frac{1}{2}\,X^1_{t\wedge T} - \frac{1}{2}\,\varphi(t\wedge T)\Big\}\exp\Big\{-X^1_{t\wedge T} + \frac{1}{2}\,t\wedge T\Big\} \\
&= \mathsf{E}_{\widetilde{\mathsf{Q}}}\exp\Big\{-\frac{1}{2}\,\big(X^1_{t\wedge T} - t\wedge T\big) - \frac{1}{2}\,\varphi(t\wedge T)\Big\} \\
&\ge \mathsf{E}_{\widetilde{\mathsf{Q}}}\exp\Big\{\frac{1}{2}\,(-X^1_t + t) - \frac{1}{2}\,\varphi(t)\Big\}I(T=\infty).
\end{aligned}$$

Note that, by Girsanov's theorem, the process $-X_t + t$ is a $\widetilde{\mathsf{Q}}$-Brownian motion. Suppose that (4.2) holds. Then, according to Lemma 4.2 below, $\widetilde{\mathsf{Q}}\{T=\infty\} = 0$. Applying Theorem 3.2, we get the desired result. □

Lemma 4.2. *Let B be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Suppose that φ is a lower function of a Brownian motion. Let $A \in \mathcal{F}$ be a set with $\mathsf{P}(A) > 0$. Then*

$$\limsup_{t\to\infty} \mathsf{E}\exp\Big\{\frac{1}{2}\,B_t - \frac{1}{2}\,\varphi(t)\Big\}I(A) = \infty. \tag{4.3}$$

Proof. *Step 1.* Let us first prove that there exists a sequence $(t_n)_{n=1}^{\infty}$ such that $n \le t_n < n+1$ and

$$\limsup_{n\to\infty}\bigl(B_{t_n} - \varphi(t_n)\bigr) = \infty \quad \text{a.s.} \tag{4.4}$$

To this end, we choose for each $n \in \mathbb{N}$ a number $t_n \in [n, n+1)$ such that

$$\varphi(t_n) \le \inf_{t\in[n,n+1)} \varphi(t) + 1$$

(we may assume that this infimum is finite for all sufficiently large n, or else the statement of Step 1 is trivial). Let us prove that (4.4) is satisfied for these numbers t_n.

Set

$$S_n = \sup_{t\in[0,1]} |B_{n+t} - B_n|.$$

Then $\mathsf{E}S_n^4 < \infty$, and, by Chebyshev's inequality,

$$\sum_{n=1}^{\infty} \mathsf{P}\{S_n > n^{1/3}\} < \infty.$$

According to the Borel-Cantelli lemma,

$$\mathsf{P}\bigl\{\exists N : \forall n \ge N,\ S_n \le n^{1/3}\bigr\} = 1. \tag{4.5}$$

For any $t \in [n, n+1)$, we can write

$$B_t - \varphi(t) \le B_{t_n} - \varphi(t_n) + 2S_n + 1.$$

Combining this with (4.5) and keeping inequality $t_n \ge n$ in mind, we get

$$\limsup_{t\to\infty}(B_t - \varphi(t)) \le \limsup_{n\to\infty}\bigl(B_{t_n} - \varphi(t_n) + 3t_n^{1/3}\bigr) \quad \text{a.s.}$$

Applying Lemma 2.2, we arrive at

$$\limsup_{n\to\infty}\bigl(B_{t_n} - \varphi(t_n) + 3t_n^{1/3}\bigr) = \infty \quad \text{a.s.} \tag{4.6}$$

On the other hand, there exists an absolutely continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(0) = 0$, $h(t) = 3t^{1/3}$ for $t > 1$ and

$$\int_0^{\infty} (h'(t))^2 dt < \infty.$$

For this function h, we have

$$\mathrm{Law}(B_t;\ t \ge 0) \sim \mathrm{Law}\bigl(B_t + h(t);\ t \ge 0\bigr)$$

(see [6; Ch. IV, (4.23)]). This, together with (4.6), yields (4.4).

Step 2. Suppose that condition (4.3) is violated. Then there exists $\gamma > 0$ such that, for any sufficiently large n,

$$\mathsf{E}\exp\Bigl\{\frac{1}{2}\,B_{t_n} - \frac{1}{2}\,\varphi(t_n)\Bigr\} \le \gamma,$$

where the numbers t_n satisfy (4.4). Hence,

$$\mathsf{P}\big(\{B_{t_n}-\varphi(t_n)>3t_n^{1/3}\}\cap A\big)\le\gamma\exp\Big\{-\frac{3}{2}\,t_n^{1/3}\Big\}\le\frac{\gamma}{n^2}.$$

By the Borel-Cantelli lemma,

$$\limsup_{n\to\infty}\big(B_{t_n}(\omega)-\varphi(t_n)-3t_n^{1/3}\big)\le 0$$

for P-almost every $\omega\in A$. Recall that $\mathsf{P}(A)>0$.

Let h be the function described in Step 1. Then

$$\mathrm{Law}(B_t;\,t\ge 0)\sim\mathrm{Law}\big(B_t-h(t);\,t\ge 0\big),$$

and condition (4.4) shows that

$$\limsup_{n\to\infty}\big(B_{t_n}-\varphi(t_n)-3t_n^{1/3}\big)=\infty\quad\text{a.s.}$$

The obtained contradiction completes the proof. $\square$

Theorem 4.1 implies the following statement (it was proved in [9; p. 159]).

Corollary 4.3. *Suppose that φ is a lower function such that the function $t/2-\varphi(t)$ is increasing. Then the condition*

$$\mathsf{E}\exp\Big\{\frac{1}{2}\,\tau-\varphi(\tau)\Big\}<\infty\tag{4.7}$$

is sufficient for (1.1).

5 Some Examples

1. The first example shows that condition (1.6) is strictly weaker than Novikov's condition (1.2).

Example 5.1. Let Ω be the space $C(\mathbb{R}_+,\mathbb{R})$ equipped with the Wiener measure Q. Set

$$\tau=\inf\{t\ge 0: X_t\ge 1+\sqrt{t}-t\},$$

where X is the canonical process on $C(\mathbb{R}_+,\mathbb{R})$. Let $\widetilde{\mathsf{Q}}=\mathrm{Law}(B_t+t;\,t\ge 0)$ and

$$\widetilde{\tau}=\inf\{t\ge 0: X_t\ge 1+\sqrt{t}\},\qquad \sigma=\inf\{t\ge 0: X_t\ge 1\}.$$

Then

$$\begin{aligned}
&\mathsf{E}_{\mathsf{Q}}\exp\Big\{\frac{1}{2}\,\tau\Big\}=\mathsf{E}_{\widetilde{\mathsf{Q}}}\exp\Big\{\frac{1}{2}\,\widetilde{\tau}\Big\}=\mathsf{E}_{\widetilde{\mathsf{Q}}}\exp\Big\{\frac{1}{2}\,\widetilde{\tau}\Big\}I(\widetilde{\tau}<\infty)\\
&=\mathsf{E}_{\mathsf{Q}}\exp\Big\{\frac{1}{2}\,\widetilde{\tau}\Big\}\exp\Big\{X_{\widetilde{\tau}}-\frac{1}{2}\,\widetilde{\tau}\Big\}I(\widetilde{\tau}<\infty)=\mathsf{E}_{\mathsf{Q}}\exp\{X_{\widetilde{\tau}}\}\\
&=\mathsf{E}_{\mathsf{Q}}\exp\{1+\sqrt{\widetilde{\tau}}\}\ge\mathsf{E}_{\mathsf{Q}}\exp\{1+\sqrt{\sigma}\}=\infty
\end{aligned}$$

(we used (3.8) in the last equality). Thus, condition (1.2) is violated.

On the other hand, the computations similar to those given above show that

$$\mathsf{E}_{\mathsf{Q}}\exp\Big\{\frac{1}{2}\,\tau-\sqrt{\tau}\Big\}=\mathsf{E}_{\mathsf{Q}}\exp\{X_{\widetilde{\tau}}-\sqrt{\widetilde{\tau}}\}=e<\infty.\qquad\square$$

2. The second example shows that condition (4.2) is strictly weaker than Kazamaki's condition (1.5).

Example 5.2. Set

$$\tau = \inf\{t \ge 0 : B_t \ge 1 + \sqrt{t}\}.$$

Consider also the stopping time

$$\sigma = \inf\{t \ge 0 : B_t \ge 1\}.$$

We have

$$\mathsf{E}\exp\Big\{\frac{1}{2}\,B_{t\wedge\tau}\Big\} \ge \mathsf{E}\exp\Big\{\frac{1}{2}(1+\sqrt{\tau})\Big\}I(\tau \le t) \ge \mathsf{E}\exp\Big\{\frac{1}{2}(1+\sqrt{\sigma})\Big\}I(\tau \le t).$$

Since τ is finite a.s., the last term converges (as $t \to \infty$) to

$$\mathsf{E}\exp\Big\{\frac{1}{2}\,(1+\sqrt{\sigma})\Big\} = \infty.$$

Thus, condition (1.6) is violated.

On the other hand,

$$\mathsf{E}\exp\Big\{\frac{1}{2}\,B_{t\wedge\tau} - \frac{1}{2}\,\sqrt{t\wedge\tau}\Big\} \le e^{1/2} < \infty. \qquad \square$$

3. The third example shows that the possible weakening of conditions (1.5), (4.2) to the condition

$$\mathsf{E}\exp\Big\{\frac{1}{2}\,B_\tau\Big\} < \infty \tag{5.1}$$

(together with the assumption $\tau < \infty$ a.s.) does not guarantee (1.1).

Example 5.3. Set

$$\tau = \inf\{t \ge 1 : B_t = 1\}.$$

Here, (5.1) is trivially satisfied. On the other hand, (1.1) is violated. Indeed, let us consider the stopping time

$$\sigma = \inf\{t \ge 0 : B_t = 1\}.$$

Applying Kazamaki's criterion, we deduce that (1.1) holds for σ. Furthermore, $B_\tau = B_\sigma = 1$ while $\tau \ge \sigma$ and $\mathsf{P}\{\tau > \sigma\} > 0$. Therefore,

$$\mathsf{E}\exp\Big\{B_\tau - \frac{1}{2}\,\tau\Big\} < \mathsf{E}\exp\Big\{B_\sigma - \frac{1}{2}\,\sigma\Big\} = 1. \qquad \square$$

4. The fourth example shows that the monotonicity assumption in Corollary 4.3 is essential.

Example 5.4. Consider the stopping time

$$\tau_0 = \inf\{t \ge 0 : B_t = -1\}.$$

Let $\tau = n+1$ on the set $\{n \le \tau_0 < n+1\}$ and let $\tau = \infty$ on the set $\{\tau_0 = \infty\}$. In view of Theorem 3.3, condition (1.1) is violated.

On the other hand, condition (4.7) is satisfied with the (discontinuous) lower function

$$\varphi(t) = \begin{cases} 0 & \text{if } t \notin \mathbb{N}, \\ t/2 & \text{if } t \in \mathbb{N}. \end{cases}$$

Obviously, one can also construct a *continuous* lower function φ for which (4.7) is true. $\square$

6 Stochastic Exponentials of Continuous Local Martingales

Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\big)$ with the quadratic variation $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$. Let us formulate a version of the Dambis-Dubins-Schwarz theorem (see [13; Ch. V,(1.6), (1.7)]).

Proposition 6.1. *There exist an enlargement $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathsf{P}})$ of $(\Omega, \mathcal{F}, \mathsf{P})$, a filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ on this space and a $(\widetilde{\mathcal{F}}_t)$-Brownian motion $B = (B_t)_{t\geq 0}$ such that $M_t = B_{\langle M\rangle_t}$. Moreover, for each $s \geq 0$, the random variable $\langle M\rangle_s$ is a $(\widetilde{\mathcal{F}}_t)$-stopping time.*

Let us consider the stochastic exponential of M:

$$Z_t = \exp\Big\{M_t - \frac{1}{2}\,\langle M\rangle_t\Big\}.$$

By Itô's formula, Z is a $(\mathcal{F}_t)$-local martingale. Being positive, it is a supermartingale (this is a consequence of Fatou's lemma). By Doob's convergence theorem, there exists (a.s.) the limit $Z_\infty = \lim_{t\to\infty} Z_t$.

Let $\langle M\rangle_\infty = \lim_{t\to\infty}\langle M\rangle_t$. On the set $\{\langle M\rangle_\infty < \infty\}$, there exists (a.s.) the limit $M_\infty = \lim_{t\to\infty} M_t$ (this is a consequence of Proposition 6.1). Obviously, on this set we have

$$Z_\infty = \exp\Big\{M_\infty - \frac{1}{2}\,\langle M\rangle_\infty\Big\} \quad \text{a.s.}$$

Proposition 6.1, combined with the property $\lim_{t\to\infty}\exp\{B_t - t/2\} = 0$ a.s., shows that $Z_\infty = 0$ on the set $\{\langle M\rangle_\infty = \infty\}$.

Let now B be the Brownian motion given by Proposition 6.1. Set $\tau = \langle M\rangle_\infty$. Note that $\tau(= \lim_{n\to\infty}\langle M\rangle_n)$ is a $(\widetilde{\mathcal{F}}_t)$-stopping time. In view of the convention $\exp\{B_\tau - \tau/2\} = 0$ on the set $\{\tau = \infty\}$, we have

$$Z_\infty = \exp\Big\{B_\tau - \frac{1}{2}\,\tau\Big\} \quad \text{a.s.} \tag{6.1}$$

As Z is a positive supermartingale, the uniform integrability of Z is equivalent to the condition $\mathsf{E}Z_\infty = 1$. Thanks to (6.1), this is, in turn, equivalent to (1.1).

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