Inhomogeneous spatial point processes

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Abstract

Recent models for inhomogeneous spatial point processes with interaction are reviewed. The focus is on models derived from homogeneous Markov point processes. For some of the models, the interaction is location dependent. A new type of transformation related model with this property is also suggested. The statistical inference based on likelihood and pseudolikelihood is discussed for the different models. In particular, it is shown that for transformation models, the pseudolikelihood function can be decomposed in a similar fashion as the likelihood function.

Keywords: Cox point processes; Gibbs point processes; inhomogeneity; interaction; likelihood; Markov point processes; Papangelou conditional intensity; Poisson point processes; pseudolikelihood; thinning; transformation

1. Introduction

In recent years, models for inhomogeneous point processes with interaction have been suggested by several authors. We will in the present paper concentrate on three ways of introducing inhomogeneity into a Markov model, i.e. inhomogeneity induced by a non-constant first-order interaction (Stoyan and Stoyan (1998); see also Ogata and Tanemura (1986)), by thinning of a homogeneous Markov point process (Baddeley et al. (2000)) and by transformation of a homogeneous Markov point process (Jensen and Nielsen (2000)). The aim is to give a unified exposition of these models in order to be able to assess their relative merits and point to research problems that remain to be solved in this area.

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For any of the three models, the inhomogeneity may be described by a function λ defined on the same set as the points. In the case where the inhomogeneous point process is Poisson, λ is the ordinary intensity function. In addition to the point pattern, explanatory variables may be observed at each point, for the purpose of explaining the inhomogeneity. Such information may be included in any of the models. The interaction specified in the models may or may not be location dependent.

In Section 2, inhomogeneous Poisson and Cox point processes are considered. In Section 3.1, a short summary of homogeneous Markov point processes is given, followed in Section 3.2 by a formal description of the three types of inhomogeneous point processes derived from homogeneous Markov point processes. In Section 3.3, parametric specification of the inhomogeneity is discussed, while in Section 3.4 parametric statistical inference is outlined for each of the three model types. Section 4 contains a discussion and some considerations concerning future research.

2. Poisson and related point processes

We will throughout the paper consider point processes on a $k-$dimensional manifold $\mathcal{X}$ in R^m. Often, $\mathcal{X}$ will be full-dimensional such that $k = m$. (Formally, we will call $\mathcal{X}$ full-dimensional if $\mathcal{X}$ is a regular compact, that is a non-empty compact subset of R^m which is the closure of its interior.) But $\mathcal{X}$ may for instance also be a planar curve or a spatial surface, cf. Figure 1. We will assume that $0 < \lambda_m^k(\mathcal{X}) < \infty$ where λ_m^k is the $k-$dimensional volume measure in R^m (Hausdorff measure). We let $\mathcal{B}_0$ be the bounded Borel subsets of $\mathcal{X}$.

It is easy to introduce inhomogeneity within the class of Poisson point processes. Let μ be a locally finite, non-atomic measure on $\mathcal{X}$ with density λ with respect to λ_m^k and let $n(\cdot)$ be the number of elements in $\cdot$. A point process X on $\mathcal{X}$ is then said to be a Poisson point process with *intensity*

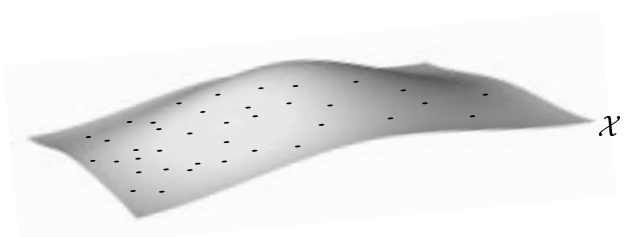


Figure 1: Illustration of point process on a 2-dimensional manifold $\mathcal{X}$ in R^3.

function λ if

- $\forall A \in \mathcal{B}_0 : n(X \cap A) \sim Po(\int_A \lambda(u) du^k)$
- $\forall A_1, \cdots, A_s \in \mathcal{B}_0$ disjoint : $n(X \cap A_1), \cdots, n(X \cap A_s)$ independent

We have here used the short notation du^k for $\lambda_m^k(du)$. It can be shown that the first requirement is enough, cf. e.g. Stoyan et al. (1995).

If λ is constant the Poisson point process is said to be homogeneous, otherwise the process is inhomogeneous. A homogeneous Poisson point process is often used as a reference (null) model. The reason is the following result

- Let X be a homogeneous Poisson point process on $\mathcal{X}$ and let $A \in \mathcal{B}_0$. Then, given $n(X \cap A) = n, X \cap A$ is a binomial process with n points, i.e. $X \cap A$ is distributed as $\{X_1, \ldots, X_n\}$ where $X_1, \ldots, X_n$ are independent and identically uniformly distributed on A.

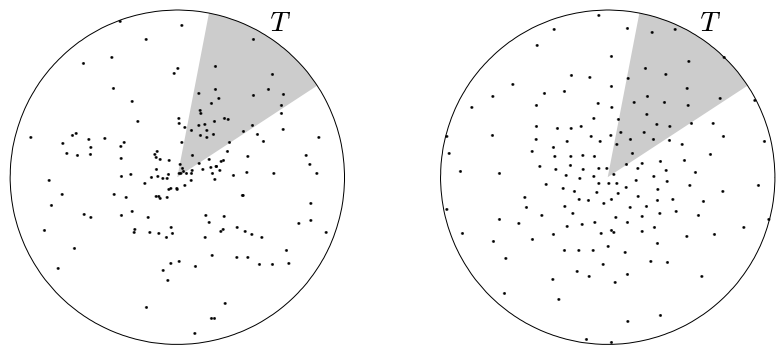
The independence property of the homogeneous Poisson point process ensures that there is no interaction in the process, the uniformity means that the process is homogeneous.

The inhomogeneity of a point process may depend on explanatory variables. One simple geometric example is an intensity function of the form

$$\lambda(\eta) = g(d_C(\eta)), \quad \eta \in \mathcal{X},$$

where $d_C(\eta)$ is the distance from η to a reference structure C. For a point process in the Euclidean plane, the reference structure may be a point or a planar curve. In Figure 2, C is a point (centre of a circle) while, in Figure 3, C is a straight line (centre of a linear band). For a point pattern in Euclidean space, the reference structure may be a point, a spatial curve or a spatial surface. See also Berman (1986) and references therein. Points lying *on* curves in two or three dimensions or points lying *on* spatial surfaces may also show inhomogeneity. In Figure 4 and 5, point processes on the unit circle S^1 and unit sphere S^2 are shown. (Points are represented as directions in Figure 4.) In any of the Figures 2 to 5, Poisson point processes are shown to the left while corresponding processes with inhibition between points are shown to the right (for details, see Section 3.3 below).

Statistical inference for inhomogeneous Poisson processes with a parametrically specified intensity function can be performed as follows. Let X be an inhomogeneous Poisson point process with intensity function $\lambda_\theta, \theta \in$



(a) Poisson (b) Strauss

Figure 2: Realizations of inhomogeneous point processes in the unit circular disc with inhomogeneity function $\lambda(\eta) \propto \mathrm{e}^{\theta d_C(\eta)}$, where $d_C(\eta)$ is the distance from η to C, the centre of the disc. The point pattern to the left is inhomogeneous Poisson, i.e. no interaction between points, whereas the point pattern to the right is inhomogeneous Strauss with $\gamma = 0.01$ and therefore shows inhibition between points. For details, see Section 3.3 and Appendix III. The number of points in the Poisson process have been chosen to equal the number of points in the Strauss process. For both processes, the distribution of the points in the shaded sampling window T remains the same if T is rotated around C.

$\Theta \subseteq R^l$. Then, the likelihood function of θ with respect to the homogeneous Poisson point process with intensity 1 takes the form

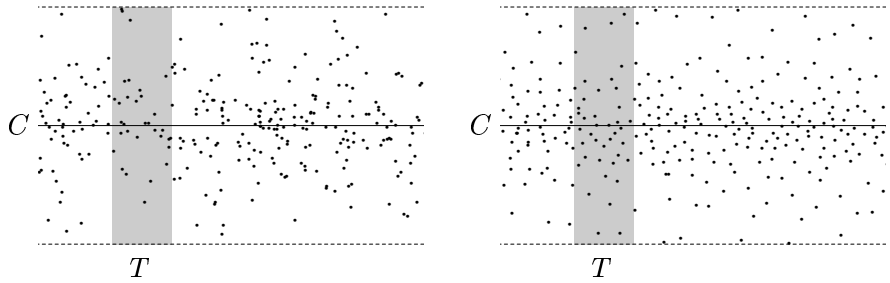
$$L_0(\theta; x) = \exp(-\int_{\mathcal{X}} [\lambda_\theta(u) - 1] du^k) \prod_{\eta \in x} \lambda_\theta(\eta). \tag{1}$$

We use index 0 in this likelihood because later it enters into more complicated likelihoods. In Berman and Turner (1992), log-linear models for λ_θ are discussed,

$$\lambda_\theta(\eta) \propto \mathrm{e}^{\theta \cdot \tau(\eta)}, \quad \eta \in \mathcal{X},$$

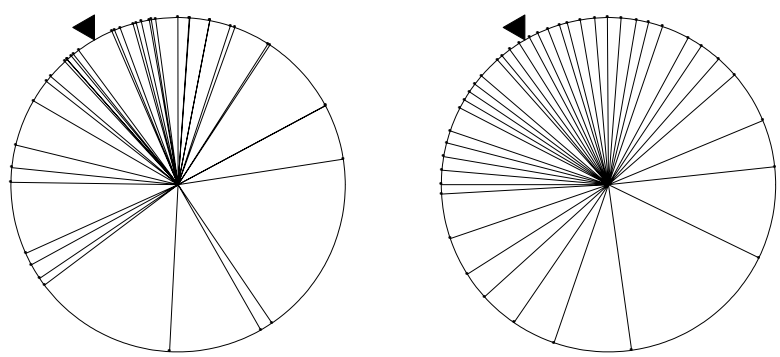
where $\tau(\eta) = (\tau_1(\eta), \ldots, \tau_l(\eta))$ is a list of explanatory variables evaluated at η and $\cdot$ indicates Euclidean inner product. After approximation of the integral by a finite sum, the likelihood takes the same analytical form as the likelihood of a generalized linear model with Poisson responses and standard software can be used to analyze the model. See also Rathbun (1996).

Alternatively, the intensity function can be estimated non-parametrically, using kernel estimation (Silverman (1986)) or a Bayesian method (Heikkinen and Arjas (1998)).



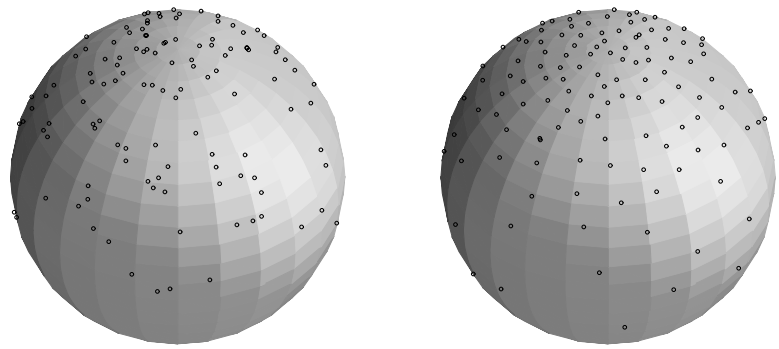
(a) Poisson (b) Strauss

Figure 3: Realizations of inhomogeneous point processes in the unit band $\{\eta \in \mathbb{R}^2 : d_C(\eta) < 1\}$, where $d_C(\eta)$ is the distance from η to C, the full-drawn horizontal line. The inhomogeneity function is as in Figure 2 and likewise the right hand-side point pattern is inhomogeneous Strauss and the left is inhomogeneous Poisson with the same number of points. The distribution of points in the shaded sampling window T remains the same under horizontal translations of T.



(a) Poisson (b) Strauss

Figure 4: Realizations of inhomogeneous point processes on the unit circle S^1. The situation is the same as in Figure 2 except that $d_C(\eta)$ is the distance along the circle from η to the point $C = (\cos(4\pi/3), \sin(4\pi/3))$ marked with an arrow. The points are shown as directions.



(a) Poisson (b) Strauss

Figure 5: Realizations of inhomogeneous point processes on the unit sphere S^2. The situation is the same as in Figure 2 except that $d_C(\eta)$ is the geodesic distance from η to C which is the north pole.

As a generalization one may consider inhomogeneous Cox processes, i.e. inhomogeneous doubly stochastic Poisson point processes. The definition of a Cox process is as follows. Let Λ be a random intensity function on $\mathcal{X}$. Then, X is a Cox process if, given $\Lambda = \lambda$, X is a Poisson point process with intensity function λ, cf. Stoyan et al. (1995, p. 154).

In Møller et al. (1998) and Brix and Møller (1998), log Gaussian Cox processes are discussed, i.e. Cox processes for which $\lambda = e^Y$ and $Y = \{Y_s\}_{s \in \mathcal{X}}$ is a Gaussian field. Inhomogeneity is introduced by letting the mean-value of Y_s depend on s. *Clustered* inhomogeneous point patterns may be modelled by this process and this appears to be a natural model if the aggregation is due to stochastic environmental heterogeneity. This type of model has in Brix and Møller (1998) been used to describe the development of two types of weeds in an organic barley field.

3. Markov point processes

If one wants to describe inhibition in addition to clustering then the class of Markov point processes is useful, cf. Ripley and Kelly (1977), Baddeley and Møller (1989), the recent monograph van Lieshout (2000) and references therein. Let us start by recalling a few preliminaries for Markov point processes.

3.1. Homogeneous Markov point processes

Let $\sim$ be a reflexive and symmetric relation on $\mathcal{X}$. Two points $\xi, \eta \in \mathcal{X}$ are called neighbours if $\xi \sim \eta$. A finite subset x of $\mathcal{X}$ is called a *clique* if all points of x are neighbours. By convention, sets of 0 and 1 points are cliques. The set of cliques is denoted $\mathcal{C}$.

If $\mathcal{X} \subseteq R^m$ is full-dimensional then the relation induced by Euclidean distance is often used. If $\mathcal{X}$ is a planar curve, distances along the curve may be more natural. If $\mathcal{X}$ is the unit sphere S^{m-1}, such that the observed points in fact are directions, then geodesic distance is natural.

Let X be a point process with density f with respect to the homogeneous Poisson point process with intensity 1. If X is a Markov point process, its Papangelou conditional intensity

$$\lambda(\eta; x) = \frac{f(x \cup \eta)}{f(x)}, \quad \eta \notin x,$$

depends only on those points in x which are neighbours of η.

Markov point processes are characterized by the Hammersley-Clifford theorem, cf. Ripley and Kelly (1977). This theorem states that a point process X on $\mathcal{X}$ is a Markov point process if and only if

$$f(x) = \prod_{y \subseteq x} \varphi(y), \tag{2}$$

where φ is an interaction function with respect to $\sim$, i.e. $\varphi(x) = 1$ unless $x \in \mathcal{C}$.

Let φ_k be the restriction of φ to subsets consisting of k points. A pairwise interaction process is then a process for which $\varphi_k \equiv 1$ for $k > 2$. The famous Strauss process, cf. Strauss (1975) and Kelly and Ripley (1976), is the pairwise interaction process with

$$\varphi_k(x) = \begin{cases} \alpha & k = 0 \\ \beta & k = 1 \\ \gamma & k = 2, \ x \in \mathcal{C}. \end{cases}$$

The density of the Strauss process becomes, cf. (2),

$$f(x) = \alpha \beta^{n(x)} \gamma^{s(x)},$$

where $s(x)$ is the number of neighbour pairs in x. If the neighbourhood relation is given by

$$\eta \sim \xi \Longleftrightarrow ||\eta - \xi|| < R,$$

then the process is called a Strauss process with interaction radius R.

If the containing manifold $\mathcal{X} \subseteq R^m$ is full-dimensional, a Markov point process X on $\mathcal{X}$ is said to be homogeneous if φ is translation invariant, cf. e.g. Stoyan and Stoyan (1998) and Baddeley et al. (2000). (We assume that φ is defined on all finite subsets of R^m.) Other definitions of homogeneity are of course possible, cf. Jensen and Nielsen (2000). Note that translation invariance implies that φ_1 is constant and for $k > 1$ $\varphi_k(y)$ only depends on the relative positions of the k points in y. A homogeneous pairwise interaction process has a density of the form

$$f(x) = \alpha\beta^{n(x)} \prod_{\{\eta,\xi\}\subseteq x}^{\neq} u(\eta - \xi), \tag{3}$$

where $\neq$ indicates that η and ξ are different. For lower dimensional manifolds $\mathcal{X}$, homogeneity may be defined in terms of invariance under other types of transformations. For instance, for $\mathcal{X} = S^{m-1}$ a natural set of transformations are the rotations. Recall that the group $O(m)$ of rotations consists of $m \times m$ real matrices

$$O(m) = \{A | AA^T = A^T A = I_m\}.$$

A homogeneous, with respect to this choice, pairwise interaction process on S^{m-1} has a density of the form

$$f(x) = \alpha\beta^{n(x)} \prod_{\{\eta,\xi\}\subseteq x}^{\neq} u(\eta \cdot \xi).$$

Recall that $\eta \cdot \xi = \cos\theta$, where θ is the angle between η and ξ.

3.2. Introducing inhomogeneity

Throughout this section, X is a homogeneous Markov point process with respect to $\sim$ on $\mathcal{X}$ and density

$$f_X(x) = \prod_{y\subseteq x} \varphi(y). \tag{4}$$

Note that φ_1 is then constant. Below, we describe three ways of introducing inhomogeneity into the model. The resulting inhomogeneous point process is denoted by Y and is a point process on a $k-$dimensional manifold $\mathcal{Y}$ in R^d, say. For the first two ways of constructing inhomogeneity, $\mathcal{Y} = \mathcal{X}$, i.e. the

homogeneous process and the associated inhomogeneous process are defined on the same space.

The inhomogeneity is described by a function $\lambda(\eta)$, $\eta \in \mathcal{Y}$, which we will call the inhomogeneity function. Common to each of the three constructions is the feature that if X is a homogeneous Poisson point process, then the associated inhomogeneous point process Y is Poisson with intensity function proportional to λ. The three inhomogeneous models are therefore extensions of the inhomogeneous Poisson model.

An obvious way of introducing inhomogeneity is by making the first-order interaction non-constant. We will call this type I inhomogeneity. The associated inhomogeneous Markov point process has then a density of the form

$$f_Y(y) \propto \prod_{\eta \in y} \lambda(\eta) \prod_{z \subseteq y} \varphi(z). \tag{5}$$

This type of model is natural if the points are centres of non-overlapping particles of constant size. This set-up has been studied in Ogata and Tanemura (1986), Stoyan and Stoyan (1998) and Baddeley and Turner (2000), among others. In Ogata and Tanemura (1986), $\log \lambda(\eta)$ is a polynomium in Cartesian coordinates, while in Stoyan and Stoyan (1998), a piecewise (region-wise) constant function is studied. It is also interesting to note that in the hierarchical point process models described in Högmander and Särkkä (1999), densities of the form (5) appear.

Type II inhomogeneity is obtained by using an independent inhomogeneous thinning of the homogeneous Markov point process. Let us suppose that the inhomogeneity function $\lambda(\eta)$, $\eta \in \mathcal{Y}$, is bounded by $\lambda_{\max}$, and let $p(\eta) = \lambda(\eta)/\lambda_{\max}$, $\eta \in \mathcal{Y}$. The inhomogeneous process is then obtained by thinning with p,

$$Y = \{x_i \in X : U_i \leq p(x_i)\},$$

where $\{U_i\}$ is a sequence of independent and identically uniformly distributed random variables in $[0, 1]$. In Baddeley et al. (2000), this approach is suggested and studied in detail. According to them this model is natural if p can be interpreted as the probability of survival of a plant or the probability of observing an animal in a wildlife population. A possibly less appealing property of the thinned Markov process is that it is non-Markovian except if X is Poisson. However, this does not complicate the likelihood inference, cf. Baddeley et al. (2000).

A third way of introducing inhomogeneity is by applying a 1-1 transformation on a homogeneous Markov point process, cf. Jensen and Nielsen (2000). This is type III inhomogeneity. The idea of using transformations

to introduce inhomogeneity has also been used for the modelling of the covariance structure of a non-stationary spatial process, cf. Perrin (1997) and references therein.

Let h be a 1-1 differentiable mapping of the $k-$dimensional differentiable manifold $\mathcal{X}$ in R^m onto a $k-$dimensional manifold $\mathcal{Y}$ in R^d. We consider on $\mathcal{Y}$ the induced relation

$$\eta_1 \approx \eta_2 \Longleftrightarrow h^{-1}(\eta_1) \sim h^{-1}(\eta_2), \quad \eta_1, \eta_2 \in \mathcal{Y}.$$

The induced relation has the property that interactions are weaker in regions where the transformation makes the intensity of points high. This will be appealing in some applications, for instance in ecology. Note that we in this case have inhomogeneity both in the intensity and the strength of the interaction. The transformation approach may be extended by using a series of transformations.

It can be shown, cf. Jensen and Nielsen (2000, Corollary 3.3), that

$$h(X) = \{h(\xi) : \xi \in X\}$$

is Markov with respect to $\approx$ on $\mathcal{Y}$ and has the density

$$f_Y(y) = \exp(-\int_{\mathcal{Y}} [\lambda(\eta) - 1] d\eta^k) \prod_{\eta \in y} \lambda(\eta) \prod_{z \subseteq y} \varphi(h^{-1}(z)), \tag{6}$$

where $\lambda(\eta) = Jh^{-1}(\eta)$, the Jacobian of the inverse transformation h^{-1}. This transformation result can be proved by the coarea formula in geometric measure theory, cf. Jensen (1998).

Note that if the homogeneous process is Poisson, then the last product in (6) becomes of the form

$$\exp(-(\beta - 1)\lambda_m^k(\mathcal{X}))\beta^{n(y)},$$

and therefore Y is an inhomogeneous Poisson point process with intensity function $\beta \cdot \lambda(\cdot)$.

It is not always easy to find an appropriate transformation which introduces an inhomogeneity of a given form. (The problem to be solved is to find h such that $Jh^{-1} = \lambda$ where λ is a given inhomogeneity function.) It is therefore useful to construct approximate transformation models with the same qualitative properties as the original transformation models. Let us suppose that $d = m = k$ and the manifolds $\mathcal{X}$ and $\mathcal{Y}$ are full-dimensional. Furthermore, suppose that the original process is a homogeneous pairwise

interaction process, cf. (3). Then, the density of the transformed point process is, cf. (6),

$$f_Y(y) \propto \beta^{n(y)} \prod_{\eta \in y} \lambda(\eta) \prod_{\{\eta,\xi\} \subseteq y}^{\neq} u(h^{-1}(\eta) - h^{-1}(\xi)).$$

Recalling that for a transformation model $Jh^{-1} = \lambda$, an obvious way of avoiding to construct the transformation is to replace

$$h^{-1}(\eta) - h^{-1}(\xi) \tag{7}$$

by an expression of the form

$$\lambda(\eta)^{\nu} \cdot \lambda(\xi)^{\nu}(\eta - \xi), \tag{8}$$

where $\nu \geq 0$ is some suitably chosen power. The density of the transformation related model becomes

$$f_Y(y) \propto \beta^{n(y)} \prod_{\eta \in y} \lambda(\eta) \prod_{\{\eta,\xi\} \subseteq y}^{\neq} u(\lambda(\eta)^{\nu} \lambda(\xi)^{\nu}(\eta - \xi)). \tag{9}$$

This type of model has also been considered in Baddeley and Turner (2000). Note that for point processes on the real line ($k = 1$), (8) can for η and ξ close be regarded as an approximation to (7), if $\nu = 1/2$. This will generally not be the case for $k > 1$.

3.3. Exponential inhomogeneity

The inhomogeneity function may be modelled parametrically or non-parametrically or both. If no prior knowledge is available about the inhomogeneity, non-parametric modelling may be useful, at least initially. With knowledge of the inhomogeneity (e.g. monotone decreasing in a known direction) then it can be worthwhile to consider parametrically modelled inhomogeneity such as that of exponential form

$$\lambda_\theta(\eta) = \alpha(\theta) \mathrm{e}^{\theta \cdot \tau(\eta)}, \quad \eta \in \mathcal{Y},$$

where $\theta \in \Theta \subseteq R^l$ and $\tau(\eta) \in R^l$.

Let us concentrate on a comparison between type I and type III exponential inhomogeneity. If the inhomogeneity is of type I, then the density of the inhomogeneous point process takes the form, cf. (5),

$$f_Y(y;\theta) \propto \alpha(\theta)^{n(y)} \mathrm{e}^{\theta \cdot t(y)} \prod_{z \subseteq y} \varphi(z), \tag{10}$$

where $t(y) = \sum_{\eta \in y} \tau(\eta)$. Note that if the homogeneous Markov point process is an exponential family model then the associated inhomogeneous model is too. In particular, if the homogeneous model is a Strauss model then

$$f_Y(y;\theta,\beta,\gamma) \propto \mathrm{e}^{\theta \cdot t(y)} (\alpha(\theta)\beta)^{n(y)} \gamma^{s(y)}. \tag{11}$$

This is a nice three parameter exponential family model.

If instead the transformation approach is used, we need to find a parametrized class of transformations $h_\theta, \theta \in \Theta$ such that

$$Jh_\theta^{-1}(\eta) = \alpha(\theta)\mathrm{e}^{\theta \cdot \tau(\eta)}, \quad \eta \in \mathcal{Y}. \tag{12}$$

Let us give a fairly general geometric example where this problem has a simple solution.

Example 3.1 The example concerns inhomogeneity for point patterns in R^d which depends on the distance d_C to a $p-$dimensional linear subspace C in R^d, $p = 0, 1, \ldots, d-1$. We will define the transformation h_θ on the whole set $\{\eta \in R^d : d_C(\eta) \leq 1\}$ and let $\tau(\eta)$ only depend on the distance of η to C, i.e. $\tau(\eta) = \tilde{\tau}(d_C(\eta))$, say. The cases $(d,p) = (2,0)$ and $(2,1)$ with $\tilde{\tau}$ the identity are illustrated in Figure 2 and 3, respectively.

Then, (12) has a unique solution among transformations of the form

$$h_\theta(\eta) = p_C(\eta) + g_\theta(d_C(\eta)) \frac{\eta - p_C(\eta)}{d_C(\eta)}, \tag{13}$$

where $p_C(\eta)$ is the orthogonal projection onto C and g_θ is an increasing function of $[0,1]$ onto itself. The solution is given by, cf. Appendix I,

$$g_\theta^{-1}(r) = \left[\frac{\int_0^r u^{d-p-1} \mathrm{e}^{\theta \cdot \tilde{\tau}(u)} du}{\int_0^1 u^{d-p-1} \mathrm{e}^{\theta \cdot \tilde{\tau}(u)} du} \right]^{1/(d-p)}. \tag{14}$$

It follows also from Appendix I that for h_θ defined by (13) and (14),

$$Jh_\theta^{-1}(\eta) = \alpha(\theta)\mathrm{e}^{\theta \cdot \tilde{\tau}(d_C(\eta))},$$

where

$$\alpha(\theta) = \left[(d-p) \int_0^1 u^{d-p-1} \mathrm{e}^{\theta \cdot \tilde{\tau}(u)} du \right]^{-1}.$$

For $p > 0$, this model may be used locally also in the case where C is curved. $\square$

Likewise, it is possible to construct transformations on S^1 or S^2 for the case of exponential inhomogeneity with $\tau(\eta) = d_C(\eta)$, where C is a point on S^1 or S^2 and d_C is the geodesic distance to C, cf. Jensen and Nielsen (2000). Illustrations are given in Figure 4 and 5. For general functions τ, the construction of an appropriate set of transformations may be difficult. An example is the inhomogeneity of the hickory tree data from Stoyan and Stoyan (1998, Figure 1).

The density of a type III exponential inhomogeneous point process becomes

$$f_Y(y;\theta) \propto \alpha(\theta)^{n(y)} \mathrm{e}^{\theta \cdot t(y)} \prod_{z \subseteq y} \varphi(h_\theta^{-1}(z)),$$

compare with (10). In particular, if the homogeneous model is a Strauss model then

$$f_Y(y;\theta,\beta,\gamma) \propto \mathrm{e}^{\theta \cdot t(y)} (\alpha(\theta)\beta)^{n(y)} \gamma^{s(h_\theta^{-1}(y))}. \tag{15}$$

Note that in contrast to type I inhomogeneity, θ is a nuisance parameter, cf. (11).

The simulations in Figure 2 to 5, right-hand sides, are from (15). Parameter values and other details of the simulations are given in Appendix III.

The transformation related approach yields densities of the form

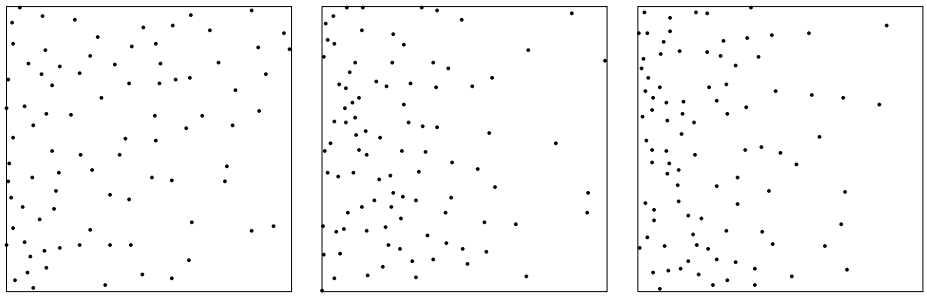
$$f_Y(y;\theta) \propto \mathrm{e}^{\theta \cdot t(y)} (\alpha(\theta)\beta)^{n(y)} \prod_{\{\eta,\xi\} \subseteq y}^{\neq} u(\alpha(\theta)^{2\nu} \mathrm{e}^{\nu(\tau(\eta)+\tau(\xi)) \cdot \theta} [\eta - \xi]).$$

In particular, in the Strauss case we get

$$f_Y(y;\theta,\beta,\gamma) \propto \mathrm{e}^{\theta \cdot t(y)} (\alpha(\theta)\beta)^{n(y)} \gamma^{s_\theta(y)}, \tag{16}$$

where $s_\theta(y)$ is the number of $\sim_\theta$ −neighbours of y. If two points η, ξ are related in the homogeneous Strauss process when $||\eta - \xi|| < R$, then the relation $\sim_\theta$ is defined by

$$\eta \sim_\theta \xi \Leftrightarrow \alpha(\theta)^{2\nu} \mathrm{e}^{\nu(\tau(\eta)+\tau(\xi)) \cdot \theta} \|\eta - \xi\| < R.$$



(a) Type I (b) Type III (c) Type III related

Figure 6: Realizations of inhomogeneous point processes on the unit square. The densities used are, from left to right, (11), (15) and (16), respectively, all with $\tau(\eta_1, \eta_2) = \eta_1$. The parameter values used are $\beta = 200$, $\gamma = 0.01$, R(interaction radius)$= 0.05$ and $\theta = -3$. In (16), the exponent is $\nu = 1/4$. The number of observed points are, from left to right, $n(y) = 87, 95$ and 93, respectively.

The three models (11), (15) and (16) are compared by simulation in Figure 6. Note that the type I process appears somewhat more homogeneous than the other processes, because the relation is for this process not location dependent. This feature becomes more pronounced if more points are forced into the point patterns by increasing β.

Furthermore, the intensity in the type I point process appears to be lower than the intensity in the other two point processes. These two are, however, similar both regarding the relation and the point intensity. Thus, the inhomogeneity function and the parameters from the associated homogeneous process play a quite different role in the actual point intensity and point interaction in the different types of inhomogeneous models. These issues have of course to be examined in more detail.

3.4. Statistical inference

3.4.1. Likelihood inference

If a Markov point process is observed in a sampling window T, cf. Figure 2 and 3, the conditional density of the point pattern observed in T, given the remaining points, may be used for inference. As in Baddeley et al. (2000),

let for disjoint point patterns y and x

$$\chi(y \mid x) = \prod_{z \subseteq y \cup x : z \cap y \neq \emptyset} \varphi(z). \tag{17}$$

For a homogeneous Markov point process with density (4), the conditional density is then of the form

$$f(x_T \mid x_{T^c}) \propto \chi(x_T \mid x_{\partial T}).$$

where $x_T = x \cap T$ and $\partial T = \{\xi \in T^c | \exists \eta \in T : \eta \sim \xi\}$. Note that this density depends on x_{T^c} only via $x_{\partial T}$.

Let us suppose that the interaction function φ can be parametrized by $\psi \in \Psi$. Then, the likelihood function based on observation of the homogeneous process in T becomes

$$L_T(\psi; x) = c_T(\psi; x_{\partial T}) \chi(x_T; \psi \mid x_{\partial T}), \tag{18}$$

where $c_T(\psi; x_{\partial T})$ is a normalizing constant and $\chi(\cdot; \psi | \cdot)$ is defined as in (17) with φ parametrized by ψ.

In the Strauss case, we get

$$L_T(\beta, \gamma; x) = c_T(\beta, \gamma; x_{\partial T}) \beta^{n(x_T)} \gamma^{s(x_T) + s(x_T; x_{\partial T})}, \tag{19}$$

where for disjoint point patterns x and y

$$s(x; y) = \sum_{\eta \in x} \sum_{\xi \in y} 1[\eta \sim \xi].$$

Likelihood inference based on these likelihood functions requires Markov chain Monte Carlo, since the normalizing constant is not known explicitly, cf. Geyer (1999) and Møller (1999).

For type I inhomogeneity, the conditional density is given by

$$f_Y(y_T \mid y_{T^c}) \propto \prod_{\eta \in y_T} \lambda(\eta) \cdot \chi(y_T \mid y_{\partial T}),$$

where χ refers to the homogeneous process. If $\lambda_\theta(\eta) = \alpha(\theta) \mathrm{e}^{\theta \cdot \tau(\eta)}$, and φ is parametrized by ψ, we get

$$L_T(\theta, \psi; y) = c_T(\theta, \psi; y_{\partial T}) \mathrm{e}^{\theta \cdot t(y_T)} \chi(y_T; \psi \mid y_{\partial T}).$$

In particular, if the homogeneous process is a Strauss process we have

$$L_T(\theta, \beta, \gamma; y) = c_T(\theta, \beta, \gamma; y_{\partial T}) \mathrm{e}^{\theta \cdot t(y_T)} \beta^{n(y_T)} \gamma^{s(y_T) + s(y_T; y_{\partial T})}.$$

Again, MCMC is required for the analysis.

For type II inhomogeneity, it has been suggested to use a missing data likelihood, cf. Baddeley et al. (2000, Proposition 1). If $\lambda_\theta(\eta) = \alpha(\theta)\mathrm{e}^{\theta\cdot\tau(\eta)}$, $p_\theta(\eta) = e^{\theta\cdot\tau(\eta)}/\max_{\eta\in\mathcal{Y}} e^{\theta\cdot\tau(\eta)}$ and φ is parametrized by ψ, then this likelihood takes the form

$$L_T(\theta,\psi;y) = \prod_{\eta\in y_T} p_\theta(\eta) E_\psi[\prod_{\eta\in X_T}(1-p_\theta(\eta))\chi(y_T;\psi|X)], \tag{20}$$

where the mean value is with respect to the homogeneous point process X with distribution parametrized by ψ. In the Strauss case, $\psi = (\beta,\gamma)$ and

$$\chi(y_T;(\beta,\gamma)|X) = \beta^{n(y_T)}\gamma^{s(y_T)+s(y_T;X)}.$$

Statistical inference in the case of type III inhomogeneity is based on the following conditional density, derived from (6),

$$f_Y(y_T \mid y_{T^c}) \propto \prod_{\eta\in y_T} \lambda(\eta)\cdot\chi(h^{-1}(y_T) \mid h^{-1}(y_{\partial T})).$$

If h_θ is chosen such that $Jh_\theta^{-1} = \lambda_\theta$ and φ is parametrized by ψ, we get, cf. Appendix II,

$$L_T(\theta,\psi;y) = L_0(\theta;y_T)L_{h_\theta^{-1}(T)}(\psi;h_\theta^{-1}(y)), \tag{21}$$

where $L_0(\cdot;y_T)$ is the likelihood (1) of an inhomogeneous Poisson point process with intensity function λ_θ and $L_{h_\theta^{-1}(T)}(\cdot;h_\theta^{-1}(y))$ is the likelihood function (18) for the homogeneous Markov point process with observation $h_\theta^{-1}(y)$, observed in $h_\theta^{-1}(T)$. Recall that (18) reduces to (19) in the Strauss case. Note that for the transformations derived in Example 3.1 and windows T as shown in Figure 2 and 3, $h_\theta^{-1}(T)$ does not depend on θ. In fact, $h_\theta(T) = T$.

Likelihood inference is simpler for type I than for type III models since the inhomogeneity parameter is a nuisance parameter in the latter case. However, simulation studies indicate that the estimate $\hat\theta_0$ of θ based on $L_0(\cdot;y_T)$, cf. (21), is close to $\hat\theta$, cf. Nielsen et al. (2000). In that case, the interaction parameter ψ can be estimated on the basis of

$$L_{h_{\hat\theta_0}^{-1}(T)}(\cdot;h_{\hat\theta_0}^{-1}(y))$$

and the analysis will be no more complicated than the analysis of a homogeneous Markov point process. It still remains to study in detail likelihood inference for a type II process based on (20).

3.4.2. Pseudolikelihood inference

A less demanding inference procedure is based on the pseudolikelihood function which is the likelihood function for a Poisson point process with intensity function equal to the Papangelou conditional intensity of the process. Recently, pseudolikelihood inference has been discussed by Baddeley and Turner (2000).

If the homogeneous process is parametrized by ψ the pseudolikelihood function based on observation in T becomes

$$PL_T(\psi; x) = \exp(-\int_T [\lambda_\psi(u; x) - 1] du^k)[\prod_{\eta \in x_T} \lambda_\psi(\eta; x \backslash \eta)]$$

where

$$\lambda_\psi(\eta; x) = \frac{f_X(x \cup \eta; \psi)}{f_X(x; \psi)}, \quad \eta \notin x.$$

If the homogeneous process is a Strauss process, then

$$\lambda_{\beta,\gamma}(\eta; x) = \beta \gamma^{s(\eta; x)}, \quad \eta \notin x,$$

and

$$PL_T(\beta, \gamma; x) = \exp(-\int_T [\beta \gamma^{s(u;x)} - 1] du^k) \beta^{n(x_T)} \gamma^{2s(x_T) + s(x_T; x_{T^c})}.$$

Compared to likelihood inference the normalizing constant is much simpler.

Let us now look at pseudolikelihood inference for the two Markovian inhomogeneous processes, viz. type I and III. For type I inhomogeneity of exponential form, the pseudolikelihood function takes the form

$$PL_T(\theta, \psi; y) = \exp(-\int_T [\mathrm{e}^{\theta \cdot \tau(u)} \lambda_\psi(u; y) - 1] du^k) \mathrm{e}^{\theta \cdot t(y_T)} [\prod_{\eta \in y_T} \lambda_\psi(\eta; y \backslash \eta)].$$

In the Strauss case, we get

$$\begin{aligned} PL_T(\theta, \beta, \gamma; y) &= \exp(-\int_T [\beta \mathrm{e}^{\theta \cdot \tau(u)} \gamma^{s(u;y)} - 1] du^k) \\ &\quad \times \mathrm{e}^{\theta \cdot t(y_T)} \beta^{n(y_T)} \gamma^{2s(y_T) + s(y_T; y_{T^c})}. \end{aligned}$$

Note that for fixed θ and γ the maximum pseudolikehood estimate of β is known explicitly. This is an example where the Papangelou conditional intensity is of log-linear form and the analysis suggested by Baddeley and Turner (2000) can therefore be used. Using the approximations described

in Berman and Turner (1992) and Baddeley and Turner (2000) one should, however, be careful when choosing the dummy points involved in the approximation.

In the case of type III inhomogeneity, it can be shown, cf. the Appendix II,

$$PL_T(\theta, \psi; y) = L_0(\theta; y_T) PL_{h_\theta^{-1}(T)}(\psi; h_\theta^{-1}(y)). \tag{22}$$

The pseudolikelihood function thus decomposes as the likelihood function, cf. (21). Pseudolikelihood inference for type III processes appears to be more complicated but again it is expected that the inference can be split into two parts.

4. Discussion

In the present paper, we have discussed three types of inhomogeneous point processes, derived from homogeneous Markov point processes. It is of course also of interest to study how inhomogeneity can be introduced into other of the classical classes of point processes. For instance, one may consider inhomogeneous Neyman-Scott point processes (the Poisson point process of the mothers is inhomogeneous), inhomogeneous Matérn hard-core processes (the un-thinned Poisson point process is inhomogeneous), inhomogeneous simple sequential inhibition point processes (the size of the circular region around each point depends on the position of the point) and inhomogeneous Gibbs processes (e.g. transformations of homogeneous Gibbs processes). See Clausen et al. (2000).

The emphasis has in the present paper been on parametrically modelled inhomogeneity. This is a new approach for type II processes. Dually, it will also be of interest to study non-parametric estimation of the transformation involved in type III models. This is the subject of Iovleff et al. (2000).

Summary statistics like the $K-$, $F-$ and $G-$functions have been developed for the initial study of the interaction in homogeneous point processes and for checking of models for homogeneous point processes, cf. Stoyan et al. (1995). In Baddeley et al. (2000), an analogue of the $K-$function is suggested for the inhomogeneous case. For a type II process, this analogue has the nice property of being identical to the $K-$function of the un-thinned process. It still remains, however, to find versions of the $F-$ and $G-$functions that can be used in the inhomogeneous case. For type III processes, an alternative is to estimate the transformation, either parametrically or non-parametrically, and then use the traditional summary statistics for the homogeneous case on the inversely transformed point pattern.

Type III processes have the special feature that the neighbourhood relation induced by the transformation is location dependent. The relation is generally not isotropic in the sense that relationship only depends on the distance between the points. The same is true for the transformation related processes defined by densities of the form (9). Another quite promising idea is to introduce inhomogeneity in Markov point processes by location dependent scaling, cf. Hahn et al. (2000).

5. Acknowledgements

The authors want to thank Ute Hahn, Jesper Møller and Aila Särkkä for fruitful discussions while preparing this paper. This research has been supported by the Centre for Mathematical Physics and Stochastics (MaPhySto), funded by a grant from the Danish National Research Foundation.

References

Baddeley, A. J. and Møller, J. (1989). Nearest-neighbour Markov point processes and random sets. *Int. Statist. Rev.*, 57:89–121.

Baddeley, A. J., Møller, J., and Waagepetersen, R. (2000). Non- and semi-parametric estimation of interaction in inhomogeneous point patterns. *Statistica Neerlandica*. To appear.

Baddeley, A. J. and Turner, R. (2000). Practical maximum pseudolikelihood for spatial point patterns. *Australian and New Zealand Journal of Statistics*. To appear.

Berman, M. (1986). Testing for spatial association between a point process and another stochastic process. *Appl. Statist.*, 35:54–62.

Berman, M. and Turner, T. R. (1992). Approximating point process likelihoods with GLIM. *Appl. Statist.*, 41:31–38.

Brix, A. and Møller, J. (1998). Space-time multitype log Gaussian Cox processes with a view to modelling weed data. *Research Report R–98–2012, Department of Mathematical Sciences, Aalborg University.*

Clausen, W. H. O., Rasmussen, H. H., and Rasmussen, M. H. (2000). *Inhomogeneous Point Processes.* Master Thesis, Department of Mathematical Sciences, Aalborg University.

Geyer, C. J. (1999). Likelihood inference for spatial point processes. In Barndorff-Nielsen, O. E., Kendall, W. S., and van Lieshout, M. N. M., editors, *Stochastic Geometry: Likelihood and Computation*, pages 79–140, London. Chapman and Hall/CRC.

Hahn, U., Jensen, E. B. V., and Nielsen, L. S. (2000). Inhomogeneous Markov point processes by location dependent scaling. *Research Report, Laboratory for Computational Stochastics, University of Aarhus.* In preparation.

Heikkinen, J. and Arjas, E. (1998). Non-parametric Bayesian estimation of spatial Poisson intensity. *Scand. J. Statist.*, 25:435–450.

Högmander, H. and Särkkä, A. (1999). Multitype spatial point patterns with hierarchical interactions. *Biometrics*, 55:1051–1058.

Iovleff, S., Jensen, E. B. V., Nielsen, L. S., and Perrin, O. (2000). Semi-parametric inference in transformation models for point processes. *Research Report, Laboratory for Computational Stochastics, University of Aarhus.* In preparation.

Jensen, E. B. V. (1998). *Local Stereology.* World Scientific, Singapore.

Jensen, E. B. V. and Nielsen, L. S. (2000). Inhomogeneous Markov point processes by transformation. *Bernoulli.* To appear.

Kelly, F. P. and Ripley, B. D. (1976). A note on Strauss' model for clustering. *Biometrika*, 63:357–360.

Møller, J. (1999). Markov chain Monte Carlo and spatial point processes. In Barndorff-Nielsen, O. E., Kendall, W. S., and van Lieshout, M. N. M., editors, *Stochastic Geometry: Likelihood and Computation*, pages 79–140, London. Chapman and Hall/CRC.

Møller, J., Syversveen, A. R., and Waagepetersen, R. P. (1998). Log Gaussian Cox processes. *Scand. J. Statist.*, 25:451–482.

Nielsen, L. S., Baddeley, A. J., Hahn, U., and Jensen, E. B. V. (2000). Estimation in transformation models for point processes. *Research Report 12, Laboratory for Computational Stochastics, University of Aarhus.* In preparation.

Ogata, Y. and Tanemura, M. (1986). Likelihood estimation of interaction potentials and external fields of inhomogeneous spatial point patterns. In

Francis, I. S., Manly, B. F. J., and Lam, F. C., editors, *Proc. Pacific Statistical Congress – 1985*, pages 150–154. Amsterdam, Elsevier.

Perrin, O. (1997). *Modèle de Covariance d'un Processus Non-Stationaire par Déformation de l'Espace et Statistique.* Ph.D Thesis, Université de Paris I Panthéon-Sorbonne.

Rathbun, S. L. (1996). Estimation of Poisson intensity using partially observed concomitant variables. *Biometrics*, 52:226–242.

Ripley, B. D. and Kelly, F. P. (1977). Markov point processes. *J. London Math. Soc.*, 15:188–192.

Silverman, B. W. (1986). *Density Estimation for Statistics and Data Analysis.* Chapman and Hall, London.

Stoyan, D., Kendall, W. S., and Mecke, J. (1995). *Stochastic Geometry and its Statistical Applications.* Wiley, Chichester, second edition.

Stoyan, D. and Stoyan, H. (1998). Non-homogeneous Gibbs process models for forestry – a case study. *Biometrical Journal*, 40:521–531.

Strauss, D. J. (1975). A model for clustering. *Biometrika*, 62:467–475.

van Lieshout, M. N. M. (2000). *Markov point processes and their applications.* World Scientific. To appear.

Appendix I

Let us start by finding Jh_θ^{-1} in the case $p = 0$ where (13) reduces to

$$h_\theta(\eta) = g_\theta(\|\eta\|)\frac{\eta}{\|\eta\|}.$$

Let $B_d(O,1)$ be the unit ball in R^d. Using polar decomposition twice we get for an arbitrary function f on $B_d(O,1)$

$$\begin{aligned}&\int_{B_d(O,1)} f(h_\theta(\eta))d\eta^d \\ &\quad = \int_{S^{d-1}}\int_0^1 f(g_\theta(t)\omega)t^{d-1}dtd\omega^{d-1}\end{aligned}$$

$$= \int_{S^{d-1}} \int_0^1 f(u\omega)(g_\theta^{-1}(u))^{d-1}(g_\theta^{-1})'(u)du d\omega^{d-1}$$
$$= \int_{B_d(O,1)} f(\eta)(g_\theta^{-1}(\|\eta\|))^{d-1}(g_\theta^{-1})'(\|\eta\|)\|\eta\|^{-(d-1)} d\eta^d.$$

Therefore,

$$Jh_\theta^{-1}(\eta) = (g_\theta^{-1}(\|\eta\|))^{d-1}(g_\theta^{-1})'(\|\eta\|)\|\eta\|^{-(d-1)} \text{ for } p = 0. \tag{23}$$

This result can now be used to find Jh_θ^{-1} for general p. Let

$$T_d(O,1) = \{\eta \in R^d : d_C(\eta) \leq 1\}.$$

For an arbitrary function f on $T_d(O,1)$ we then get

$$\int_{T_d(O,1)} f(h_\theta(\eta))d\eta^d$$
$$= \int_{C^\perp} \int_C f(x + g_\theta(\|y\|)\frac{y}{\|y\|})1\{\|y\| \leq 1\}dx^p dy^{d-p}$$
$$= \int_{C^\perp} \int_C f(x+y)1\{\|y\| \leq 1\}(g_\theta^{-1}(\|y\|))^{d-p-1}(g_\theta^{-1})'(\|y\|)\|y\|^{-(d-p-1)}dx^p dy^{d-p}$$
$$= \int_{T_d(O,1)} f(\eta)(g_\theta^{-1}(d_C(\eta)))^{d-p-1}(g_\theta^{-1})'(d_C(\eta))d_C(\eta)^{-(d-p-1)}d\eta^d,$$

where we at the second equality sign have used (23). It follows that

$$Jh_\theta^{-1}(\eta) = (g_\theta^{-1}(d_C(\eta)))^{d-p-1}(g_\theta^{-1})'(d_C(\eta))d_C(\eta)^{-(d-p-1)}.$$

Since we also have

$$Jh_\theta^{-1}(\eta) = \alpha(\theta)\mathrm{e}^{\theta\cdot\tilde{\tau}(d_C(\eta))},$$

g_θ must satisfy

$$\frac{1}{d-p}[(g_\theta^{-1}(u))^{d-p}]' = \alpha(\theta)u^{d-p-1}\mathrm{e}^{\theta\cdot\tilde{\tau}(u)}, u \in [0,1]. \tag{24}$$

Since g_θ is increasing 1-1 of [0,1] onto itself, the unique solution of (24) is (14).

Appendix II

In order to derive (21), we need to find the constant $c_T(\theta, \psi; y_{\partial T})$ in the expression for the conditional density

$$f_Y(y_T; \theta, \psi | y_{T^c}) = c_T(\theta, \psi; y_{\partial T})[\prod_{\eta \in y_T} \lambda_\theta(\eta)]\chi(h_\theta^{-1}(y_T); \psi | h_\theta^{-1}(y_{\partial T})).$$

Since this is a density with respect to the Poisson point process on T with intensity measure λ_d^k we get, using that $\lambda_\theta(\eta) = Jh_\theta^{-1}(\eta)$ and the well-known expansion of the distribution of the Poisson point process, cf. Møller (1999, Section 2),

$$\begin{aligned}
&c_T(\theta, \psi; y_{\partial T})^{-1} \\
&\quad = \mathrm{e}^{-\lambda_d^k(T)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_T \cdots \int_T \prod_{i=1}^{n} Jh_\theta^{-1}(y_i) \\
&\qquad\qquad \times \chi(\{h_\theta^{-1}(y_1), \ldots, h_\theta^{-1}(y_n)\}; \psi | h_\theta^{-1}(y_{\partial T})) dy_1^k \cdots dy_n^k \\
&\quad = \mathrm{e}^{-\lambda_d^k(T)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{h_\theta^{-1}(T)} \cdots \int_{h_\theta^{-1}(T)} \chi(\{x_1, \ldots, x_n\}; \psi | h_\theta^{-1}(y_{\partial T})) dx_1^k \cdots dx_n^k \\
&\quad = \mathrm{e}^{-\lambda_d^k(T)} \mathrm{e}^{\lambda_m^k(h_\theta^{-1}(T))} c_{h_\theta^{-1}(T)}(\psi; h_\theta^{-1}(y_{\partial T}))^{-1}.
\end{aligned}$$

The result now follows by noting that

$$\lambda_m^k(h_\theta^{-1}(T)) = \int_{h_\theta^{-1}(T)} d\xi^k = \int_T Jh_\theta^{-1}(\eta) d\eta^k = \int_T \lambda_\theta(\eta) d\eta^k.$$

The proof of (22) is obtained as follows. The Papangelou conditional intensity of the transformed process becomes, cf. (6),

$$\frac{f_Y(y \cup \eta; \theta, \psi)}{f_Y(y; \theta, \psi)} = \lambda_\theta(\eta) \lambda_\psi(h_\theta^{-1}(\eta); h_\theta^{-1}(y)), \eta \notin y,$$

where λ_ψ is the Papangelou conditional intensity of the un-transformed process. The pseudolikelihood function of the transformed process therefore becomes

$$\begin{aligned}
&PL_T(\theta, \psi; y) \\
&\quad = \exp(-\int_T [\lambda_\theta(u) \lambda_\psi(h_\theta^{-1}(u); h_\theta^{-1}(y)) - 1] du^k) \\
&\qquad \times \prod_{\eta \in y_T} [\lambda_\theta(\eta) \lambda_\psi(h_\theta^{-1}(\eta); h_\theta^{-1}(y \setminus \eta))]
\end{aligned}$$

$$
\begin{aligned}
&= [\prod_{\eta \in y_T} \lambda_\theta(\eta)] \exp(-\int_T [\lambda_\theta(u) - 1] du^k) \\
&\quad \times \exp(-\int_T \lambda_\theta(u)[\lambda_\psi(h_\theta^{-1}(u); h_\theta^{-1}(y)) - 1] du^k) \\
&\quad \times \prod_{\eta \in y_T} \lambda_\psi(h_\theta^{-1}(\eta); h_\theta^{-1}(y \setminus \eta)).
\end{aligned}
$$

The result is now obtained by noting that

$$
\int_T \lambda_\theta(u)[\lambda_\psi(h_\theta^{-1}(u); h_\theta^{-1}(y)) - 1] du^k = \int_{h_\theta^{-1}(T)} [\lambda_\psi(v; h_\theta^{-1}(y)) - 1] dv^k.
$$

Appendix III

Simulations from the inhomogeneous Strauss point process (15) are shown in the right hand-sides of Figure 2 to 5. In Table 1 the model parameters and the resulting number of points in the simulated point patterns are given. Note, however, that the number of points in Figure 3, $n(x) = 355$, is for a 33% wider rectangle. A larger area was used to avoid edge problems.

The point patterns shown in Figure 2 to 5, right, and the three point patterns in Figure 6 have been simulated using Metropolis-Hastings birth-death algorithm with 500000 iterations, cf. e.g. Møller (1999).

Figure	β	γ	R	$n(y)$	θ
2	1000	0.01	0.1	163	-3
3	400	0.01	0.1	355	-3
4	70	0.01	0.1	40	-1
5	100	0.01	0.2	143	-2

Table 1: Parameters used for simulation and the resulting number of points for the point patterns in the right-hand sides of Figure 2 to 5.