Multivariate Type $\mathcal{G}$ Distributions

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Abstract

The class of multivariate distributions all of whose one-dimensional projections are of type G is discussed and examples of such distributions presented. In the course of this, we introduce a new representation of the cumulant function of the symmetric multivariate stable laws as well as multivariate extensions of the Inverse Gaussian and the symmetric Normal Inverse Gaussian laws. A concept of weak infinite divisibility of random matrices is introduced and this leads to further examples.

Key words: multivariate stable laws; random positive definite matrices; type G; weak infinite divisibility

1. Introduction

Marcus (1987) introduced the concept of type G random variables and processes and this has been further studied by Rosinski (1991) (see also Maejima and Samorodnitsky (1999) and references therein). In the present paper we consider a related concept that we term type $\mathcal{G}$. This coincides with type G in the case of one-dimensional random variates.

Recall that a random variable x is said to be of type G if in law x is of the form $z\sqrt{s}$ where z and $s > 0$ are independent random variables with s being infinitely divisible and z having the standard normal distribution. We refer to $z\sqrt{s}$ as a *G-representation* of x.

The notion of random variables of type G is closely related to that of subordination of Brownian motion, and it is of considerable import in connection with statistical modelling for instance in finance, cf. Barndorff-Nielsen and Shephard (2000a,b).

We shall say that a random vector $x = (x_1, ..., x_m)$ and its distribution are of type $\mathcal{G}$ if for any deterministic vector $c = (c_1, ..., c_m)$ the law of $c \cdot x = c_1x_1 + ... + c_mx_m$ is of type G. Our aim is to study properties of the class of multivariate type $\mathcal{G}$ distributions.

If a multivariate distribution D is of type $\mathcal{G}$ and if for all $c \in \mathbf{R}^d$ the Lévy measure of s_c in the G-representation $z\sqrt{s_c}$ of $c \cdot x$ is known, then insight into the nature of D can be obtained by simulating the one-dimensional projected laws of D using the techniques discussed in the surveys given by Asmussen (1998; Sect. VIII.2) and Rosinski (2000).

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Section 2 provides various elementary properties and examples of type $\mathcal{G}$ random vectors including a new representation of the characteristic function of the symmetric multivariate stable law as well as multivariate extensions of the Inverse Gaussian and the Normal Inverse Gaussian laws. Section 3 extends the discussion to stochastic processes of type $\mathcal{G}$ and includes examples of subordinated and stationary processes of this type. In Section 4 we introduce a concept of weak infinite divisibility of nonnegative definite random matrices and show that it provides additional examples of type $\mathcal{G}$ laws covering some multivariate distributions commonly used in statistics.

In the sequel we shall use the following notation for characteristic functions and cumulant transforms of a random variate x

$$\chi\{\zeta \ddagger x\} = \mathrm{E}\{e^{i\zeta x}\}$$

$$\mathrm{C}\{\zeta \ddagger x\} = \log \mathrm{E}\{e^{i\zeta x}\}$$

$$\bar{\mathrm{K}}\{\theta \ddagger x\} = \log \mathrm{E}\{e^{-\theta x}\}$$

with straightforward extensions of the notation to more general random variates. Vectors will be taken as row vectors.

2. Type $\mathcal{G}$: elementary properties and examples

The definition of distributions and random vectors of type $\mathcal{G}$ was given in Section 1. Let $\mathcal{G}_m$ denote the family of m-dimensional type $\mathcal{G}$ laws and let $\mathcal{G} = \cup_1^\infty \mathcal{G}_m$. If x has law in $\mathcal{G}_m$ then we say that x is of type $\mathcal{G}_m$.

The following proposition summarizes elementary properties of type $\mathcal{G}$ laws.

Proposition 2.1

a) Linear transformations of type $\mathcal{G}$ distributions are again type $\mathcal{G}$.

b) Let $x \in \mathcal{G}_m$ and $y \in \mathcal{G}_n$ and suppose that x and y are independent. Then $(x, y) \in \mathcal{G}_{m+n}$.

c) Sums of independent type $\mathcal{G}$ variates are type $\mathcal{G}$.

d) The class $\mathcal{G}$ is closed under convergence in law.

PROOF The proofs of a) and b) are trivial.

c) It suffices to consider the sum of just two variates, x and y, of class $\mathcal{G}_m$. We must show that $c \cdot (x + y) = c \cdot x + c \cdot y$ is of class $\mathcal{G}_1$ and one sees that it is enough to prove that if x and y are one-dimensional with independent G-representations $z_1\sqrt{s_1}$ and $z_2\sqrt{s_2}$, respectively, then $x + y$ is type $\mathcal{G}$. But

$$x + y \overset{law}{=} z_1\sqrt{s_1} + z_2\sqrt{s_2} \overset{law}{=} z(s_1 + s_2)^{1/2}$$

with z a normal standard random variate independent of (s_1, s_2).

d) Consider a sequence of vectors x_n of class $\mathcal{G}_m$, converging in law to a random vector x, and let $z_n\sqrt{s_n}$ be a G-representation of x_n. Then, by assumption,

$$\chi\{\zeta \ddagger x_n\} \to \chi\{\zeta \ddagger x\}$$

for all $\zeta \in \mathbf{R}^m$ and on the other hand we have

$$\mathrm{C}\{\zeta \ddagger x_n\} = \bar{\mathrm{K}}\{\frac{1}{2}|\zeta|^2 \ddagger s_n\}$$

It follows that $\bar{\mathrm{K}}\{\theta \ddagger s_n\}$ converges for all $\theta \geq 0$ to some function $\kappa(\theta)$ where $\kappa(0) = \mathrm{C}\{0 \ddagger x\} = 0$. Hence κ is necessarily the cumulant function of a nonnegative random variable s, i.e. $\kappa(\theta) = \bar{\mathrm{K}}\{\theta \ddagger s\}$, which implies

$$\mathrm{C}\{\zeta \ddagger x\} = \bar{\mathrm{K}}\{\frac{1}{2}|\zeta|^2 \ddagger s\}$$

i.e. x is of type $\mathcal{G}$. □

Let $\mathbf{M}_m^+$ be the closed cone of $m \times m$ nonnegative definite matrices. Let S be a nonnegative definite random matrix with law Q and let $S^{1/2}$ be its symmertric square root. Then, for z a standard normal vector in $\mathbf{R}^m$ independent of S, a random vector x such that $x \overset{law}{=} zS^{1/2}$ will have probability density

$$p(x) = \int_{\mathbf{M}_m^+} \varphi_m(x; \Sigma) Q(\mathrm{d}\Sigma) \tag{2.1}$$

where $\varphi_m(x; \Sigma)$ denotes the probability density function of the m-dimensional normal distribution with mean 0 and variance matrix Σ. If furthermore S is infinitely divisible and K is the cumulant function of S, then for every $\zeta \in \mathbf{R}^m$

$$\mathrm{C}\{\zeta \ddagger x\} = \bar{\mathrm{K}}\{\frac{1}{2}\zeta^T\zeta \ddagger S\}. \tag{2.2}$$

In this case, which constitutes an immediate generalization of type G, we say that x has *G-representation* $zS^{1/2}$. The proposition below shows, inter alia, that if an m-dimensional random vector x has a G-representation then it is of type $\mathcal{G}_m$. Note that if x has G-representation $zS^{1/2}$ with S diagonal then the law of $c \cdot x$ depends on c through $|c|^2$ only.

Proposition 2.2 Let x be an m-dimensional random vector having a G-representation $zS^{1/2}$. Suppose that the infinitely divisible nonnegative definite random matrix S with law Q has Lévy measure V concentrated on $\mathbf{M}_m^+$ and that there is no nonnegative definite matrix $A \neq 0$ such that $P\{S \geq A\} = 1$. Then x is a type $\mathcal{G}$ infinitely divisible random vector with probability density p of the form (2.1) and Lévy density u given by

$$u(x) = \int_{\mathbf{M}_m^+} \varphi_m(x; \Sigma) V(\mathrm{d}\Sigma). \tag{2.3}$$

Proof It follows trivially that for $c \in \mathbf{R}^m$, $c \cdot x$ is one dimensional type G and therefore x is type $\mathcal{G}$.

Since S is infinitely divisible there exists an $\mathbf{M}_m^+$-valued Lévy process $S(t)$ such that $S(1) \stackrel{law}{=} S$. For each $t > 0$, by formula (2.2)

$$\begin{aligned} t\mathrm{C}\{\zeta \ddagger x\} &= t\bar{\mathrm{K}}\{\frac{1}{2}\zeta^T\zeta \ddagger S\} \\ &= \bar{\mathrm{K}}\{\frac{1}{2}\zeta^T\zeta \ddagger S(t)\} \\ &= \mathrm{C}\{\zeta \ddagger x(t)\} \end{aligned}$$

where $x(t) \stackrel{law}{=} zS(t)^{1/2}$, for z a standard normal vector independent of $S(t)$. This verifies the infinite divisibility of x.

In proving (2.3), without loss of generality we may assume that x has no Gaussian component, i.e. the characteristic triplet of x is of the form $(0, 0, U)$. Now, on the one hand, since x is a symmetric infinitely divisible random vector we have

$$C\{c \ddagger x\} = \int_{\mathbf{R}^m} (\cos(c \cdot x) - 1) U(\mathrm{d}x). \tag{2.4}$$

On the other hand, by the Lévy-Khintchine representation of infinitely divisible distributions on cones as given by Skorohod (1991, p. 156-157), we have

$$\bar{\mathrm{K}}\{\frac{1}{2}c^T c \ddagger S\} = -\int_{\mathbf{M}_m^+} (1 - e^{-\frac{1}{2}Tr(c^T c\Sigma)}) V(\mathrm{d}\Sigma).$$

Recalling further that for a given $\Sigma \in \mathbf{M}_m^+$

$$e^{-\frac{1}{2}c\Sigma c^T} = \int_{\mathbf{R}^m} \cos(c \cdot x)\varphi_m(x; \Sigma)\mathrm{d}x \tag{2.5}$$

which implies

$$1 - e^{-\frac{1}{2}c\Sigma c^T} = \int_{\mathbf{R}^m} (1 - \cos(c \cdot x))\varphi_m(x; \Sigma)\mathrm{d}x$$

we find

$$\begin{aligned} \bar{\mathrm{K}}\{\frac{1}{2}c^T c \ddagger S\} &= -\int_{\mathbf{M}_m^+}\int_{\mathbf{R}^m} (1 - \cos(c \cdot x))\varphi_m(x; \Sigma)\mathrm{d}x V(\mathrm{d}\Sigma) \\ &= \int_{\mathbf{R}^m} (\cos(c \cdot x) - 1) \int_{\mathbf{M}_m^+} \varphi_m(x; \Sigma) V(\mathrm{d}\Sigma)\mathrm{d}x. \end{aligned} \tag{2.6}$$

The result then follows from (2.2), (2.4), (2.6) and the uniqueness of the Lévy measure. □

Remark 2.1 Let x be an m-dimensional random vector having a G-representation $zS^{1/2}$. Then for any $n \times m$ matrix C the random vector $zS^{1/2}C^T$ has the G-representation

$z_n S_{n,C}^{1/2}$ where $S_{n,C} = CSC^T$ is an infinitely divisible $n \times n$ nonnegative definite symmetric random matrix.

Below we present some important examples of infinitely divisible nonnegative definite random matrices and of type $\mathcal{G}$ random vectors having the $G-$representation.

Example 2.1 *Matrix $\alpha-$stable laws* We now consider results on stable random matrices. Let $\mathbf{M}_m$ be the Hilbert space of $m \times m$ real symmetric matrices with inner product $< A, B >= tr(AB)$. An $\mathbf{M}_m-$valued random element S is said to have an α-stable distribution (see Samorodnitsky and Taqqu (1994, Remark 3, p. 66)) if and only if $0 < \alpha \leq 2$ and for every $\Sigma \in \mathbf{M}_m$

$$\mathrm{C}\{\Sigma \ddagger S\} = \int_{\mathbf{SM}_m} \int_0^\infty \left[e^{irtr(\Sigma\Theta)} - 1 - ir\frac{tr(\Sigma\Theta)}{1+r^2} \right] \frac{dr}{r^{1+\alpha}} \Gamma(\mathrm{d}\Theta) + itr(\Sigma\Theta_0),$$

where $\mathbf{SM}_m$ is the unit sphere of $\mathbf{M}_m, \Theta_0 \in \mathbf{M}_m$ and Γ is a finite measure on $\mathbf{SM}_m$.

For $0 < \alpha < 1$, one has $\int_0^\infty \frac{1}{1+r^2} \frac{dr}{r^\alpha} < \infty$, and the centering term can be absorbed in the constant term. Supposing further that Γ is concentrated on the unit sphere $\mathbf{SM}_m^+$ of the closed cone $\mathbf{M}_m^+$, as a special case of the above formula for $\mathrm{C}\{\Sigma \ddagger S\}$ we therefore have for every $\Sigma \in \mathbf{M}_m^+$

$$\mathrm{C}\{\Sigma \ddagger S\} = \int_{\mathbf{SM}_m^+} \int_0^\infty \left(e^{irtr(\Sigma\Theta)} - 1 \right) \frac{dr}{r^{1+\alpha}} \Gamma(\mathrm{d}\Theta) = \int_{\mathbf{M}_m^+} \left(e^{itr(\Sigma\Lambda)} - 1 \right) V(\mathrm{d}\Lambda), \qquad (2.7)$$

where $V(\mathrm{d}\Lambda) = \frac{dr}{r^{1+\alpha}}\Gamma(\mathrm{d}\Theta), \Lambda = r\Theta, r = \|\Lambda\| = \{tr(\Lambda\Lambda)\}^{1/2}$, $\Theta = \Lambda/r$ and V is such that $\int_{\|\Lambda\|\leq 1} \|\Lambda\| V(\mathrm{d}\Lambda) < \infty$. Then, from Skorohod (1991, p. 156-157), (2.7) is the cumulant function of an infinitely divisible symmetric nonnegative definite random matrix S such that for $\Sigma \in \mathbf{M}_m^+$

$$\bar{\mathrm{K}}\{\Sigma \ddagger S\} = -\int_{\mathbf{SM}_m^+} \int_0^\infty \left(1 - e^{-rtr(\Sigma\Theta)} \right) \frac{dr}{r^{1+\alpha}} \Gamma(\mathrm{d}\Theta).$$

Finally, by usual integration techniques we have $\int_0^\infty \left[e^{-rtr(\Sigma\Theta)} - 1 \right] \frac{dr}{r^{1+\alpha}} = -[tr(\Sigma\Theta)]^\alpha$.

Thus we have shown that for any $0 < \alpha < 2$ there exists an $\alpha/2-$stable $-$and hence infinitely divisible$-$ nonnegative definite symmetric random matrix S such that for $\Sigma \in \mathbf{M}_m^+$

$$\bar{\mathrm{K}}\{\Sigma \ddagger S\} = -\int_{\mathbf{SM}_m^+} [tr(\Sigma\Theta)]^{\alpha/2} \Gamma(\mathrm{d}\Theta). \qquad (2.8)$$

Example 2.2 *Multivariate symmetric $\alpha-$stable laws* have a $G-$representation as we now show. Let S be an $\alpha/2-$stable nonnegative definite symmetric random matrix with cumulant function (2.8) and take $x = zS^{1/2}$ for z a standard normal vector in $\mathbf{R}^m$ independent of S. Then, by Proposition 2.2, x is a type $\mathcal{G}$ random vector and, by (2.2) and (2.8) we find that x is a multivariate symmetric $\alpha-$stable random vector such that for every $\zeta \in \mathbf{R}^m$

$$\mathrm{C}\{\zeta \ddagger x\} = \bar{\mathrm{K}}\{\frac{1}{2}\zeta^T\zeta \ddagger S\} = -\frac{1}{2^{\alpha/2}}\int_{\mathbf{SM}_m^+}[\zeta\Theta\zeta^T)]^{\alpha/2}\Gamma(\mathrm{d}\Theta). \tag{2.9}$$

In particular, when S is of the form $s\mathrm{I}$ (*i.e.* Γ is concentrated on the identity matrix), where s an $\alpha/2-$stable positive variable, we obtain the multivariate sub-Gaussian law as presented in Samorodnitsky and Taqqu (1994, Section 2.5).

We observe that (2.9) gives a new parametrization of the multivariate symmetric $\alpha-$stable distributions in terms of the measure Γ on $\mathbf{SM}_m^+$, as an alternative to the usual representation

$$\mathrm{C}\{\zeta \ddagger x\} = \bar{\mathrm{K}}\{\frac{1}{2}\zeta^T\zeta \ddagger S\} = -\frac{1}{2^{\alpha/2}}\int_{\mathbf{S}_m}|\zeta\cdot\theta|^{\alpha}\,\sigma(\mathrm{d}\theta),$$

where σ is a symmetric measure on the unit sphere $\mathbf{S}_m$ of $\mathbf{R}^m$(see Samorodnitsky and Taqqu (1994, Th. 2.4.3).

Example 2.3 *Multivariate extensions of the Inverse Gaussian and symmetric Normal Inverse Gaussian laws*[1]

A matrix extension of the Inverse Gaussian distribution may now be obtained as follows. Let S be an $\alpha/2-$stable nonnegative definite random matrix with cumulant function (2.8), let $\Sigma \in \mathbf{M}_m^+$ and let R be a random matrix with density relative to the law of S given by

$$p(r;\Sigma) = \exp\left\{-tr(r\Sigma) - \bar{\mathrm{K}}\{\Sigma \ddagger S\}\right\} \tag{2.10}$$

and cumulant transform

$$\bar{\mathrm{K}}\{\Phi \ddagger R\} = \bar{\mathrm{K}}\{\Phi + \Sigma \ddagger S\} - \bar{\mathrm{K}}\{\Sigma \ddagger S\}. \tag{2.11}$$

In other words, the law of R is determined by exponential tilting from the law of S.

For any natural exponential family of probability laws $\mathcal{P}$, an element of $\mathcal{P}$ is infinitely divisible if and only if all elements of $\mathcal{P}$ are infinitely divisible. Hence R is an infinitely divisible nonnegative definite symmetric random matrix. Furthermore,

[1]For information on the Inverse Gaussian and Normal Inverse Gaussian distributions see for instance Barndorff-Nielsen and Shephard (2000a,b) and references therein.

$$
\begin{aligned}
\bar{\mathrm{K}}\{\Phi \ddagger R\} &= \int_{\mathbf{SM}_m^+} [tr(\Sigma\Theta)]^{\alpha/2}\Gamma(\mathrm{d}\Theta) - \int_{\mathbf{SM}_m^+} [tr((\Sigma+\Phi)\Theta)]^{\alpha/2}\Gamma(\mathrm{d}\Theta) \\
&= \int_{\mathbf{SM}_m^+}\int_0^\infty \left(1 - e^{-rtr(\Sigma\Theta)}\right)\frac{dr}{r^{1+\alpha/2}}\Gamma(\mathrm{d}\Theta) \\
&\quad - \int_{\mathbf{SM}_m^+}\int_0^\infty \left(1 - e^{-rtr((\Sigma+\Phi)\Theta)}\right)\frac{dr}{r^{1+\alpha/2}}\Gamma(\mathrm{d}\Theta) \\
&= -\int_{\mathbf{SM}_m^+}\int_0^\infty \left(1 - e^{-rtr(\Phi\Theta)}\right)e^{-rtr(\Sigma\Theta)}\frac{dr}{r^{1+\alpha/2}}\Gamma(\mathrm{d}\Theta) \\
&= -\int_{\mathbf{M}_m^+}\left(1 - e^{-tr(\Phi\Omega)}\right)V(\mathrm{d}\Omega) \qquad (2.12)
\end{aligned}
$$

where $V(\mathrm{d}\Omega) = \frac{e^{-rtr(\Sigma\Theta)}}{r^{1+\alpha/2}}\mathrm{d}r\Gamma(\mathrm{d}\Theta)$ is a Lévy measure, with $r^2 = tr(\Omega^T\Omega)$ and $\Theta = \Omega/r$.

In the case $\alpha = 1$ R follows a matrix extension of the Inverse Gaussian distribution, where $\Sigma \in \mathbf{M}_m^+$ and Γ, a finite measure on the sphere $\mathbf{SM}_m^+$, are parameters. We note that if C is a nonsingular $m \times m$ deterministic matrix then the random matrix CRC^T follows again this matrix extension of the Inverse Gaussian distribution but now with parameters $(C^{-1})^T\Sigma C^{-1}$ and $\Gamma(C^{-1}\mathrm{d}\cdot(C^{-1})^T)$. This generalises an elementary rule for the Inverse Gaussian (in one dimension).

Using Proposition 2.2 we have that if z is a standard normal vector in $\mathbf{R}^m$ independent of R, then $x = zR^{1/2}$ has a multivariate type $\mathcal{G}$ distribution with cumulant function

$$
\mathrm{C}\{\zeta \ddagger x\} = \int_{\mathbf{SM}_m^+} [tr(\Sigma\Theta)]^{\alpha/2}\Gamma(\mathrm{d}\Theta) - \int_{\mathbf{SM}_m^+} [tr((\Sigma + \frac{1}{2}\zeta^T\zeta)\Theta)]^{\alpha/2}\Gamma(\mathrm{d}\Theta). \qquad (2.13)
$$

and Lévy density

$$
u(x;\Sigma,\Gamma) = \int_{\mathbf{SM}_m^+}\int_0^\infty \varphi_m(x; r\Theta)\frac{e^{-rtr(\Sigma\Theta)}}{r^{1+\alpha/2}}\mathrm{d}r\Gamma(\mathrm{d}\Theta). \qquad (2.14)
$$

The resulting law constitutes a generalization of the symmetric Normal Inverse Gaussian distribution to dimensions greater than one.

We finally observe that for a real τ

$$
\mathrm{C}\{\tau \ddagger c\cdot x\} = \int_{\mathbf{SM}_m^+} [tr(\Sigma\Theta)]^{\alpha/2}\Gamma(\mathrm{d}\Theta) - \int_{\mathbf{SM}_m^+} [tr((\Sigma + \frac{1}{2}\tau^2 c^T c)\Theta)]^{\alpha/2}\Gamma(\mathrm{d}\Theta). \qquad (2.15)
$$

In particular, taking Γ concentrated on the matrix $d^{-1/2}\mathrm{I}$ ($\Gamma(\{\mathrm{I}\}) = \delta$), $\Sigma = \gamma^2\mathrm{I}$ and $\alpha = 1$ we obtain

$$
\mathrm{C}\{\tau \ddagger c\cdot x\} = \delta\gamma[1 - \{1 + \frac{1}{2}\gamma^{-2}d^{-1/2}cc^T\tau^2\}^{1/2}], \qquad (2.16)
$$

which shows that in this case the one-dimensional marginal distributions are Normal Inverse Gaussian (in other words, $x \overset{law}{=} zs^{1/2}$, where s has a one dimensional Inverse Gaussian distribution independent of the standard normal variate z).

3. Type $\mathcal{G}$ stochastic processes

An n-dimensional stochastic process $x(t)$ will be said to be of type $\mathcal{G}$ if all finite dimensional laws of $x(t)$ are type $\mathcal{G}$, in other words if for any $t_1 < ... < t_r$ we have that $(x(t_1), ..., x(t_r)) \in \mathcal{G}_{rn}$.

3.1. Subordinated processes

Multivariate subordinated processes are studied systematically in Barndorff–Nielsen, Pedersen and Sato (2000). Let $\{T(t)\}_{t \geq 0}$ be a d-dimensional subordinator, i.e. T is a Lévy process with values in $\mathbf{R}^d_{0+}$ (where $\mathbf{R}_{0+} = [0, \infty)$). Furthermore, let $\{X(t)\}_{t \geq 0}$ where $X(t) = (X_1(t), ..., X_d(t))$ and the components $X_1(t), ..., X_d(t)$ are independent Lévy processes, of dimension $n_1, ..., n_d$, respectively. Then

$$Y(t) = (X_1(T_1(t)), ..., X_d(T_d(t)))$$

constitutes a Lévy process of dimension $n_1 + \cdots + n_d$.

If $X_i(t)$ is n_i-dimensional Brownian motion, for $i = 1, ..., d$, then the multivariate subordinated process $Y(t)$ is of type $\mathcal{G}$ and for each $t, Y(t)$ has a G-representation, as is easily checked.

3.2. Stationary processes of type $\mathcal{G}$

In this section we consider a class of strictly stationary processes constructed in Barndorff -Nielsen and Pérez-Abreu (1999), showing that they are of type $\mathcal{G}$.

Let F be an arbitrary one-dimensional distribution function and suppose (v, w) is a bivariate stochastic process of the form $y(F(\cdot))$ where $y(t) = (y_1(t), y_2(t))$ is a bivariate Lévy process such that $y(1)$ has a G-representation $(z_1, z_2)\sqrt{s}$ where s is one-dimensional. Then a process $x = \{x(t)\}_{t \in \mathbf{R}}$ is well-defined by

$$x(t) = \int_{-\infty}^{\infty} \cos(\lambda t) v(\mathrm{d}\lambda) + \int_{-\infty}^{\infty} \sin(\lambda t) w(\mathrm{d}\lambda) \tag{3.1}$$

where the integrals are interpreted in the sense of integration with respect to independently scattered random measures. It was proved in Barndorff -Nielsen and Pérez-Abreu (1999) that the process x is strictly stationary and infinitely divisible and that for $t_1 < \cdots < t_m$

$$\mathrm{C}\{c_1, ..., c_m \ddagger x(t_1), ..., x(t_m)\} = \int_{-\infty}^{\infty} \bar{\mathrm{K}} \left\{ \frac{1}{2} \psi(c, \lambda) \ddagger s \right\} F(\mathrm{d}\lambda) \tag{3.2}$$

where $c = (c_1, ..., c_m)$,

$$\psi(c, \lambda) = c \Omega_\lambda(t_1, ..., t_m) c^T$$

and

$$\Omega_\lambda(t_1, ..., t_m) = \{\cos(\lambda(t_j - t_k)); i, j = 1, ...m\}$$

is a symmetric nonnegative definite matrix.

Moreover, for any one-dimensional distribution D of type G there exists a strictly stationary process x of the above kind having D as the one-dimensional marginal law.

We now show that the finite dimensional distributions of any such process are of type $\mathcal{G}$. Let $x = (x(t_1), ..., x(t_m))$, then by (3.2)

$$\mathrm{C}\{\zeta \ddagger c \cdot x\} = \int_{-\infty}^{\infty} \bar{\mathrm{K}} \left\{ \frac{1}{2}\zeta^2 \ddagger \psi(c, \lambda)s \right\} F(\mathrm{d}\lambda) \tag{3.3}$$

and the result follows by Proposition 2.1.

Since $c \cdot x$ in (3.3) is type G is has a G-representation $z\sqrt{s_c}$. An expression for the Lévy measure V_c of s_c in terms of the Lévy measure V of s is obtained by noting that, writing θ for $\frac{1}{2}\zeta^2$,

$$\bar{\mathrm{K}}\{\theta \ddagger s\} = -\int_0^{\infty} (1 - e^{-\theta\xi}) V(\mathrm{d}\xi)$$

which together with (3.3) yields

$$\begin{aligned} \bar{\mathrm{K}}\{\theta \ddagger s_c\} &= \int_{-\infty}^{\infty} \bar{\mathrm{K}}\{\psi(c, \lambda)\theta \ddagger s\} F(\mathrm{d}\lambda) \\ &= -\int_{-\infty}^{\infty} \int_0^{\infty} (1 - e^{-\theta\psi(c,\lambda)\xi}) V(\mathrm{d}\xi) F(\mathrm{d}\lambda) \\ &= -\int_0^{\infty} (1 - e^{-\theta\xi}) \int_{-\infty}^{\infty} V(\psi(c, \lambda)^{-1}\mathrm{d}\xi) F(\mathrm{d}\lambda) \end{aligned}$$

from which it follows that

$$V_c(\mathrm{d}\xi) = \int_{-\infty}^{\infty} V(\psi(c, \lambda)^{-1}\mathrm{d}\xi) F(\mathrm{d}\lambda). \tag{3.4}$$

By the usual Lamperti transformation, given a strictly stationary process x, one can construct an $H-$selfsimilar process $\mathrm{x}^*(t) = t^H x(\log t)$. Then if x is given by (3.1), the selfsimilar process x^* is itself of type $\mathcal{G}$. For a suitable choice of the distribution function F, it was shown in Barndorff-Nielsen and Perez-Abreu (1999) that the corresponding selfsimilar process x^* has second order stationary increments.

3.3. A general class and its relation to type G processes

We now present a class of examples that includes the type G processes studied by Rosinski (1991) and references therein, showing that they are as well type $\mathcal{G}$ processes in our sense. It is enough to look at the multivariate distributions of the finite dimensional distributions.

Let ν be a σ-finite measure on the space of nonnegative definite matrices $\mathbf{M}_m^+$ and let s be a an infinitely divisible positive random variable with cumulant function $\bar{\mathrm{K}}\{\cdot \ddagger s\}$

and Lévy measure V. Suppose that there exists a random vector x in R^m whose cumulant transform is given by

$$\mathrm{C}\{\zeta \ddagger x\} = \int_{\mathbf{M}_m^+} \bar{\mathrm{K}}\left\{\frac{1}{2}\zeta\Omega\zeta^T \ddagger s\right\} \nu(\mathrm{d}\Omega). \tag{3.5}$$

Note that the expression (2.9) for the cumulant function of the multivariate symmetric α-stable law is a special case of (3.5).

We may rewrite (3.5) as

$$\mathrm{C}\{\zeta \ddagger x\} = \int_{\mathbf{M}_m^+} \bar{\mathrm{K}}\left\{\frac{1}{2}\zeta\zeta^T \ddagger s\Omega\right\} \nu(\mathrm{d}\Omega).$$

and, by the infinite divisibility of s, there must exist an infinitely divisible nonnegative definite random matrix S such that

$$\bar{\mathrm{K}}\left\{\frac{1}{2}\zeta^T\zeta \ddagger S\right\} = \int_{\mathbf{M}_m^+} \bar{\mathrm{K}}\left\{\frac{1}{2}\zeta^T\zeta \ddagger s\Omega\right\} \nu(\mathrm{d}\Omega).$$

It follows that x has a G-representation $zS^{1/2}$ and is therefore of type $\mathcal{G}$ (cf. Proposition 2.2).

Furthermore, the Lévy measure of x is given by

$$u(x) = \int_{\mathbf{M}_m^+} \int_0^\infty \varphi_m(x; \xi\Omega) V(\mathrm{d}\xi)\nu(\mathrm{d}\Omega).$$

In the case where ν is concentrated on matrices of the form $y^T y$, ν may be thought of as a measure on R^m and we then have

$$\mathrm{C}\{\zeta \ddagger x\} = \int_{R^m} \bar{\mathrm{K}}\left\{\frac{1}{2}(\zeta y^T)^2 \ddagger s\right\} \nu(\mathrm{d}y), \tag{3.6}$$

which is the definition of a type G multivariate distribution as given by Rosinski (1991). The corresponding Lévy measure of x is

$$u(x) = \int_{R^m} \int_0^\infty \prod_{j=1}^m \varphi(x_j; \xi y_j^2) V(\mathrm{d}\xi)\nu(\mathrm{d}y).$$

4. Weak infinite divisibility and type $\mathcal{G}$ laws

As shown by Proposition 2.2, if S is an $m \times m$ nonnegative definite symmetric random matrix and z is an m-dimensional standard normal random vector independent of z then $zS^{1/2}$ is of type $\mathcal{G}$ provided S is infinitely divisible. However, many useful nonnegative definite random matrices are not infinitely divisible. The Wishart matrix is a well known example. A weaker condition on S than infinite divisibility is sufficient to imply the type $\mathcal{G}$ property.

Definition 4.1 A random $m \times m$ matrix S and its distribution are said to be *weakly infinitely divisible* if for any constant vector c the one dimensional random variable cSc^T is infinitely divisible.

Trivially, infinite divisibility of S implies weak infinite divisibility.

Example 4.1 *Wishart distribution* It was shown by Lévy (1948) that the Wishart distribution is not infinitely divisible (see also Gindikin (1975) and Shanbhag (1988)). However, the distribution is weakly infinitely divisible. To show this it is enough to consider the standard Wishart distribution. Thus let $z_1, ..., z_n$ be independent and m-dimensional random vectors having the standard normal distribution and let $W_n = z_1^T z_1 + ... + z_n^T z_n$. To see that W_n is weakly infinitely divisible it is enough to prove that $z_1^T z_1$ is weakly infinitely divisible, which follows since for any $c \in \mathbf{R}^m$ we have $cz_1^T z_1 c^T = (cz_1^T)^2$, which follows a Gamma distribution.

Proposition 4.1 A multivariate random variable x is of type $\mathcal{G}$ if it has the representation $x \overset{law}{=} zS^{1/2}$ where S is a weakly infinitely divisible random matrix with values in $\mathbf{M}_m^+$ and z is a standard normal vector in $\mathbf{R}^m$ independent of S.

Proof Since cSc^T is an infinitely divisible positive random variable, via conditioning on S we find

$$\begin{aligned} \chi\{\zeta \ddagger x \cdot c\} &= \chi\{\zeta \ddagger zS^{1/2}c^T\} \\ &= \mathrm{E}\exp\{-\frac{1}{2}\zeta^2 cSc^T\} \\ &= \exp \bar{\mathrm{K}}\{\frac{1}{2}\zeta^2 \ddagger cSc^T\} \end{aligned}$$

which shows that $c \cdot x \overset{law}{=} w(cSc^T)^{1/2}$, for w a standard normal variable independent of S, and hence x is of type $\mathcal{G}$. $\square$

Example 4.1 *Wishart distribution (continued)* Let $S = W_n$ where W_n was defined above. Then, by Proposition 4.1, $x = zW_n^{1/2}$ is of type $\mathcal{G}$.

The cumulant function of W_n is

$$\bar{\mathrm{K}}\{\Theta \ddagger W_n\} = -\frac{n}{2}\log\det(\mathrm{I}_d + 2\Theta)$$

and hence for the cumulant function of x we find

$$\mathrm{C}\{\zeta \ddagger x\} = \bar{\mathrm{K}}\{\frac{1}{2}\zeta^T\zeta \ddagger W_n\} = -\frac{n}{2}\log\det(\mathrm{I}_d + \zeta^T\zeta) = -\frac{n}{2}\log(1 + |\zeta|^2). \tag{4.1}$$

For $n = 2$ this is the cumulant function of the bivariate Laplace distribution.

Since (4.1) depends on ζ through $|\zeta|^2$ only and since the one-dimensional Laplace distribution is of type G, with a G-representation $z\sqrt{s}$ where s follows the generalized inverse

Gaussian law $GIG(1,0,1)$ (cf. Barndorff-Nielsen (1977)), we find that x possesses a G-representation $zS^{1/2}$ with $S = sI$ where I is the identity matrix.

Example 4.2 The *Inverse Wishart distribution* is weakly infinitely divisible. In fact, if S is a nondegenerated nonnegative definite random $m \times m$ matrix such that S^{-1} follows a Wishart distribution, i.e. S itself has an inverse Wishart law, then for any $r \times m$ matrix C of rank r ($r \leq m$) the law of CSC^T is inverse Wishart, cf. for instance Eaton (1983; p. 330). In particular then, if $C = c$, a vector of dimension of m, we have that cSc^T has a reciprocal gamma distribution and this is infinitely divisible (cf. Barndorff-Nielsen and Halgreen (1977), see also Halgreen (1979)). Furthermore, it follows from these remarks and those at the end of the previous example that also in the present setting we have that x has a G-representation of the form $zS^{1/2}$ with $S = sI$.

Remark 4.1 It was pointed out to us by M. Eaton, that a nonnegative definite random matrix S which is orthogonally invariant ($OSO^T \overset{law}{=} S$ for every orthonormal matrix O) is weakly infinitely divisible if and only if the first diagonal element s_{11} of S is infinitely divisible. Then, since the Wishart and the Inverse Wishart distributions are orthogonally invariant, this provides another proof of the weak infinite divisibility of these laws.

Example 4.3 The *multivariate t distribution* (as defined for instance in Johnson and Kotz (1972; p.144)) is the law of a random vector x having a G-representation $zS^{-1/2}$ where S follows a Wishart distribution. In consequence of the result of the previous example and Proposition 4.1 we find that x is of type $\mathcal{G}$.

Example 4.4 Let (w_1, w_2) be a two dimensional Gaussian random vector with zero mean and covariance matrix Σ. It has been shown in Vere Jones (1968) that the bivariate Gamma distribution of (w_1^2, w_2^2) is infinitely divisible. Let s be a positive 1/2 stable random variable and let $y = s^{1/2}(w_1^2, w_2^2)$. Let $y_1, ..., y_n$ be n independent copies of y. Then $S = y_1^T y_1 + ... + y_n^T y_n$ is a weakly infinitely divisible nonnegative definite random matrix. Indeed, it is enough to show that $cy^T yc^T = (c \cdot y)^2$ is infinitely divisible. Observing that $(c \cdot y)^2 = s(c_1 w_1^2 + c_2 w_2^2)^2 = sq^2$ where q is positive and infinitely divisible it follows, by a result of Barndorff-Nielsen, Pedersen and Sato (2000), that the product sq^2 is infinitely divisible.

In its general form, the probability density function of the bivariate gamma law is

$$\frac{(\psi_1\psi_2 - 1)^\lambda}{2^{\lambda+1}\Gamma(\lambda)} I_{\lambda-1}(2\sqrt{x_1 x_2})(x_1 x_2)^{-(\lambda-1)/2} e^{-\psi_1 x_1 - \psi_2 x_2}.$$

The exponential form of this implies that if x is defined as $zS^{1/2}$ with S as above then the characteristic function of $c \cdot x$ does not depend on c solely through $|c|^2$ (in contrast to what was the case in Examples 4.1 and 4.2).

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