LIGHT, ATOMS, AND SINGULARITIES

OLE E. BARNDORFF-NIELSEN, FRED ESPEN BENTH AND JENS LEDET JENSEN

ABSTRACT. Motivated by questions concerning cooling and trapping of atoms using counterpropagating laser beams we consider continuous time Markovian jump processes with singular jump intensities and some related stochastic processes, modeling the atomic momentum. The asymptotic behaviour of these processes as time goes to infinity is studied, in particular drawing on methods from renewal theory. Results from this type of study have been instrumental for the understanding and improvement of the efficiency of the cooling schemes.

www.maphysto.dk
ISSN 1398-599X MPS-RR 2000-19

1. INTRODUCTION

One of the exciting areas of present day Physics is the study of the interaction of light and particles, in particular the cooling of atoms and molecules, and investigations flowing from this.

The technique(s) of cooling and trapping of clouds of atoms were developed over a period of about ten years, from the mid Eighties to the mid Nineties, resulting in the award of the 1997 Nobel price in Physics to the three pioneer physicists in the area, Steven Chu, William Phillips and Claude Cohen-Tannoudji.

As will be indicated below, a detailed study of the stochastic elements in the cooling processes has been essential for the understanding of a key element of the physics and has led to dramatic improvements of the cooling techniques, so that at present the most efficient techniques are capable of achieving temperatures at a staggeringly low level, at the order of nano Kelvins. The study was carried out by Cohen-Tannoudji and some of his collaborators in a tour de force mathematical analysis in which they reinvented and extended parts of the classical renewal theory of Probability. See Bardou, Bouchaud, Emile, Aspect and Cohen-Tannoudji [5], Bardou [6], Bardou and Castin [7] and in particular the extensive survey paper by Bardou, Bouchaud, Aspect and Cohen-Tannoudji [2].

A compact review of these achievements, as viewed from Stochastics and with some further work, has been given in Barndorff-Nielsen and Benth [3] and Barndorff-Nielsen, Benth and Jensen [4]. A focal point in that further work was a discussion of continuous time Markov jump processes with singularities in the jump intensities.

The present paper outlines the above-mentioned developments and adds some results. We start in Section 2 by describing the physical background for our study. Section 3 deals with continuous time jump models where the waiting times are mixtures or sums of exponential distributions, generalizing the results in Barndorff-Nielsen et al [4]. The relation between laser cooling and Bessel processes is discussed in Section 4, and, finally, in Section 5, some other ramifications, concerning time-dependent and supersingular jump rates, are considered.

2. Background

The cooling techniques mentioned in the Introduction have opened the way for a multitude of fundamental studies in physics concerning the interaction of light and matter and for a variety of high tech applications, for instance to atom optics, atom lithography, atomic clocks, atomic lasers, Bose-Einstein condensation (see Townsend, Ketterle and Stringari [18], Ketterle [14] and Burnett, Edwards and Clark [9]), and slowing the speed of light (Hau, Harris, Dutton and Behroozi [12][1]; see also Marangos [15]).

Briefly, the cooling is achieved by directing three pairs of counterpropagating laser beams towards a chosen point in space where a cloud of the atoms is initially trapped by means of an inhomogeneous magnetic field (see for instance Aspect and Dalibard [1]). The operating effect of the lasers, which have to be suitably tuned, can only be properly understood at a basic quantum physical level.

However, the stochastic phenomenon that explains a key part of the efficiency of the methods can to a reasonable degree of realism be described as follows[2]. The momentum (or velocity) of the atom, viewed as a vector in one, two or three dimensions, behaves over time as a Markov jump process with a singularity at the origin. In other words, when the atom arrives at a position x in momentum space it remains there for an exponentially distributed time with a mean value $\lambda(x)^{-1}$, the jump rate $\lambda(x)$ being a continuous function with $\lambda(0) = 0$ and $\lambda(x) > 0$ for $x \neq 0$. Furthermore, when the atom shifts momentum the shift vector is, at least in the neighbourhood of the origin, stochastically independent of position[3]. Experimentally important cases are of the form $\lambda(x) = const.|x|^{\gamma}$, with $\gamma = 2$ or 4 in one dimension. In such cases atoms arriving near the origin will, due to the singularity, stay there for a time whose distribution is heavy tailed and belongs to the domain of attraction of a stable law with index $\alpha = d/\gamma$ where d denotes dimension. As is well known, the maximum af n positive, independent and identically distributed random stable variables is of the same magnitude as the sum of the variables, implying that the longer the experiment lasts the greater the number of atoms with a very low velocity.

In Barndorff-Nielsen, Benth and Jensen [4] continuous state Markov jump processes x_t with a singularity, in the sense indicated above, are studied in general[4], the main topic being the behaviour of x_t near the origin for large t. In particular, results are derived concerning the limiting distributions, after suitable normalisations, of the momentum distribution near the origin and of the time spent in present state.

3. Generalization of the waiting time distribution

The models discussed in Barndorff-Nielsen et al [4] capture the main aspects of the cooling process. However, as mentioned in Saubaméa, Leduc and Cohen-Tannoudji [17], the description of the waiting time distribution as an exponential distribution with mean $\lambda(x)^{-1}$ is only an approximation. We will in this section

[1] The front page of *Nature* referred to this result by the title "*Cycling at the speed of light*". The techniques have now been improved to velocities of light around 2km/h.

[2] For more details, see Bardou, Bouchaud, Aspect and Cohen-Tannoudji [2] or, for a brief account, Barndorff-Nielsen and Benth [3].

[3] The shifts occur when the atom emits a photon, previously absorbed from one of the laser beams.

[4] entailing extensions of classical renewal theory

consider a model that allows for a more general waiting time distribution. We do this by extending the state space B of the Markov process x_t of the momentum by a variable having only a finite number of values. In this way we can accomodate a waiting time distribution that is either a mixture of exponential distributions (subsection 3.1) or a sum of exponentially distributed terms (subsection 3.2). From Kolmogorov's forward equation we derive a renewal type equation for a transform of the density of x_t, which in turn allows us to find the asymptotic form of the density for large t.

We consider a Markov process (x_t, I_t) with state space $B \times \{1, 2, \dots, K\}$, where B is a region in $\mathbf{R}^d$. The intensity of leaving the state (x, j) is $\lambda(x, j)$. We assume that there exist positive constants c_1, c_2 such that

$$\begin{array}{rclll} c_1 & \leq & \lambda(x,j) & \leq c_2, & x \in B, j \in \{1,..,K-1\} \\ & & \lambda(x,K) & \leq c_2, & x \in B, \end{array} \tag{1}$$

and where we have in mind that $\lambda(0, K) = 0$. Given that we jump from state $(x_{t-}, I_{t-}) = (y, l)$ let $p(\cdot|y, l)$ be the probability density function of the new position. Furthermore, given also that the new position is x let $\alpha_j(x, y, l)$ be the probability that the new value of I_t is j. In the present setup if we want the waiting time distribution to be a sum of exponentials this can be achieved by letting some of the jumps of (x_t, I_t) involve a change of I_t only. We will therefore assume that there exists $\tilde{K} \in \{0, 1, \dots, K-1\}$ such that if $I_t \leq \tilde{K}$ the next jump will change I_t only. The special case $\tilde{K} = 0$ means that all jumps will change both x_t and I_t and the waiting time distribution is a mixture of exponentials. When $l \leq \tilde{K}$ we use the notation $\alpha_j(y, l)$ instead of $\alpha_j(y, y, l)$. Finally, let $p(x, j; t)$ be the probability density of (x_t, I_t) with initial density $a(x, j) = p(x, j; 0)$, and let $p_t(x, j; t)$ be derivative of $p(x, j; t)$ with respect to t.

Kolmogorov's forward equation is

$$\begin{aligned} p_t(x,j;t) &= -\lambda(x,j)p(x,j;t) + \sum_{l=1}^{\tilde{K}} \lambda(x,l)\alpha_j(x,l)p(x,l;t) \\ &\quad + \sum_{l=\tilde{K}+1}^{K} \int_B \lambda(y,l)p(x|y,l)\alpha_j(x,y,l)p(y,l;t)dy. \end{aligned} \tag{2}$$

Define

$$h_j(x,t) = \sum_{l=\tilde{K}+1}^{K} \int_B \lambda(y,l)p(x|y,l)\alpha_j(x,y,l)p(y,l;t)dy \tag{3}$$

We will not analyze (2) in its full generality, but only consider the two cases $\tilde{K} = 0$ and $\tilde{K} = K - 1$.

3.1. **The case $\tilde{K} = 0$.** We now assume that $\tilde{K} = 0$, that is, all jumps change the position x_t. This models the situation where the waiting time for a jump is a mixture of exponential waiting times. We will also assume that

$$p(y|x,l)\alpha_j(x,y,l) \leq k_1, \quad \forall\, x, y \in B, \ \forall\, l, j, \tag{4}$$

for some konstant k_1. We first derive a general renewal type equation for h_j and then specialize to a simple model, corresponding to the 'simple model' in Barndorff-Nielsen et al. [4].

When the functions h_j are known the solution to (2) can be written

$$p(x,j;t) = a(x,j)\mathrm{e}^{-t\lambda(x,j)} + \int_0^t h_j(x,\tau)\mathrm{e}^{-(t-\tau)\lambda(x,j)}d\tau. \tag{5}$$

From this we derive

$$\begin{aligned} h_j(x,t) &= \sum_{l=1}^{K}\int_B \lambda(y,l)p(x|y,l)\alpha_j(x,y,l) \\ &\qquad \times \left[a(y,l)\mathrm{e}^{-t\lambda(y,l)} + \int_0^t h_l(y,\tau)\mathrm{e}^{-(t-\tau)\lambda(y,l)}d\tau\right]dy \\ &= v_j(x,t) + \sum_{l=1}^{K}\int_B\int_0^t h_l(y,t-w)f_{lj}(x,y,w)dwdy \end{aligned} \tag{6}$$

with

$$f_{lj}(x,y,s) = p(x|y,l)\alpha_j(x,y,l)\lambda(y,l)\mathrm{e}^{-s\lambda(y,l)}$$

and

$$v_j(x,t) = \sum_{l=1}^{K}\int_B a(y,l)f_{lj}(x,y,t)dy.$$

We next define the n-fold convolution

$$f_{ab}^{n*}(x,y,t) = \sum_l \int_B\int_0^t f_{al}^{(n-1)*}(z,y,t-w)f_{lb}(x,z,w)dwdz.$$

One can now repeat the proof of Proposition 2.1. in Barndorff-Nielsen et al. [4] using the bound

$$\begin{aligned} &f_{l_n l_0}^{n*}(z_0,z_n,w_n) \\ &= \sum_{l_{n-1},\dots,l_1}\int_B\cdots\int_B\int_0^{w_n}\cdots\int_0^{w_2}\prod_{i=1}^{n} f_{l_i,l_{i-1}}(z_{i-1},z_i,w_i-w_{i-1})dz_{n-1}\cdots dz_1 \\ &\qquad \times dw_{n-1}\cdots dw_1 \\ &\le \frac{c_2^n t^{n-1}}{(n-1)!}\sum_{l_{n-1},\dots,l_1}\int_B\cdots\int_B\prod_{i=1}^{n}[p(z_i|z_{i-1},l_{i-1})\alpha_{l_i}(z_i,z_{i-1},l_{i-1})]dz_{n-1}\cdots dz_1 \\ &\le \frac{c_2^n t^{n-1}}{(n-1)!}k_1 \end{aligned}$$

for $w_n \le t$. We are therefore able to express the solution to (6) as an integral of v_j times the renewal density $\sum_{n=1}^{\infty} f_{ab}^{n*}(x,y,t)$.

We now make some further assumptions to get back to some more simple equations. We first assume that the new state j only depends on the new position x,

$$\alpha_j(x,y,l) = \alpha_j(x). \tag{7}$$

Then $h_j(x,t) = \alpha_j(x)h(x,t)$ with

$$h(x,t) = \sum_{l=1}^{K}\int_B p(x|y,l)\lambda(y,l)p(y,l;t)dy.$$

Instead of (6) we get

$$h(x,t) = v(x,t) + \int_B \int_0^t h(y,t-w) f(x,y,w)\,dw\,dy \tag{8}$$

with

$$f(x,y,s) = \sum_{l=1}^{K} \alpha_l(y) p(x|y,l) \lambda(y,l) \mathrm{e}^{-s\lambda(y,l)} \tag{9}$$

and

$$v(x,t) = \sum_{l=1}^{K} \int_B a(y,l) p(x|y,l) \lambda(y,l) \mathrm{e}^{-t\lambda(y,l)}\, dy.$$

If we make the further assumption that the new position x is independent of the previous position y and previous state l,

$$p(x|y,l) = b(x) \tag{10}$$

we find $h(x,t) = b(x)g(t)$ with

$$g(t) = \sum_{l=1}^{K} \int_B \lambda(y,l) p(y,l;t)\,dy.$$

Instead of (8) we obtain

$$g(t) = v(t) + \int_0^t g(t-w) u(w)\,dw \tag{11}$$

with

$$u(s) = \sum_{l=1}^{K} \int_B b(y) \alpha_l(y) \lambda(y,l) \mathrm{e}^{-s\lambda(y,l)}\, dy$$

and

$$v(t) = \sum_{l=1}^{K} \int_B a(y,l) \lambda(y,l) \mathrm{e}^{-t\lambda(y,l)}\, dy.$$

Under the conditions (7) the only difference to the general case in Barndorff-Nielsen et al. [4] is that f in (9) now consists of a sum of terms instead of one term only. The results from Barndorff-Nielsen et al. [4] therefore carry over, especially, Proposition 2.1 of that paper is still valid. The Proposition states that the solution to (8) is given as an integral of v with respect to a generalized renewal density obtained from f. Similarly, under the conditions (7) and (10) the situation is as for the 'simple model' in Barndorff-Nielsen et al. [4] where we have only one state and the new position is independent of the previous position. Thus from the form of $\lambda(y,l)$ for y close to zero we find the asymptotic form of $u(t)$ for $t \to \infty$. This in turn gives us the asymptotic form of $g(t)$ and finally also the asymptotic form of $p(x,j;t)$. As a concrete example assume that the $\lambda(y,l)$'s are such that $u(t) \sim ct^{-(1+\xi)}$ with $\frac{1}{2} < \xi < 1$. Then $g(t) \sim \xi\{c\Gamma(\xi)\Gamma(1-\xi)\}^{-1} t^{-(1-\xi)}$ for $t \to \infty$. Since $h_j(x,t) = \alpha_j(x) b(x) g(t)$ we get from (4) and Proposition 3.2 in Barndorff-Nielsen et al. [4] that

$$p(x,j;t) = t^{\xi} \frac{\xi \alpha_j(x) b(x)}{c\Gamma(\xi)\Gamma(1-\xi)} \Psi(t\lambda(x,j),\xi) + \frac{t^{\xi}}{1+t\lambda(x,j)} o(1),$$

where $\Psi(z,\xi)=\int_0^1(1-s)^{\xi-1}\mathrm{e}^{-sz}ds$.

3.2. **The case $\tilde{K}=K-1$.** We now consider the case where $\tilde{K}=K-1$ and when we jump from the state $l<K$ we jump to $l+1$ and when we jump from $l=K$ we jump to $l=1$. This models the situation where the waiting time is a sum of exponential waiting times.

The Kolmogorov forward equation (2) becomes

$$p_t(x,j;t)=\begin{cases}-\lambda(x,j)p(x,j;t)+\lambda(x,j-1)p(x,j-1;t) & j>1\\ -\lambda(x,1)p(x,1;t)+\int_B\lambda(y,K)p(x|y,K)p(y,K;t)dy & j=1.\end{cases}\tag{12}$$

Defining

$$h(x,t)=\int_B p(x|y,K)\lambda(y,K)p(y,K;t)dy\tag{13}$$

we find from (12) with $j=1$

$$p(x,1,t)=a(x,1)\mathrm{e}^{-t\lambda(x,1)}+\int_0^t h(x,\tau_0)\mathrm{e}^{-(t-\tau_0)\lambda(x,1)}d\tau_0.\tag{14}$$

We may now use this together with (12) with $j=2$ to find $p(x,2,t)$,

$$\begin{aligned}p(x,2,t)&\\ =\;& a(x,2)\mathrm{e}^{-t\lambda(x,2)}\\ &+\int_0^t\lambda(x,1)\left[a(x,1)\mathrm{e}^{-\tau_1\lambda(x,1)}+\int_0^{\tau_1}h(x,\tau_0)\mathrm{e}^{-(\tau_1-\tau_0)\lambda(x,1)}d\tau_0\right]\mathrm{e}^{-(t-\tau_1)\lambda(x,2)}d\tau_1\\ =\;& a(x,2)\mathrm{e}^{-t\lambda(x,2)}+\lambda(x,1)a(x,1)\int_0^t\mathrm{e}^{-\tau_1\lambda(x,1)-(t-\tau_1)\lambda(x,2)}d\tau_1\\ &+\lambda(x,1)\int_0^t\int_0^{\tau_1}h(x,\tau_0)\mathrm{e}^{-(\tau_1-\tau_0)\lambda(x,1)-(t-\tau_1)\lambda(x,2)}d\tau_0d\tau_1.\end{aligned}\tag{15}$$

Proceeding in this way we find with $\tau_j=t$

$$\begin{aligned}p(x,j,t)&\\ =\;& a(x,j)\mathrm{e}^{-t\lambda(x,j)}\\ &+\sum_{r=1}^{j-1}a(x,r)\prod_{s=r}^{j-1}\lambda(x,s)\int_0^{\tau_j}\cdots\int_0^{\tau_{r+1}}\mathrm{e}^{-\tau_r\lambda(x,r)-\sum_{s=r+1}^{j}(\tau_s-\tau_{s-1})\lambda(x,s)}\prod_{s=r}^{j-1}d\tau_s\\ &+\prod_{s=1}^{j-1}\lambda(x,s)\int_0^{\tau_j}\cdots\int_0^{\tau_1}h(x,\tau_0)\mathrm{e}^{-\sum_{s=1}^{j}(\tau_s-\tau_{s-1})\lambda(x,s)}\prod_{s=0}^{j-1}d\tau_s.\end{aligned}\tag{16}$$

Inserting (16) with $j=K$ in (13) we obtain a renewal equation for $h(x,t)$. In the special case with $K=2$ this equation becomes

$$\begin{aligned}&h(x,t)=\\ &\int_B p(x|y,2)\lambda(y,2)\left[a(y,2)+\lambda(y,1)a(y,1)\frac{1-\mathrm{e}^{-t(\lambda(y,1)-\lambda(y,2))}}{\lambda(y,1)-\lambda(y,2)}\right]\mathrm{e}^{-t\lambda(y,2)}dy\\ &+\int_B\int_0^t h(y,\tau_0)p(x|y,2)\lambda(y,2)\lambda(y,1)\mathrm{e}^{-(t-\omega)\lambda(y,2)}\frac{1-\mathrm{e}^{-(t-\tau_0)(\lambda(y,1)-\lambda(y,2))}}{\lambda(y,1)-\lambda(y,2)}d\tau_0dy.\end{aligned}\tag{17}$$

In the simple model with $p(x|y,2) = b(x)$ we have $h(x,t) = b(x)g(t)$ and instead of (17) we find

$$g(t) = \int_B \lambda(y,2)\left[a(y,2) + \lambda(y,1)a(y,1)\frac{1-\mathrm{e}^{-t(\lambda(y,1)-\lambda(y,2))}}{\lambda(y,1)-\lambda(y,2)}\right]\mathrm{e}^{-t\lambda(y,2)}dy + \int_0^t g(t-\tau_0)\left[\int_B b(y)\lambda(y,2)\lambda(y,1)\mathrm{e}^{-\tau_0\lambda(y,2)}\frac{1-\mathrm{e}^{-\tau_0(\lambda(y,1)-\lambda(y,2))}}{\lambda(y,1)-\lambda(y,2)}dy\right]d\tau_0.$$

This renewal equation for g can be analyzed as in Barndorff-Nielsen et al. [4]. We can then establish the asymptotic form of $g(t)$ for $t \to \infty$, and using this in (14) and (15) we find the asymptotic form of $p(x,j,t)$ for $t \to \infty$. The arguments are as in Barndorff-Nielsen et al. [4]. As an example assume that $\lambda(y,2) \le \lambda(y,1) \le c_1$, $\lambda(y,1) \ge c_2 > 0$ and that $\lambda(y,2) = |y|^\delta$ for small values of y with $\frac{1}{2} < d/\delta < 1$. Then

$$\begin{aligned} u(t) &= \int_B b(y)\lambda(y,2)\lambda(y,1)\mathrm{e}^{-t\lambda(y,2)}\frac{1-\mathrm{e}^{-t(\lambda(y,1)-\lambda(y,2))}}{\lambda(y,1)-\lambda(y,2)}dy \\ &\sim t^{-(1+d/\delta)}b(0)\frac{C_d\Gamma(1+d/\delta)}{\delta}, \end{aligned}$$

and from Proposition 3.1 in Barndorff-Nielsen et al. [4] we get $g(t) \sim c_3 t^{-(1-d/\delta)}$. From Proposition 3.2 in Barndorff-Nielsen et al. [4] and (14) we see that

$$p(x,1;t) = t^\xi c_4 b(x)\Psi(t\lambda(x,1),\xi) + \frac{t^\xi}{1+t\lambda(x,1)}\,o(1)$$

and from (15)

$$p(x,2;t) = t^\xi \frac{c_5 b(x)}{\lambda(x,1)-\lambda(x,2)}\Psi(t\lambda(x,2),\xi) + \frac{t^\xi}{1+t\lambda(x,2)}\,o(1),$$

with $\xi = d/\delta$.

4. Relation to Bessel processes

In this Section we consider the different laser cooling schemes presented in Bardou et al. [2] in the context of continuous-time stochastic processes. Many of the different cooling schemes can be modelled, at least approximately, within the framework of diffusions. As we shall see, one is naturally lead to Langevin diffusions and Bessel processes, which capture many of the important features of laser cooling. We restrict our consideration to one space dimension throughout the Section.

In laser cooling mechanisms based on friction there are two effects, which together results in atomic temperatures down to the level of the recoil limit [5]. Friction forces lead to a drift of the atomic momentum towards zero, while spontaneous and random emission of fluorescence photons introduces a 'diffusion' of momentum. In the discussion of Bardou et al. [2] the authors say that the competition between friction forces and diffusion leads to a steady state momentum distribution, where the effective temperature of the cooling process can be expressed in terms of the half-width (or the standard deviation) of the stationary distribution. Let us now consider the

[5] The recoil limit is a temperature of the order of 1 microKelvin, and is the limit of the cooling schemes known as 'Doppler' and 'Sisyphus'. These rely on different kinds of friction mechanisms.

standard cooling schemes in the framework of continuous-time stochastic processes. A natural model in light of the above presentation is the Langevin diffusion:

$$dX_t = -aX_t\,dt + c\,dB_t\,, \tag{18}$$

where X_t is the atomic momentum at time t and the constants a and c are positive and B_t is a standard Brownian motion. This model has a friction proportional to the momentum and a spontaneous emission of photons with rate c. It is well-known that X_t has a normal distribution with expectation $X_0\exp(-at)$ and variance $\frac{c^2}{2a}(1-\exp(-2at))$. Hence, X_t will reach a normally distributed steady state with expectation zero and variance $c^2/2a$. The 'half-width' of the stationary distribution is therefore $c/\sqrt{2a}$, which we can relate to the effective temperature of the cooling process described by the Langevin diffusion.

As Bardou et al. [2, Sect. 2.2.1] describe, it is possible to cool atoms without friction. Instead of pushing the atomic momentum towards zero with a friction force, one resorts to cooling principles where the spontaneous rate of emission of photons[6] depends on the momentum and vanishes at zero. Such cooling schemes are in fact more effective than the standard ones, since one may circumvent the recoil limit[7]. In such cooling mechanisms the random walk in momentum space has an inhomogeneous diffusion coefficient dependent on the current momentum and vanishing at zero. If we model the emission rate at momentum x by $\lambda(x) := c^2x^\delta$, for positive constants c and δ (cf. Section 2), a diffusion approximation of the subrecoil cooling scheme may be

$$dX_t = cX_t^{\delta/2}\,dB_t \tag{19}$$

where X_t is the atomic momentum at time t. The invariant measure of X_t is $\lambda^{-1}(x) = c^{-2}x^{-\delta}$, the inverse of the fluorescence rate. Observe that the invariant measure is not a probability density when $\delta \geq 1$. We will investigate the asymptotic behaviour of such diffusions and compare with results for subrecoil cooling schemes like Raman cooling (where $\delta = 4$, see Bardou et al. [2]).

First, notice that X_t given as in (19) will never cross zero. If it reaches zero, the process will remain trapped ever after [8]. Recall that this is not in agreement with the actual laser cooling experiment, where discrete jumps in momentum across zero may take place.

The study of (19) naturally leads us to Bessel processes. In order to establish the connection, we consider the following class of models with $\delta > 0$

$$dX_t = \mathrm{sgn}(2-\delta)cX_t^{\delta/2}\,dB_t \tag{20}$$

where c is a positive constant and we use the convention $\mathrm{sgn}(0) = 1$. The reasons for introducing $\mathrm{sgn}(2-\delta)$ are purely technical and has nothing to do with the physical model. Note that when $\delta > 2$, we can always change the Brownian motion by $\tilde{B}_t = -B_t$ in order to remove the negative sign from the model. Shifting to $\tilde{B}_t$ will define the same stochastic process (in distribution) since W_t again is a Brownian motion. Assume from now on that $X_0 = a > 0$.

Let Y_t be the pathwise unique solution of the stochastic differential equation

$$dY_t = 2\sqrt{Y_t^+}\,dB_t + \frac{2(1-\delta)}{2-\delta}\,dt$$

[6] The spontaneous rate of emission of photons is called the fluorescence rate.

[7] The schemes are therefore called subrecoil laser cooling.

[8] The probability of hitting zero will be discussed further below.

where $Y^+ = \max(y,0)$. The process Y_t is called a $\mathrm{BES}^2(\alpha)$-process with $\alpha = \frac{2(1-\delta)}{2-\delta}$ in the notation of Rogers and Williams, [16]. Furthermore, $R_t := \sqrt{Y_t}$ is known as an α-dimensional Bessel process. We need to have $\alpha > 0$, which is achieved if either $\delta \in [0,1)$ or $\delta > 2$. In the latter case we know from [16] that $Y_t > 0$ for all $t \geq 0$ since $\alpha > 2$. When $\delta \in [0,1)$, on the other hand, Y_t may become zero in finite time. Note that Y_t is non-explosive in the sense that it does not reach infinity in finite time with a positive probability. Define

$$X_t = \big(\frac{1}{4}c^2(2-\delta)^2 Y_t\big)^{1/(2-\delta)}$$

and let $Y_0 = \frac{4a^{2-\delta}}{c^2(2-\delta)^2} > 0$. When $\delta > 2$, X_t is well-defined (does not explode) since Y_t is positive for all times. An easy application of Itô's Formula shows that when $\delta \in [0,1)$,

$$dX_t = cX_t^{\delta/2}\,dB_t$$

and when $\delta > 2$,

$$dX_t = -cX_t^{\delta/2}\,dB_t$$

In the latter case, we have that X_t is non-explosive and positive for all times due to the properties of Y_t, while for $\delta \in [0,1)$, there is a positive probability for X_t of eventually being trapped at zero in finite time. Recall again that the minus sign in the latter model is introduced for technical reasons (see the discussion above).

We prove pathwise uniqueness of solutions of (19) for a given Brownian motion B_t. Consider the case $\delta > 2$, the other is similar: Let X_t and $\hat{X}_t$ be two solutions of

$$dX_t = -cX_t^{\delta/2}\,dB_t$$

which are positive and non-explosive. Using Itô's Formula, we have that $Y_t = g(X_t)$ and $\hat{Y}_t = g(\hat{X}_t)$ with $g(x) = (4/c^2(2-\delta)^2)x^{2-\delta}$ are both positive solutions of

$$dY_t = 2\sqrt{Y_t}\,dB_t + \frac{2(1-\delta)}{(2-\delta)}\,dt$$

with the same initial condition. By pathwise uniqueness of this equation (see e.g. Rogers and Williams [16]), $X_t^{2-\delta} = \hat{X}_t^{2-\delta}$ and uniqueness thus follows.

When $\frac{2(1-\delta)}{(2-\delta)} = n \in \mathbb{N}$ the solution Y_t can be explicitly constructed when B_t is given in a special manner. Let $W(t)$ be an n-dimensional Brownian motion with $|W(0)|^2 = b > 0$. Define

$$dB_t = |W(t)|^{-1}W(t)\cdot dW(t)$$

which is seen to be a Brownian motion on $\mathbb{R}$ by Lévy's Theorem (see e.g. Rogers and Williams [16]). Moreover, $Y_t = |W(t)|^2$ is a solution to

$$dY_t = 2\sqrt{Y_t}\,dB_t + n\,dt\,,$$

i.e., for such δ we can produce explicit representations of X_t as a function of the modulus of an n-dimensional Brownian motion. In the example below we demonstrate this for Raman cooling, where $\delta = 4$.

Example: Choose $\delta = 4$ which gives $n = 3$. Hence, let $W(t)$ be a 3-dimensional Brownian motion and $Y_t = |W(t)|^2$. With $|W(0)| = (ac)^{-1}$ we have that

$$X_t = \frac{1}{c}|W(t)|^{-1}$$

is a solution to

$$dX_t = -cX_t^2\,dB_t$$

and $X_0 = a$.

We turn to the study of asymptotic properties of X_t when t goes to infinity. Only the case $\delta > 2$ will be treated, since this is the most interesting from the laser cooling point of view. Let, for a natural number $n \geq 3$, $\frac{2(\delta-1)}{\delta-2} = n$ or $\delta(n) := \delta = \frac{2(n-1)}{(n-2)}$. Letting n run through the natural numbers, we see that $\delta(n)$ decreases from $\delta(3) = 4$ to $\delta(\infty) = 2$. $\delta(3) = 4$ will correspond to Raman cooling and $\delta(\infty) = 2$ to VSCPT cooling[9], since $\lambda(x) = c^2x^4$ and $\lambda(x) = c^2x^2$, resp. For a given $n \geq 3$, we have that

$$Y_t = W_1(t)^2 + \ldots + W_n^2(t)$$

To simplify our considerations, we assume from now on that the Brownian motion $W(t)$ starts at the origin[10]. It is well-known that

$$Y_t \sim f_{\frac{1}{2t},\frac{n}{2}}(x)$$

where

$$f_{\gamma,\nu} = \frac{\gamma^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\gamma x}$$

This implies that

$$X_t = \left(\frac{1}{4}c^2(2-\delta(n))^2 Y_t\right)^{-1/(\delta(n)-2)}$$

has a probability distribution with density $p(x,t)$ given by

$$\begin{aligned} p(x,t) &= -f_{\frac{1}{2t},\frac{n}{2}}(k^{-2}x^{-(\delta(n)-2)}) \cdot k^{-2}(-1)(\delta(n)-2)x^{-(\delta(n)-2)-1} \\ &= k^{-n}\frac{\delta(n)-2}{2^{n/2}\Gamma(\frac{n}{2})} t^{-n/2} x^{-\frac{n}{2}\delta(n)+n-1} \exp\left(-(2tk^2x^{\delta(n)-2})^{-1}\right) \\ &= \frac{2^{1-n/2}}{k^n(n-2)\Gamma(\frac{n}{2})} t^{-\frac{n}{2}} x^{-2\frac{n-1}{n-2}} \exp\left(-(2k^2tx^{\frac{2}{n-2}})^{-1}\right) \end{aligned}$$

where $k = \frac{1}{4}c^2(2-\delta(n))^2$. Hence, when $t \to \infty$,

$$t^{\frac{n}{2}} p(x,t) \sim \frac{2^{1-\frac{n}{2}}}{k^n(n-2)\Gamma(\frac{n}{2})} \cdot x^{-2\frac{n-1}{n-2}} \tag{21}$$

We conclude that the renormalized probability distribution of X_t has power law tails for big t, with power $\delta(n) \in (2,4], n \geq 3$. It is interesting to compare this with the asymptotic results of Bardou et al. [2]. Even though they do not consider the same fluorescence rate as here, there are close connections in the asymptotics. In Sect. 6.3.2 they show that for large times the momentum distribution has tails which are proportional to $t^{-1+1/\delta}x^{-\delta}$ (see eq. (6.35) in [2]). The Bessel model has the same tail behaviour. The rate of t is, on the other hand, quite different. From the above considerations it is seen to be $t^{-1-1/(\delta-2)}$ in the Bessel model. However,

[9] *V*elocity *S*elective *C*oherent *P*opulation *T*rapping, a subrecoil laser cooling scheme. See Bardou et al. [2]

[10] Of course, the transformation to X_t is then not well defined for all $t \geq 0$, since it implies that $X_0 = \infty$. However, we may form X_t from Y_t when $t > 0$, and the assumption simplifies the asymptotic considerations considerably.

our fluorescence function is unbounded outside the trapping region, while [2] use a flat or confined function.

We can easily calculate the moments of X_t for different n: A straightforward integration shows, for $m \geq 1$,

$$\begin{aligned}
\mathrm{E}\big[X_t^m\big] &= \frac{2^{1-\frac{n}{2}}}{k^n(n-2)\Gamma(\frac{n}{2})} t^{-\frac{n}{2}} \int_0^\infty x^{m-2\frac{n-1}{n-2}} \exp\big(-(2k^2 t x^{\frac{2}{n-2}})^{-1}\big)\, dx \\
&= \frac{2^{1+m(1-\frac{n}{2})}}{k^{m(n-2)}(n-2)\Gamma(\frac{n}{2})} \cdot t^{m(1-\frac{n}{2})} \int_0^\infty u^{(1-\frac{1}{2}(n-2)(m-1))-1} e^{-u}\, du
\end{aligned}$$

which is finite if and only if $1 - \frac{1}{2}(n-2)(m-1) > 0$, or

$$m < 1 + \frac{2}{n-2} \tag{22}$$

¿From this we see that X_t has finite expectation and variance when $n = 3$, while for $n > 3$, X_t has only finite expectation. More precisely,

$$t^{\frac{n}{2}-1}\mathrm{E}\big[X_t\big] = \frac{2^{2-\frac{n}{2}}}{k^{n-2}(n-2)\Gamma(\frac{n}{2})}$$

while for $n = 3$,

$$t\mathrm{E}\big[X_t^2\big] = \frac{2}{k^2}$$

Recall that $n = 3$ corresponds to $\delta = 4$ (Raman cooling). For this case we see that X_t has finite expectation and variance, converging to zero at the rate $t^{-1/2}$ and t^{-1}, respectivly, when t tends to infinity. Hence, X_t will eventually be 'trapped' at the origin almost surely. On the other hand, when $n > 3$ (i.e. $\delta \in (2, 4)$) we have only finite expectation. X_t converges to zero in expectation at the rate $t^{-(n/2-1)}$, while its higher order moments are infinite.

In this section we have seen that diffusion processes in connection with laser cooling show many of the characteristics of Lévy statistics. Momentum distributions with power tails and non-existence of higher moments, signifying laser cooling schemes, may be reproduced in a diffusion framework using Bessel processes.

5. Some other ramifications

5.1. **Time dependent λ.** In Barndorff-Nielsen et al. [4] we considered jump intensities $\lambda(x)$ dependent on the current position x only. We will here generalize to Markov processes where the jump intensity depends on time as well and derive the corresponding renewal-type equations. The idea is to model situations where the experimental conditions are changed as the experiment is carried out.

Assume the state space B to be a measurable set of $\mathbb{R}^d$ and let $a(x)$ be the initial probability distribution of the Markov process. The probability of jumping from $y \in B$ to $x \in B$ is denoted $p(x|y)$. We assume that $\lambda(t, x) \leq \Lambda$ on $[0, T] \times B$ for some constant Λ, where T may be either finite of infinite.

5.1.1. *General model.* The Kolmogorov backward equation is,

$$p_t(x, t) = -\lambda(t, x)p(x, t) + \int \lambda(t, y)p(x|y)p(y, t)\, dy$$

Define as before

$$h(x, t) = \int \lambda(t, y)p(x|y)p(y, t)\, dy \tag{23}$$

and obtain, by using the backward equation, the following expression for $p(x,t)$:

$$p(x,t) = a(x)\mathrm{e}^{-\int_0^t \lambda(s,x)\,ds} + \int_0^t h(x,s)\mathrm{e}^{-\int_s^t \lambda(u,x)\,du}\,ds$$

By inserting this relation into (23) we get a renewal type equation for $h(x,t)$,

$$\begin{aligned} h(x,t) &= \int \lambda(t,y)p(x|y)\Big\{a(y)\mathrm{e}^{-\int_0^t \lambda(s,y)\,ds} + \int_0^t h(y,s)\mathrm{e}^{-\int_s^t \lambda(u,y)\,du}\,ds\Big\}\,dy \\ &= \int a(y)p(x|y)\lambda(t,y)\mathrm{e}^{-\int_0^t \lambda(s,y)\,ds}\,dy \\ &\quad + \int_0^t \int h(y,s)p(x|y)\lambda(t,y)\mathrm{e}^{-\int_s^t \lambda(u,y)\,du}\,dy\,. \end{aligned}$$

Observe that it is not possible to write the last integral as a time convolution. We can therefore not resort to Laplace transformation techniques to study this renewal-type equation in further detail.

5.1.2. *Simple model.* To simplify matters, consider $p(x|y) = b(x)$, i.e. the jump distribution is independent of the current state of the process. The Kolmogorov backward equation now becomes,

$$p_t(x,t) = -\lambda(t,x)p(x,t) + b(x)\int \lambda(t,y)p(y,t)\,dy$$

Define

$$g(t) = \int \lambda(t,y)p(y,t)\,dy$$

and obtain

$$p(x,t) = a(x)\mathrm{e}^{-\int_0^t \lambda(s,x)\,ds} + b(x)\int_0^t g(s)\mathrm{e}^{-\int_s^t \lambda(u,x)\,du}\,ds$$

This gives a relation for g:

$$\begin{aligned} g(t) &= \int \lambda(t,y)\Big\{a(y)\mathrm{e}^{-\int_0^t \lambda(s,y)\,ds} + b(y)\int_0^t g(s)\mathrm{e}^{-\int_s^t \lambda(u,y)\,du}\Big\}\,dy \\ &= \int a(y)\lambda(t,y)\mathrm{e}^{-\int_0^t \lambda(s,y)\,ds}\,dy \\ &\quad + \int_0^t g(s)\Big\{\int b(y)\lambda(t,y)\mathrm{e}^{-\int_s^t \lambda(u,y)\,du}\,dy\Big\}\,ds \end{aligned}$$

Letting

$$v(t) = \int a(y)\lambda(t,y)\mathrm{e}^{-\int_0^t \lambda(s,y)\,ds}\,dy \tag{24}$$

and

$$u(t,s) = \int b(y)\lambda(t,y)\mathrm{e}^{-\int_s^t \lambda(u,y)\,du}\,dy \tag{25}$$

we get a renewal type equation for $g(t)$:

$$g(t) = v(t) + \int_0^t g(s)u(t,s)\,ds \tag{26}$$

In the terminology of Gripenberg, Londen and Staffans [11] equation (26) is a non-convolution Volterra equation of the second kind. They provide a rather extensive treatment of existence and uniqueness results for such problems under various integrability hypotheses on u and v. With the boundedness condition on λ, an explicit solution can be constructed in the following manner: Let $u^{(1)}(t,s) = u(t,s)$ and define the functions $u^{(n)}(t,s)$ for $n = 2,3,4,\ldots$ inductively by

$$u^{(n)}(t,s) = \int_s^t u^{(n-1)}(t,\tau)u(\tau,s)\,d\tau \tag{27}$$

We have the following proposition:

Proposition 5.1. *The solution of* (26) *is*

$$g(t) = v(t) + \int_0^t v(s)Q_t(ds) \tag{28}$$

where $Q_t(ds) = \sum_{n=1}^{\infty} u^{(n)}(t,s)\,ds$ *and the* $u^{(n)}(t,s)$ *are defined in* (27)*. Moreover,* $g(t)$ *is bounded on every compact subset of* $[0,T]$*. The solution is unique in the class of non-negative functions being bounded on every compact subset of* $[0,T]$.

Proof. We prove the result directly instead of appealing to the general theory of Gripenberg et al. [11]. Our argument is a simplified version of the proof of Proposition 2.1 in Barndorff-Nielsen et al. [4]:

We start by showing that $g(t)$ is well-defined: By iterating the definition of $u^{(n)}(t,s)$ we get the following bound,

$$\begin{aligned}
u^{(n)}(t,s) &= \int_s^t \int_{\tau_1}^t \cdots \int_{\tau_{n-2}}^t u(t,\tau_{n-1})u(\tau_{n-1},\tau_{n-2})\cdots u(\tau_1,s)\,d\tau_{n-1}\cdots d\tau_1 \\
&= \int_s^t \int_{\tau_1}^t \cdots \int_{\tau_{n-2}}^t \Big(\int_B b(y)\lambda(t,y)e^{-\int_{\tau_{n-1}}^t \lambda(\theta,y)\,d\theta}\,dy\Big)\cdots \\
&\qquad \Big(\int_B b(y)\lambda(\tau_1,y)e^{-\int_{\tau_1}^s \lambda(\theta,y)\,d\theta}\,dy\Big)\,d\tau_{n-1}\cdots d\tau_1 \\
&\le \Lambda^n \int_s^t \int_{\tau_1}^t \cdots \int_{\tau_{n-2}}^t d\tau_{n-1}\cdots d\tau_1 \\
&\le \frac{\Lambda^n (t-s)^{n-1}}{(n-1)!}
\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} u^{(n)}(t,s) \le \Lambda \mathrm{e}^{\Lambda(t-s)}$ for all $t > 0$. We can then bound the integral on the right hand side of (28) by,

$$\int_0^t v(s)Q_t(ds) \le \Lambda \int_0^t v(s)\mathrm{e}^{\Lambda(t-s)}\,ds \le \Lambda \mathrm{e}^{\Lambda t}\,.$$

This means that $g(t)$ in (28) is well-defined. Moreover, $\sup_{s\le t} g(s) \le \Lambda + \Lambda\mathrm{e}^{\Lambda t}$ which implies that $g(t)$ is bounded on every compact subset of $[0,T]$.

It is a straightforward calculation to show that $g(t)$ solves (26). We turn to uniqueness of the solution: Assume $\tilde{g}(t)$ is a non-negative solution of (28) which is bounded on every compact subset of $[0,T]$. Then,

$$g(t) - \tilde{g}(t) = \int_0^t \big(g(s) - \tilde{g}(s)\big)u(t,s)\,ds$$

Iterating this expression gives

$$g(t) - \tilde{g}(t) = \int_0^t \big(g(s) - \tilde{g}(s)\big)u^{(n)}(t,s)\,ds$$

Hence,

$$
\begin{aligned}
|g(t) - \tilde{g}(t) &\leq \frac{\Lambda^n}{(n-1)!} \int_0^t |g(s) - \tilde{g}(s)|(t-s)^{n-1}\, ds \\
&\leq \big(\sup_{s \leq t} g(s) + \sup_{s \leq t} \tilde{g}(s)\big) \frac{\Lambda^n t^n}{n!}
\end{aligned}
$$

Letting $n \to \infty$ yields $g(t) = \tilde{g}(t)$. □

Proposition 5.1 tells us that the equation (26) has a solution that formally looks like the solution to an ordinary renewal equation. However, it is not clear how to extend these results to also get the asymptotic form of $g(t)$ as $t \to \infty$, a result that is needed to find the asymptotic form of the density $p(x,t)$.

5.2. **Supersingular jump rate.** In this subsection we return to the setting of Barndorff-Nielsen et al [4] where the jump intensity λ depends on the current state x only. In Barndorff-Nielsen et al [4] all the possibilities considered had $\lambda(x) = |x|^\alpha$ for x small and with $\alpha > 0$. Here we will consider an example outside this class. We take the state space to be $B = [-1,1]$ and consider $\lambda(x) = \exp(-|x|^{-1})$. The transition density is $p(y|x) = b(y)$ with b a continuous function.

The density of a typical waiting time is

$$
u(t) = \int_B b(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}\, dy = \int_{-1}^1 b(y)\mathrm{e}^{-|y|^{-1}} \exp(-t\mathrm{e}^{-|y|^{-1}})dy.
$$

For $t \to \infty$ we find

$$
\begin{aligned}
u(t) &\sim 2b(0) \int_0^{\mathrm{e}^{-1}} (\log v)^{-2} \mathrm{e}^{-tv}\, dv \\
&= 2b(0) \int_0^{\mathrm{e}^{-1}} \mathrm{e}^{-tv}\, dK(v),
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
K(v) &= \int_0^v (\log w)^{-2} dw = \frac{v}{(\log v)^2} \int_0^1 \left(1 + \frac{\log z}{\log v}\right)^{-2} dz \\
&\sim v(\log v)^{-2} \quad \text{for } v \to 0.
\end{aligned}
\tag{30}
$$

Comparing (29) and (30) we find, by Tauberian theory (cf. Feller [10] Theorem 3, p. 445)

$$
u(t) \sim 2b(0)(\log t)^{-2} t^{-1} \quad \text{for } t \to \infty. \tag{31}
$$

For the cases considered in Barndorff-Nielsen and Benth [4] one had $u(t) \sim ct^{-(1+\alpha)}$ with $\alpha > 0$. The case (31) is therefore a limiting case corresponding to $\alpha = 0$. Referring, for instance, to Loéve [13, p. 334] we find that the distribution with density u is not in the domain of attraction of any (stable) law.

We can now proceed with the analysis as in Section 3 in Barndorff-Nielsen et al. [4]. From (31) we see that the tail probability is

$$
1 - U(t) = \int_t^\infty u(s)ds \sim 2b(0)(\log t)^{-1}, \quad \text{for } t \to \infty,
$$

and this implies (see Bingham, Goldie and Teugels [8, Cor. 8.1.7])

$$
1 - \hat{U}(s) \sim 2b(0)(-\log s)^{-1}, \quad s \to 0,
$$

As in Barndorff-Nielsen et al. [4] we then get

$$\hat{G}(s) \sim \frac{-\log s}{2b(0)} \quad \text{and} \quad G(t) \sim \frac{\log t}{2b(0)}, \tag{32}$$

where G appears in

$$p(x,t) = a(x)\mathrm{e}^{-t\lambda(x)} + b(x)\left\{G(t) - \int_0^t G(\tau)\lambda(x)\mathrm{e}^{-(t-\tau)\lambda(x)}d\tau\right\}. \tag{33}$$

To analyze the asymptotic form of (33) we rewrite (32) as

$$G(t) = \frac{1}{2b(0)}\log(1+t)(1+m(t)), \quad m(t) \to 0 \text{ for } t \to \infty.$$

In the asymptotic analysis we assume that $\xi = t\lambda(x)$ is bounded. From

$$\begin{aligned}
&\int_0^t \log(1+\tau)\lambda(x)\mathrm{e}^{-(t-\tau)\lambda(x)}d\tau \\
&= \xi\int_0^1 [\log(t) + \log(\frac{1}{t} + 1 - u)]\mathrm{e}^{-u\xi}du \\
&= \log(t)(1-\mathrm{e}^{-\xi}) + O(\xi).
\end{aligned}$$

we find

$$p(x,t) = \frac{1}{2}\log(t)\{\mathrm{e}^{-\xi} + o(1)\} + O\left(\int_0^t \log(1+\tau)m(\tau)\lambda(x)\mathrm{e}^{-(t-\tau)\lambda(x)}d\tau\right).$$

We split the last integral into the integral from 0 to s and from s to t. The first part is of order $O(\frac{s}{t}\log(t))$, and the second part is of order $O(m(s)\log(t))$. We therefore take $s = \sqrt{t}$ so that both terms are $o(\log(t))$ and we have

$$p(x,t) = \frac{1}{2}\log(t)\{\mathrm{e}^{-t\lambda(x)} + o(1)\}.$$

As compared with the results in Section 3 of Barndorff-Nielsen et al. [4] we see that the scaling here is with $\log t$ instead of t^α for some $\alpha > 0$. Also, the function $\Psi(\xi,\alpha)$ has here been replaced by $\mathrm{e}^{-\xi}$.

References

[1] Aspect, A. and Dalibard, J. (1994). Le refroidissement des atomes par laser. *La Recherche* **261** 30-37.

[2] Bardou, F., Bochaud, J.-P., Aspect, A. and Cohen-Tannoudji, C. (2000). *Non-ergodic cooling: subrecoil laser cooling and Lévy statistics*, Manuscript.

[3] Barndorff-Nielsen, O.E. and Benth, F.E. *Laser cooling and stochastics.* In M.C.M. de Gunst, C.A.J. Klaassen and A.W. van der Vaart (Eds.): *State of the Art in Probability and Statistics; Festschrift for Willem R. van Zwet.* Lecture Notes - Monogragraph Series. Institute of Mathematical Statistics. (To appear.)

[4] Barndorff-Nielsen, O.E., Benth, F.E. and Jensen, J.L. (2000). *Markov processes with a singularity*, Preprint.

[5] Bardou, F., Bouchaud, J.P., Emile, O., Aspect, A. and Cohen-Tannoudji, C. (1994). Subrecoil laser cooling and Lévy flights. *Physical Review Letters* **72** 203-206.

[6] Bardou, F. (1995). Refroidissement laser sub-recul par résonances noires. *Ph D Thesis.*

[7] Bardou, F. and Castin, Y. (1998). Le refroidissement laser subrecul: fonctions d'onde stochastiques et vols de Lévy. *Images de la Physique (Revue de CNRS)*, 7-14.

[8] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular variation.* Encyclopedia of Mathematics and its Applications, Cambridge: Cambridge University Press.

[9] Burnett, K., Edwards M. and Clark, C.W. (1999). The theory of Bose-Einstein condensation of dilute gases. *Physics Today, December* 37-42.

[10] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications, Vol II.* Second Edition. New York: J. Wiley & Sons.

[11] Gripenberg, G., Londen, S.-O. and Staffans, O. (1990). *Volterra Integral and Functional Equations.* Encyclopedia of Mathematics and its Applications, Cambridge: Cambridge University Press.

[12] Hau, L.V., Harris, S.E., Dutton, Z. and Behroozi, C.H. (1999): *Nature* **397**, 594-598.

[13] Loéve, M. (1963). *Probability Theory.* Toronto: van Nostrand.

[14] Ketterle, W. (1999) Experimental studies of Bose-Einstein condensation. *Physics Today, December* 30-35.

[15] Marangos, J. (1999): *Nature* **397**, 559-560.

[16] Rogers, L.C.G. and Williams, D. (1987). *Diffusions, Markov Processes and Martingales* Vol 2, London: J. Wiley & Sons.

[17] Saubaméa, B., Leduc, M. and Cohen-Tannoudji, C. (1999). Experimental investigations of non-ergodic effects in subrecoil laser cooling. *Physical Review Letters* **83** 3796-3799.

[18] Townsend, C., Ketterle W. and Stringari S. (1997). Bose-Einstein condensation. *Physics World, March* **10** 29-34.

MaPhySto[11],
Department of Mathematical Sciences,
University of Aarhus,
DK-8000 Århus, Denmark
E-mail address: oebn@imf.au.dk

[11] MaPhySto - Centre for Mathematical Physics and Stochastics, funded by a grant from the Danish National Research Foundation.