Probability densities and Lévy densities

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Abstract

For positive Lévy processes (i.e. subordinators) formulae are derived that express the probability density or the distribution function in terms of power series in time t. The applicability of the results to finance and to turbulence is briefly indicated.

Contents

1. Introduction

For the infinitely divisible laws there are a number of intriguing and useful relations and points of similarity between the probability measures and probability densities of the laws on the hand and their associated Lévy measures and Lévy densities on the other.

The recent comprehensive monograph by Sato (1999) contains many instances of this. Some examples are the relation between unimodality properties of the two types of densities, Sato (1999; Section 52), and the behaviour under exponential tilting (or Esscher transformation). Another instance is the result that if U is the Lévy measure of an infinitely divisible law on $\mathbf{R}^d$ with associated Lévy process X_t and if P^t denotes the law of X_t then

$$\lim_{t\downarrow 0} t^{-1} \int_{\mathbf{R}^d} f(x) P^t(\mathrm{d}x) = \int_{\mathbf{R}^d} f(x) U(\mathrm{d}x) \tag{1.1}$$

for any function f in the space $C_{\#}$ of bounded countinuous functions on $\mathbf{R}^d$ vanishing in a neighbourhood of 0, Sato (1999; Corollary 8.9). See also Léandre (1987), Ishikawa (1994) and Picard (1997) who, partly in the wider setting of pure jump processes, study cases where the transition density exists and behaves as a power of t for $t \downarrow 0$.

The present paper considers extensions of the result (1.1), but for simplicity the discussion is largely restricted to one-dimensional (i.e. $d = 1$) distributions on the positive halfline. In

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particular, a formula is proposed that expresses the probability density of such a distribution in terms of the corresponding Lévy density.

Formulae of that kind are of interest, in particular, in connection with recent work on stochastic modelling of asset returns in finance and velocity differences in turbulence. The approach taken there is to seek to capture the observed distributional behaviour by specification of Lévy densities rather than probability densities. For some case studies see Novikov (1994), Koponen (1995), Cont, Potters and Bouchaud (1997), Mantegna and Stanley (1999) and Barndorff-Nielsen and Shephard (1999, 2000). Boyarchenko and Levendorskiĭ (2000) develops this approach considerably, including applications to option pricing.

Section 2 provides some preliminaries, and then in Section 3 it is proved that for subordinators X_t the complementary distribution function $F^+(x;t) = 1 - F(x;t)$ of X_t is the limit as $\varepsilon \downarrow 0$ of a power series in t whose coefficients depend on ε and are determined from convolutions of approximations to the Lévy density u. This indicates the existence of general power series expressions for $F^+(x;t)$ and its probability density $f(x;t)$ and in the rest of the paper (Sections 4-6) the nature and applicability of these power series is discussed and illustrated. The determination of the coefficients in the series involves an intricate phenomenon of cancellation of singular terms.

2. Preliminaries

For any function f defined on $\mathbf{R}_+ = (0,\infty)$ we write $\bar{f}$ for the function defined by $\bar{f}(x) = xf(x)$.

The n-fold convolution of a probability density q is denoted by q^{*n} and similarly for distribution functions and other measures.

In the sequel we shall use the following notation for the cumulant transform of a positive random variate y

$$\bar{\mathrm{K}}\{\theta \ddagger y\} = \log \mathrm{E}\{e^{-\theta y}\}$$

We shall say that a random variable y, or its distribution, is of *class* $\mathcal{P}_+$ if y is positive and infinitely divisible and the infimum of the support of y is 0. In this case

$$\bar{\mathrm{K}}\{\theta \ddagger y\} = -\int_{\mathbf{R}_+} (1 - e^{-\theta x}) U(\mathrm{d}x) \tag{2.1}$$

where the Lévy measure U satisfies

$$\int_{\mathbf{R}_+} \min\{1, x\} U(\mathrm{d}x) < \infty$$

A stochastic process $\{z(t)\}_{t\geq 0}$ is said to be a Lévy process if it has independent increments and cadlag sample paths and is continuous in probability. If the increments are stationary $\{z(t)\}$ is said to be homogeneous. In the following, as is customary, we take the term Lévy process to mean a homogeneous Lévy process $\{z(t)\}$ such that $z(t) \xrightarrow{p} 0$ as $t \downarrow 0$. We say that $\{z(t)\}$ is of class $\mathcal{P}_+$ if the law of $z(t)$ is of class $\mathcal{P}_+$ for all $t > 0$.

Unless otherwise stated, $\{z(t)\}$ is taken to be of class $\mathcal{P}_+$. Furthermore, we shall mainly consider the case where for each $t > 0$ the law of the random variable $z(t)$ has a probability density, which we will denote by $f(x;t)$. We shall need the following

Lemma 2.1 For $m = 1, 2, ...$ we have

$$\sum_{\nu=0}^{m} (-1)^{m-\nu} \binom{m}{\nu} (\nu+1)^q = \begin{matrix} 0 & \text{for} & q < m \\ \\ q! & \text{for} & q = m \end{matrix} \tag{2.2}$$

□

PROOF We have

$$\begin{aligned} \sum_{\nu=0}^{m} (-1)^{m-\nu} \binom{m}{\nu} (\nu+1)^q &= \sum_{\tau=0}^{q} \tbinom{q}{\tau} \sum_{\nu=0}^{m} (-1)^{m-\nu} \tbinom{m}{\nu} \nu^\tau \\ &= \sum_{\tau=1}^{q} \tbinom{q}{\tau} \sum_{\nu=1}^{m} (-1)^{m-\nu} \tbinom{m}{\nu} \nu^\tau \end{aligned}$$

and therefore (2.2) follows from the fact that

$$\sum_{\nu=1}^{m} (-1)^{m-\nu} \tbinom{m}{\nu} \nu^\tau = \begin{matrix} 0 & \text{for} & \tau < m \\ \\ \tau! & \text{for} & \tau = m \end{matrix} \tag{2.3}$$

To verify the latter let

$$q(\theta) = (e^\theta - 1)^m - (-1)^m$$

and note that

$$q(\theta) = \sum_{\nu=1}^{m} (-1)^{m-\nu} \tbinom{m}{\nu} e^{\nu\theta}$$

Thus on the one hand, for $\tau = 1, 2, ...,$

$$\left. \frac{\mathrm{d}q(\theta)}{\mathrm{d}\theta^\tau} \right|_{\theta=0} = \sum_{\nu=1}^{m} (-1)^{m-\nu} \tbinom{m}{\nu} \nu^\tau$$

while on the other

$$\left. \frac{\mathrm{d}q(\theta)}{\mathrm{d}\theta^\tau} \right|_{\theta=0} = \begin{matrix} 0 & \text{for} & \tau < m \\ \\ m! & \text{for} & \tau = m \end{matrix}$$

and this implies (2.3). □

3. Probability densities and Lévy densities on $\mathbf{R}_+$

Let z be a Lévy process of class $\mathcal{P}_+$ and suppose that the Lévy measure U of $z(1)$ has density u with

$$\int_0^\infty u(x)\mathrm{d}x = \infty \tag{3.1}$$

Furthermore, for every $\varepsilon > 0$, let U_ε be a Lévy measure on $\mathbf{R}_+$ and suppose that U_ε has a density u_ε such that for every $x \in \mathbf{R}_+$

$$\int_0^\infty u_\varepsilon(x)\mathrm{d}x < \infty$$

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon(x) = u(x)$$

and such that for all $x \in \mathbf{R}_+$

$$\lim_{\varepsilon \downarrow 0} \int_x^\infty u_\varepsilon(\xi)\mathrm{d}\xi = \int_x^\infty u(\xi)\mathrm{d}\xi \tag{3.2}$$

For instance, we may take $u_\varepsilon(x) = 1_{(\varepsilon,\infty)} u(x)$. Define

$$c(\varepsilon) = \int_0^\infty u_\varepsilon(x)\mathrm{d}x \tag{3.3}$$

and

$$a_\varepsilon(x) = c(\varepsilon)^{-1} u_\varepsilon(x)$$

so that a_ε is the density function of a probability measure on $\mathbf{R}_+$ with distribution function

$$A_\varepsilon(x) = \int_0^x a_\varepsilon(\xi)\mathrm{d}\xi$$

Finally, let

$$U_{n\varepsilon}^+(x) = c(\varepsilon)^n \sum_{\nu=1}^n (-1)^{n-\nu} \binom{n}{\nu} (A_\varepsilon^{*\nu})^+(x) \tag{3.4}$$

where $(A_\varepsilon^{*\nu})^+(x) = 1 - A_\varepsilon^{*\nu}(x)$.

Theorem 3.1 Let $F(x;t)$ be the distribution function of $z(t)$ where $\{z(t)\}_{t \geq 0}$ is a Lévy process of class $\mathcal{P}_+$, and let $F^+(x;t) = 1 - F(x;t)$ Assume that the Lévy measure U of $z(1)$ has a density u satisfying $\int_0^\infty u(x)\mathrm{d}x = \infty$. Then, for every $x, t \in \mathbf{R}_+$,

$$F^+(x;t) = \lim_{\varepsilon \downarrow 0} \sum_{n=1}^\infty \frac{t^n}{n!} U_{n\varepsilon}^+(x) \tag{3.5}$$

□

To establish this theorem we first show

Lemma 3.1 The relation (3.4) is reexpressible as

$$(A_\varepsilon^{*n})^+(x) = \sum_{\nu=1}^n \binom{n}{\nu} \tilde{U}_{\nu\varepsilon}^+(x) \tag{3.6}$$

where

$$\tilde{U}_{\nu\varepsilon}^+(x) = c(\varepsilon)^{-\nu} U_{\nu\varepsilon}^+(x) \tag{3.7}$$

□

Proof Inserting (3.6) in the right hand side of (3.4) we find

$$
\begin{aligned}
\sum_{\nu=1}^{n}(-1)^{n-\nu}\binom{n}{\nu}(A_\varepsilon^{*\nu})^+(x) &= \sum_{\nu=1}^{n}(-1)^{n-\nu}\binom{n}{\nu}\sum_{m=1}^{\nu}\binom{\nu}{m}\tilde{U}_{\nu\varepsilon}^+(x) \\
&= \sum_{m=1}^{n}\binom{n}{m}\tilde{U}_{m\varepsilon}^+(x)\sum_{\nu=m}^{n}(-1)^{n-\nu}\frac{(n-m)!}{(n-\nu)!(\nu-m)!} \\
&= \sum_{m=1}^{n}\binom{n}{m}\tilde{U}_{m\varepsilon}^+(x)\sum_{\nu=0}^{n-m}(-1)^{n-m-\nu}\binom{n-m}{\nu} \\
&= \tilde{U}_{n\varepsilon}^+(x)
\end{aligned}
$$

□

Proof of Theorem 3.1 Let

$$
F_\varepsilon(x;t) = \frac{e^{-c(\varepsilon)t}}{1-e^{-c(\varepsilon)t}}\sum_{n=1}^{\infty}\frac{(c(\varepsilon)t)^n}{n!}A_\varepsilon^{*n}(x) \tag{3.8}
$$

Clearly, for every $t>0$, $F_\varepsilon(x;t)$ is the distribution function of a truncated compound Poisson law and, letting

$$
\lambda_\varepsilon(\theta) = \int_0^\infty e^{-\theta x}u_\varepsilon(x)\mathrm{d}x
$$

we find, by (3.3) and for $\theta \geq 0$,

$$
\begin{aligned}
\int_0^\infty e^{-\theta x}\mathrm{d}F_\varepsilon(x;t) &= \frac{e^{-c(\varepsilon)t}}{1-e^{-c(\varepsilon)t}}\sum_{n=1}^{\infty}\frac{(\lambda_\varepsilon(\theta)t)^n}{n!} \\
&= \frac{e^{-c(\varepsilon)t}}{1-e^{-c(\varepsilon)t}}\left(e^{\lambda_\varepsilon(\theta)t}-1\right) \\
&= \frac{1}{1-e^{-c(\varepsilon)t}}\exp\left\{-t\int_0^\infty(1-e^{-\theta x})u_\varepsilon(x)\mathrm{d}x\right\} - \frac{e^{-c(\varepsilon)t}}{1-e^{-c(\varepsilon)t}}
\end{aligned}
$$

It follows, in view of (3.2) and the fact that $c(\varepsilon)\to\infty$ for $\varepsilon\downarrow 0$, that

$$
\begin{aligned}
\int_0^\infty e^{-\theta x}\mathrm{d}F_\varepsilon(x;t) &\to \exp\left\{-t\int_0^\infty(1-e^{-\theta x})u(x)\mathrm{d}x\right\} \\
&= \int_0^\infty e^{-\theta x}\mathrm{d}F(x;t)
\end{aligned}
$$

Moreover, by Theorem 27.4 in Sato (1999) $F(x;t)$ is continuous in x and hence

$$
\lim_{\varepsilon\downarrow 0}F_\varepsilon(x;t)\to F(x;t) \tag{3.9}
$$

for all $x>0$.

On the other hand, substituting (3.6) in (3.8) we find

$$
\begin{aligned}
F_\varepsilon^+(x;t) &= \frac{e^{-c(\varepsilon)t}}{1-e^{-c(\varepsilon)t}} \sum_{n=1}^{\infty} \frac{(c(\varepsilon)t)^n}{n!} \sum_{\nu=1}^{n} \binom{n}{\nu} \tilde{U}_{\nu\varepsilon}^+(x) \\
&= \frac{e^{-c(\varepsilon)t}}{1-e^{-c(\varepsilon)t}} \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} c(\varepsilon)^\nu \tilde{U}_{\nu\varepsilon}^+(x) \sum_{n=\nu}^{\infty} \frac{(c(\varepsilon)t)^{n-\nu}}{(n-\nu)!} \\
&= \frac{1}{1-e^{-c(\varepsilon)t}} \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} U_{\nu\varepsilon}^+(x) \qquad (3.10)
\end{aligned}
$$

The interchange of summation is justified by absolute summability; in fact we have, by (3.7) and (3.4) and since $A_\varepsilon(x)$ is a distribution function,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{(c(\varepsilon)t)^n}{n!} \sum_{\nu=1}^{n} \binom{n}{\nu} |\tilde{U}_{\nu\varepsilon}^+(x)| &\leq \sum_{n=1}^{\infty} \frac{(c(\varepsilon)t)^n}{n!} \sum_{\nu=1}^{n} \binom{n}{\nu} \sum_{s=1}^{\nu} \binom{\nu}{s} |(A_\varepsilon^{*\nu})^+(x)| \\
&\leq \sum_{n=1}^{\infty} \frac{(c(\varepsilon)t)^n}{n!} \sum_{\nu=1}^{n} \binom{n}{\nu} \sum_{s=1}^{\nu} \binom{\nu}{s} \\
&\leq \sum_{n=1}^{\infty} \frac{(c(\varepsilon)t)^n}{n!} \sum_{\nu=1}^{n} \binom{n}{\nu} 2^\nu \\
&\leq \sum_{n=1}^{\infty} \frac{(3c(\varepsilon)t)^n}{n!} < \infty
\end{aligned}
$$

The formulae (3.9) and (3.10) together imply the validity of (3.5). □

In view of the conclusion in Theorem 3.1, it is plausible conjecture that for each $n = 1, 2, ...$ and for all $x \in \mathbf{R}_+$ the function $U_{n\varepsilon}^+(x)$ has a limit $U_n(x)$ for ε tending to 0 and that

$$
F^+(x;t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} U_n^+(x) \qquad (3.11)
$$

We have not been able to establish this in general, but see Section 4. Note also that letting t be a function of ε that converges sufficiently fast to 0 as $\varepsilon \downarrow 0$ we have

$$
t^{-n}\{F_\varepsilon^+(x;t) - \sum_{\nu=1}^{n} \frac{t^\nu}{\nu!} U_{\nu\varepsilon}^+(x)\} \to 0
$$

In fact, by (3.4) and since $A_\varepsilon^{*\nu}$ is a distribution function,

$$
\begin{aligned}
|F_\varepsilon^+(x;t) - \sum_{\nu=1}^{n} \frac{t^\nu}{\nu!} U_{\nu\varepsilon}^+(x)| &\leq \sum_{\nu=n+1}^{\infty} \frac{(c(\varepsilon)t)^\nu}{\nu!} \sum_{s=1}^{\nu} \binom{\nu}{s} \\
&\leq \sum_{\nu=n+1}^{\infty} \frac{(2c(\varepsilon)t)^\nu}{\nu!} = t^n \sum_{\nu=n+1}^{\infty} \frac{(2c(\varepsilon))^\nu t^{\nu-n}}{\nu!}
\end{aligned}
$$

and it suffices, for instance, to take $t = t(\varepsilon) = c(\varepsilon)^{-(n+2)}$.

The density version of (3.11) is[1]

$$f(x;t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} u_n(x) \tag{3.12}$$

where $u_n(x) = \lim_{\varepsilon \downarrow 0} u_{n\varepsilon}(x)$ and

$$\begin{aligned} u_{n\varepsilon}(x) &= c(\varepsilon)^n \sum_{\nu=1}^{n} (-1)^{n-\nu} \binom{n}{\nu} a_\varepsilon^{*\nu}(x) \\ &= \sum_{\nu=1}^{n} (-1)^{n-\nu} \binom{n}{\nu} c(\varepsilon)^{n-\nu} u_\varepsilon^{*\nu}(x) \end{aligned} \tag{3.13}$$

The first four of these functions are

$$u_{1\varepsilon}(x) = u_\varepsilon(x)$$

$$u_{2\varepsilon}(x) = u_\varepsilon^{*2}(x) - 2c(\varepsilon)u_\varepsilon(x)$$
$$u_{3\varepsilon}(x) = u_\varepsilon^{*3}(x) - 3c(\varepsilon)u_\varepsilon^{*2}(x) + 3c(\varepsilon)^2 u_\varepsilon(x)$$
$$u_{4\varepsilon}(x) = u_\varepsilon^{*4} - 4c(\varepsilon)u_\varepsilon^{*3} + 6c(\varepsilon)^2 u_\varepsilon^{*2}(x) - 4c(\varepsilon)^3 u_\varepsilon(x)$$

Further, (3.13) may be reexpressed as

$$u_\varepsilon^{*n}(x) = \sum_{\nu=1}^{n} \binom{n}{\nu} c(\varepsilon)^{n-\nu} u_{\nu\varepsilon}(x) \tag{3.14}$$

Another useful variant of (3.13) and (3.14) is

$$u_{n\varepsilon}(x) = u_\varepsilon^{*n}(x) - \sum_{s=1}^{n-1} \binom{n}{s} c(\varepsilon)^{n-s} u_{s\varepsilon}(x) \tag{3.15}$$

In particular, we have

$$u_{3\varepsilon}(x) = u_\varepsilon^{*3}(x) - 3c(\varepsilon)u_{2\varepsilon}(x) - 3c(\varepsilon)^2 u_\varepsilon(x) \tag{3.16}$$

$$u_{4\varepsilon}(x) = u_\varepsilon^{*4} - 4c(\varepsilon)u_{3\varepsilon} - 6c(\varepsilon)^2 u_{2\varepsilon}(x) - 4c(\varepsilon)^3 u_\varepsilon(x) \tag{3.17}$$

We have not been able to establish a general verification of (3.11) and (3.12), but in the following we discuss and illustrate these formulae from various points of view.

Note 3.1 If for any given Lévy density $u(x)$, satisfying the conditions for validity of Theorem 3.1, one establishes that for every $n = 2, 3, ...$ and every $x \in \mathbf{R}_+$ we have that $u_n(x) = \lim_{\varepsilon \downarrow 0} u_{n\varepsilon}(x)$ and $U_n^+(x) = \lim_{\varepsilon \downarrow 0} U_{n\varepsilon}^+(x)$ exist then (3.11) and (3.12) hold. $\square$

Remark 3.1 *Relation to the case of compound Poisson distributions.* It was assumed for Theorem 3.1 that the Lévy density u integrates to ∞ over $\mathbf{R}_+$. In case u is integrable the

[1] The existence of the probability density $f(x;t)$ for $z(t)$ is a consequence of the assumed existence of the Lévy density $u(x)$ and condition (3.1), cf. Sato (1999; Theorem 27.7).

process $z(t)$ is a compund Poisson process. We now comment on the relation to the formulae (3.5) and (3.11).

Suppose without loss of generality that u is a probability density function, let U denote the corresponding distribution function, and define $u_n(x)$ by

$$u_n(x) = \sum_{\nu=1}^{n} (-1)^{n-\nu} \binom{n}{\nu} u^{*\nu}(x) \tag{3.18}$$

Inspection of the proof of Theorem 3.1, but now with $u_\varepsilon(x) = u(x)$, $c(\varepsilon) = 1$ and $U_\nu(x) = \int_0^x u_\nu(\xi)\mathrm{d}\xi)$ (so that $U^+_{\nu\varepsilon}(x) = U^+_\nu(x)$ and $F^+_\varepsilon(x;t) = F^+(x;t)$), shows that

$$F(x;t) = \frac{1}{1-e^{-t}} \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} U_\nu(x) \tag{3.19}$$

is a distribution function, in fact the distribution function for the conditional law of $z(t)$ given $z(t) > 0$ where now $z(t)$ denotes the compound Poisson process with distribution function

$$P\{z(t) \le x\} = e^{-t} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} U^{*\nu}(x) \tag{3.20}$$

In other words,

$$e^{-t} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} U^{*\nu+}(x) = \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} U^+_\nu(x) \tag{3.21}$$

However, this relation is not valid when $\int_0^\infty u(x)\mathrm{d}x = \infty$ (in this case the convolutions $U^{*\nu}$ are not defined). □

Formula (3.12) implies in particular that

$$\lim_{t \downarrow 0} t^{-1} f(x;t) = u(x) \tag{3.22}$$

in consistency with formula (1.1).

The following examples illustrate (3.22).

Example 3.1 *Gamma distribution* Suppose z is the gamma Lévy process with $z(1)$ having law $\Gamma(\lambda, \alpha)$, so that the probability density of $z(t)$ is

$$f(x;t) = \frac{\alpha^{t\lambda}}{\Gamma(t\lambda)} x^{t\lambda-1} e^{-\alpha x} \tag{3.23}$$

with corresponding Lévy density for $z(1)$

$$u(x) = \lambda x^{-1} e^{-\alpha x} \tag{3.24}$$

In view of (3.23), this formula for u may be seen as an immediate consequence of (3.22) and the fact that $t\Gamma(t) \to 1$ for $t \downarrow 0$. □

Example 3.2 *Inverse Gaussian distribution* For the inverse Gaussian distribution $IG(\delta, \gamma)$ with probability density function

$$f(x;t) = (2\pi)^{-1/2} t\delta e^{t\delta\gamma} x^{-3/2} e^{-(t^2\delta^2 x^{-1} + \gamma^2 x)/2} \tag{3.25}$$

the cumulant transform and the Lévy density are

$$\bar{\mathrm{K}}\{\theta\} = -t\delta\gamma[1 - \{1 + 2\theta/\gamma^2\}^{1/2}] \tag{3.26}$$

$$u(x) = t(2\pi)^{-1/2} \delta x^{-3/2} e^{-\gamma^2 x/2} \tag{3.27}$$

and (3.27) follows directly from (3.22) and (3.25). □

Example 3.3 *Bessel distribution* For the Bessel distribution with probability density function

$$f(x;t) = tx^{-1} e^{-x} I_t(x) \tag{3.28}$$

the cumulant transform and the Lévy density are

$$\bar{\mathrm{K}}\{\theta\} = -t \log[\theta + 1 - \{(\theta + 1)^2 - 1\}^{1/2}] \tag{3.29}$$

$$u(x) = tx^{-1} e^{-x} I_0(x) \tag{3.30}$$

cf. Feller (1971; p. 437, 451 and 502). Again, the formula for u may be seen as an immediate consequence of (3.22). □

Example 3.4 *Mittag-Leffler Lévy process* Let $\tau(t)$ and $z(t)$ both be subordinators, with $\tau(t)$ the gamma Lévy motion for which $\mathcal{L}\{\tau(1)\} = \Gamma(\alpha, \alpha)$ and $z(t)$ the positive stable process with $\bar{\mathrm{K}}\{\theta; \tau(1)\} = \theta^\alpha$, $\alpha \in]0, 1[$, and let $x = z \circ \tau$ be the subordination of z by τ. Direct calculation shows that

$$\bar{\mathrm{K}}\{\theta; x(t)\} = t \log(1 + \theta^\alpha) \tag{3.31}$$

and this equals the cumulant transform of a probability law on $\mathbf{R}_+$ with distribution function

$$F(x \ddagger x(t)) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(t+k) x^{\alpha(t+k)}}{\Gamma(t) k! \Gamma(1 + \alpha(t+k))} \tag{3.32}$$

and probability density

$$f(x;t) = \alpha x^{-1} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(1 + t + k) x^{\alpha(t+k)}}{\Gamma(t) k! \Gamma(1 + \alpha(t+k))} \tag{3.33}$$

For $t = 1$ we have

$$F(x \ddagger x(1)) = 1 - E_\alpha(-x^\alpha)$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \tag{3.34}$$

defined for complex z and all $\alpha > 0$, is the Mittag-Leffler function[2]. Accordingly, (3.33) is known as the *Mittag-Leffler Lévy distribution*[3] and the process x is termed the *Mittag-Leffler Lévy process.*

The Lévy measure of the process x is absolutely continuous with density

$$u(x) = \alpha x^{-1} \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{\Gamma(1+\alpha k)} \tag{3.35}$$

in agreement with (3.22) and (3.33). $\square$

Above, formula (3.22) was considered for Lévy processes of class $\mathcal{P}_+$ but it applies more generally. An illustration of this is provided by the following example.

Example 3.5 *Normal inverse Gaussian distribution* The normal inverse Gaussian distribution with parameters α, β, μ and δ is denoted $NIG(\alpha, \beta, \mu, \delta)$ and is the distribution on $\mathbf{R}$ having density function

$$g(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q(\frac{x-\mu}{\delta})^{-1} K_1\{\delta \alpha q(\frac{x-\mu}{\delta})\} e^{\beta x} \tag{3.36}$$

where $q(x) = \surd(1+x^2)$ and

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1} \alpha e^{\delta \surd(\alpha^2 - \beta^2) - \beta \mu} \tag{3.37}$$

and where K_1 is the modified Bessel function of the third kind and index 1. The domain of variation of the parameters is given by $\mu \in \mathbf{R}$, $\delta \in \mathbf{R}_+$, and $0 \leq \beta < \alpha$.

If z is the Lévy process with $z(1)$ distributed as $NIG(\alpha, \beta, 0, \delta)$ then the cumulant transform of $z(t)$ is given by

$$\mathrm{K}\{\theta \ddagger z(t)\} = t\delta[\{\alpha^2 - \beta^2\}^{1/2} - \{\alpha^2 - (\beta+\theta)^2\}^{1/2}] + t\mu\theta \tag{3.38}$$

from which it immediately follows that $z(t)$ has density function

$$f(x; t) = a(\alpha, \beta, t\mu, t\delta) q(\frac{x-t\mu}{t\delta})^{-1} K_1\{t\delta \alpha q(\frac{x-t\mu}{t\delta})\} e^{\beta x} \tag{3.39}$$

It was shown in Barndorff-Nielsen (1998) that $z(1)$ has Lévy density

$$u(x) = \pi^{-1} \delta \alpha |x|^{-1} K_1(\alpha |x|) e^{\beta x} \tag{3.40}$$

Using the fact that

$$K_1(x) \sim x^{-1} \quad \text{for} \quad x \downarrow 0$$

one sees that (3.40) occurs from (3.39) by formal application of (3.22). $\square$

[2]For a discussion of the properties of the Mittag-Leffler function $E_\alpha(z)$, see Erdélyi, Magnus, Oberhettinger and Tricomi (1955; Vol. 3, Section 18.1).

[3]cf. Pillai (1990)

4. Nature of limit relations

We proceed to discuss the nature of the limit relations

$$u_{n\varepsilon}(x) \to u_n(x)$$

The next section exemplifies the convergence of $u_{n\varepsilon}(x)$ to a function $u_n(x)$. Note that such convergence implies a subtle cancellation of singularities, cf. formula (3.13) (see the first few instances of that formula, listed after (3.13) and recall that $c(\varepsilon) \to \infty$ as $\varepsilon \downarrow 0$.

To gain an understanding of how this cancellation occurs, note first that for $n = 0, 1, 2, ...$ we have

$$u_\varepsilon^{*(n+1)}(x) = (n+1)x^{-1} \int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \tag{4.1}$$

This may be shown by induction. In fact, for $n = 0$ the statement is trivial and assuming validity up till $n-1$ we find

$$\begin{aligned}
xu_\varepsilon^{*(n+1)}(x) &= (x-y+y)\int_0^x u_\varepsilon^{*n}(x-y)u_\varepsilon(y)\mathrm{d}y \\
&= \int_0^x (x-y)u_\varepsilon^{*n}(x-y)u_\varepsilon(y)\mathrm{d}y + \int_0^x u_\varepsilon^{*n}(x-y)yu_\varepsilon(y)\mathrm{d}y \\
&= \int_0^x yu_\varepsilon^{*n}(y)u_\varepsilon(x-y)\mathrm{d}y + \int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \\
&= n\int_0^x u_\varepsilon(x-y)\int_0^y u_\varepsilon^{*(n-1)}(y-z)\bar{u}_\varepsilon(z)\mathrm{d}z\mathrm{d}y + \int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \\
&= n\int_0^x \bar{u}_\varepsilon(z)\int_z^x u_\varepsilon(x-y)u_\varepsilon^{*(n-1)}(y-z)\mathrm{d}y\mathrm{d}z + \int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \\
&= xu_\varepsilon^{*(n+1)}(x) = (n+1)\int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y
\end{aligned}$$

Furthermore,

$$u_{n+1\varepsilon}(x) = (n+1)x^{-1}\int_0^x u_{n\varepsilon}(x-\xi)\bar{u}_\varepsilon(\xi)\mathrm{d}\xi + (-1)^n c(\varepsilon)^n u_\varepsilon(x) \tag{4.2}$$

as follows by the calculation

$$\begin{aligned}
\int_0^x u_{n\varepsilon}(x-\xi)\bar{u}_\varepsilon(\xi)\mathrm{d}\xi &= c(\varepsilon)^n \sum_{\nu=1}^n (-1)^{n-\nu}\binom{n}{\nu}\int_0^x a_\varepsilon^{*\nu}(x-\xi)\bar{u}_\varepsilon(\xi)\mathrm{d}\xi \\
&= c(\varepsilon)^{n+1}x\sum_{\nu=1}^n (-1)^{n-\nu}\binom{n}{\nu}(\nu+1)^{-1}a_\varepsilon^{*(\nu+1)}(x) \\
&= \frac{1}{n+1}xc(\varepsilon)^{n+1}\sum_{\nu=1}^n (-1)^{n+1-(\nu+1)}\binom{n+1}{\nu+1}a_\varepsilon^{*(\nu+1)}(x) \\
&= \frac{1}{n+1}\bar{u}_{n+1\varepsilon}(x) - (-1)^n c(\varepsilon)^n \bar{u}_\varepsilon(x)
\end{aligned}$$

Next we discuss the limiting behaviour of $u_{n\varepsilon}(x)$ as $\varepsilon \downarrow 0$. Consider first the case $n = 2$, and let

$$U_\varepsilon^+(x) = \int_x^\infty u_\varepsilon(\xi)\mathrm{d}\xi$$

and

$$U^+(x) = \int_x^\infty u(\xi)\mathrm{d}\xi$$

Using (4.2) and noting that

$$U_\varepsilon^+(0) = c(\varepsilon) = \int_0^x u_\varepsilon(\xi)\mathrm{d}\xi + U_\varepsilon^+(x) \tag{4.3}$$

we may rewrite $u_{2\varepsilon}(x)$ as

$$\begin{aligned} u_{2\varepsilon}(x) &= 2x^{-1}\left\{\int_0^x u_\varepsilon(\xi)\bar{u}_\varepsilon(x-\xi)\mathrm{d}\xi - c(\varepsilon)\bar{u}_\varepsilon(x)\right\} \\ &= 2x^{-1}\left\{\int_0^x u_\varepsilon(\xi)\{\bar{u}_\varepsilon(x-\xi) - \bar{u}_\varepsilon(x)\}\mathrm{d}\xi - \bar{u}_\varepsilon(x)U_\varepsilon^+(x)\right\} \end{aligned}$$

Hence we have

Proposition 4.1 Suppose the Lévy density u is differentiable and satisfies (3.1). Then

$$u_2(x) = 2x^{-1}\left\{\int_0^x u(\xi)\{\bar{u}(x-\xi) - \bar{u}(x)\}\mathrm{d}\xi - \bar{u}(x)U^+(x)\right\} \tag{4.4}$$

with the integral existing and being finite. □

We note that (4.4) may be reexpressed as

$$\frac{1}{2}\bar{u}_2(x) = \int_0^x u(\xi)\{\bar{u}(x-\xi) - \bar{u}(x)\}\mathrm{d}\xi - \bar{u}(x)U^+(x) \tag{4.5}$$

A similar approach does not seem to work for $n > 2$, but see the examples in the next section.

Now, let

$$U_2^+(x) = \int_x^\infty u_2(\xi)\mathrm{d}\xi$$

with $u_2(x)$ as given by (4.4). We may expect that, under the conditions in Theorem 3.1 and Proposition 4.1, we have

$$t^{-2}\{F^+(x;t) - tU^+(x) - \frac{t^2}{2}U_2^+(x)(x)\} = O(t) \tag{4.6}$$

for $t \downarrow 0$. However, again a general proof is lacking, but see Note 3.1 above.

5. Examples

In the following examples the Lévy density u is not integrable at 0 and hence we are outside the field of compound Poisson distributions.

Example 5.1 *Gamma law* Consider again the gamma Lévy process $z(t)$ of Example 3.1 and suppose for simplicity of notation that $\alpha = \lambda = 1$. Then the probability density of $z(t)$ is

$$\frac{1}{\Gamma(t)}x^{t-1}e^{-x} \tag{5.1}$$

and $z(1)$ has Lévy density

$$u(x) = x^{-1}e^{-x} \tag{5.2}$$

We illustrate formula (3.12) by showing how (5.1) is derivable from (5.2) by formal application of (3.12).

Defining $u_\varepsilon(x)$ by

$$u_\varepsilon(x) = x^\varepsilon u(x) = x^{\varepsilon-1}\bar{u}(x)$$

we have

$$c(\varepsilon) = \Gamma(\varepsilon)$$

and

$$u_\varepsilon^{*\nu}(x) = c(\varepsilon)^\nu c(\nu\varepsilon)^{-1}x^{\nu\varepsilon-1}e^{-x}$$

and, using (3.13), we find

$$u_{n\varepsilon}(x) = x^{-1}e^{-x}c(\varepsilon)^n \sum_{\nu=1}^{n}(-1)^{n-\nu}\binom{n}{\nu}c(\nu\varepsilon)^{-1}x^{\nu\varepsilon} \tag{5.3}$$

The reciprocal of the function $\bar{\Gamma}(x) = x\Gamma(x)$ possesses a power series expansion around 0 of the form

$$\bar{\Gamma}(x)^{-1} = 1 + \sum_{i=1}^{\infty} c_i x^i \tag{5.4}$$

(cf., for instance, Abramowitz and Stegun, 1972; p. 256).

Using (5.4) in (5.3) we obtain

$$\begin{aligned} u_{n\varepsilon}(x) &= \varepsilon c(\varepsilon)^n x^{\varepsilon-1}e^{-x}\sum_{\nu=1}^{n}(-1)^{n-\nu}\binom{n}{\nu}\nu\bar{\Gamma}(\nu\varepsilon)^{-1}x^{(\nu-1)\varepsilon} \\ &= n\varepsilon c(\varepsilon)^n x^{\varepsilon-1}e^{-x}S_n(x,\varepsilon) \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} S_n(x,\varepsilon) &= \sum_{\nu=1}^{n}(-1)^{n-1-(\nu-1)}\binom{n-1}{\nu-1} \\ &\quad \cdot\{1+\sum_{r=1}^{\infty}c_r\nu^r\varepsilon^r\}\{1+\sum_{s=1}^{\infty}\frac{1}{s!}(\nu-1)^s(\log x)^s\varepsilon^r\} \\ &= \sum_{\nu=1}^{n}(-1)^{n-1-(\nu-1)}\tbinom{n-1}{\nu-1}\sum_{q=0}^{\infty}d_q\frac{\nu^q}{q!}\varepsilon^q \\ &= \sum_{\nu=0}^{n-1}(-1)^{n-1-\nu}\tbinom{n-1}{\nu}\sum_{q=0}^{\infty}d_q\frac{(\nu+1)^q}{q!}\varepsilon^q \end{aligned}$$

with

$$d_q = \sum_{r=0}^{q} c_r r!\tbinom{q}{r}(\log x)^{q-r} \tag{5.6}$$

and $c_0 = 1$. Hence

$$\begin{aligned} S_n(x,\varepsilon) &= \sum_{q=0}^{\infty} d_q \frac{\varepsilon^q}{q!} \sum_{\nu=0}^{n-1} (-1)^{n-1-\nu} \binom{n-1}{v} \nu^q \\ &= \sum_{q=0}^{n-1} d_q \frac{\varepsilon^q}{q!} \sum_{\nu=0}^{n-1} (-1)^{n-1-\nu} \binom{n-1}{v} (\nu+1)^q + O(\varepsilon^n) \end{aligned}$$

or, equivalently,

$$S_n(x,\varepsilon) = d_{n-1}\varepsilon^{n-1} + O(\varepsilon^n) \tag{5.7}$$

where for the last step we have used Lemma 2.1.

Now, combining (5.5), (5.6) and (5.7) and letting $\varepsilon \downarrow 0$ we obtain

$$u_n(x) = x^{-1}e^{-x} n! \sum_{s=0}^{n-1} c_s \frac{\log^{n-1-s} x}{(n-1-s)!}$$

and inserting this in (3.12) gives

$$\begin{aligned} f(x;t) &= x^{-1}e^{-x} \sum_{n=1}^{\infty} t^n \sum_{s=0}^{n-1} c_s \frac{\log^{n-1-s} x}{(n-1-s)!} \\ &= t x^{-1} e^{-x} \sum_{s=0}^{\infty} c_s t^{n-s} \sum_{n=s}^{\infty} \frac{(t\log x)^{n-s}}{(n-s)!} \\ &= \frac{1}{\Gamma(t)} x^{t-1} e^{-x} \end{aligned}$$

as was to be demonstrated. □

Example 5.2 *Positive α − stable laws* Up to a location-scale change a stable law with index $\alpha < 1$ and skewness parameter γ has density

$$p(x;\alpha,\gamma) = \frac{1}{\pi} x^{-1} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k\alpha+1)}{k!} x^{-k\alpha} \sin \frac{k\pi}{2}(\gamma-\alpha) \tag{5.8}$$

cf., for instance, Feller (1971; Section XVII.6) or Sato (1999; p. 88). The densities $p(x;\alpha)$ of the positive stable laws correspond to the case $\gamma = -\alpha$, i.e.

$$p(x;\alpha) = \frac{1}{\pi} x^{-1} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k\alpha+1)}{k!} x^{-k\alpha} \sin k\pi\alpha \tag{5.9}$$

The corresponding Laplace transform and Lévy density are, respectively,

$$e^{-\theta^\alpha} \tag{5.10}$$

and

$$u(x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha} \tag{5.11}$$

If $z(t)$ is the Lévy process for which $z(1)$ has density (5.9) then $z(t)$ has density

$$p_t(x;\alpha) = \frac{1}{\pi} x^{-1} \sum_{k=1}^{\infty} (-1)^{k-1} t^k \frac{\Gamma(k\alpha+1)}{k!} x^{-k\alpha} \sin k\pi\alpha \tag{5.12}$$

as follows simply from (5.9), (5.10) and (5.11).

Using the identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \tag{5.13}$$

which holds for $0 < \alpha < 1$, we find that if $\alpha < \frac{1}{2}$ the first two terms of (5.12) are

$$p(x;\alpha) = t\alpha\Gamma(1-\alpha)^{-1} x^{-1-\alpha} - \frac{t^2}{2!} 2\alpha\Gamma(1-2\alpha)^{-1} x^{-1-2\alpha} + ... \tag{5.14}$$

In agreement with (5.14) and the definition (3.12) we have

$$u_1(x) = \alpha\Gamma(1-\alpha)^{-1} x^{-1-\alpha} = u(x) \tag{5.15}$$

Next, by the formulae (4.4) and (5.15), we find

$$\begin{aligned} u_2(x) &= 2\alpha^2\Gamma(1-\alpha)^{-2} x^{-1} \left\{ \int_0^x \xi^{-1-\alpha}\{(x-\xi)^{-\alpha} - x^{-\alpha}\} \mathrm{d}\xi - \alpha^{-1} x^{-2\alpha} \right\} \\ &= 2\alpha\Gamma(1-\alpha)^{-2} x^{-1-2\alpha} \left\{ \alpha \int_0^1 s^{-1-\alpha}\{(1-s)^{-\alpha} - 1\} \mathrm{d}s - 1 \right\} \\ &= -2\alpha\Gamma(1-2\alpha)^{-1} x^{-1-2\alpha} \end{aligned} \tag{5.16}$$

in consistency (3.12) with (5.14). The last step in (5.16) follows from the formula

$$\int_0^1 s^{-1-\alpha}\{(1-s)^{-\alpha} - 1\} \mathrm{d}s = \alpha^{-1} - \alpha^{-1} \frac{\Gamma(1-\alpha)^2}{\Gamma(1-2\alpha)} \tag{5.17}$$

To verify (5.17) we first note that, for $0 < \delta < 1$ and by partial integration,

$$\int_\delta^1 s^{-\alpha}(1-s)^{-\alpha} \mathrm{d}s = (1-\alpha)^{-1}\delta^{-\alpha}(1-\delta)^{1-\alpha} - \alpha(1-\alpha)^{-1} \int_\delta^1 s^{-1-\alpha}(1-s)^{1-\alpha} \mathrm{d}s$$

i.e.

$$\int_\delta^1 s^{-1-\alpha}(1-s)^{1-\alpha} \mathrm{d}s = \alpha^{-1}\delta^{-\alpha}(1-\delta)^{1-\alpha} - \alpha^{-1}(1-\alpha) \int_\delta^1 s^{-\alpha}(1-s)^{-\alpha} \mathrm{d}s \tag{5.18}$$

Moreover, we have

$$\begin{aligned} \int_\delta^1 s^{-1-\alpha}(1-s)^{-\alpha} \mathrm{d}s &= \int_\delta^1 (s+1-s) s^{-1-\alpha}(1-s)^{-\alpha} \mathrm{d}s \\ &= \int_\delta^1 s^{-\alpha}(1-s)^{-\alpha} \mathrm{d}s + \int_\delta^1 s^{-1-\alpha}(1-s)^{1-\alpha} \mathrm{d}s \end{aligned} \tag{5.19}$$

and combining (5.18) and (5.19) we find

$$\int_\delta^1 s^{-1-\alpha}(1-s)^{-\alpha} \mathrm{d}s = \alpha^{-1}\delta^{-\alpha}(1-\delta)^{1-\alpha} - \alpha^{-1}(1-2\alpha) \int_\delta^1 s^{-\alpha}(1-s)^{-\alpha} \mathrm{d}s$$

Hence, for $\delta \downarrow 0$,

$$\begin{aligned}\int_\delta^1 s^{-1-\alpha}\{(1-s)^{-\alpha}-1\}\mathrm{d}s &= \alpha^{-1}\left\{\delta^{-\alpha}(1-\delta)^{1-\alpha}-\delta^{-\alpha}+1\right\} \\ &\quad -\alpha^{-1}(1-2\alpha)\int_\delta^1 s^{-\alpha}(1-s)^{-\alpha}\mathrm{d}s \\ &\to \alpha^{-1}-\alpha^{-1}(1-2\alpha)\int_0^1 s^{-\alpha}(1-s)^{-\alpha}\mathrm{d}s \\ &= \alpha^{-1}-\alpha^{-1}(1-2\alpha)\frac{\Gamma(1-\alpha)^2}{\Gamma(2-2\alpha)} \\ &= \alpha^{-1}-\alpha^{-1}\frac{\Gamma(1-\alpha)^2}{\Gamma(1-2\alpha)}\end{aligned}$$

implying (5.17).

In the case $\alpha=\frac{1}{2}$ the formulae (5.11) and (5.12) reduce to

$$u(x)=\frac{1}{2\sqrt{\pi}}x^{-3/2} \tag{5.20}$$

and

$$p_t(x;\alpha)=\frac{1}{\sqrt{\pi}}x^{-3/2}\sum_{m=0}^{\infty}(-1)^m\frac{t^{2m+1}}{(2m+1)!}\Gamma(m+\frac{3}{2})x^{-m} \tag{5.21}$$

and a derivation of (5.21) from (5.20) via (3.12) may be accomplished by choosing

$$u_\varepsilon(x)=u(x)e^{-\frac{\varepsilon^2}{4}x^{-1}}=\frac{1}{2\sqrt{\pi}}x^{-3/2}e^{-\frac{\varepsilon^2}{4}x^{-1}} \tag{5.22}$$

With this choice

$$c(\varepsilon)=\varepsilon^{-1}$$

and, since $c(\varepsilon)^{-1}u_\varepsilon(x)$ equals the probability density of a $\frac{1}{2}$-stable law, we furthermore have

$$u_\varepsilon^{*n}(x)=\varepsilon^{-n}\frac{n}{2\sqrt{\pi}}x^{-3/2}e^{-n^2\frac{\varepsilon^2}{4}x^{-1}}$$

Hence, for instance, for $n=3$ we find

$$\begin{aligned}u_{3\varepsilon}(x) &= u_\varepsilon^{*3}(x)-3c(\varepsilon)u_\varepsilon^{*2}(x)+3c(\varepsilon)^2u_\varepsilon(x) \\ &= \varepsilon^{-2}\frac{1}{2\sqrt{\pi}}x^{-3/2}\left\{3e^{-9\frac{\varepsilon^2}{4}x^{-1}}-6e^{-4\frac{\varepsilon^2}{4}x^{-1}}+3e^{-\frac{\varepsilon^2}{4}x^{-1}}\right\} \\ &= -\frac{3}{4\sqrt{\pi}}x^{-5/2}+o(\varepsilon)\end{aligned}$$

i.e.

$$u_3(x)=-\frac{3}{4\sqrt{\pi}}x^{-5/2}$$

in consistency with (5.21). $\square$

6. Applicability to OU processes

Let $\{x(s)\}_{s\geq 0}$ be an OU process[4] (i.e. a process of Ornstein-Uhlenbeck type) and, for the present context, assume that $x(s)$ is positive. In other words, $x(s)$ is representable as

$$x(s) = e^{-\lambda s}x(0) + \int_0^s e^{-\lambda(s-\sigma)}\mathrm{d}z(\lambda\sigma) \tag{6.1}$$

where z is a subordinator (i.e. a positive Lévy process on $\mathbf{R}_+$) with $E\{\log(1+z(1))\} < \infty$.

Suppose we know the Lévy density q, say, of $x(s)$ and are interested in determining the law of the innovation term

$$y = \int_0^s e^{-\lambda(s-\sigma)}\mathrm{d}z(\lambda\sigma) \tag{6.2}$$

in (6.1). Writing $\kappa = e^{\lambda s/2}$ and assuming that q is positive and differentiable on $\mathbf{R}_+$ we have that y is of type $\mathcal{P}_+$ and has a Lévy density u given by

$$\bar{u}(x) = \bar{q}(x) - \bar{q}(\kappa^2 x) \tag{6.3}$$

(where $\bar{u}(x) = xu(x)$ and $\bar{q}(x) = xq(x)$). Moreover, provided $u(x)$ satisfies $\int_0^\infty u(x)\mathrm{d}x = \infty$ we have that $u(x)$ fulfills the conditions required in Theorem 3.1 and Proposition 4.1, and complete or approximate formulae for the law of y may be obtained via those two results. In particular, letting

$$r(x) = \kappa^2 q(\kappa^2 x)$$

the second order term (4.5) takes, in obvious notation, the form

$$\frac{1}{2}\bar{u}_2(x) = \frac{1}{2}\bar{q}_2(x) + \frac{1}{2}\bar{r}_2(x) - D(x)$$

where

$$\begin{aligned} D(x) \quad &= \quad \int_0^x q(\xi)\{\bar{r}(x-\xi) - \bar{r}(x)\}\mathrm{d}\xi + \int_0^x r(\xi)\{\bar{q}(x-\xi) - \bar{q}(x)\}\mathrm{d}\xi \\ &\quad\quad -\bar{q}(x)R^+(x) - \bar{r}(x)Q^+(x) \end{aligned}$$

Example 6.1 The inverse Gaussian OU process has played a substantial role in recent work relating to mathematical finance, see Barndorff-Nielsen (1998), Barndorff-Nielsen and Shephard (1999, 2000), and references given there. In that case, $x(s)$ has the inverse Gaussian law $IG(\delta, \gamma)$ with probability density

$$(2\pi)^{-1/2}\delta e^{\delta\gamma}x^{-3/2}e^{-(\delta^2 x^{-1} + \gamma^2 x)/2}$$

and Lévy density

$$q(x) = (2\pi)^{-1/2}\delta x^{-3/2}e^{-\gamma^2 x/2}$$

Thus,

$$u(x) = (2\pi)^{-1/2}\delta x^{-3/2}e^{-\gamma^2 x/2}(1 - \kappa^{-1}e^{-(\kappa^2-1)\gamma^2 x/2})$$

and $\int_0^\infty u(x)\mathrm{d}x = \infty$. $\square$

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[4]for the necessary background material on OU processes, see for instance Barndorff-Nielsen (1998)

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