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PORTFOLIO OPTIMIZATION IN A LÉVY MARKET WITH INTERTEMPORAL SUBSTITUTION AND TRANSACTION COSTS

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Abstract. We investigate an infinite horizon investment-consumption model in which a single agent consumes and distributes her wealth between a risk-free asset (bank account) and several risky assets (stocks) whose prices are governed by Lévy (jump-diffusion) processes. We suppose that transactions between the assets incur a transaction cost proportional to the size of the transaction. The problem is to maximize the total utility of consumption under Hindy-Huang-Kreps intertemporal preferences. This portfolio optimization problem is formulated as a singular stochastic control problem and is solved using dynamic programming and the theory of viscosity solutions. The associated dynamic programming equation is a second order degenerate elliptic integro-differential variational inequality subject to a state constraint boundary condition. The main result is a characterization of the value function as the unique constrained viscosity solution of the dynamic programming equation. Emphasis is put on providing a framework that allows for a general class of Lévy processes. Owing to the complexity of our investment-consumption model, it is not possible to derive closed form solutions for the value function. Hence the optimal policies cannot be obtained in closed form from the first order conditions for the dynamic programming equation. Therefore we have to resort to numerical methods for computing the value function as well as the associated optimal policies. In view of the viscosity solution theory, the analysis found in this paper will ensure the convergence of a large class of numerical methods for the investment-consumption model in question.

1. Introduction

We investigate an infinite horizon investment-consumption model that captures the effects of intertemporal substitution and possible jumps in the (multi-dimensional) stock market. Moreover, in the model it is supposed that transactions between the assets incur a transaction fee proportional to the size of the transaction. In many classical as well as recent studies (see, for example, Akian, Menaldi, and Sulem [1], Davis and Norman [15], Merton [33], Shreve and Soner [41] and Zariphopoulou [45, 46, 47]) of investment-consumption models with and without transaction costs, the investor derives utility directly from the present (rate of) consumption. Hindy, Huang, and Kreps [25, 23] have shown that such preferences exclude the possibility of intertemporal substitution, the reason being that the rate of consumption reacts too sensitively to small changes in the (life time) consumption plan. We recall that *intertemporal substitution* in continuous time is the notion that consumption at one time reduces marginal utility at nearby times. To overcome the deficiencies of the standard choices of preferences, Hindy, Huang, and Kreps [25, 23] replaced the present rate of consumption with some level of satisfaction, described by an exponentially weighted average of past consumption. As demonstrated by Hindy and Huang [24], this feature is the key to representing the notion of intertemporal substitution. With such preferences they showed that an agent will consume periodically (or in gulps). Thus, the agent regards consumption at adjacent dates as similar alternatives. Hindy-Huang-Kreps preferences may also be interpreted as a model for irreversible purchases of a durable good. The satisfaction process mentioned above is now understood as the agent's service flow, which is given by the exponentially weighted average of the total purchase of the good. We remark that in both interpretations, periods of absence in

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consumption or purchases will lead to a decrease in the agent's level of satisfaction or service flow. In the latter case, this is due to deterioration in the stock of the good.

In [24], Hindy and Huang used their preference structure in an investment-consumption model without transaction costs. In particular, they provided explicit consumption and allocation choices for an investor having HARA (Hyperbolic Absolute Risk Aversion) utility in a geometric Brownian market. The model studied in [24] was extended to general utility functions by Alvarez [2] and Hindy, Huang, and Zhu [26], and later by Bank and Riedel [4] and Benth, Karlsen and Reikvam [8, 9] to stock markets with jumps. Whereas the analysis in [4] relied on a stochastic version of the Kuhn-Tucker approach, the authors of [2, 8, 9, 26] (see also [24]) used the dynamic programming approach and the theory of viscosity solutions for nonlinear partial differential equations.

Geometric Brownian motion enjoys popularity in portfolio optimization problems (see, e.g., [1, 15, 41, 45, 46, 47]) since it is analytically tractable and has an economically interpretable dynamical structure. However, and this is a major concern in risk management, it predicts normally distributed logarithmic price increments (also known as logreturns). Empirical studies of logreturns show that the normal assumption must be rejected, at least for logreturns based on daily or weekly data, see Eberlein and Keller [18], Rydberg [37], and Prause [35]. To have stock price models relevant for the market one needs to consider more general price dynamics than generated by geometric Brownian motion or even continuous-time diffusions. Diffusion models driven by Lévy processes seem to provide a flexible class of models which capture statistical and economical properties of market data and yet being mathematically tractable, see Barndorff-Nielsen [7] or Eberlein and Keller [18]. Recently, several papers have investigated portfolio optimization problems where stock prices are driven by Lévy processes (see Bank and Riedel [4], Benth, Karlsen, and Reikvam [8, 9, 10], Framstad, Øksendal, and Sulem [20, 21], and Kallsen [28]).

This paper considers a multi-dimensional geometric jump-diffusion model for the stock price dynamics that includes a fairly general class of Lévy processes. In fact, we only impose a growth restriction on the tail of the Lévy measure which is satisfied by many Lévy processes of interest in finance. We will restrict our attention to a market where borrowing of money or short-selling of stocks are not allowed. The investor's consumption and transactions of wealth between the assets are understood as cumulative processes which may be singular with respect to the Lebesgue measure. The problem of maximizing the investor's expected utility over these controls is therefore a *singular* stochastic control problem. The market assumption of no borrowing of money nor short-selling of stocks imposes restrictions on the set of admissible consumption and transaction policies. In particular, it introduces a *state space constraint* into our control problem.

To investigate the investment-consumption model we use Bellman's *dynamic programming* method (see, e.g., [19]). Provided that the value function is sufficiently regular, it is well known that the associated Hamilton-Jacobi-Bellman equation can be derived using the dynamic programming principle. However, due to degeneracy and market imperfections such as trading constraints and transaction costs, it is often difficult to show that the value function in question is sufficiently smooth so as to solve the dynamic programming equation in the classical sense. The by now standard approach is to weaken the concept of solution and prove instead that the value function is a *viscosity solution* of the dynamic programming equation. The notion of viscosity solutions was introduced in the early eighties by Crandall and Lions [14] (see also Crandall, Evans, and Lions [12]) for first order Hamilton-Jacobi equation and extended by Lions [31] to fully nonlinear second order partial differential equations. We refer to Crandall, Ishii, and Lions [13] for a general overview of the theory of viscosity solutions. One of the main merits of this theory lies of course in the fact that it allows merely continuous functions to be (unique) solutions of fully nonlinear second order degenerate partial differential equations. The observation that the dynamic programming principle is intimately connected to the notion of viscosity solutions goes back to Lions [31]. After his work it became apparent that the concept of viscosity solutions was well suited for analysing stochastic control problems. We refer to the book by Fleming and Soner [19] for an up-to-date account on the applications of viscosity solution theory to stochastic control problems. For an overview of the use of viscosity solutions in the area of portfolio management and derivative pricing (with emphasis on transaction costs), we refer to Zariphopoulou [47].

For our investment-consumption model, the dynamic programming equation is a second order degenerate elliptic *integro-differential variational inequality*. The non-local operator arises because we model the stock market by diffusion processes which may have jumps in their sample paths. On the other hand, the fact that we allow the controls (i.e., consumption and transaction policies) to be singular implies that the dynamic programming equation takes the form of a variational inequality. Moreover, due to the state space constraint, this variational inequality is augmented with a so-called *state constraint* boundary condition. Consequently, we need to consider *constrained* viscosity solutions. We refer to Section 4 for a discussion of (constrained) viscosity solutions in the context of integro-differential operators and an overview of the available literature.

The main contribution of this paper is a characterization of the value function as the unique constrained viscosity solution of the dynamic programming equation associated with our singular control problem. Roughly speaking, this characterization is obtained in three steps. In the the first step, we prove that the value function satisfies several monotonicity and growth properties as well as being uniformly continuous on its unbounded domain. In the second step, we prove that the value function is a constrained viscosity solution of an integro-differential variational inequality. Here the situation is complicated by the fact that both the singular controls and the diffusion processes can make the state process jump out of a small ball for small times. Our investment-consumption model combines several difficulties such as gradient and state constraints as well as a highly singular non-local operator. Consequently, a comparison principle that fits our needs cannot be found (directly!) in the literature. Therefore, as the third and final step in the characterization of the value function, we prove a comparison principle between unbounded semicontinuous sub- and supersolutions of the state constraint problem for a class of degenerate elliptic integro-differential variational inequalities. In particular, this result ensures that the characterization of the value function as a constrained viscosity solution is *unique*. In proving the comparison principle, we adopt the uniqueness machinery for second order partial differential equations, which relies on the maximum principle for semicontinuous functions (see, e.g., [13]).

From the point of view of applications, it is of course equally or even more important to obtain the optimal investment and consumption policies (i.e., the optimal controls) than the value function itself. Owing to the complexity of our investment-consumption model, it is not possible to derive closed form solutions for the value function and hence the optimal policies cannot be obtained in closed form from the first order conditions for the Hamilton-Jacobi-Bellman equation. Therefore we have to resort to numerical methods for computing the value function as well as the associated optimal policies. The construction and analysis of numerical methods is, however, outside the scope of this paper and will instead be the topic of future work. In fact, we will in future work present a Markov chain approximation method for computing the value function and the optimal policies. As is well known by now (see, e.g., [1, 13, 16, 19, 45, 46, 48]), the viscosity solution theory provides a very flexible and powerful framework for proving convergence of numerical methods. However, to take advantage of this framework, we strongly need the analysis found in the present paper. In particular, the characterization of the value function as the unique constrained viscosity solution of an integro-differential variational inequality is of fundamental importance for the convergence analysis of a large class of (monotone, stable, and consistent) numerical methods for the investment-consumption model studied herein.

The rest of this paper is organized as follows: In Section 2, we discuss in more detail Lévy processes as basic models for stock prices. In Section 3, we give a precise formulation of the stochastic control problem as well as a statement of the associated dynamic programming equation and the main result of this paper. In Section 4, we introduce a proper viscosity solution framework for degenerate elliptic integro-differential variational inequalities. Basic monotonicity, growth, and continuity properties of the value function are proved in Section 5. A proof of the constrained viscosity solution property is given in Section 6. In this section we also prove a strong comparison result, which eventually leads to the characterization of the value function as the unique constrained viscosity solution of the dynamic programming equation.

2. Stock price models and Lévy processes

The standard model for the time dynamics of a stock price is a geometric Brownian motion,

$$S_t = S_0 e^{\mu t + \sigma W_t},$$

where μ is the drift coefficient (or mean return rate), σ is the volatility, and W_t is a Brownian motion. This model produces logreturns which are normally distributed with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$. Here we have let Δt denote the chosen time window, which is measured, for instance, in weeks, days, minutes, etc. However, empirical studies of logreturn data from stock markets show a large deviation from normality when small time windows are chosen (e.g., daily data). The tails of the logreturn data are typically fatter (heavier) than what can be captured by the normal distribution. Other deviations from normality like asymmetry or long-range dependency may also be detected. Several authors have proposed stochastic models for the stock price dynamics which take into account these non-normal effects. The canonical extension of the standard geometric model is to substitute the Brownian motion by a Lévy process. This has been suggested by, e.g., Barndorff-Nielsen [7], Eberlein and Keller [18], Gerber and Shiu [22], and Mandelbrot [32]. From a statistical point of view, this way of modelling stock prices seems to be the most appealing approach, since the logreturns are distributed as $\mu + \sigma L_1$ (choosing a time window of size 1). From general theory of Lévy processes, the distribution of L_1 belongs to the class of infinite divisible distributions. Mandelbrot [32] proposed to use stable Pareto laws as a model for the logreturns. However, empirical work indicates that this class is not suited for stock price modelling (see discussion and references in [18]). However, the class of generalized hyperbolic distributions introduced by Barndorff-Nielsen [6] seems to fit logreturn data very well. We refer to Barndorff-Nielsen [7], Eberlein and Keller [18], and Rydberg [37] for applications and empirical studies of this class of infinite divisible distributions in the context of finance.

Rather than directly writing up the stock price model as an exponential of some Lévy process, another possibility is to start out with the stochastic differential of a geometric Brownian motion,

$$dS_t = aS_t\,dt + \sigma S_t\,dW_t,$$

and use a Lévy process as the driving noise instead. This seems to be the preferred modelling approach in most works dealing with stochastic control in markets with jumps (see, e.g., the initial work by Merton [33]). A rather frequently used model is

$$dS_t = aS_t\,dt + \sigma S_t\,dW_t + S_{t-}\int_{-1}^{\infty} z\,\tilde{N}(dz,dt),$$

where $\tilde{N}$ is a compensated Poisson random measure. Note that in order to ensure that the stock price remains positive, one has to consider a Lev́y process with jumps strictly bigger than -1, which explains why the Poisson random measure is integrated only from -1. The distribution of the logreturn data imposed by this model is not so apparent, even though the solution of the stochastic differential equation can be written up explicitly in terms of the well-known Dooleans-Dade exponential, see Protter [36].

Both the above models will fit the framework chosen in this paper, since we model the stock prices as

$$dS_t = aS_t\,dt + \sigma S_t\,dW_t + S_{t-}\int_{\mathbb{R}\setminus\{0\}} \eta(z)\tilde{N}(dz,dt), \tag{2.1}$$

for some (Borel measurable) function $\eta(z) > -1$. Under certain conditions on the Lévy measure associated with L_t, the pricing model

$$S_t = S_0 e^{\mu t + L_t}$$

can be written on the form (2.1) with $\eta(z) = e^z - 1$. In [8, 9, 10], we used such a stock price model in a related optimal consumption and portfolio selection problem without transaction costs.

In the present paper, we develop a viscosity solution framework in which it is possible to treat a rather general class of Lévy processes. Recall that any Lévy process can be decomposed by the

Lévy-Khintchine formula as

$$L_t = at + \sigma W_t + \int_0^t \int_{|z|<1} z\tilde{N}(dt, dz) + \int_0^t \int_{|z|\geq 1} zN(dt, dz), \tag{2.2}$$

where N is a Poisson random measure on the Borel sets of $\mathbb{R}\backslash\{0\}$ being independent of the Wiener process W. The compensator of N takes the form $dt \times n(dz)$, where $n(dz)$ is a σ-finite measure on the Borel sets of $\mathbb{R}\backslash\{0\}$. In (2.2), $\tilde{N}$ denotes the compensated Poisson random measure, and a, σ are given constants. The Lévy-Khintchine representation decomposes any Lévy process into a Wiener process with drift, a compound Poisson process having jumps of size at least one, and a pure-jump martingale with jumps strictly less than one (the "small jump" part). The measure $n(dz)$ is usually called the Lévy measure, and satisfies the integrability condition

$$\int_{\mathbb{R}\backslash\{0\}} \min\big(1, z^2\big)\, n(dz) < \infty.$$

We will assume here that the Lévy measure associated to the Poisson random measure integrates $\eta^2(z)$ in a neighborhood around zero (usually taken to be the ball with radius one). Outside this neighborhood, we suppose that the Lévy measure integrates $|\eta(z)|$.

If $\eta(z) = O(z)$ near the origin, we have imposed no extra condition on the Lévy measure, i.e., we can treat Lévy processes with paths of unbounded variation. If the Lévy measure only integrates z near zero, we have a Lévy process where the "small jump" part have paths of bounded variation. Finally, a Lévy process with a measure integrating constants over the origin have a "small jump" part belonging to the family of compound Poisson processes.

If $\eta(z) = e^z - 1$, we see from a Taylor expansion that the condition on the Lévy measure around zero is trivially fulfilled by the general integrability property of Lévy measures. Thus, all Lévy processes are included. However, the condition outside a neighborhood of zero does not hold for a general Lévy process. For instance, the Lévy measure of an α-stable Lévy motion does not integrate $e^z - 1$ at infinity. The Normal inverse Gaussian process, on the other hand, will for certain parameters integrate $e^z - 1$ at infinity, see [37] for the Lévy measure associated to this specific Lévy process.

We refer to Bertoin [11] and Sato [38] for a general treatment of Lévy processes and their properties.

3. Formulation of the problem

Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\big)$ be a filtered complete probability space satisfying the usual hypotheses. We consider a single investor who divides her wealth between one risk-free asset (bank account) paying a fixed interest rate $r > 0$ and $n \geq 1$ risky assets (stocks). We denote by $X_0(t)$ the amount of money the investor has in the bank account and X_i the amount of money the investor has in the ith stock, $i = 1, \ldots, n$. We assume that the investor holdings have the following dynamics:

$$\begin{cases} X_0(t) = x_0 - C(t) + \displaystyle\int_0^t rX_0(s)\, ds - \sum_{i=1}^n \Big[(1+\lambda_i)L_i(t) - (1-\mu_i)M_i(t)\Big], \\ X_i(t) = x_i + \displaystyle\int_0^t a_i X_i(s)\, ds + \int_0^t \sigma_i X_i(s)\, dW_i(s) \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}\backslash\{0\}} \eta_i(z) X_i(s-)\, \tilde{N}_i(ds, dz) + L_i(t) - M_i(t), \qquad i = 1, \ldots, n, \end{cases} \tag{3.1}$$

where $a_i, \sigma_i > 0$ are constants, $C(t)$ is the cumulative consumption up to time t, $L_i(t)$ is the cumulative value of the shares *bought* up to time t from the ith stock, $M_i(t)$ is the cumulative value of the shares *sold* up to time t from the ith stock, and $\mu_i \in [0, 1]$ and $\lambda_i \geq 0$ are the proportional transaction costs of respectively selling and buying shares from the ith stock. We assume $\mu_i + \lambda_i > 0$ for all i. Moreover, $W_i(s)$ is a standard Brownian motion and N_i is a Poisson random measure on the Borel sets of $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$ with intensity $dt \times n_i(dz)$, where dt is the Lebesgue measure on the positive real line $\mathbb{R}_+$, $n_i(dz)$ is the Lévy measure on $\mathbb{R}\backslash\{0\}$, and $\tilde{N}_i$ is the compensated Poisson measure given by $\tilde{N}_i(dt, dz) = N_i(dt, dz) - dt \times n_i(dz)$. We assume that

$\{N_i\}_i$ and $\{W_i\}_i$ are independent processes and that the Lévy process is right-continuous with left limits. The functions $\eta_i(z)$, $i = 1, \dots, n$, are assumed to be Borel measurable on $\mathbb{R}\backslash\{0\}$ with the property

$$\eta_i(z) > -1, \quad i = 1, \dots, n. \tag{3.2}$$

In addition, we require the following integrability conditions on the Lévy measure:

$$\int_{|z|<1} \eta_i(z)^2 \, n_i(dz) < \infty, \qquad i = 1, \dots, n, \tag{3.3}$$

$$\int_{|z|\geq 1} |\eta_i(z)| \, n_i(dz) < \infty, \qquad i = 1, \dots, n. \tag{3.4}$$

In the case of $\eta_i(z) = z\mathbf{1}_{z>-1}$, we see that the integrability condition around zero is the usual one for Lévy measures. Depending on the form of $\eta_i(z)$, Lévy measures which are singular in zero are included in our setup.

The following basic assumption on the drift parameters of the stocks is introduced:

$$a_i \geq r, \quad i = 1, \dots, n.$$

This assumption is very natural since it states that the expected rate of return of each stock is greater than or equal to the risk-free interest rate of the bank account.

Introduce the process of average past consumption

$$dY(t) = \beta \, dC(t) - \beta Y(t) \, dt, \qquad \beta > 0. \tag{3.5}$$

This process has the explicit solution

$$Y(t) = ye^{-\beta t} + \beta e^{-\beta t} \int_{[0,t]} e^{\beta s} \, dC(s).$$

The integral is interpreted pathwise in a Lebegue-Stieltjes sense. Note that Y is an exponentially weighted average of past consumption. Higher values of β imply higher emphasis on the recent past consumption and less emphasis on the distant past consumption.

The market considered here does not allow short-selling of stocks nor borrowing of money in the bank. In other words, the amount of money allocated in the bank account and the stocks must stay nonnegative. Hence the domain for the control problem is

$$\mathcal{D} = \Big\{ x = (x_0, x_1, \dots, x_n, y) \in \mathbb{R}^N \mid y, x_i > 0, \, i = 0, \dots, n \Big\}, \quad N := n + 2.$$

Remark. From now on, we shall use the convention of numbering the coordinates of elements in $\mathcal{D}$ from zero to $N - 1 = n + 1$; that is, if $x \in \mathcal{D}$, then the zeroth coordinate of x is x_0, while the $(N-1)$th coordinate is y. For later use, we shall also need the ith ($i = 0, \dots, N-1$) unit vector of $\mathbb{R}^N$. We denote this vector by $e_i = (0, \dots, 0, 1, 0, \dots, 0)$.

Let $L = (L_1, \dots, L_n)$ and $M = (M_1, \dots, M_n)$. We refer to $\Pi = (C, L, M)$ as a policy for investment and consumption if Π belongs to the set $\mathcal{A}_x$ of *admissible controls.* For $x \in \overline{\mathcal{D}}$, we say that $\Pi \in \mathcal{A}_x$ if the following conditions hold:

(C.1) The processes $C(t), L(t), M(t)$ are adapted, nondecreasing, and right-continuous with left limits. Moreover, $C(0-) = M(0-) = L(0-) = 0$, i.e, we allow for an initial jump. Finally, $\mathbb{E}[C(t)] < \infty$, $\mathbb{E}[L(t)] < \infty$, and $\mathbb{E}[M(t)] < \infty$ for all $t \geq 0$.

(C.2) The state process $X(t) = (X_0(t), X_1(t), \dots, X_n(t), Y(t))$ respects the state-space constraint $X(t) = X^{\Pi}(t) \in \overline{\mathcal{D}}$ for all $t \geq 0$.

Note that thanks to (C.1),

$$\begin{cases} X_0(0) = x_0 - C(0) - \displaystyle\sum_{i=1}^{n} \Big[(1+\lambda_i) L_i(0) - (1-\mu_i) M_i(0) \Big], \\ X_i(0) = x_i + L_i(0) - M_i(0), \qquad i = 1, \dots, n, \end{cases}$$

may differ from $X_0(0-)$, $X_i(0-)$, $i = 1, \dots, n$, because of possible consumption/transaction(s) at time $t = 0$. Moreover, since $\eta_i(z) > -1$, we will have $X_i^{x_i,0}(t) \geq 0$ for all $t \geq 0$ whenever $x_i \geq 0$. This implies that $0 \in \mathcal{A}_x$ for all $x \in \overline{\mathcal{D}}$.

The objective of the investor is to maximize her expected utility over an infinite investment horizon. The functional to be optimized is

$$\mathcal{J}(x; \Pi) = \mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(Y^\Pi(t))\, dt\Big], \quad x \in \overline{\mathcal{D}},$$

where U is the investor's utility function and $\delta > 0$ is the discount factor. We introduce the following assumptions on the utility function:

(U.1) $U(z)$ is a continuous, nondecreasing, and concave function on $[0, \infty)$ with $U(0) = 0$.

(U.2) There exist $\gamma \in (0, 1)$ and constant $K > 0$ such that $U(z) \leq K(1 + z)^\gamma$ for all $z \in [0, \infty)$.

In addition, we require that

$$\delta > a := \max_{1 \leq i \leq n} a_i \tag{3.6}$$

and

$$\delta > \rho(\gamma) := \gamma\Big[r + \frac{1}{2(1-\gamma)} \sum_{i=1}^n \Big(\frac{a_i - r}{\sigma_i}\Big)^2\Big]. \tag{3.7}$$

Remark. We need condition (3.7) to construct a strict supersolution of our Hamilton-Jacobi-Bellman equation, and *not* to prove that the value function of the control problem is finite. When the stock price processes are geometric Brownian motions, the optimal portfolio selection and consumption problem without transaction costs (known as Merton's problem) is well-defined under condition (3.7), see Davis and Norman [15] or Akian, Menaldi, and Sulem [1] for further details.

We define the value function as

$$V(x) = \sup_{\Pi \in \mathcal{A}_x} \mathcal{J}(x; \Pi), \qquad x \in \overline{\mathcal{D}}. \tag{3.8}$$

We are facing a singular stochastic control problem, which will be studied using the dynamic programming method. Without giving a proof, we will assume throughout this paper that the following dynamic programming principle holds:

Proposition 3.1 (Dynamic Programming Principle)**.** *For any stopping time τ and $t \geq 0$, the value function satisfies:*

$$V(x) = \sup_{\Pi \in \mathcal{A}_x} \mathbb{E}\Big[\int_0^{t \wedge \tau} e^{-\delta s} U(Y^\Pi(s))\, ds + e^{-\delta(t\wedge\tau)} V(X^\Pi(t \wedge \tau))\Big]. \tag{3.9}$$

For $x \in \overline{\mathcal{D}}$, let us define a second order degenerate elliptic integro-differential operator $\mathcal{A}$ by

$$\begin{aligned}\mathcal{A}v(x) = {} & -\beta y v_y + r x_0 v_{x_0} + \sum_{i=1}^n a_i x_i v_{x_i} + \frac{1}{2}\sum_{i=1}^n \sigma_i^2 x_i^2 v_{x_i x_i} \\ & + \sum_{i=1}^n \int_{\mathbb{R}\setminus\{0\}} \Big(v(x + \eta_i(z) x_i e_i) - v(x) - \eta_i(z) x_i v_{x_i}(x)\Big)\, n_i(dz).\end{aligned}$$

The dynamic programming (or Hamilton-Jacobi-Bellman) equation associated with our control problem is a second order degenerate elliptic integro-differential variational inequality of the form

$$\max\Big(U(y) - \delta v + \mathcal{A}v, \max_{0 \leq i \leq n}\big[-(1+\lambda_i) v_{x_0} + v_{x_i}\big], \max_{0 \leq i \leq n}\big[(1-\mu) v_{x_0} - v_{x_i}\big], -v_{x_0} + \beta v_y\Big) = 0.$$

We denote by $D_x v$ the gradient of v with respect to x, $D_x^2 v$ the Hessian of v with respect to x, $\mathcal{C}(D_x v) = -v_{x_0} + \beta v_y$, and

$$\mathcal{L}(D_x v) = \max_{1 \leq i \leq n} \mathcal{L}_i(D_x v), \qquad \mathcal{L}_i(D_x v) = -(1+\lambda_i) v_{x_0} + v_{x_i},$$
$$\mathcal{M}(D_x v) = \max_{1 \leq i \leq n} \mathcal{M}_i(D_x v), \qquad \mathcal{M}_i(D_x v) = (1-\mu_i) v_{x_0} - v_{x_i}.$$

Moreover, we introduce the drift vector $b(x) = (rx_0, a_1x_1, \dots, a_nx_n, -\beta y) \in \mathbb{R}^N$, the $N \times N$ diffusion matrix $\sigma(x) = \operatorname{diag}(0, \sigma_1x_1, \dots, \sigma_nx_n, 0)$, the ith jump vector $\eta_i(x,z) = \eta_i(z)x_ie_i \in \mathbb{R}^N$ for $i = 1, \dots, n$, and the non-local operator

$$\mathcal{B}(x, v, D_xv) = \sum_{i=1}^{n} \int_{\mathbb{R}\setminus\{0\}} \Big(v(x + \eta_i(x,z)) - v(x) - \langle \eta_i(x,z), D_xv\rangle\Big)\, n_i(dz).$$

Finally, we introduce the operator

$$\begin{aligned}&\mathcal{F}(x, v, D_xv, D_x^2v, \mathcal{B}(x, v, D_xv))\\ &\qquad = U(y) - \delta v + \langle b(x), D_xv\rangle + \frac{1}{2}\operatorname{Tr}\big(\sigma(x)^2D_x^2v\big) + \mathcal{B}(x, v, D_xv).\end{aligned}$$

Note that $b(x)$ and $\sigma(x)$ are uniformly Lipschitz continuous and that the operator $\mathcal{F}$ is degenerate elliptic in the sense that

$$\mathcal{F}(x, r, p, A, \mathcal{B}(x, r, p)) \le \mathcal{F}(x, r, p, \tilde{A}, \mathcal{B}(x, r, p)) \text{ whenever } A \le \tilde{A},$$

where $r \in \mathbb{R}$, $x, p \in \mathbb{R}^N$, $A, \tilde{A} \in \mathbb{S}^N$, and $\mathbb{S}^N$ denotes the set of $N \times N$ symmetric matrices with the usual partial ordering; that is, $A \le \tilde{A}$ means $\langle A\xi, \xi\rangle \le \langle \tilde{A}\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^N$. Here and in what follows, $\langle \cdot, \cdot\rangle$ denotes the Euclidean inner product.

Now our Hamilton-Jacobi-Bellman equation can be written more compactly as

$$\max\Big(\mathcal{F}(x, v, D_xv, D_x^2v, \mathcal{B}(x, v, D_xv)), \mathcal{L}(D_xv), \mathcal{M}(D_xv), \mathcal{C}(D_xv)\Big) = 0 \text{ in } \mathcal{D}. \tag{3.10}$$

To have a well-posed problem, we need to augment our dynamic programming equation (3.10) with a suitable boundary condition. Condition (C.2) is a state-space constraint which translates naturally into the following so-called state constraint boundary condition (see Section 4 for details)

$$\max\Big(\mathcal{F}(x, v, D_xv, D_x^2, \mathcal{B}(x, v, D_xv)), \mathcal{L}(D_xv), \mathcal{M}(D_xv), \mathcal{C}(D_xv)\Big) \le 0 \text{ on } \partial\mathcal{D}. \tag{3.11}$$

Our main results are stated in the following theorem.

Theorem 3.2. *The value function V defined in* (3.8) *is well defined, concave, nondecreasing, it satisfies the sublinear growth condition*

$$0 \le V(x) \le \text{Const} \cdot \Big(1 + \sum_{i=0}^{n} x_i + y\Big)^{\gamma}, \qquad x = (x_0, x_1, \dots, x_n, y) \in \overline{\mathcal{D}},$$

and it is uniformly continuous on $\overline{\mathcal{D}}$. Moreover, the value function V is the unique viscosity solution of (3.10)-(3.11) *in the class of sublinearly growing solutions.*

Theorem 3.2 is a consequence of the results stated and proved in Section 5 and Section 6.

4. Viscosity solutions

We now introduce a proper notion of constrained viscosity solutions for integro-differential variational inequalities. Via the dynamic programming method, this notion of weak solutions will be our main tool for analysing the investment-consumption model described in Section 3. A notion of viscosity solutions for integro-differential equations was first used by Soner [43, 44] and Sayah [39, 40] for problems involving a first order local operator. Alvarez and Tourin [3] and Barles, Buckdahn, and Pardoux [5] later used this notion for integro-differential equations involving a second order local operator, while Pham [34] used this notion for second order integro-differential quasi-variational inequalities associated with the optimal stopping time problem for controlled jump-diffusion processes. All the papers cited so far prove various existence and uniqueness results for the "whole space" case and hence do not take into account boundary conditions.

As already mentioned several times, for our investment-consumption problem we need the notion of constrained viscosity solutions since we do not allow for short-selling of stocks nor borrowing of money in the bank. The notion of constrained viscosity solutions was first introduced by Soner [42] and later Capuzzo-Dolcetta and Lions [17] for first order partial differential equations, see also Lasry and Lions [30], Lions and Ishii [27], and Katsoulakis [29] for second order partial differential equations. In [43], Soner used the notion of constrained viscosity solutions for certain first order integro-differential equations associated with piecewise deterministic processes with jumps. More recently, Benth, Karlsen, and Reikvam [8, 9, 10] used the notion of constrained viscosity solutions for first and second order integro-differential variational inequalities.

Let $\mathcal{O} \subset \overline{\mathcal{D}}$. In what follows, we let $C^p(\mathcal{O})$ denote the usual space of $p \geq 0$ times continuously differentiable functions on $\mathcal{O}$. We shall also need the following spaces of semicontinuous functions on $\mathcal{O}$:

$$USC(\mathcal{O}) = \Big\{ v : \mathcal{O} \to \mathbb{R} \,\Big|\, v \text{ is upper semicontinuous} \Big\},$$
$$LSC(\mathcal{O}) = \Big\{ v : \mathcal{O} \to \mathbb{R} \,\Big|\, v \text{ is lower semicontinuous} \Big\}.$$

For notational convenience, whenever v belongs to $C^p(\mathcal{O})$ and for some $\nu > 0$ satisfies

$$\sup_{x \in \mathcal{O}} \frac{|v(x)|}{\left(1 + \sum_{i=0}^n x_i + y\right)^\nu} < \infty, \tag{4.1}$$

we shall signify this by writing $v \in C^p_\nu(\mathcal{O})$. Similarly, we write $v \in USC_\nu(\mathcal{O})$ $(LSC_\nu(\mathcal{O}))$ if v satisfies (4.1) and belongs to $USC(\mathcal{O})$ $(LSC(\mathcal{O}))$.

Although we prove later that the value function is continuous, we will formulate our "viscosity-related" results and in particular a strong comparison principle (Theorem 6.6) in terms of semicontinuous viscosity sub- and supersolutions. The main reason for working with semicontinuous functions is the need for such in the study of numerical methods for our investment-consumption model, which is the topic of future work. In particular, convergence analysis of numerical methods for this model relies heavily on the strong comparison principle proved in this paper.

In what follows, it will be useful to distinguish the singularities at zero and infinity. To this end, we introduce two operators $\mathcal{B}_\kappa, \mathcal{B}^\kappa$. For $\kappa \in (0,1)$ and $v \in C^2(\overline{\mathcal{D}})$, we define

$$\mathcal{B}_\kappa(x, v, D_x v) = \sum_{i=1}^n \int_{|z| \leq \kappa} \Big(v(x + \eta_i(x,z)) - v(x) - \langle \eta_i(x,z), D_x v \rangle \Big)\, n_i(dz).$$

Keeping in mind that $\eta_i(x,z) = \eta_i(z) x_i e_i$, we can rewrite $\mathcal{B}_\kappa$ to obtain

$$\begin{aligned}
\mathcal{B}_\kappa(x, v, D_x v) &= \sum_{i=1}^n \int_0^1 \int_{|z| \leq \kappa} \Big\langle (1-\theta) D_x^2 v(x + \theta \eta_i(x,z)) \eta_i(x,z), \eta_i(x,z) \Big\rangle\, n_i(dz)\, d\theta \\
&\leq \mathrm{Const}\big(x, D_x^2 v, \kappa\big) \sum_{i=1}^n \int_{|z| \leq \kappa} \eta_i(z)^2\, n_i(dz),
\end{aligned}$$

and thus $\mathcal{B}_\kappa$ is convergent thanks to (3.3). Furthermore, (3.3) implies

$$\lim_{\kappa \to 0+} \mathcal{B}_\kappa(x, v, D_x v) = 0. \tag{4.2}$$

For $\kappa \in (0,1)$ and $v \in C^1_1(\overline{\mathcal{D}})$, we define

$$\mathcal{B}^\kappa(x, v, D_x v) = \sum_{i=1}^n \int_{|z| > \kappa} \Big(v(x + \eta_i(x,z)) - v(x) - \langle \eta_i(x,z), D_x v \rangle \Big)\, n_i(dz).$$

The integrand of $\mathcal{B}^\kappa$ is bounded by $\mathrm{Const}\big(x, D_x v, \kappa\big)\, \big(1 + |\eta_i(z)|\big)$ and thus (3.4) implies that $\mathcal{B}^\kappa$ is convergent for every positive κ.

Note that for $v \in C^2_1(\overline{\mathcal{D}})$, we can write

$$\mathcal{B}(x, v, D_x v) = \mathcal{B}_\kappa(x, v, D_x v) + \mathcal{B}^\kappa(x, v, D_x v). \tag{4.3}$$

We thus conclude that the dynamic programming equation (3.10) is well defined for all $v \in C^2_1(\overline{\mathcal{D}})$. However, in many applications the solution of (3.10) is not C^2 or even C^1. Consequently, the dynamic programming equation (3.10) should be interpreted in the sense of viscosity solutions.

Definition 4.1. (i) Let $\mathcal{O} \subset \overline{\mathcal{D}}$. A locally bounded function $v \in USC(\overline{\mathcal{D}})$ $(LSC(\overline{\mathcal{D}}))$ is a *viscosity subsolution* (*supersolution*) of (3.10) in $\mathcal{O}$ if and only if $\forall \phi \in C_1^2(\mathcal{O})$ we have:

$$\begin{cases} \text{for each } x \in \mathcal{O} \text{ being a global maximizer (minimizer) relative to } \mathcal{O} \text{ of } v - \phi, \\ \max\Big(\mathcal{F}(x, v, D_x\phi, D_x^2\phi, \mathcal{B}(x, \phi, D_x\phi)), \mathcal{L}(D_x\phi), \mathcal{M}(D_x\phi), \mathcal{C}(D_x\phi)\Big) \geq 0\ (\leq 0). \end{cases} \tag{4.4}$$

(ii) A function $v \in C(\overline{\mathcal{D}})$ is a *constrained viscosity solution* of (3.10) if and only if v is a viscosity supersolution of (3.10) in $\mathcal{D}$ and v is a viscosity subsolution of (3.10) in $\overline{\mathcal{D}}$.

Hereafter we use the terms *subsolution* and *supersolution* instead of viscosity subsolution and viscosity supersolution, respectively. Furthermore, a viscosity solution of (3.10)-(3.11) is of course the same as a constrained viscosity solution of (3.10).

For $\kappa > 0$, $\phi \in C^2(\overline{\mathcal{D}})$, and $v \in USC_1(\overline{\mathcal{D}})$ or $v \in LSC_1(\overline{\mathcal{D}})$, let us introduce the notation

$$\begin{aligned} &\mathcal{F}(x, v, D_x\phi, D_x^2\phi, \mathcal{B}_\kappa(x, \phi, D_x\phi), \mathcal{B}^\kappa(x, v, D_x\phi)) \\ &\quad = U(y) - \delta v + \langle b(x), D_x v\rangle + \frac{1}{2}\mathrm{Tr}\big(\sigma(x)^2 D_x^2 v\big) + \mathcal{B}_\kappa(x, \phi, D_x\phi) + \mathcal{B}^\kappa(x, v, D_x\phi). \end{aligned}$$

Note that $\mathcal{B}_\kappa(x, \phi, D_x\phi)$ and $\mathcal{B}^\kappa(x, v, D_x\phi)$ are well defined. We now have the following equivalent formulation of viscosity solutions.

Lemma 4.1. *Let $\mathcal{O} \subset \overline{\mathcal{D}}$ and fix any $\kappa > 0$. A function $v \in USC_1(\overline{\mathcal{D}})(LSC_1(\overline{\mathcal{D}}))$ is a subsolution (supersolution) of* (3.10) *in $\mathcal{O}$ if and only if $\forall \phi \in C^2(\overline{\mathcal{D}})$ we have:*

$$\begin{cases} \textit{for each } x \in \mathcal{O} \textit{ being a global maximizer (minimizer) relative to } \mathcal{O} \textit{ of } v - \phi, \\ \max\Big(\mathcal{F}(x, v, D_x\phi, D_x^2\phi, \mathcal{B}_\kappa(x, \phi, D_x\phi), \mathcal{B}^\kappa(x, v, D_x\phi)), \mathcal{L}(D_x\phi), \mathcal{M}(D_x\phi), \mathcal{C}(D_x\phi)\Big) \geq 0\ (\leq 0). \end{cases} \tag{4.5}$$

Proof. We prove the statement only for the subsolutions, the supersolution case can be proved similarly. Suppose v satisfies

$$\mathcal{F}(x, v, D_x\phi, \mathcal{B}_\kappa(x, \phi, D_x\phi), \mathcal{B}^\kappa(x, v, D_x\phi)) \geq 0, \tag{4.6}$$

where $x \in \mathcal{O}$ is a global maximizer relative to $\mathcal{O}$ for $v - \phi$, $\phi \in C_1^2(\overline{\mathcal{D}})$. Then

$$v(\tilde{x}) - v(x) \leq \phi(\tilde{x}) - \phi(x)$$

for all $\tilde{x} \in \mathcal{O}$. Consequently, since $\mathcal{B}^\kappa(x, \phi, D_x\phi) \geq \mathcal{B}^\kappa(x, v, D_x\phi)$, we can use (4.3) and (4.6) to conclude that

$$\mathcal{F}(x, v, D_x\phi, \mathcal{B}(x, \phi, D_x\phi)) = \mathcal{F}(x, v, D_x\phi, \mathcal{B}_\kappa(x, \phi, D_x\phi), \mathcal{B}^\kappa(x, \phi, D_x\phi)) \geq 0.$$

From this observation we eventually conclude that v is a subsolution of (3.10) in $\mathcal{O}$ if (4.5) holds.

Conversely, let $x \in \mathcal{O}$ be a global maximizer relative to $\mathcal{O}$ for $v - \phi$, $\phi \in C^2(\overline{\mathcal{D}})$. With $t = (0, t_1, \ldots, t_n, 0)$, let $\mathcal{N}(x,t) \subset \mathbb{R}^N$ denote the open hyperellipse centered in x with semiaxis $0, t_1, \ldots, t_n, 0$ and let $\mathcal{N}_\mathcal{O}(x,t) = \mathcal{N}(x,t) \cap \mathcal{O}$. With $\nu(x,k) = \big(0, \frac{x_1}{k}, \ldots, \frac{x_n}{k}, 0\big)$, let χ_k be a smooth function satisfying $0 \leq \chi_k \leq 1$, $\chi_k = 1$ in $\mathcal{N}_\mathcal{O}\big(x, \eta(x,\kappa) - \nu(x,k)\big)$, and $\chi_k = 0$ in $\mathcal{O} \backslash \mathcal{N}_\mathcal{O}\big(x, \eta(x,\kappa)\big)$. Note that as $k \to \infty$, we have $\nu(x,k) \to 0$ and hence $\chi_k \to 1$ in $\mathcal{N}_\mathcal{O}\big(x, \eta(x,\kappa)\big)$. If we choose a suitable $v_k \in C_1^\infty(\overline{\mathcal{D}})$ such that $v_k \uparrow v$ a.e. as $k \to \infty$, then the function

$$\psi_k(\tilde{x}) = \chi_k(\tilde{x})\phi(\tilde{x}) + (1 - \chi_k(\tilde{x}))v_k(\tilde{x})$$

belongs to $C_1^2(\overline{\mathcal{D}})$ and x is a global maximizer relative to $\mathcal{O}$ of $v - \psi_k$. Moreover, $\psi_k \to \phi$ in $\mathcal{N}_\mathcal{O}\big(x, \eta(x,\kappa)\big)$ as $k \to \infty$ and $\psi_k = v_k$ in $\mathcal{O} \backslash \mathcal{N}_\mathcal{O}\big(x, \eta(x,\kappa)\big)$. In fact, we have $\psi_k = \phi$ in $\mathcal{N}_\mathcal{O}\big(x, \eta(x,\kappa) - \nu(x,k)\big)$ so that $D_x\psi_k(x) = D_x\phi(x)$. From these properties and Lebesgue's dominated convergence theorem, we get

$$\mathcal{B}(x, \psi_k) = \mathcal{B}_\kappa(x, \psi_k, D_x\phi) + \mathcal{B}^\kappa(x, \psi_k, D_x\phi) \to \mathcal{B}_\kappa(x, \phi, D_x\phi) + \mathcal{B}^\kappa(x, v, D_x\phi) \text{ as } k \to \infty.$$

Choosing ψ_k as test function in the definition (4.4) of a subsolution and then sending $k \to \infty$, we see that (4.5) holds if v is a subsolution of (3.10) in $\mathcal{O}$. $\square$

We remark that Lemma 4.1 is an adaption of a similar lemma in Soner [42], see also Sayah [39] and [8]. It will be convenient to use Definition 4.1 when proving existence of a constrained viscosity solution (see Theorem 6.1). On the other hand, a formulation of viscosity solutions based on the notion of jets is more convenient when proving uniqueness (see Theorem 6.6).

Definition 4.2. Let $x \in \mathcal{O} \subset \overline{\mathcal{D}}$. For a function $v \in USC(\mathcal{O})$ $(LSC(\mathcal{O}))$, the second order *superjet* (*subjet*) $J^{2,+(-)}_{\mathcal{O}} v(x)$ is the set of $(p, A) \in \mathbb{R}^N \times \mathbb{S}^N$ such that

$$v(\tilde{x}) \leq (\geq)\, v(x) + \langle p, \tilde{x} - x\rangle + \tfrac{1}{2}\langle A(\tilde{x} - x), \tilde{x} - x\rangle + o(|\tilde{x} - x|^2) \text{ as } \mathcal{O} \ni \tilde{x} \to x.$$

The closure $\overline{J}^{2,+(-)}_{\mathcal{O}} v(x)$ is the set of $(p, A) \in \mathbb{R}^N \times \mathbb{S}^N$ for which there exists a sequence $(p^k, A^k) \in J^{2,+(-)}_{\mathcal{O}} v(x^k)$ such that $(x^k, v(x^k), p^k, A^k) \to (x, v(x), p, A)$ as $k \to \infty$.

In view of Lemma 4.1 and Definition 4.2, we have now the following formulation of viscosity solutions based on jets, which is similar to the formulation used in Pham [34].

Lemma 4.2. *Let $\mathcal{O} \subset \overline{\mathcal{D}}$ and fix any $\kappa > 0$. Let $v \in USC_1(\overline{\mathcal{D}})$ $(LSC_1(\overline{\mathcal{D}}))$ be a subsolution (supersolution) of* (3.10) *in $\mathcal{O}$. Then, for each (p, A) that belongs to $\overline{J}^{2,+(-)}_{\mathcal{O}} v(x)$ with $x \in \mathcal{O}$, there exists a test function $\phi \in C^2(\overline{\mathcal{D}})$ such that*

$$\max\Big(\mathcal{F}(x, v, p, A, \mathcal{B}_\kappa(x, \phi, D_x\phi), \mathcal{B}^\kappa(x, v, p)), \mathcal{L}(p), \mathcal{M}(p), \mathcal{C}(p)\Big) \geq 0\, (\leq 0). \tag{4.7}$$

The test function ϕ is such that $v - \phi$ has a global maximum (minimum) at x^k relative to $\mathcal{O}$ with $x^k \to x$ as $k \to \infty$.

Proof. Let $(p, A) \in \overline{J}^{2,+(-)}_{\mathcal{O}} v(x)$. Then there exists $(p^k, A^k) \in J^{2,+(-)}_{\mathcal{O}} v(x^k)$ such that $(p^k, A^k) \to (p, A)$ and $(x^k, v(x^k)) \to (x, v(x))$ as $k \to \infty$. Using standard arguments (see, e.g., [19]), one can prove that $(p^k, A^k) \in J^{2,+(-)}_{\mathcal{O}} v(x^k)$ if and only if there exists $\phi \in C^2(\overline{\mathcal{D}})$ such that $\phi(x^k) = v(x^k)$, $D_x\phi(x^k) = p^k$, $D^2_x\phi(x^k) = A^k$, and $v - \phi$ has a global maximum (minimum) relative to $\mathcal{O}$ at x^k. Therefore, (4.7) holds with $x = x^k$, $p = p^k$, $A = A^k$. The lemma now follows by sending $k \to \infty$ and using continuity of the equation. □

Later we shall prove a comparison principle for (3.10)-(3.11). To this end, we need the following maximum principle for semicontinuous functions taken from [13]:

Theorem 4.3 (Crandall, Ishii, and Lions [13])**.** *Let $\mathcal{O}$ be a locally compact subset of $\mathbb{R}^N$. Let $u_1, -u_2 \in USC(\mathcal{O})$ and $\varphi \in C^2(\mathcal{O} \times \mathcal{O})$. Suppose $(x_\varphi, \tilde{x}_\varphi) \in \mathcal{O} \times \mathcal{O}$ is a local maximizer of $u_1(x) - u_2(\tilde{x}) - \varphi(x, \tilde{x})$. Then for every $\nu > 0$ there exist two matrices $A, \tilde{A} \in \mathbb{S}^N$ such that*

$$\big(D_x\varphi(x_\varphi, \tilde{x}_\varphi), A\big) \in \overline{J}^{2,+}_{\mathcal{O}} u_1(x_\varphi), \qquad \big(-D_y\varphi(x_\varphi, \tilde{x}_\varphi), \tilde{A}\big) \in \overline{J}^{2,-}_{\mathcal{O}} u_2(\tilde{x}_\varphi),$$

and

$$-\Big(\frac{1}{\nu} + \|D^2\varphi(x_\varphi, \tilde{x}_\varphi)\|\Big) I \leq \begin{pmatrix} A & 0 \\ 0 & -\tilde{A} \end{pmatrix} \leq D^2\varphi(x_\varphi, \tilde{x}_\varphi) + \nu\big(D^2\varphi(x_\varphi, \tilde{x}_\varphi)\big)^2. \tag{4.8}$$

The norm of a symmetric matrix A is $\|A\| = \sup\Big\{|\langle A\xi, \xi\rangle| \;\Big|\; \xi \in \mathbb{R}^N,\ |\xi| \leq 1\Big\}$.

5. Properties of the value function

In this section, we prove that the value function possesses some basic monotonicity, growth, and continuity properties. The techniques used to prove these properties are by now rather standard in the literature. In particular, the results stated (and proved) in this section are inspired by the corresponding results in Zariphopoulou [45, 46] (see also [47]), who study a related investment-consumption model with transaction costs in the case of geometric Brownian motion and a standard time-additive (von Neumann-Morgenstern) utility functional.

Proposition 5.1. *The value function V is nonnegative, nondecreasing, and concave on its unbounded domain $\overline{\mathcal{D}}$ with $V(0) = 0$.*

Proof. Since U is nonnegative, V is obviously nonnegative. Furthermore, it is easily seen that $\Pi = (0,0,0)$ is the only admissible control when starting at the origin. From the definition of the processes $X_0(t)$, $X_i(t)$, $i = 1, \dots, n$, and $Y(t)$, we see that they will remain at the origin if they start there and thus $V(0) = 0$. We next prove the monotonicity of V. Let $\hat{x}_i \geq x_i$ and $\hat{y} \geq y$ for $i = 0, 1, \dots, n$, and assume $\Pi \in \mathcal{A}_x$. Then

$$\begin{aligned}\hat{Y}(t) := Y^{\hat{y},\Pi}(t) &= \hat{y}e^{-\beta t} + \beta e^{-\beta t} \int_{[0,t]} e^{\beta s}\, dC(s) \\ &= (\hat{y} - y)e^{-\beta t} + Y^{y,\Pi}(t) \geq Y^{y,\Pi}(t).\end{aligned}$$

Consider now $Z_i(t) = \hat{X}_i(t) - X_i(t)$, $i = 0, 1, \dots, n$, where $\hat{X}_i := X_i^{\hat{x}_i,\Pi}$ and $X_i := X_i^{x_i,\Pi}$. A direct calculation shows that

$$Z_0(t) = (\hat{x}_0 - x_0) + \int_0^t r Z_0(s)\, ds,$$

which yields $Z_0(t) \geq 0$. Furthermore,

$$Z_i(t) = (\hat{x}_i - x_i) + \int_0^t a_i Z_i(s)\, ds + \int_0^t \sigma_i Z_i(s)\, dW_i(s) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \eta_i(z) Z_i(s-)\, \tilde{N}_i(ds, dz).$$

Hence, $Z_i(t)$ is a stochastic (Dooleans-Dade) exponential with initial condition $\hat{x}_i - x_i \geq 0$. Since $\eta_i(z) > -1$, we get $Z_i(t) \geq 0$ (see, e.g., Protter [34]). Thus, $\Pi \in \mathcal{A}_{\hat{x}}$, and since U is nondecreasing,

$$\mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(Y(t))\, dt\Big] \leq \mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(\hat{Y}(t))\, dt\Big] \leq V(\hat{x}).$$

Finally, taking the supremum over the set of $\mathcal{A}_x$ - controls gives the desired monotonicity of V.

We now prove the concavity of V. For $\hat{x}, \tilde{x} \in \overline{\mathcal{D}}$, consider two arbitrary controls $\hat{\Pi} \in \mathcal{A}_{\hat{x}}$ and $\tilde{\Pi} \in \mathcal{A}_{\tilde{x}}$. Let $\theta \in [0,1]$ be a fixed number and define the control Π to be $\Pi = \theta\hat{\Pi} + (1-\theta)\tilde{\Pi}$. We now prove that $\Pi \in \mathcal{A}_x$, where $x = \theta\hat{x} + (1-\theta)\tilde{x}$. The concavity of V will follow from this and the assumption that U is concave. We calculate

$$\begin{aligned}\theta\hat{X}_0(t) + (1-\theta)\tilde{X}_0(t) &= x_0 - \big(\theta\hat{C}(t) + (1-\theta)\tilde{C}(t)\big) + \int_0^t r\big(\theta\hat{X}_0(s) + (1-\theta)\tilde{X}_0(s)\big)\, ds \\ &\quad - \sum_{i=1}^n \Big(\theta(1+\lambda_i)\hat{L}_i(t) + (1-\theta)(1+\lambda_i)\tilde{L}_i(t)\Big) \\ &\quad + \sum_{i=1}^n \theta(1-\mu_i)\hat{M}_i(t) + (1-\theta)(1-\mu_i)\tilde{M}_i(t) \\ &= x_0 - C(t) + \int_0^t r\big(\theta\hat{X}_0(s) + (1-\theta)\tilde{X}_0(s)\big)\, ds \\ &\quad - \sum_{i=1}^n \Big((1+\lambda_i)L_i(t) - (1-\mu_i)M_i(t)\Big).\end{aligned}$$

Hence, by uniqueness of the paths,

$$X_0^\Pi(t) = \theta\hat{X}_0^{\hat{\Pi}}(t) + (1-\theta)\tilde{X}_0^{\tilde{\Pi}}(t).$$

A similar calculation yields

$$\begin{aligned}\theta\hat{X}_i(t) + (1-\theta)\tilde{X}_i(t) &= x_i + \int_0^t a_i\big(\theta\hat{X}_i(s) + (1-\theta)\tilde{X}_i(s)\big)\, ds + L_i(t) - M_i(t) \\ &\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \eta_i(z)\big(\theta\hat{X}_i(s) + (1-\theta)\tilde{X}_i(s)\big)\, \tilde{N}_i(ds, dz) \\ &\quad + \int_0^t \sigma_i\big(\theta\hat{X}_i(s) + (1-\theta)\tilde{X}_i(s)\big)\, dW_i(s).\end{aligned}$$

Again by uniqueness of the paths,

$$X_i^{\Pi}(t) = \theta \hat{X}_i^{\hat{\Pi}}(t) + (1-\theta)\tilde{X}_i^{\tilde{\Pi}}(t).$$

Finally,

$$Y(t) = y + \beta\big(\theta\hat{C}(t) + (1-\theta)\tilde{C}(t)\big) - \beta\int_0^t \big(\theta\hat{Y}(s) + (1-\theta)\tilde{Y}(s)\big)\,ds$$

and thus

$$Y^{\Pi}(t) = \theta\hat{Y}^{\hat{\Pi}}(t) + (1-\theta)\tilde{Y}^{\tilde{\Pi}}(t).$$

In conclusion, $\Pi \in \mathcal{A}_x$. We therefore easily see, using the concavity of U,

$$\begin{aligned}\theta\mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(\hat{Y}^{\hat{\Pi}}(t))\,dt\Big] &+ (1-\theta)\mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(\tilde{Y}^{\tilde{\Pi}}(t))\,dt\Big] \\ &\leq \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(Y^{\Pi}(t))\,dt\Big] \leq V(x).\end{aligned}$$

Taking the supremum over $\hat{\Pi} \in \mathcal{A}_{\hat{x}}$ and $\tilde{\Pi} \in \mathcal{A}_{\tilde{x}}$ yields the desired concavity of V. □

We next show that the value function is dominated by the utility function U.

Proposition 5.2. *For $x = (x_0, x_1, \dots, x_n, y) \in \overline{\mathcal{D}}$ and $z = x_0 + \sum_{i=1}^n x_i + y/\beta$, we have*

$$V(x) \leq \frac{1}{\delta}U(z).$$

Proof. Introduce the process

$$Z(t) = X_0(t) + \sum_{i=1}^n X_i(t) + Y(t)/\beta$$

with initial condition $Z(0) = z$. Using (3.1) and (3.5), a direct calculation yields

$$\begin{aligned}Z(t) = x_0 - C(t) &+ \int_0^t rX_0(s)\,ds - \sum_{i=1}^n\Big((1+\lambda_i)L_i(t) - (1-\mu_i)M_i(t)\Big) \\ &+ \sum_{i=1}^n x_i + \int_0^t\sum_{i=1}^n a_iX_i(s)\,ds + \int_0^t\sum_{i=1}^n \sigma_iX_i(s)\,dW_i(s) \\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}}\sum_{i=1}^n \eta_i(z)X_i(s-)\,\tilde{N}_i(ds,dz) \\ &+ \sum_{i=1}^n L_i(t) - \sum_{i=1}^n M_i(t) + y/\beta - \int_0^t Y(s)\,ds + C(t) \\ = z &+ \int_0^t\Big(rX_0(s) + \sum_{i=1}^n a_iX_i(s)\Big)\,ds + \int_0^t\sum_{i=1}^n\sigma_iX_i(s)\,dW_i(s) \\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}}\sum_{i=1}^n \eta_i(z)X_i(s-)\,\tilde{N}_i(ds,dz) \\ &+ \sum_{i=1}^n(1-(1+\lambda_i))L_i(t) - \sum_{i=1}^n\mu_iM_i(t) - \int_0^t Y(s)\,ds \\ \leq z &+ \int_0^t \bar{a}Z(s)\,ds + \int_0^t\sum_{i=1}^n\sigma X_i(s)\,dW_i(s) \\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}}\sum_{i=1}^n \eta_i(z)X_i(s-)\,\tilde{N}_i(ds,dz) - \int_0^t Y(s)\,ds,\end{aligned}$$

where we have used $\bar{a} := \max_i a_i \geq r$ and $Y(s) \geq 0$ to derive the inequality from the second equality. Taking the expectation on both sides, we get

$$\mathbb{E}\big[Z(t)\big] \leq z + \bar{a} \int_0^t \mathbb{E}\big[Z(s)\big]\, ds - \int_0^t \mathbb{E}\big[Y(s)\big]\, ds.$$

From Gronwall's inequality it follows that

$$\mathbb{E}\big[Z(t)\big] \leq z e^{\bar{a}t} - e^{\bar{a}t}\mathbb{E}\Big[\int_0^t e^{-\bar{a}s} Y(s)\, ds\Big],$$

or, equivalently since $Z(t) \geq 0$,

$$\mathbb{E}\Big[\int_0^t e^{-\bar{a}s} Y(s)\, ds\Big] \leq z - e^{-\bar{a}t}\mathbb{E}\big[Z(t)\big] \leq z.$$

From the assumption $\delta \geq a_i$ for all $i = 1, \dots, n$, we obtain

$$\mathbb{E}\Big[\int_0^\infty e^{-\delta s} Y(s)\, ds\Big] \leq z.$$

Since U is concave, we apply Jensen's inequality to get

$$\mathbb{E}\Big[\int_0^\infty e^{-\delta s} U(Y(s))\, ds\Big] \leq \frac{1}{\delta} U\Big(\mathbb{E}\Big[\int_0^\infty e^{-\delta s} Y(s)\, ds\Big]\Big) \leq \frac{1}{\delta} U(z),$$

where we have used that U is nondecreasing. This concludes the proof of the proposition. □

From the general bound of the value function in terms of the investor's utility, we immediately get a sublinear growth estimate and continuity at the origin.

Corollary 5.3. *For all $x \in \overline{\mathcal{D}}$ there exists a positive constant K such that*

$$0 \leq V(x) \leq K\Big(1 + x_0 + \sum_{i=1}^n x_i + y\Big)^\gamma. \tag{5.1}$$

Proof. This follows from condition (U.2) on the utility function and Proposition 5.2. □

Corollary 5.4. *V is continuous at the origin.*

Proof. Let $x^k \in \mathcal{D}$ be a sequence such that $x^k \to 0$ as $k \to \infty$. Furthermore, let $z^k = x_0^k + \sum_{i=1}^n x_i^k + y^k/\beta$ and use Proposition 5.2 to obtain $V(x^k) \leq U(\delta z^k)/\delta$. But since $z^k \to 0$ as $k \to \infty$ and U is continuous at the origin with $U(0) = 0$, we get $V(x^k) \to 0$. Hence, the corollary follows since $V(0) = 0$. □

We now show that the value function is uniformly continuous on the closure of $\mathcal{D}$.

Proposition 5.5. *The value function V is uniformly continuous on $\overline{\mathcal{D}}$.*

Proof. Since V is concave, it is continuous on $\mathcal{D}$. It remains to prove that V is continuous on $\partial\mathcal{D}$. Without loss of generality, we will consider only the case of one risky asset ($n = 1$).

Consider a sequence $(x_0^k, x_1^k, y^k) \in \mathcal{D}$ such that $(x_0^k, x_1^k, y^k) \to (x_0, x_1, 0)$ as $k \to \infty$, where $x_0, x_1 > 0$. By the triangle inequality

$$\begin{aligned}|V(x_0^k, x_1^k, y^k) - V(x_0, x_1, 0)| &\leq |V(x_0^k, x_1^k, y^k) - V(x_0, x_1^k, y^k)| \\ &\quad + |V(x_0, x_1^k, y^k) - V(x_0, x_1, y^k)| + |V(x_0, x_1, y^k) - V(x_0, x_1, 0)|.\end{aligned}$$

The concavity of the value function implies that V is locally Lipschitz continuous on $\mathcal{D}$, and therefore the two first terms on the right hand side of the inequality can be controlled as $k \to \infty$. Hence, to show continuity it is sufficient to show that for any given $\varepsilon > 0$, there exists a natural number N_ε such that $|V(x_0, x_1, y^k) - V(x_0, x_1, 0)| \leq \varepsilon$ for $k \geq N_\varepsilon$. But since V is increasing this amounts in showing that for a given $\varepsilon > 0$ there exists a natural number N_ε such that

$$V(x_0, x_1, y^k) \leq V(x_0, x_1, 0) + \varepsilon,$$

for $k \geq N_\varepsilon$. We show this as follows. Let $\Pi^k \in \mathcal{A}_{(x_0,x_1,y^k)}$ be an ε - optimal control and introduce $\bar{x}_0^k = x_0 + \frac{1}{\beta}y^k$, $d\bar{C}^k(t) = dC^k(t) + \frac{1}{\beta}y^k\delta_0(t)$, where $\delta_0(t) = 0$ if $t \neq 0$ and $\delta_0(t) = 1$ if $t = 0$. We use the notation $\bar{\Pi}^k := (\bar{C}^k, L^k, M^k)$. A straightforward calculation shows that

$$X_0^{\bar{x}_0^k,\bar{\Pi}^k}(t) = x_0 + \frac{1}{\beta}y^k - C^k(t) - \frac{1}{\beta}y^k + \int_0^t rX_0^{\bar{x}_0^k,\bar{\Pi}^k}(s)\,ds$$
$$- \sum_{i=1}^n \Big((1+\lambda_i)L_i^k(t) - (1-\mu_i)M_i^k(t)\Big) = X_0^{x_0,\Pi^k}(t)$$

by uniqueness of the paths. Therefore, $X_0^{\bar{x}_0^k,\bar{\Pi}^k}(t) \geq 0$ for $t \geq 0$. Obviously, $X_1^{x_1,\bar{\Pi}^k}(t) = X_1^{x_1,\Pi^k}(t)$. Inserting the definition of $\bar{\Pi}^k$ gives

$$Y^{0,\bar{\Pi}^k}(t) = \beta e^{-\beta t}\frac{1}{\beta}y^k + \beta e^{-\beta t}\int_{[0,t]} e^{\beta s}\,dC^k(s) = Y^{y^k,\Pi^k}(t),$$

again by uniqueness of the paths. Hence, $Y^{0,\bar{\Pi}^k}(t) \geq 0$ for all $t \geq 0$. We conclude that $\bar{\Pi}^k \in \mathcal{A}_{(\bar{x}_0^k,x_1,0)}$. By ε - admissibility,

$$V(x_0,x_1,y^k) \leq \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(Y^{y^k,\Pi^k}(t))\,dt\Big] + \varepsilon$$
$$= \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(Y^{0,\bar{\pi}^k}(t))\,dt\Big] + \varepsilon = V(\bar{x}_0^k, x_1, 0) + \varepsilon.$$

Since V is continuous on $\mathcal{D}$, there exists a N_ε such that $V(\bar{x}_0^k,x_1,0) \leq V(x_0,x_1,0)+\varepsilon$ for $k \geq N_\varepsilon$. This is true since $\bar{x}_0^k \to x_0$ as $k \to \infty$. In conclusion,

$$V(x_0,x_1,y^k) \leq V(x_0,x_1,0) + 2\varepsilon \tag{5.2}$$

for $k \geq N_\varepsilon$ and hence V is continuous at $(x_0,x_1,0)$.

Consider now a sequence $(x_0^k,x_1^k,y^k) \in \mathcal{D}$ such that $(x_0^k,x_1^k,y^k) \to (0,x_1,y)$ as $k \to \infty$, where $x_1, y > 0$. By a similar argument as above, it is sufficient to show that

$$V(x_0^k,x_1,y) \leq V(0,x_1,y) + 2\varepsilon, \tag{5.3}$$

for k large enough. Let $\Pi^k = (C^k,L^k,M^k)$ be an ε - admissible control for (x_0^k,x_1,y). Define $d\bar{M}^k(t) = dM^k(t) + \frac{x_0^k}{1-\mu}\delta_0(t)$ and $\bar{x}_1^k = x_1 + \frac{x_0^k}{1-\mu}$. Note that $\bar{M}^k(t)$ is increasing. Introduce the control $\bar{\Pi}^k := (C^k,L^k,\bar{M}^k)$. By pathwise uniqueness, we get

$$X_0^{0,\bar{\Pi}^k}(t) = -C^k(t) + \int_0^t rX_0^{0,\bar{\Pi}^k}(s)\,ds - (1-\lambda)L^k(t)$$
$$+ (1-\mu)M^k(t) + \frac{x_0^k}{1-\mu}(1-\mu)$$
$$= x_0^k - C^k(t) + \int_0^t rX_0^{0,\bar{\Pi}^k}(s)\,ds - (1-\lambda)L^k(t) + (1-\mu)M^k(t) = X_0^{x_0^k,\Pi^k}(t),$$

which implies $X_0^{0,\bar{\Pi}^k}(t) \geq 0$ for all $t \geq 0$. Similarly,

$$X_1^{\bar{x}_1^k,\bar{\Pi}^k}(t) = x_1 + \frac{x_0}{1-\mu} + \int_0^t aX_1^{\bar{x}_1^k,\bar{\Pi}^k}(s)\,ds + \int_0^t \sigma X_1^{\bar{x}_1^k,\bar{\Pi}^k}(s)\,dW(s)$$
$$+ \int_0^t\int_{\mathbb{R}\setminus\{0\}} \eta(z)X_1^{\bar{x}_1^k,\bar{\Pi}^k}(s-)\,\tilde{N}(ds,dz) + L^k(t) - M^k(t) - \frac{x_0^k}{1-\mu} = X_1^{x_1,\Pi^k}(t),$$

which gives $X_1^{\bar{x}_1^k,\bar{\Pi}^k}(t) \geq 0$ for every $t \geq 0$. Obviously, $Y^{y,\bar{\Pi}^k}(t) = Y^{y,\Pi^k}(t)$, and we therefore have $\bar{\Pi}^k \in \mathcal{A}_{(0,\bar{x}_1^k,y)}$. Repeating the argument that produced (5.2), we get (5.3) and hence continuity for the part of the boundary where $x_0 = 0$.

Finally, consider a sequence $(x_0^k, x_1^k, y^k) \in \mathcal{D}$ such that $(x_0^k, x_1^k, y^k) \to (x_0, 0, y)$ as $k \to \infty$, where $x_0, y > 0$. Again it is sufficient to show

$$V(x_0, x_1^k, y) \le V(x_0, 0, y) + \varepsilon, \tag{5.4}$$

for k large enough. As usual, let $\Pi^k = (C^k, L^k, M^k)$ be an ε - admissible control for (x_0, x_1^k, y). Define $d\bar{L}^k(t) = dL^k(t) + x_1^k \delta_0(t)$ and $\bar{x}_0^k = x_0 + (1+\lambda)x_1^k$. Let $\bar{\Pi}^k := (C^k, \bar{L}^k, M^k)$. Pathwise uniqueness gives

$$\begin{aligned} X_0^{\bar{x}_0^k, \bar{\Pi}^k}(t) = x_0 + (1+\lambda)x_1^k - C(t) + \int_0^t r X_0^{\bar{x}_0^k, \bar{\Pi}^k}(s)\, ds \\ - (1+\lambda)L^k(t) - (1+\lambda)x_1^k + (1-\mu)M^k(t) = X_0^{x_0, \pi^k}(t), \end{aligned}$$

and hence $X_0^{\bar{x}_0^k, \bar{\Pi}^k}(t) \ge 0$ for all $t \ge 0$. Similarly,

$$\begin{aligned} X_1^{0, \bar{\Pi}^k}(t) = \int_0^t a X_1^{0, \bar{\Pi}^k}(s)\, ds + \int_0^t \sigma X_1^{0, \bar{\Pi}^k}(s)\, dW(s) \\ + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \eta(z) X_1^{0, \bar{\Pi}^k}(s-)\, \tilde{N}(ds, dz) + L^k(t) + x_1^k - M^k(t) = X_1^{x_1^k, \Pi^k}(t) \end{aligned}$$

and hence $X_1^{0, \bar{\Pi}^k}(t) \ge 0$ for all $t \ge 0$. Obviously, $Y^{y, \bar{\Pi}^k}(t) = Y^{y, \Pi^k}(t)$ and we therefore have $\bar{\Pi}^k \in \mathcal{A}_{(\bar{x}_0^k, 0, y)}$. The same argument as before yields (5.4) and thus continuity for the part of the boundary where $x_1 = 0$.

From Corollary 5.4, we know that V is continuous at the origin. Hence, V is continuous on $\overline{\mathcal{D}}$ and uniformly continuous on every compact subset of $\overline{\mathcal{D}}$. By the concavity property, we know that V is locally Lipschitz continuous on $\mathcal{D}$. Consequently, V is uniformly continuous on $\overline{\mathcal{D}}$. $\square$

6. Proofs of the main results

In this section, we first prove that the value function of our control problem can be characterized as a constrained viscosity solution of the associated dynamic programming equation, i.e., the integro-differential variational inequality (3.10). As already mentioned in the introduction, for singular control problems the classical proof (see Lions [31]) of the viscosity property fails. The reason being that the state process may jump due to the singular controls and thus it needs not to stay in a small ball for small t. Our situation is further complicated by the fact that we work with diffusion processes whose (uncontrolled) sample paths may be discontinuous. The problem associated with singular controls has usually been circumvented by either relying on the existence of optimal controls (see, e.g., [16, 26, 46]) or by establishing appropriate estimates for the state process (see, e.g., [19, 41]). Here we use a more direct argument to show that the value function possesses the (constrained) viscosity property. Our argument is similar to the one used by Benth, Karlsen, and Reikvam [8, 9], see also Alvarez [2].

Theorem 6.1. *The value function V defined in* (3.8) *is a constrained viscosity solution of the dynamic programming equation* (3.10).

We divide the proof of this theorem into two propositions.

Proposition 6.2. *The value function V is a supersolution of* (3.10) *in $\mathcal{D}$.*

Proof. Let $\phi \in C_1^2(\overline{\mathcal{D}})$ and $x \in \mathcal{D}$ be a global minimizer of $V - \phi$. Without loss of generality we may assume that $(V-\phi)(x) = 0$. Choosing $C(0) = 0$, $M_i(0) = 0$, $L_i(0) = l_i > 0$ for any $l_i \in (0, x_i]$, $i = 1, \dots, n$, and $t = 0$ in the dynamic programming principle (3.9), we get

$$\phi(x) = V(x) \ge V(x - (1+\lambda_i)l_i e_0 + l_i e_i) \ge \phi(x - (1+\lambda_i)l_i e_0 + l_i e_i),$$

where we recall that e_i denotes the ith unit vector in $\mathbb{R}^N$, $i = 0, \dots, n+1$. Dividing by l_i and sending $l_i \to 0$, we conclude that

$$\mathcal{L}_i(D_x)\phi(x) = -(1+\lambda_i)\phi_{x_0} + \phi_{x_i} \le 0, \qquad i = 1, \dots, n. \tag{6.1}$$

Choosing $C(0) = 0$, $M_i(0) = m_i$, for any $m_i \in (0, x_i]$, $L_i(0) = 0$, $i = 1, \ldots, n$, and $t = 0$ in the dynamic programming principle (3.9), we get

$$\phi(x) = V(x) \geq V(x + (1-\mu_i)m_i e_0 - m_i e_i) \geq \phi(x + (1-\mu_i)m_i e_0 - m_i e_i).$$

Dividing by m_i and sending $m_i \to 0$, we conclude that

$$\mathcal{M}_i(D_x)\phi(x) = (1-\mu_i)\phi_{x_0}(x) - \phi_{x_i}(x) \leq 0, \qquad i = 1, \ldots, n. \tag{6.2}$$

Choosing $C(0) = c$, for any $c \in (0, x_0]$, $M_i(0) = 0$, $L_i(0) = 0$, $i = 1, \ldots, n$, and $t = 0$ in the dynamic programming principle (3.9), we get

$$\phi(x) = V(x) \geq V(x - ce_0 + \beta c e_{n+1}) \geq \phi(x - ce_0 + \beta c e_{n+1}).$$

Dividing by c and sending $c \to 0$, we conclude that

$$\mathcal{C}(D_x\phi)(x) = -\phi_{x_0}(x) + \beta\phi_y(x) \leq 0. \tag{6.3}$$

With $\Pi \in \mathcal{A}_x$, let τ_ρ be the exit time of $X(t) = X^\Pi(t)$ from the closed ball $\mathcal{N}_\rho$ with radius ρ and center at x. By choosing ρ small enough, $\mathcal{N}_\rho \subset \mathcal{D}$. Using the dynamic programming principle (3.9) with $h \wedge \tau_\rho$, $\Pi \equiv 0$, the inequality $V \geq \phi$, and Dynkin's formula, we obtain

$$\begin{aligned}
0 &\geq \mathbb{E}\Big[\int_0^{h\wedge\tau_\rho} e^{-\delta t} U(Y(t))\,dt + e^{-\delta(h\wedge\tau_\rho)}\phi(X(h\wedge\tau_\rho))\Big] - \phi(x) \\
&\geq \mathbb{E}\Big[\int_0^{h\wedge\tau_\rho} e^{-\delta t}\Big\{U(Y(t)) - \delta\phi(X(t)) + \mathcal{A}\phi(X(t))\Big\}\,dt\Big] \\
&\geq \mathbb{E}\Big[\frac{1 - e^{-\delta(h\wedge\tau_\rho)}}{\delta}\Big] \inf_{\tilde{x}\in\mathcal{N}_\rho}\Big\{U(\tilde{y}) - \delta\phi(\tilde{x}) + \mathcal{A}\phi(\tilde{x})\Big\}.
\end{aligned}$$

By the right continuity of the paths, $\tau_\rho > 0$ a.s. Hence, by Lebesgue's dominated convergence theorem, $\lim_{h\to 0}\mathbb{E}\Big[\frac{1-e^{-\delta(h\wedge\tau_\rho)}}{h}\Big] = \delta$. Dividing the above inequality by h, sending $h \to 0$ and then $\rho \to 0$, we obtain

$$U(y) - \delta V(x) + \mathcal{A}\phi(x) \leq 0. \tag{6.4}$$

From (6.1), (6.2), (6.3), and (6.4), we conclude that V is a supersolution. □

To prove the subsolution property, we shall need the following easy result.

Lemma 6.3. *If $x' \in \overline{\mathcal{D}}$ can be reached from $x \in \overline{\mathcal{D}}$ by an investment and consumption policy, i.e.,*

$$x' = x - \Big(c + \sum_{i=1}^n [(1+\lambda_i)l_i - (1-\mu_i)m_i]\Big)e_0 + \sum_{i=1}^n (l_i - m_i)e_i + \beta c e_{n+1},$$

for some nonnegative constants c, l_i, m_i, $i = 1, \ldots, n$, then $V(x) \geq V(x')$.

Proposition 6.4. *The value function V is a subsolution of* (3.10) *in $\overline{\mathcal{D}}$.*

Proof. Let $\phi \in C_1^2(\overline{\mathcal{D}})$ and $x \in \overline{\mathcal{D}}$ be a global maximizer of $V - \phi$. Without loss of generality we may assume $(V - \phi)(x) = 0$ and that the maximum is strict. Arguing by contradiction, we suppose that the subsolution inequality (4.4) is violated at x. By continuity, we can find an $\varepsilon > 0$ and a nonempty open neighborhood $\mathcal{N}_\mathcal{D}$ ($\mathcal{N}_\mathcal{D} = \mathcal{N} \cap \mathcal{D}$ for some nonempty open ball $\mathcal{N} \subset \mathbb{R}^N$ centered in x) such that $V \leq \phi - \varepsilon$ on $\partial\mathcal{N}_\mathcal{D}$, and in $\overline{\mathcal{N}_\mathcal{D}}$ one has the following series of inequalities

$$\begin{cases}
-(1+\lambda_i)\phi_{x_0} + \phi_{x_i} \leq 0, \ i = 1, \ldots, n, \\
(1-\mu_i)\phi_{x_0} - \phi_{x_i} \leq 0, \ i = 1, \ldots, n, \\
\beta\phi_y - \phi_{x_0} \leq 0, \\
U(y) - \delta V + \mathcal{A}\phi \leq -\varepsilon\delta.
\end{cases} \tag{6.5}$$

With $\Pi = (C, L, M) \in \mathcal{A}_x$, let τ^* be the exit time of the state process $X(t)$ from $\overline{\mathcal{N}_\mathcal{D}}$. We also introduce τ_L, the first time the state process jumps because of the Lévy process, and recall that $\tau_L > 0$ a.s. Define the stopping time $\tau = \tau^* \wedge \tau_L$. If necessary, we truncate τ by a constant to make it finite. Let $A := \{\tau_L = 0\}$ and note that this is a set of zero probability.

If $\tau^* < \tau_L$ we know with positive probability that the control $\Pi(t) = (C(t), L(t), M(t))$ has made the state process jump out of $\overline{\mathcal{N}_{\mathcal{D}}}$. Let x' be the intersection point between $\partial\mathcal{N}_{\mathcal{D}}$ and the line between $X(\tau-)$ and $X(\tau)$. Since this line can be written as a linear combination of the vectors

$$-e_0 + \beta e_{n+1}, \qquad -(1+\lambda_i)e_0 + e_i,\ i = 1, \dots, n, \qquad (1-\mu_i)e_0 - e_i,\ i = 1, \dots, n,$$

ϕ is nonincreasing along this line in $\overline{\mathcal{N}_{\mathcal{D}}}$. Thanks to Lemma 6.3, we also know that V is nonincreasing along this line in $\overline{\mathcal{D}}$. Hence,

$$V(X(\tau^*)) \le V(x') \le \phi(x') - \varepsilon \le \phi(X(\tau^*-)) - \varepsilon. \tag{6.6}$$

In what follows, we let $C^c(t)$, $M^c(t)$, and $L^c(t)$ denote the continuous parts of $C(t)$, $L(t)$, and $M(t)$, respectively. Moreover, we let $\Delta C(t) = C(t) - C(t-)$, $\Delta L(t) = L(t) - L(t-)$, and $\Delta M(t) = M(t) - M(t-)$. Using (6.5), (6.6), and Itô's formula for semimartingales, we get

$$\begin{aligned}
&\int_0^{\tau^*} e^{-\delta t} U(Y(t))\,dt + e^{-\delta\tau^*} V(X(\tau^*)) \\
&\quad \le \int_0^{\tau^*} e^{-\delta t} U(Y(t))\,dt + e^{-\delta\tau^*}\big(\phi(X(\tau^*-)) - \varepsilon\big) \\
&\quad \le \phi(x) - \varepsilon e^{-\delta\tau^*} + \int_0^{\tau^*} e^{-\delta t}\Big\{U(Y(t)) - \delta V + \mathcal{A}\phi(X(t))\Big\}\,dt \\
&\qquad + \sum_{i=1}^n \int_0^{\tau^*} e^{-\delta t}\big(-(1+\lambda_i)\phi_{x_0} + \phi_{x_i}\big)\,dL_i^c(t) \\
&\qquad + \sum_{i=1}^n \int_0^{\tau^*} e^{-\delta t}\big((1-\mu_i)\phi_{x_0} - \phi_{x_i}\big)\,dM_i^c(t) + \int_0^{\tau^*} e^{-\delta t}\big(-\phi_{x_0} + \beta\phi_y\big)\,dC_t^c \\
&\qquad + \sum_{t\in[0,\tau^*)} e^{-\delta t}\Delta^{L,M,C}\phi(t) + \sum_{i=1}^n \int_0^{\tau^*}\int_{\mathbb{R}\setminus\{0\}} \Delta_i^\eta \phi(t)\,\tilde{N}_i(dt,dz) \\
&\quad \le \phi(x) - \varepsilon e^{-\delta\tau^*} + (1 - e^{-\delta\tau^*}) \le \phi(x) - \varepsilon,
\end{aligned} \tag{6.7}$$

where we have introduced the short-hand notations

$$\Delta_i^\eta \phi(t) = \phi(X_{t-} + \eta_i(x,z)) - \phi(X_{t-})$$

and

$$\begin{aligned}
\Delta^{L,M,C}\phi(t) = \phi\Big(X(t-) + \Big[\Delta C(t) + \sum_{i=1}^n \big[(1+\mu_i)\Delta M_i(t) - (1+\lambda_i)\Delta L_i(t)\big]\Big]e_0 \\
+ \sum_{i=1}^n \big[\Delta L_i(t) - \Delta M_i(t)\big]e_i + \beta\Delta C(t)e_{n+1}\Big) - \phi(X(t-)).
\end{aligned}$$

On the set $\{\tau* \geq \tau_L\} \cap A^c$, we have $\tau = \tau_L$ and calculate as follows

$$
\begin{aligned}
&\int_0^{\tau_L} e^{-\delta t} U(Y_t)\,dt + e^{-\delta\tau_L} V(X_{\tau_L}) \\
&\quad\leq \int_0^{\tau_L} e^{-\delta t} U(Y_t)\,dt + e^{-\delta\tau_L}\phi(X_{\tau_L}) \\
&\quad\leq \phi(x) + \int_0^{\tau_L} e^{-\delta t}\Big\{U(Y(t)) - \delta V + \mathcal{A}\phi(X(t))\Big\}\,dt \\
&\qquad + \sum_{i=1}^n \int_0^{\tau_L} e^{-\delta t}\big(-(1+\lambda_i)\phi_{x_0} + \phi_{x_i}\big)\,dL_i^c(t) \\
&\qquad + \sum_{i=1}^n \int_0^{\tau_L} e^{-\delta t}\big((1-\mu_i)\phi_{x_0} - \phi_{x_i}\big)\,dM_i^c(t) + \int_0^{\tau_L} e^{-\delta t}\big(-\phi_{x_0} + \beta\phi_y\big)\,dC_t^c \\
&\qquad + \sum_{t\in[0,\tau_L]} e^{-\delta t}\Delta^{L,M,C}\phi(t) + \sum_{i=1}^n \int_0^{\tau_L}\int_{\mathbb{R}\setminus\{0\}} \Delta_i^\eta\phi(t)\,\tilde{N}_i(dt,dz) \\
&\quad\leq \phi(x) - \varepsilon e^{-\delta\tau_L} + \varepsilon(1-e^{-\delta\tau_L}) + \sum_{i=1}^n \int_0^{\tau_L}\int_{\mathbb{R}\setminus\{0\}} \Delta_i^\eta\phi(t)\,\tilde{N}_i(dt,dz) \\
&\quad\leq \phi(x) - \varepsilon + \sum_{i=1}^n \int_0^{\tau_L}\int_{\mathbb{R}\setminus\{0\}} \Delta_i^\eta\phi(t)\,\tilde{N}_i(dt,dz).
\end{aligned}
\tag{6.8}
$$

To derive the third inequality in (6.8) from the second, we have assumed that

$$
\begin{aligned}
\Big(X(\tau^*-) + \Big[\Delta C(\tau^*) + \sum_{i=1}^n \big[(1+\mu_i)\Delta M_i(\tau^*) - (1+\lambda_i)\Delta L_i(\tau^*)\big]\Big]e_0 \\
+ \sum_{i=1}^n \big[\Delta L_i(\tau^*) - \Delta M_i(\tau^*)\big]e_i + \beta\Delta C(\tau^*)e_{n+1}\Big) \in \overline{\mathcal{N}_{\mathcal{D}}},
\end{aligned}
$$

so that $\Delta^{C,L,M}\phi(t) \leq 0$. In view of Lemma 6.3, we can make such an assumption without loss of generality. From (6.7) and (6.8), we finally get

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^{\tau} e^{-\delta t} U(Y_t)\,dt + e^{-\delta\tau} V(X_\tau)\Big] \\
&\quad\leq \mathbb{E}\Big[\mathbf{1}_{\tau^*<\tau_L}\Big(\int_0^{\tau^*} e^{-\delta t} U(Y_t)\,dt + e^{-\delta\tau^*} V(X_{\tau^*})\Big)\Big] \\
&\qquad + \mathbb{E}\Big[\mathbf{1}_{\tau^*\geq\tau_L}\Big\{\int_0^{\tau_L} e^{-\delta t} U(Y_t)\,dt + e^{-\delta\tau_L} V(X_{\tau_L})\Big\}\Big] \\
&\quad\leq \mathbb{E}\Big[\mathbf{1}_{\tau^*<\tau_L}\Big\{\phi(x) - \varepsilon + \int_0^{\tau^*}\int_{\mathbb{R}\setminus\{0\}} \Delta_i^\eta\phi\,\tilde{N}(dt,dz)\Big\}\Big] \\
&\qquad + \mathbb{E}\Big[\mathbf{1}_{\tau^*\geq\tau_L}\Big\{\phi(x) - \varepsilon(1-e^{-\delta\tau_L}) + \sum_{i=1}^n \int_0^{\tau_L}\int_{\mathbb{R}\setminus\{0\}} \Delta_i^\eta\phi\,\tilde{N}_i(dt,dz)\Big\}\Big] \\
&\quad\leq \phi(x) - \varepsilon\mathbb{E}\Big[1 - \mathbf{1}_{\tau^*\geq\tau_L}e^{-\delta\tau_L}\Big] + \mathbb{E}\Big[\sum_{i=1}^n \int_0^{\tau}\int_{\mathbb{R}\setminus\{0\}} \Delta_i^\eta\phi\,\tilde{N}_i(dt,dz)\Big] \\
&\quad\leq \phi(x) - \varepsilon\mathbb{E}\Big[1 - e^{-\delta\tau_L}\Big].
\end{aligned}
$$

The proof is now finished after observing that the dynamic programming principle (3.9) gives $V(x) < \phi(x)$, which is a contradiction since $(V-\phi)(x) = 0$. □

Definition 6.1. Let $\mathcal{O} \subset \overline{\mathcal{D}}$. A locally bounded function $v \in LSC(\overline{\mathcal{D}})$ is a *strict* supersolution of (3.10) in $\mathcal{O}$ if and only if $\forall \phi \in C^2_1(\mathcal{O})$ we have:

$$\begin{cases} \text{for each } x \in \mathcal{O} \text{ being a global minimizer relative to } \mathcal{O} \text{ of } v - \phi, \\ \max\Big(\mathcal{F}(x, v, D_x\phi, D_x^2\phi, \mathcal{B}(x,\phi,D_x\phi)), \mathcal{L}(D_x\phi), \mathcal{M}(D_x\phi), \mathcal{C}(D_x\phi)\Big) \leq -\vartheta, \end{cases}$$

for some constant $\vartheta > 0$.

Remark. Notice that Lemmas 4.1 and 4.2 still hold for a strict supersolution.

We demonstrate next that it is possible to construct strict supersolutions of (3.10) in any bounded subset of $\mathcal{D}$.

Lemma 6.5. *For $\gamma \in (0,1)$ such that $\delta > \rho(\gamma)$, let $v \in LSC_\gamma(\overline{\mathcal{D}})$ be a supersolution of (3.10) in $\mathcal{D}$. Then there exists $\overline{\gamma} \in (\gamma, 1)$ such that $\delta > \rho(\overline{\gamma})$ and, for a suitable constant $K > 0$,*

$$w = K + \chi^{\overline{\gamma}} \in C^\infty_{\overline{\gamma}}(\mathcal{D}) \cap C_{\overline{\gamma}}(\overline{\mathcal{D}}), \qquad \chi = \Big(1 + \sum_{i=0}^n x_i + \frac{y}{2\beta}\Big),$$

is a strict supersolution of (3.10) in any bounded set $\mathcal{O} \subset \mathcal{D}$.

Moreover, for any $\theta \in (0,1]$, the function

$$v^\theta = (1-\theta)v + \theta w \in LSC_{\overline{\gamma}}(\overline{\mathcal{D}})$$

is a strict supersolution of (3.10) in any bounded set $\mathcal{O} \subset \mathcal{D}$.

Proof. Throughout this proof, we let $\mathcal{O}$ be a fixed (but arbitrary) bounded subset of $\mathcal{D}$. Observe first that the quantity $\rho(\gamma)$ in (3.7) is continuously increasing in γ. Hence, we can find $\overline{\gamma} \in (\gamma, 1)$ such that (3.7) still holds for $\overline{\gamma}$, i.e., $\delta > \rho(\overline{\gamma})$. We first claim that

$$\max\Big(F(x, w, D_x w, D_x^2 w, \mathcal{B}(x, w, D_x w)), \mathcal{L}(D_x w), \mathcal{M}(D_x w), \mathcal{C}(D_x w)\Big) \leq -f, \tag{6.9}$$

for some $f \in C(\overline{\mathcal{D}})$ that is strictly positive in $\mathcal{O}$. By direct calculations, we observe first that

$$\mathcal{L}(D_x w) = -\overline{\gamma} \max_{1 \leq i \leq n} \lambda_i \, \chi^{\overline{\gamma}-1}, \quad \mathcal{M}(D_x w) = -\overline{\gamma} \max_{1 \leq i \leq n} \mu_i \, \chi^{\overline{\gamma}-1}, \quad \mathcal{C}(D_x w) = -\overline{\gamma}\,\frac{1}{2}\,\chi^{\overline{\gamma}-1}.$$

Consider now $\mathcal{A}w$. Since w obviously is concave, $\mathcal{B}(x, w, D_x w) \leq 0$. This can be easily seen by Taylor expanding $w(x)$ up to second order in each argument x_i ($i = 1, \dots, n$):

$$\begin{aligned} \int_{\mathbb{R}\setminus\{0\}} &\Big(w(x + \eta_i(z)x_i e_i) - w(x) - \eta_i(z) x_i w_{x_i}(x)\Big)\, n_i(dz) \\ &\leq \overline{\gamma}(\overline{\gamma}-1)\chi^{\overline{\gamma}-2}(\tilde{x}) \int_{\mathbb{R}\setminus\{0\}} \eta_i(z)^2 \, n_i(dz) \leq 0, \end{aligned}$$

for some suitable $\tilde{x}$. Hence $\mathcal{A}w(x) \leq \mathcal{A}_0 w(x)$, where $\mathcal{A}_0 = \mathcal{A} - \mathcal{B}$, i.e., $\mathcal{A}_0$ is the second order differential operator part of $\mathcal{A}$. A direct calculation yields

$$\begin{aligned} &-\delta w(x) + \mathcal{A}_0 w(x) \\ &= -\delta K + \chi^{\overline{\gamma}}(x)\Big\{-\delta - \overline{\gamma}\frac{y}{\chi(x)} + r\overline{\gamma}\frac{x_0}{\chi(x)} + \overline{\gamma}\sum_{i=1}^n a_i \frac{x_i}{\chi(x)} + \frac{1}{2}\overline{\gamma}(\overline{\gamma}-1)\sum_{i=1}^n \sigma_i^2\Big(\frac{x_i}{\chi(x)}\Big)^2\Big\} \\ &\leq -\delta K - \chi^{\overline{\gamma}}(x)\Big\{\delta - \overline{\gamma}\Big[r\frac{x_0}{\chi(x)} + \sum_{i=1}^n a_i \frac{x_i}{\chi(x)} - \frac{1}{2}(1-\overline{\gamma})\sum_{i=1}^n \sigma_i^2\Big(\frac{x_i}{\chi(x)}\Big)^2\Big]\Big\}, \end{aligned}$$

where we have used that $y/\chi(x) \geq 0$ in the last inequality. Now, define $\pi_j = x_j/\chi(x)$ for $j = 0, 1, \dots, n$. We have by definition of $\chi(x)$ that $\pi_j \in [0,1]$ for all j. Moreover,

$$\sum_{j=0}^n \pi_j = \frac{x_0 + \sum_{i=1}^n x_i}{x_0 + \sum_{i=1}^n x_i + \frac{y}{2\beta}} < 1.$$

Define the second order polynomial

$$p(\pi_0,\dots,\pi_n) = r\pi_0 + \sum_{i=1}^{n} a_i\pi_i - \frac{1}{2}(1-\overline{\gamma})\sum_{i=1}^{n}\sigma_i^2\pi_i^2. \tag{6.10}$$

This function has a finite positive maximum and we get for some positive constant c,

$$-\delta w(x) + \mathcal{A}_0 w(x) \le -\delta K - c\chi^{\overline{\gamma}}(x),$$

provided $\delta > \overline{\gamma}\sup p(\pi_0,\dots,\pi_n)$. The sup is taken over all $\pi_j \in [0,1]$ such that $\sum_{j=0}^{n}\pi_j \le 1$. The maximal value of p over the set $\left\{\pi_j \in [0,1], \sum_{j=0}^{n}\pi_j \le 1\right\}$ is dominated by the maximum over $\left\{\sum_{j=0}^{n}\pi_j \le 1\right\}$. Hence, by straightforward optimization, we see that p has a maximal value over the set $\left\{\pi_j \in [0,1], \sum_{j=0}^{n}\pi_j \le 1\right\}$ dominated by $\rho(\overline{\gamma})/\overline{\gamma}$. Therefore, under the assumption $\delta > \rho(\overline{\gamma})$, we get

$$\mathcal{A}_0 w(x) \le -c\chi(x)^{\overline{\gamma}} \text{ and thus } \mathcal{A}w(x) \le -c\chi(x)^{\overline{\gamma}}.$$

Summing up,

$$\begin{aligned}&\mathcal{F}(x, w, D_x w, D_x^2 w, \mathcal{B}(x, w, D_x w))\\ &\qquad = U(y) - \delta w(x) + \mathcal{A}w(x) \le U(y) - \delta K - c\chi^{\overline{\gamma}}(x) \le -1\end{aligned}$$

by choosing, e.g., $\delta K = 1 + \sup_{\overline{\mathcal{D}}}\left[U(y) - c\chi^{\overline{\gamma}}(x)\right]$. Since c is positive and U has sublinear growth of order $\gamma < \overline{\gamma}$ by assumption, our claim (6.9) holds provided we set

$$f(x) = \min\Big(1, \overline{\gamma}\min_{1\le i\le n}\lambda_i\,\chi^{\overline{\gamma}-1}(x), \,\overline{\gamma}\min_{1\le i\le n}\mu_i\,\chi^{\overline{\gamma}-1}(x), \,\overline{\gamma}\frac{1}{2}\,\chi^{\overline{\gamma}-1}(x)\Big).$$

Next, we claim that v^θ is a strict supersolution in $\mathcal{O}$. Note that for any $\phi \in C_1^2(\overline{\mathcal{D}})$, $x \in \mathcal{O}$ is a global minimizer relative to $\mathcal{O}$ of $v - \phi$ if and only if x is a global minimizer relative to $\mathcal{O}$ of $v^\theta - \phi^\theta$ with $\phi^\theta = (1-\theta)\phi + \theta w$. Since v is a supersolution and by linearity of the differential operators, we get

$$\begin{aligned}
\mathcal{L}_i(D_x\phi^\theta) &= (1-\theta)\mathcal{L}_i(D_x\phi) + \theta\mathcal{L}_i(D_x w) \le -\theta\overline{\gamma}\lambda_i\chi^{\overline{\gamma}-1} \le -\theta f,\\
\mathcal{M}_i(D_x\phi^\theta) &= (1-\theta)\mathcal{M}_i(D_x\phi) + \theta\mathcal{M}_i(D_x w) \le -\theta\overline{\gamma}\mu_i\chi^{\overline{\gamma}-1} \le -\theta f,\\
\mathcal{C}(D_x\phi^\theta) &= (1-\theta)\mathcal{C}(D_x\phi) + \theta\mathcal{C}(D_x w) \le -\theta\overline{\gamma}\frac{1}{2}\chi^{\overline{\gamma}-1} \le -\theta f,
\end{aligned}$$

and

$$\begin{aligned}&\mathcal{F}(x, v^\theta, D_x\phi^\theta, D_x^2\phi^\theta, \mathcal{B}(x, v^\theta, D_x\phi^\theta)\\ &\quad = (1-\theta)\mathcal{F}(x, v, D_x\phi, D_x^2\phi, \mathcal{B}(x, v, D_x\phi)) + \theta\mathcal{F}(x, w, D_x w, D_x^2 w, \mathcal{B}(x, w, D_x w)) \le -\theta.\end{aligned}$$

Hence

$$\max\Big(\mathcal{F}(x, v^\theta, D_x\phi^\theta, D_x^2\phi^\theta, \mathcal{B}(x, v^\theta, D_x\phi^\theta), \mathcal{L}(D_x\phi^\theta), \mathcal{M}(D_x\phi^\theta), \mathcal{C}(D_x\phi^\theta)\Big) \le -\theta f.$$

This concludes the proof of the lemma.

□

Adopting the uniqueness machinery for viscosity solutions of second order partial differential equations (see, e.g., [13, 19]), we now prove a strong comparison principle for the state constraint problem for the integro-differential variational inequality (3.10). This comparison result ensures that the characterization in Theorem 6.1 is unique.

As already mentioned, our investment-consumption model combines several difficulties such as gradient constraints, a state constraint boundary condition, as well as a highly singular non-local operator. Consequently, the existing comparison results [43, 44, 3, 5, 34, 8] for problems involving an integro-differential operator do not apply (directly!) in our context. Having said this, we do not hesitate to point that the comparison result stated and proved below is nevertheless inspired by these results and in particular by the one in [8].

Let us be a bit more precise about the proof of the comparison principle. First of all, we handle the gradient constraint by producing strict supersolutions (Lemma 6.5) that are close to the supersolution being compared. This approach is inspired by Ishii and Lions [27] and has been used in [2, 8, 9, 16, 45, 46] for singular stochastic control problems. To handle the state constraint boundary condition we follow Soner [42, 43] when building a test function such that the minimum associated with the supersolution cannot be on the boundary. Finally, let us mention that when dealing with unbounded domains, it is well known that one has to specify the asymptotic behaviour of the functions being compared. However, here we can take advantage of our choice of strict supersolutions and "localize" the comparison proof to a bounded domain. This idea was also exploited in Alvarez [2] and Benth, Karlsen, and Reikvam [8].

Theorem 6.6. *Let $\gamma' \in (0,1)$ be such that $\delta > \rho(\gamma')$. Assume $\underline{v} \in USC_{\gamma'}(\overline{\mathcal{D}})$ is a subsolution of* (3.10) *in $\overline{\mathcal{D}}$ and $\overline{v} \in LSC_{\gamma'}(\overline{\mathcal{D}})$ is a supersolution of* (3.10) *in $\mathcal{D}$. Then*

$$\underline{v} \leq V \leq \overline{v} \text{ in } \overline{\mathcal{D}}, \tag{6.11}$$

where V is the value function (3.8)*. In particular, the dynamic programming equation* (3.10) *admits at most one constrained viscosity solution in the class of sublinearly growing solutions.*

Proof. It is sufficient to prove (6.11) under the assumption that the subsolution $\underline{v}$ or the supersolution $\overline{v}$ is continuous on $\overline{\mathcal{D}}$. Such a comparison result imply that $\underline{v} \leq V$ for $\underline{v} \in USC_{\gamma'}(\overline{\mathcal{D}})$ as well as $V \leq \overline{v}$ for $\overline{v} \in LSC_{\gamma'}(\overline{\mathcal{D}})$, and we can immediately conclude that the theorem holds. In what follows, we assume for definiteness that the supersolution $\overline{v}$ is continuous on $\overline{\mathcal{D}}$. One can easily modify the proof below so that it works under the assumption that the subsolution is continuous instead of the supersolution (see also the remark given after the proof).

By Lemma 6.5, there exist $\overline{\gamma} \in (\gamma', 1)$ such that $\delta > \rho(\overline{\gamma})$ and a function $\overline{v}^\theta \in LSC_{\overline{\gamma}}(\overline{\mathcal{D}})$ which is a strict supersolution of (3.10) in any bounded subset of $\mathcal{D}$. Moreover, $\overline{v}^\theta \to \overline{v}$ as $\theta \to 0+$. Instead of comparing $\underline{v}$ and $\overline{v}$, we will compare $\underline{v}$ and $\overline{v}^\theta$. Sending $\theta \to 0+$, we obtain the desired result $\underline{v} \leq \overline{v}$ in $\overline{\mathcal{D}}$. Observe that

$$\underline{v}(x) - \overline{v}^\theta(x) \to -\infty \text{ as } x \to \infty,$$

which implies that $R > 0$ can be chosen so large that

$$\underline{v} \leq \overline{v}^\theta \text{ in } \left\{ x \in \overline{\mathcal{D}} \,\middle|\, x_0, x_1, \dots, x_n, y \geq R \right\}. \tag{6.12}$$

For later use, let $\Gamma_{\mathrm{SC}} = [0,R)^N \backslash (0,R)^N$ denote the state constraint boundary of $\overline{\mathcal{D}}$ restricted strictly by R. In view of (6.12), we will "localize" our attention to the bounded domain

$$\mathcal{K} = \left\{ x \in \overline{\mathcal{D}} \,\middle|\, 0 < x_0, y < R,\ 0 < x_i < R + R\eta_i(R,1),\ i = 1, \dots, n \right\} \tag{6.13}$$

and prove that $\underline{v} \leq \overline{v}^\theta$ in $\overline{\mathcal{K}}$, which in turn follows if we can prove that $\underline{v} \leq \overline{v}^\theta$ in $[0,R)^N$. To this end, we assume to the contrary that

$$M := \max_{\overline{\mathcal{K}}} (\underline{v} - \overline{v}^\theta) = (\underline{v} - \overline{v}^\theta)(x_{\max}) > 0 \text{ for some } x_{\max} \in [0,R)^N. \tag{6.14}$$

The maximum point $x_{\max}$ exists in view of the compactness of $\overline{\mathcal{K}}$ and the upper semicontinuity of $\underline{v} - \overline{v}^\theta$. For later use, notice that

$$(\underline{v} - \overline{v}^\theta)(x) \leq M \text{ for all } x \in \overline{\mathcal{D}}. \tag{6.15}$$

To overcome the lack of regularity of $\underline{v}, \overline{v}^\theta$, we employ the classical "doubling of variables" device [14, 12, 13] and approximate the maximum in (6.14) by the (penalized) maximum of the function

$$\Phi(x, \tilde{x}) = \underline{v}(x) - \overline{v}^\theta(\tilde{x}) - \varphi(x, \tilde{x}), \qquad (x, \tilde{x}) \in \overline{\mathcal{K}} \times \overline{\mathcal{K}},$$

where $\varphi(x, \tilde{x})$ is a properly chosen penalization term. For some constants $t_0, d > 0$, let $\zeta : \overline{\mathcal{K}} \to \mathbb{R}^N$ be a uniformly continuous map satisfying

$$\mathcal{N}(x + t\zeta(x), td) \subset \mathcal{K} \text{ for all } x \in \overline{\mathcal{K}} \text{ and } t \in (0, t_0]. \tag{6.16}$$

Since $\partial\mathcal{K}$ is piecewise linear, such a map can certainly be found. For our problem, a suitable choice of penalization term takes the form

$$\varphi(x,\tilde{x}) = \begin{cases} |\alpha(x-\tilde{x})+\varepsilon\zeta(x_{\max})|^2+\varepsilon|x-x_{\max}|^2, & \text{if } x_{\max}\in\Gamma_{\mathrm{SC}} \text{ (Case I)},\\ \frac{\alpha}{2}|x-\tilde{x}|^2, & \text{if } x_{\max}\in(0,R)^N \text{ (Case II)}, \end{cases} \tag{6.17}$$

where $\alpha>1$ and $0<\varepsilon<1$ are parameters that eventually will be sent, respectively, to ∞ and 0.

The next step in the classical viscosity solution technique [14, 13] is to look at maxima of the function Φ. Letting

$$M_\alpha = \max_{\overline{\mathcal{K}}\times\overline{\mathcal{K}}}\Phi(x,\tilde{x}),$$

we have $M_\alpha\geq M>0$ for any $\alpha>1$ (and sufficiently small $\varepsilon>0$ in Case I). Note that Φ is upper semicontinuous on $\overline{\mathcal{K}}\times\overline{\mathcal{K}}$. The compactness of $\overline{\mathcal{K}}$ implies that there exists $(x_\alpha,\tilde{x}_\alpha)\in\overline{\mathcal{K}}\times\overline{\mathcal{K}}$ such that $M_\alpha=\Phi(x_\alpha,\tilde{x}_\alpha)$. Here, x_α denotes the vector $\big(x_{\alpha0},x_{\alpha1},\dots,x_{\alpha n},y_\alpha\big)$ and similarly for $\tilde{x}_\alpha$. Moreover, $(x_\alpha,\tilde{x}_\alpha)$ converges along a subsequence to some $(x,\tilde{x})\in\overline{\mathcal{K}}\times\overline{\mathcal{K}}$. In what follows, we consider Case I and Case II separately, starting with Case I.

Case I: Exploiting that the supersolution $\overline{v}^\theta$ is assumed to be continuous, it is fairly standard to show that the penalized maxima $(x_\alpha,\tilde{x}_\alpha)$ in Case I satisfy (see, e.g., [13, 19])

$$\begin{cases} \text{(I.i) } x_\alpha,\tilde{x}_\alpha\to x_{\max} \text{ as } \alpha\to\infty,\\ \text{(I.ii) } \alpha(x_\alpha-\tilde{x}_\alpha)+\varepsilon\zeta(x_{\max})\to 0 \text{ as } \alpha\to\infty,\\ \text{(I.iii) } \big(\underline{v}(x_\alpha)-\overline{v}^\theta(\tilde{x}_\alpha)\big)\to M \text{ as } \alpha\to\infty,\\ \text{(I.iv) } M_\alpha\to M \text{ as } \alpha\to\infty. \end{cases}$$

For the sake of completeness, let us prove (I.i)-(I.iv). To this end, let $\{\alpha_j\}$ be any sequence of numbers (greater than one) such that $\alpha:=\alpha_j\to\infty$ as $j\to\infty$. Moreover, assume that $x_\alpha\to x$ and $\tilde{x}_\alpha\to\tilde{x}$ as $j\to\infty$. We next note that the inequality $\Phi(x_\alpha,y_\alpha)\geq\Phi\big(x_{\max},x_{\max}+\frac{\varepsilon}{\alpha}\zeta(x_{\max})\big)$ reads

$$\begin{aligned} &|\alpha(x_\alpha-\tilde{x}_\alpha)+\varepsilon\zeta(x_{\max})|^2+\varepsilon|x_\alpha-x_{\max}|^2\\ &\quad\leq\underline{v}(x_\alpha)-\overline{v}^\theta(\tilde{x}_\alpha)-\big(\underline{v}-\overline{v}^\theta\big)(x_{\max})+\overline{v}^\theta(x_{\max}+\tfrac{\varepsilon}{\alpha}\zeta(x_{\max}))-\overline{v}^\theta(x_{\max}). \end{aligned} \tag{6.18}$$

Since $\underline{v},\overline{v}^\theta$ are bounded on $\overline{\mathcal{K}}$, $|\alpha(x_\alpha-\tilde{x}_\alpha)|$ is bounded uniformly in α and hence $x_\alpha-\tilde{x}_\alpha\to 0$ as $j\to\infty$. This gives $\tilde{x}_\alpha\to\tilde{x}\equiv x$ as $j\to\infty$ and hence $\lim_{j\to\infty}\big(\underline{v}(x_\alpha)-\overline{v}^\theta(\tilde{x}_\alpha)\big)\leq(\underline{v}-\overline{v}^\theta)(x)\leq M$. Sending $j\to\infty$ in (6.18) and using uniform continuity of $\overline{v}^\theta$ on $\overline{\mathcal{K}}$, we conclude that $x_\alpha,\tilde{x}_\alpha\to x_{\max}$ and $\alpha(x_\alpha-\tilde{x}_\alpha)+\varepsilon\zeta(x_{\max})\to 0$. From this and $M_\alpha\geq M$, we get

$$\begin{aligned} 0&=\lim_{j\to\infty}\Big\{|\alpha(x_\alpha-\tilde{x}_\alpha)+\varepsilon\zeta(x_{\max})|^2+\varepsilon|x_\alpha-x_{\max}|^2\Big\}\\ &=\lim_{j\to\infty}\Big\{v(x_\alpha)-\overline{v}^\theta(\tilde{x}_\alpha)-M_\alpha\Big\}\leq\lim_{j\to\infty}\big(v(x_\alpha)-\overline{v}^\theta(\tilde{x}_\alpha)\big)-M\leq 0. \end{aligned}$$

Therefore we get $\lim_{j\to\infty}\big(v(x_\alpha)-\overline{v}^\theta(\tilde{x}_\alpha)\big)=M$ and $M_\alpha\to M$ as $j\to\infty$. Since the sequence $\{\alpha_j\}$ was arbitrary, we can finally conclude that (I.i)-(I.iv) hold.

In view of (I.ii) and also the uniform continuity of ζ,

$$\tilde{x}_\alpha=x_\alpha+\frac{\varepsilon}{\alpha}\zeta(x_{\max})+\frac{1}{\alpha}o\Big(\frac{1}{\alpha}\Big)=x_\alpha+\frac{\varepsilon}{\alpha}\zeta(x_\alpha)+\frac{1}{\alpha}o\Big(\frac{1}{\alpha}\Big)$$

and we thus use (6.16) to get $\tilde{x}_\alpha\in\mathcal{K}$ for α large enough. In fact, we must have

$$\tilde{x}_\alpha\in(0,R)^N \text{ for } \alpha \text{ large enough as well as } x_\alpha\in[0,R)^N.$$

Using Theorem 4.3 with the penalization term $\varphi(x,\tilde{x})$ defined in (6.17) (Case I), $u_1=\underline{v}$, $u_2=\overline{v}^\theta$, and $\mathcal{O}=\overline{\mathcal{K}}$, we conclude that there exist matrices $A,\tilde{A}\in\mathbb{S}^N$ such that

$$\begin{aligned} &(p,A)\in\overline{J}^{2,+}_{\overline{\mathcal{K}}}\underline{v}(x_\alpha), && p=D_x\varphi(x_\alpha,\tilde{x}_\alpha)=2\alpha[\alpha(x_\alpha-\tilde{x}_\alpha)+\varepsilon\zeta(x_{\max})]+2\varepsilon(x_\alpha-x_{\max}),\\ &(\tilde{p},\tilde{A})\in\overline{J}^{2,-}_{\overline{\mathcal{K}}}\overline{v}^\theta(\tilde{x}_\alpha), && \tilde{p}=-D_{\tilde{x}}\varphi(x_\alpha,\tilde{x}_\alpha)=2\alpha[\alpha(x_\alpha-\tilde{x}_\alpha)+\varepsilon\zeta(x_{\max})]. \end{aligned}$$

Since $\overline{v}^\theta$ is a strict supersolution in $(0,R)^N$, there exists according to Lemma 4.2 a C^2 function ψ such that

$$\mathcal{F}(\tilde{x}_\alpha, \overline{v}^\theta, \tilde{p}, \tilde{A}, \mathcal{B}_\kappa(\tilde{x}_\alpha, \psi, D_{\tilde{x}}\psi), \mathcal{B}^\kappa(\tilde{x}_\alpha, \overline{v}^\theta, \tilde{p})) \leq -\vartheta, \tag{6.19}$$

as well as $\mathcal{L}(\tilde{p}), \mathcal{M}(\tilde{p}), \mathcal{C}(\tilde{p}) \leq -\vartheta$, for some constant $\vartheta > 0$. Assume that $\mathcal{L}(p) \geq 0$. Since $\tilde{p} - p \to 0$ as $\alpha \to \infty$, we get the contradiction $-\vartheta > \mathcal{L}(\tilde{p}) - \mathcal{L}(p) \to 0$ as $\alpha \to \infty$, thereby proving that $\mathcal{L}(p) < 0$. We can prove exactly in the same way that $\mathcal{M}(p), \mathcal{C}(p) < 0$. Then since $\underline{v}$ is a subsolution in $[0,R)^N$, there exists according to Lemma 4.2 a C^2 function ϕ such that

$$\mathcal{F}(x_\alpha, \underline{v}, p, A, \mathcal{B}_\kappa(x_\alpha, \phi, D_x\phi), \mathcal{B}^\kappa(x_\alpha, \underline{v}, p)) \geq 0. \tag{6.20}$$

Using (6.19) and (6.20), we get

(6.21)
$$\begin{aligned}
\vartheta &\leq \mathcal{F}(x_\alpha, \underline{v}, p, A, \mathcal{B}_\kappa(x_\alpha, \phi, D_x\phi), \mathcal{B}^\kappa(x_\alpha, \underline{v}, p)) - \mathcal{F}(\tilde{x}_\alpha, \overline{v}^\theta, \tilde{p}, \tilde{A}, \mathcal{B}_\kappa(\tilde{x}_\alpha, \psi, D_x\psi), \mathcal{B}^\kappa(\tilde{x}_\alpha, \overline{v}^\theta, \tilde{p})) \\
&= \big[U(x_\alpha) - U(\tilde{x}_\alpha)\big] - \delta\big[\underline{v}(x_\alpha) - \overline{v}^\theta(\tilde{x}_\alpha)\big] + \big[\langle b(x_\alpha), p\rangle - \langle b(\tilde{x}_\alpha), \tilde{p}\rangle\big] \\
&\quad + \frac{1}{2}\big[\mathrm{Tr}\big(\sigma(x_\alpha)^2 A\big) - \mathrm{Tr}\big(\sigma(\tilde{x}_\alpha)^2 \tilde{A}\big)\big] \\
&\quad + \big[\mathcal{B}_\kappa(x_\alpha, \phi, D_x\phi) - \mathcal{B}_\kappa(\tilde{x}_\alpha, \psi, D_{\tilde{x}}\psi)\big] + \big[\mathcal{B}^\kappa(x_\alpha, \underline{v}, p) - \mathcal{B}^\kappa(\tilde{x}_\alpha, \overline{v}^\theta, \tilde{p})\big].
\end{aligned}$$

Let us now estimate the various terms in (6.21). First, observe that we have

$$U(x_\alpha) - U(\tilde{x}_\alpha) \to 0, \qquad \langle b(x_\alpha), p\rangle - \langle b(\tilde{x}_\alpha), \tilde{p}\rangle \to 0, \qquad \text{as } \alpha \to \infty. \tag{6.22}$$

Second, an easy calculation reveals that

$$\begin{cases}
D^2\varphi(x_\alpha, \tilde{x}_\alpha) = 2\alpha^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\varepsilon \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \\
\big(D^2\varphi(x_\alpha, \tilde{x}_\alpha)\big)^2 = 8\alpha^4 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 8\alpha^2\varepsilon \begin{pmatrix} I & 0 \\ -I & 0 \end{pmatrix} + 4\varepsilon^2 \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.
\end{cases} \tag{6.23}$$

Therefore, by (4.8) and (6.23), we have for all $\xi, \tilde{\xi} \in \mathbb{R}^N$ the following estimate

$$\big\langle A\xi, \xi\big\rangle - \big\langle \tilde{A}\tilde{\xi}, \tilde{\xi}\big\rangle = \Big\langle \begin{pmatrix} A & 0 \\ 0 & -\tilde{A} \end{pmatrix} \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix}, \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} \Big\rangle \leq 2\alpha^2|\xi - \tilde{\xi}|^2 + 2\varepsilon|\xi|^2 + K\nu, \tag{6.24}$$

where $K = K(\alpha, \varepsilon, \xi, \tilde{\xi}) = \big[8\alpha^4|\xi - \tilde{\xi}|^2 + 8\alpha^2\varepsilon(|\xi|^2 - \xi\cdot\tilde{\xi}) + 4\varepsilon^2|\xi|^2\big]$.

Recall that $\sigma_{ii}(x) = \sigma_i x_i$ for $i = 1, \dots, n$ and $\sigma_{ii}(x) \equiv 0$ for $i = 0, n+1$. Using estimate (6.24) with $\xi_i = \sigma(x_\alpha)e_i$ and $\tilde{\xi}_i = \sigma(\tilde{x}_\alpha)e_i$, we obtain

$$\begin{aligned}
&\mathrm{Tr}\big(\sigma(x_\alpha)^2 A\big) - \mathrm{Tr}\big(\sigma(\tilde{x}_\alpha)^2 \tilde{A}\big) \\
&\quad = \sum_{i=0}^{n+1}\Big(\big\langle A\xi_i, \xi_i\big\rangle - \big\langle \tilde{A}\tilde{\xi}_i, \tilde{\xi}_i\big\rangle\Big) = O\big(|\alpha(x_\alpha - \tilde{x}_\alpha)|^2\big) + O(\varepsilon) + K(\alpha, \varepsilon)\nu,
\end{aligned} \tag{6.25}$$

for some $K(\alpha, \varepsilon) > 0$. Since $\nu > 0$ was arbitrary and $\alpha(x_\alpha - \tilde{x}_\alpha) \to -\varepsilon\zeta(x_{\max})$ as $\alpha \to \infty$, we can conclude from (6.25) that the following estimate holds

$$\lim_{\varepsilon \to 0} \lim_{\alpha \to \infty} \Big(\mathrm{Tr}\big(\sigma(x_\alpha)^2 A\big) - \mathrm{Tr}\big(\sigma(\tilde{x}_\alpha)^2 \tilde{A}\big)\Big) \leq 0. \tag{6.26}$$

Finally, let us estimate the more difficult (non-standard) integral terms in (6.21). First, (4.2) implies that

$$\mathcal{B}_\kappa(x_\alpha, \phi, D_x\phi), \mathcal{B}_\kappa(\tilde{x}_\alpha, \psi, D_{\tilde{x}}\psi) \to 0 \text{ as } \kappa \to 0 \text{ (for each fixed } \alpha < \infty). \tag{6.27}$$

Second, we write

$$\mathcal{B}^\kappa(x_\alpha, \underline{v}, p) - \mathcal{B}^\kappa(\tilde{x}_\alpha, \overline{v}^\theta, \tilde{p}) = I_1 + I_2, \tag{6.28}$$

where, for $A_1 = \{\kappa < |z| < 1\}$ and $A_2 = \{|z| \geq 1\}$,

$$I_\ell = \sum_{i=1}^{n} \int_{A_\ell} \Big(\big[\underline{v}(x_\alpha + \eta_i(x_\alpha, z)) - \overline{v}^\theta(\tilde{x}_\alpha + \eta_i(\tilde{x}_\alpha, z))\big] - \big[\underline{v}(x_\alpha) - \overline{v}^\theta(\tilde{x}_\alpha)\big] - \big[\langle \eta_i(x_\alpha, z), p\rangle - \langle \eta_i(\tilde{x}_\alpha, z), \tilde{p}\rangle\big] \Big) \, n_i(dz), \qquad \ell = 1, 2. \tag{6.29}$$

We consider first the term I_2. Since $\alpha(x_\alpha - \tilde{x}_\alpha)$ remains bounded as $\alpha \to \infty$,

$$\begin{aligned} &\big|\langle \eta_i(x_\alpha, z), p\rangle - \langle \eta_i(\tilde{x}_\alpha, z), \tilde{p}\rangle\big| \\ &\quad \leq \big|\eta_i(z)(x_{\alpha i} - \tilde{x}_{\alpha i})\big(2\alpha[\alpha(x_{\alpha i} - \tilde{x}_{\alpha i}) + \varepsilon\zeta_i(x_{\max})] + 2\varepsilon x_{\alpha i}(x_{\alpha i} - x_{\max i})\big)\big| \\ &\qquad + \big|\eta_i(z)\tilde{x}_{\alpha i} 2\varepsilon(x_{\alpha i} - x_{\max i})\big| = o\Big(\frac{1}{\alpha}\Big)|\eta_i(z)|, \quad i = 1, \ldots, n, \end{aligned} \tag{6.30}$$

where we have also taken into account (I.i) and (I.ii). Notice that

$$|(x_\alpha + \eta_i(x_\alpha, z)) - (\tilde{x}_\alpha + \eta_i(\tilde{x}_\alpha, z))| \leq |x_\alpha - \tilde{x}_\alpha| + |x_{\alpha i} - \tilde{x}_{\alpha i}|\,|\eta_i(z)| \to 0 \text{ as } \alpha \to \infty.$$

Using (I.iv), (I.i), continuity of $\overline{v}^\theta$, (6.30), (3.4), and Lebesgue's dominated convergence theorem, we get

$$I_2 \leq \sum_{i=1}^{n} \int_{|z|\geq 1} \Big(M + \overline{v}^\theta(x_\alpha + \eta_i(x_\alpha, z)) - \overline{v}^\theta(\tilde{x}_\alpha + \eta_i(\tilde{x}_\alpha, z)) - M_\alpha - \big[\langle \eta_i(x_\alpha, z), p\rangle - \langle \eta_i(\tilde{x}_\alpha, z), \tilde{p}\rangle\big] \Big) \, n_i(dz) \to 0 \text{ as } \alpha \to \infty. \tag{6.31}$$

Next we estimate I_1. To this end, recall that $x_\alpha, \tilde{x}_\alpha \in [0, R)^N$ for α large enough. We conclude that $\big(x_\alpha + \eta_i(x_\alpha, z), \tilde{x}_\alpha + \eta_i(\tilde{x}_\alpha, z)\big) \in \mathcal{K} \times \mathcal{K}$ for $z \in (-1, 1)$ and thus

$$\Phi\big(x_\alpha + \eta_i(x_\alpha, z), \tilde{x}_\alpha + \eta_i(\tilde{x}_\alpha, z)\big) - \Phi(x_\alpha, \tilde{x}_\alpha) \leq 0. \tag{6.32}$$

A calculation reveals that the ith integrand of I_1 equals

$$\Phi\big(x_\alpha + \eta_i(x_\alpha, z), \tilde{x}_\alpha + \eta_i(\tilde{x}_\alpha, z)\big) - \Phi(x_\alpha, \tilde{x}_\alpha) + \Big([\alpha(x_{\alpha i} - \tilde{x}_{\alpha i})]^2 + \varepsilon x_{\alpha i}^2\Big)\eta_i(z)^2,$$

which, thanks to (6.32), is less than or equal to $\Big([\alpha(x_{\alpha i} - \tilde{x}_{\alpha i})]^2 + \varepsilon x_{\alpha i}^2\Big)\eta_i(z)^2$. Hence, by summing over i we get

$$I_1 \leq \sum_{i=1}^{n} \Big([\alpha(x_{\alpha i} - \tilde{x}_{\alpha i})]^2 + \varepsilon x_{\alpha i}^2\Big) \int_{\kappa < |z| < 1} \eta_i(z)^2 \, n_i(dz).$$

Note that the integral on the right hand side is convergent due to assumption (3.3). Recalling that $\alpha(x_\alpha - \tilde{x}_\alpha) \to -\varepsilon\zeta(x_{\max})$ as $\alpha \to \infty$, we conclude that

$$\lim_{\varepsilon \to 0} \lim_{\alpha \to \infty} I_1 \leq 0.$$

Summing up, we have

$$\lim_{\varepsilon \to 0} \lim_{\alpha \to \infty} \big[\mathcal{B}^\kappa(x_\alpha, \underline{v}, p) - \mathcal{B}^\kappa(\tilde{x}_\alpha, \overline{v}^\theta, \tilde{p})\big] \leq 0. \tag{6.33}$$

Finally, in view of the estimates derived above and (I.iii), we can send (in that order) $\alpha \to \infty$, $\varepsilon \to 0$, and $\kappa \to 0$ in (6.21) to obtain the desired contradiction $M \leq -\vartheta/\delta < 0$.

Case II: Let us now consider the case $x_{\max} \in (0, R)^N$. First, we note that the inequality $\Phi(x_\alpha, x_\alpha) + \Phi(\tilde{x}_\alpha, \tilde{x}_\alpha) \leq 2\Phi(x_\alpha, \tilde{x}_\alpha)$ implies

$$\alpha|x_\alpha - \tilde{x}_\alpha|^2 \leq \underline{v}(x_\alpha) - \underline{v}(\tilde{x}_\alpha) + \overline{v}^\theta(x_\alpha) - \overline{v}^\theta(\tilde{x}_\alpha).$$

Similar to Case I, one can easily deduce from this inequality that the penalized maxima $(x_\alpha, \tilde{x}_\alpha)$ satisfy (see, e.g., [13])

$$\begin{cases} \text{(II.i) } x_\alpha - \tilde{x}_\alpha \to 0 \text{ as } \alpha \to \infty, \\ \text{(II.ii) } \alpha|x_\alpha - \tilde{x}_\alpha|^2 \to 0 \text{ as } \alpha \to \infty, \\ \text{(II.iii) } \big(\underline{v}(x_\alpha) - \overline{v}^\theta(\tilde{x}_\alpha)\big) \to M \text{ as } \alpha \to \infty, \\ \text{(II.iv) } M_\alpha \to M \text{ as } \alpha \to \infty. \end{cases}$$

Thanks to Case I, $\underline{v} \le \overline{v}^\theta$ on $\partial\big\{(0,R)^N\big\}$ and we conclude that any limit point of $(x_\alpha, \tilde{x}_\alpha)$ belongs to $(0,R)^N$. Hence for α large enough,

$$x_\alpha, \tilde{x}_\alpha \in (0,R)^N. \tag{6.34}$$

Using Theorem 4.3 with the penalization term $\varphi(x,\tilde{x})$ defined in (6.17) (Case II), $u_1 = \underline{v}$, $u_2 = \overline{v}^\theta$, and $\mathcal{O} = \overline{\mathcal{K}}$, we conclude that there exist $A, \tilde{A} \in \mathbb{S}^N$ such that

$$\begin{aligned} &(p, A) \in \overline{J}^{2,+}_{\overline{\mathcal{K}}}\underline{v}(x_\alpha), && p = D_x\varphi(x_\alpha, \tilde{x}_\alpha) = \alpha(x_\alpha - \tilde{x}_\alpha), \\ &(\tilde{p}, \tilde{A}) \in \overline{J}^{2,-}_{\overline{\mathcal{K}}}\overline{v}^\theta(\tilde{x}_\alpha), && \tilde{p} = -D_{\tilde{x}}\varphi(x_\alpha, \tilde{x}_\alpha) = \alpha(x_\alpha - \tilde{x}_\alpha). \end{aligned}$$

Continuing as in Case I, we can prove that (6.21) holds with $x_\alpha, \tilde{x}_\alpha, p, \tilde{p}, A, \tilde{A}, \phi, \psi$ as given in Case II. First, (II.i) implies that (6.22) holds true also in Case II. Second, notice that

$$D^2\varphi(x_\alpha, \tilde{x}_\alpha) = \alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \qquad \big(D^2\varphi(x_\alpha, \tilde{x}_\alpha)\big)^2 = 2\alpha^2\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{6.35}$$

so that, by (4.8) and (6.35), we have for all $\xi, \tilde{\xi} \in \mathbb{R}^N$ the estimate

$$\big\langle A\xi, \xi\big\rangle - \big\langle \tilde{A}\tilde{\xi}, \tilde{\xi}\big\rangle = \left\langle \begin{pmatrix} A & 0 \\ 0 & -\tilde{A} \end{pmatrix}\begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix}, \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} \right\rangle \le \alpha(1+2\nu\alpha)|\xi - \tilde{\xi}|^2.$$

Proceeding as in Case I, we obtain from this inequality the estimate

$$\begin{aligned} &\operatorname{Tr}\big(\sigma(x_\alpha)^2 A\big) - \operatorname{Tr}\big(\sigma(\tilde{x}_\alpha)^2 \tilde{A}\big) \\ &\quad \le \alpha(1+2\nu\alpha)\sum_{i=0}^{n+1}|\sigma_i(x_\alpha) - \sigma_i(\tilde{x}_\alpha)|^2 = O\big(\alpha|x_\alpha - \tilde{x}_\alpha|^2\big) + K(\alpha)\nu, \end{aligned} \tag{6.36}$$

for some $K(\alpha) > 0$. Since $\nu > 0$ was arbitrary and $\alpha|x_\alpha - \tilde{x}_\alpha|^2 \to 0$ as $\alpha \to \infty$, we can conclude that (6.26) holds also in Case II.

It remains to estimate the integral terms in (6.21), which is done by following closely the approach taken in Case I. First, estimate (6.27) is still true in Case II. Next, we estimate I_1 and I_2 in (6.28), starting with I_2. Using (II.ii), we get

$$\big|\langle \eta_i(x_\alpha, z), p\rangle - \langle \eta_i(\tilde{x}_\alpha, z), \tilde{p}\rangle\big| = \big|\eta_i(z)\alpha(x_{\alpha i} - \tilde{x}_{\alpha i})^2\big| = o\Big(\frac{1}{\alpha}\Big)|\eta_i(z)|,$$

for $i = 1, \dots, n$. Then, proceeding exactly as in (6.31), we conclude that $\lim_{\alpha\to\infty} I_2 \le 0$.

Next we estimate I_1. Thanks to (6.34), we have $x_\alpha + \eta_i(x_\alpha, z), \tilde{x}_\alpha + \eta_i(\tilde{x}_\alpha, z) \in \overline{\mathcal{K}}$ for $z \in (-1,1)$ and thus (6.32) holds also in Case II. Moreover, we observe that the ith integrand of I_2 in Case II equals

$$\Phi\big(x_\alpha + \eta_i(x_\alpha, z), x_\alpha + \eta_i(\tilde{x}_\alpha, z)\big) - \Phi(x_\alpha, \tilde{x}_\alpha) + \tfrac{\alpha}{2}(x_{\alpha i} - \tilde{x}_{\alpha i})^2\eta_i(z)^2.$$

Since (3.3) is assumed to hold and $\alpha|x_\alpha - \tilde{x}_\alpha|^2 \to 0$ as $\alpha \to \infty$, we obtain after summing over i

$$I_1 \le \sum_{i=1}^n \tfrac{\alpha}{2}(x_{\alpha i} - \tilde{x}_{\alpha i})^2 \int_{\kappa<|z|<1} \eta_i(z)^2\, n_i(dz) \to 0 \text{ as } \alpha \to \infty.$$

Finally, sending (in that order) $\alpha \to \infty$ and $\kappa \to 0$ in (6.21), we reach the desired contradiction $M \le -\vartheta/\delta$. This concludes the proof of the theorem. □

Remark. Note that if we were to carry out the above proof under the assumption that the subsolution (instead of the supersolution) was continuous, we had to modify the proof only at (6.18) and (6.31). We leave the details to the reader.

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