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When are local stereological volume estimators exact?

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Summary

In this paper, we show that local stereology and geometric tomography are closely related. Using this relationship, bodies in R^n are studied which have the property that their volume can be determined without error by the local stereological volume estimator of order (n, p, r). In such cases, the local stereological volume estimator is said to be exact. The balls in R^n have exact local stereological volume estimators of any order. For this reason, bodies in R^n with exact local stereological volume estimator of order (n, p, r) are called quasi-spherical of order (n, p, r). It is shown, using the injectivity property of the spherical Radon transform, that the class of quasi-spherical bodies of order (n, p, r) does not depend on p. Furthermore, *star-shaped* quasi-spherical bodies of order $(n, p, 0)$ are characterized by a constant $n-$chord function. This class is studied in some detail and it is shown that it contains non-spherical convex bodies as well as non-convex bodies. A formula for the variance of the local stereological volume estimators is also given.

Keywords: chord function; convex body; exact estimator; geometric tomography; local stereology; non-convex body; quasi-spherical body; section function; spherical Radon transform; star-shaped body.

1. Introduction

Local stereology is a relative new part of stereology, concerned with the estimation of quantitative parameters of spatial structures which may be regarded as neighbourhoods of points, called reference points. The estimation is based on information collected in sections through the reference points. The important example of application comes from the microscopical study of biological tissue. Here, the spatial structure is a cell which is regarded as a neighbourhood of its nucleus or nucleolus. Original derivations of local stereological estimators may be found in Jensen & Gundersen (1985, 1989). Specific local methods are treated in Gundersen (1988), Jensen & Gundersen (1993) and Tandrup et al. (1997).

A unified exposition of local stereological methods has recently been given in the research monograph Jensen (1998). See also the recent review by Jensen & Nielsen (1999). The emphasis has in the monograph been on providing the necessary mathematical background in geometric measure theory and on giving an overview of the existing local stereological estimators. Less attention has been paid to the statistical properties of the local stereological estimators, although it has been pointed out that the transitive methods of Matheron, cf. Matheron (1965, 71), can be used to provide variance approximations of local stereological estimators, associated with systematic designs. See also Gundersen et al. (1999).

In the present paper, we will take up the discussion of the statistical properties of local stereological estimators. More specifically, we will study the variances of local stereological estimators of volume. We will derive a formula for the variance and give a characterization of the bodies for which the variance of the volume estimator is equal to zero. Since local stereological estimators are unbiased, a zero variance implies that the volume estimator is equal to the actual volume, almost surely. For this reason, the local stereological volume estimator is called exact in such cases.

In studying these questions, it turns out to be important to utilize a close connection between *local stereology* and the part of tomography called *geometric tomography*. In the research monograph Gardner (1995), a lucid and self-contained exposition of geometric tomography is given. See also the recent condensed review Gardner (1999). It turns out that the local stereological estimators of volume are well-known in geometric tomography with names such as chord functions and section functions.

In Section 2, important concepts from local stereology are presented and the relation to geometric tomography is established. Bodies in R^n, for which the local stereological volume estimator of order (n, p, r) is exact, are studied in Section 3. Such bodies are called quasi-spherical of order (n, p, r). In Section 4, star-shaped quasi-spherical bodies of order $(n, p, 0)$ are discussed in further detail. In Section 5, a formula for the variance of the local stereological volume estimators is derived. Problems for further research are briefly mentioned in Section 6.

2. Local stereology and its relation to geometric tomography

Most of the integral geometric results mentioned below can be found in many text books, but are here presented in a unified way with reference to Jensen (1998), for practical reasons. For a detailed treatment of geometric tomography, the reader is referred to Gardner (1995).

Local stereological estimates of quantitative parameters are based on information collected on section planes in R^n through a reference point. Without loss of generality, the origin O can be used as reference point. A section plane through the reference point of dimension p is then a $p-$dimensional linear subspace of R^n, $p = 0, 1, \ldots, n$. For brevity, a $p-$dimensional linear subspace will be called a $p-$subspace.

It is of interest to consider $p-$subspaces, containing a fixed lower dimensional part. We let $\mathcal{L}^n_{p(r)}$ denote the set of $p-$subspaces, containing a fixed $r-$subspace L_r, $0 \leq r < p \leq n$. On $\mathcal{L}^n_{p(r)}$, a measure $\mu^n_{p(r)}$ can be constructed, which is invariant under rotations that keep L_r fixed. This measure is unique up to multiplication with a positive constant and is often constructed such that

$$\mu^n_{p(r)}(\mathcal{L}^n_{p(r)}) = c(n-r, p-r),$$

where

$$c(n,p) = \frac{\sigma_n \sigma_{n-1} \cdots \sigma_{n-p+1}}{\sigma_p \sigma_{p-1} \cdots \sigma_1}$$

and $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere S^{n-1} in R^n, cf. e.g. Jensen (1998, Propositions 3.2 and 3.4). In what follows, we will write $dL^n_{p(r)}$ as short for $\mu^n_{p(r)}(dL_p)$ and use the notation dL^n_p for $dL^n_{p(0)}$.

By normalizing $\mu^n_{p(r)}$, a probability measure can be constructed on $\mathcal{L}^n_{p(r)}$ which defines the type of random subspaces considered in the present paper.

Definition 1. An isotropic $p-$subspace in R^n, containing the fixed $r-$subspace L_r, is a random $p-$subspace with constant density f with respect to $\mu^n_{p(r)}$

$$f(L_p) = \frac{1}{c(n-r, p-r)}, L_p \in \mathcal{L}^n_{p(r)}.$$

An isotropic $p-$subspace in R^n, containing O, is simply called an isotropic $p-$subspace in R^n. $\square$

By identifying a $q-$subspace L_q in R^n with R^q it is obvious how to extend this definition to isotropic $p-$subspaces L_p (contained) in L_q, (and) containing L_r, where $0 \leq r < p < q \leq n$ and $L_r \subseteq L_q$ are fixed subspaces of the indicated dimension.

Isotropic subspaces have a number of useful properties some of which are listed below. We use the notation $L_p \ominus L_r$ for the orthogonal difference of L_p and L_r, i.e. $L_p \ominus L_r = L_p \cap L_r^{\perp}$. The indices in the list fulfil $0 \leq r < p < q \leq n$. Proofs may be found in Jensen (1998, p. 65, 68, 71, 83).

- Let L_1 be an isotropic 1–subspace in L_q. Then, L_1 is distributed as $\mathrm{span}\{\omega\}$ where ω is uniform random on $S^{n-1} \cap L_q$.
- L_p is an isotropic $p-$subspace in L_q, containing L_r, if and only if $L_q \ominus L_p$ is an isotropic $(q-p)-$subspace in $L_q \ominus L_r$.
- L_p is an isotropic $p-$subspace in L_q, containing L_r, if and only if $L_p \ominus L_r$ is an isotropic $(p-r)-$subspace in $L_q \ominus L_r$.
- An isotropic $p-$subspace L_p in R^n can be generated by first generating an isotropic $q-$subspace L_q in R^n and next an isotropic $p-$subspace L_p in L_q.

We will now introduce the local stereological estimators of volume ($n-$dimensional Lebesgue measure). Suppose that $X \subseteq R^n$ is a body in R^n, i.e. a non-empty compact subset which is its closure of its interiour. Then, the local stereological estimator of its volume $V(X)$ $(= \lambda_n(X))$, based on an isotropic $p-$subspace L_p, containing a fixed $r-$subspace L_r, takes the form

$$\widehat{V}_{n,p,r}(X \cap L_p; L_r) = \frac{\sigma_{n-r}}{\sigma_{p-r}} \int\limits_{X \cap L_p} \|\pi_{L_r^\perp} x\|^{n-p} dx^p, \tag{1}$$

where $\|\cdot\|$ is the Euclidean norm, $\pi_{L_r^\perp}$ is the orthogonal projection onto $L_r^\perp$ and dx^p is the element of $p-$dimensional Lebesgue measure on L_p. Note that (1) defines a whole class of estimators. The estimator $\widehat{V}_{n,p,r}$ is called the local stereological volume estimator of order (n, p, r). It is unbiased

$$E(\widehat{V}_{n,p,r}(X \cap L_p; L_r)) = V(X), \tag{2}$$

and can be derived as a Horvitz-Thompson type estimator, cf. Jensen (1998, p. 105, 115).

It is possible to express $\widehat{V}_{n,p,r}$ in terms of $\widehat{V}_{n-r,p-r,0}$. According to Jensen (1998, Proposition 4.6), we thus have

$$\widehat{V}_{n,p,r}(X \cap L_p; L_r) = \int\limits_{L_r} \widehat{V}_{n-r,p-r,0}((X-y) \cap (L_p \ominus L_r); O) dy^r. \tag{3}$$

For $p = 1$ and $r = 0$, (1) reduces to

$$\widehat{V}_{n,1,0}(X \cap L_1; O) = \frac{\pi^{n/2}}{\Gamma(n/2)} \int\limits_{X \cap L_1} \|x\|^{n-1} dx^1.$$

If X is star-shaped (at O), i.e. if $X \cap L_1$ is a line-segment for all $L_1 \in \mathcal{L}_1^n$, then $\widehat{V}_{n,1,0}(X \cap L_1; O)$ takes a particularly simple form. Thus, let $\rho_X(\omega), \omega \in S^{n-1}$, be the radial function of X, i.e.

$$\rho_X(\omega) = \max\{c : c\omega \in X\},$$

cf. Gardner (1995, p. 18), and let

$$\rho_{n,X}(\omega) = \begin{cases} \rho_X(\omega)^n + \rho_X(-\omega)^n & \text{for } O \in X \\ ||\rho_X(\omega)|^n - |\rho_X(-\omega)|^n| & \text{for } O \notin X, \end{cases}$$

$\omega \in S^{n-1}$, be the $n-$chord function of X at O, cf. Gardner (1995, Definition 6.1.1). Then, the estimator is proportional to the $n-$chord function

$$\widehat{V}_{n,1,0}(X \cap \mathrm{span}\{\omega\}; O) = \frac{\pi^{n/2}}{n\Gamma(n/2)} \times \rho_{n,X}(\omega), \omega \in S^{n-1}. \tag{4}$$

The $n-$chord function and its generalizations are a very well studied subject in Gardner (1995). Thus, Chapter 6 of his research monograph is devoted to this topic.

The estimators based on subspaces of different dimensions are related to each other by a so-called Rao-Blackwell procedure whereby one estimator can be obtained by a conditional mean-value operation on the other. For $0 \leq r < p_1 \leq p_2 \leq n$, we thus have

$$\widehat{V}_{n,p_2,r}(X \cap L_{p_2}; L_r) = E(\widehat{V}_{n,p_1,r}(X \cap L_{p_1}; L_r)|L_{p_2}), \tag{5}$$

where L_{p_1} is an isotropic p_1-subspace in L_{p_2}, containing L_r, cf. Jensen (1998, p. 110–111). In particular, if X is star-shaped, $r = 0$, $p_1 = 1$ and $p_2 = p$

$$\begin{aligned}\widehat{V}_{n,p,0}(X \cap L_p; O) &= E(\widehat{V}_{n,1,0}(X \cap L_1; O)|L_p) \\ &= \int_{S^{n-1}\cap L_p} \widehat{V}_{n,1,0}(X \cap \operatorname{span}\{\omega\}; O) \frac{d\omega^{p-1}}{\sigma_p},\end{aligned}$$

where we have used that L_1 is an isotropic 1–subspace in L_p. The element of surface area measure on $S^{n-1} \cap L_p$ is denoted by $d\omega^{p-1}$. In particular, if X is star-shaped, we can use (4) and get

$$\begin{aligned}\widehat{V}_{n,p,0}(X \cap L_p; O) &= \frac{\pi^{n/2}}{n\Gamma(n/2)} \frac{1}{\sigma_p} \int_{S^{n-1}\cap L_p} \rho_{n,X}(\omega) d\omega^{p-1} \\ &= \frac{\pi^{n/2}}{n\Gamma(n/2)} \frac{2p}{\sigma_p} \widetilde{V}_{n,p}(X \cap L_p),\end{aligned}$$

where $\widetilde{V}_{n,p}(X \cap \cdot)$ is the so-called section function, cf. Gardner (1995, p. 247).

For $p = 1$, the section function is equal to the $n-$chord function while for $p = n$ we have

$$\widehat{V}_{n,n,0}(X; O) = \widetilde{V}_{n,n}(X) = V(X),$$

the Lebesgue measure of X. Section functions are studied in detail in Chapter 7 of Gardner (1995).

Local stereological estimators may in general be constructed for $d-$dimensional volume measure λ_n^d (Hausdorff measure) in R^n. Recall that λ_n^0 is counting measure, λ_n^{n-1} is surface area measure and $\lambda_n^n = \lambda_n$ is the Lebesgue measure in R^n.

3. Quasi-spherical bodies of order (n, p, r)

In this section, we will use the results from the previous section to study the class of bodies X for which the local stereological volume estimator of order (n, p, r) is exact, i.e.

$$\widehat{V}_{n,p,r}(X \cap L_p; L_r) = V(X), \text{ for almost all } L_p.$$

Since local stereological volume estimators are unbiased, cf. (2), exact estimators are characterized by a zero variance.

It is easy to see that $\widehat{V}_{n,p,r}(X \cap \cdot; L_r)$ is exact if X is a ball. For this reason, bodies having an exact volume estimator will be called quasi-spherical.

Definition 2. Let X be a body in R^n. Then, X is called quasi-spherical of order (n, p, r) if and only if $\widehat{V}_{n,p,r}(X \cap L_p; L_r) = V(X)$, for almost all L_p. □

It turns out that quasi-sphericity is a property that only depends on n and r. The proof of this result uses the injectivity property of the spherical Radon transform, as stated in the lemma below. The spherical Radon transform Rf of a Borel function f on S^{n-1} is defined by

$$Rf(\omega) = \int\limits_{S^{n-1}\cap \mathrm{span}\{\omega\}^{\perp}} f(v)dv^{n-2}, w \in S^{n-1}.$$

A slightly weaker version of the lemma may be found in Gardner (1995, Theorem C.2.4). The version presented below is tantamount to saying that R is injective on distributions, cf. Goodey & Weil (1992).

Lemma 3. If f is a bounded even Borel function on S^{n-1} such that for $\lambda_n^{n-1}-$almost all $\omega \in S^{n-1}$

$$Rf(\omega) = 0,$$

then $f(\omega) = 0$ for $\lambda_n^{n-1}-$almost all $\omega \in S^{n-1}$. □

Proof. Let $g \in C_e^{\infty}(S^{n-1})$, the set of even infinitely differentiable functions on S^{n-1}. Since R is a bijection of $C_e^{\infty}(S^{n-1})$ to itself, cf. Gardner (1995, Theorem C.2.5), we can find $h \in C_e^{\infty}(S^{n-1})$ such that $g = Rh$. Using the self-adjoint property of R at $(*)$ below, cf. Gardner (1995, Theorem C.2.6), we find

$$\begin{aligned}\int\limits_{S^{n-1}} f(\omega)g(\omega)d\omega^{n-1} &= \int\limits_{S^{n-1}} f(\omega)Rh(\omega)d\omega^{n-1} \\ &\overset{(*)}{=} \int\limits_{S^{n-1}} Rf(\omega)h(\omega)d\omega^{n-1} \\ &= 0.\end{aligned}$$

Since g was chosen arbitrarily, $f(\omega) = 0$ for $\lambda_n^{n-1}-$almost all $\omega \in S^{n-1}$. □

In the proposition below, we show that quasi-sphericity is a property that only depends on n and r. The proposition is for star-shaped X and $r = 0$ closely related to Gardner (1995, Theorem 7.2.3). See also Gardner & Volcic (1994). The important tool in proving the theorem by Gardner is also the injectivity property of the spherical Radon transform.

Proposition 4. Let $0 \le r < p_1 \le p_2 \le n$ and let X be a body in R^n. Then, X is quasi-spherical of order (n, p_1, r) if and only if X is quasi-spherical of order (n, p_2, r).

Proof. Let L_{p_2} be an isotropic p_2-subspace in R^n, containing L_r, and let L_{p_1} be an isotropic p_1-subspace in L_{p_2}, containing L_r. Then, the marginal distribution of L_{p_1} is that of an isotropic p_1-subspace in R^n, containing L_r, and we find, using (5), that

$$\begin{aligned}
&\mathrm{Var}(\widehat{V}_{n,p_1,r}(X \cap L_{p_1}; L_r)) \\
&= \mathrm{Var}(E(\widehat{V}_{n,p_1,r}(X \cap L_{p_1}; L_r)|L_{p_2})) + E(\mathrm{Var}(\widehat{V}_{n,p_1,r}(X \cap L_{p_1}; L_r)|L_{p_2})) \\
&= \mathrm{Var}(\widehat{V}_{n,p_2,r}(X \cap L_{p_2}; L_r)) + E(\mathrm{Var}(\widehat{V}_{n,p_1,r}(X \cap L_{p_1}; L_r)|L_{p_2})) \\
&\geq \mathrm{Var}(\widehat{V}_{n,p_2,r}(X \cap L_{p_2}; L_r)).
\end{aligned}$$

Therefore, it immediately follows that if X is quasi-spherical of order (n, p_1, r) then X is quasi-spherical of order (n, p_2, r).

In order to prove that quasi-sphericity of order (n, p_2, r) implies quasi-sphericity of order (n, p_1, r), first note that because of what we have shown already it suffices to consider the case $p_2 = n - 1$ and $p_1 = r + 1$. We then know that

$$\widehat{V}_{n,n-1,r}(X \cap L_{n-1}; L_r) = V(X), \tag{6}$$

for almost all L_{n-1}. The random subspace L_{n-1} is an isotropic $(n-1)-$subspace in R^n, containing L_r. Therefore, L_{n-1} is distributed as

$$L_r \oplus (L_r^\perp \ominus \mathrm{span}\{\omega\}),$$

where ω is uniform random on $S^{n-1} \cap L_r^\perp$. Accordingly, the assumption (6) implies that

$$\widehat{V}_{n,n-1,r}(X \cap [L_r \oplus (L_r^\perp \ominus \mathrm{span}\{\omega\})]; L_r) = V(X), \tag{7}$$

for $\lambda_n^{n-1-r}-$almost all $\omega \in S^{n-1} \cap L_r^\perp$.

We will next rewrite $\widehat{V}_{n,n-1,r}(X \cap L_{n-1}; L_r)$. According to (5), we have

$$\widehat{V}_{n,n-1,r}(X \cap L_{n-1}; L_r) = E(\widehat{V}_{n,r+1,r}(X \cap L_{r+1}; L_r)|L_{n-1}),$$

where L_{r+1} is an isotropic $(r+1)-$subspace in L_{n-1}, containing L_r. Since L_{r+1} is distributed as $L_r \oplus \mathrm{span}\{v\}$ where v is uniform random on $S^{n-1} \cap (L_{n-1} \ominus L_r)$, we find

$$\begin{aligned}
&\widehat{V}_{n,n-1,r}(X \cap L_{n-1}; L_r) \\
&= \int\limits_{S^{n-1} \cap (L_{n-1} \ominus L_r)} \widehat{V}_{n,r+1,r}(X \cap (L_r \oplus \mathrm{span}\{v\}); L_r) \frac{dv^{n-2-r}}{\sigma_{n-1-r}}.
\end{aligned} \tag{8}$$

Using (7), (8) and the identity

$$\lambda_n^{n-2-r}(S^{n-1} \cap (L_r^\perp \ominus \mathrm{span}\{\omega\})) = \sigma_{n-1-r}, \omega \in S^{n-1} \cap L_r^\perp,$$

we finally get

$$\int\limits_{S^{n-1} \cap (L_r^\perp \ominus \mathrm{span}\{\omega\})} f(v) dv^{n-2-r} = 0,$$

for almost all $\omega \in S^{n-1} \cap L_r^\perp$, where

$$f(v) = \widehat{V}_{n,r+1,r}(X \cap (L_r \oplus \mathrm{span}\{v\}); L_r) - V(X).$$

Using Lemma 3, it follows that $f(\omega) = 0$ for $\lambda_n^{n-1-r}-$almost all $\omega \in S^{n-1} \cap L_r^\perp$ and X is quasi-spherical of order $(n, r+1, r)$. $\square$

According to Proposition 4, the class of quasi-spherical bodies of order (n, p, r) does not depend on p. This class will in what follows be denoted by $\mathcal{X}_r$.

In the next section, star-shaped bodies in $\mathcal{X}_0$ are studied in some detail.

4. Star-shaped bodies in $\mathcal{X}_0$

Let X be a star-shaped body in $\mathcal{X}_0$. Then, because X is quasi-spherical of order $(n, 1, 0)$, we find, using (4), that its $n-$chord function $\rho_{n,X}(\omega)$ is constant for $\lambda_n^{n-1}-$almost all $w \in S^{n-1}$. Note that because of the unbiasedness of the local volume estimators and (4), the constant cannot be any number. We have

$$\rho_{n,X}(\omega) = \frac{n\Gamma(n/2)}{\pi^{n/2}} V(X), \tag{9}$$

for $\lambda_n^{n-1}-$almost all $w \in S^{n-1}$.

It is clear that $\mathcal{X}_0$ is a very rich class of bodies, obeying a certain kind of anti-symmetry. Indeed, any star-shaped $X \in \mathcal{X}_0$, which contains O, is determined by its 'upper half' $X_+ = X \cap H_+$, where

$$H_+ = \{x \in R^n : x_n \geq 0\},$$

and the value of $V(X)$. The set X_+ can be quite arbitrary. In fact, let $v > 0$ and let X_+ be any star-shaped body, satisfying $O \in X_+ \subseteq H_+$ and

$$\rho_{X_+}(\omega) \leq [\frac{n\Gamma(n/2)}{\pi^{n/2}} v]^{1/n} \text{ for } \lambda_n^{n-1} - \text{almost all } \omega \in S^{n-1}. \tag{10}$$

Then, there exists a unique star-shaped $X \in \mathcal{X}_0$, which contains O, such that $V(X) = v$ and $X \cap H_+ = X_+$. The actual construction of X from knowledge of v and X_+ can be performed as follows. Since X is star-shaped, X is determined by its radial function ρ_X. It is easy to see that this function may be expressed in terms of v and the radial function ρ_{X_+} of X_+. For $\omega \in S^{n-1}$, we have

$$\rho_X(\omega) = \begin{cases} \rho_{X_+}(\omega) & \text{for } \omega_n > 0 \\ [\frac{n\Gamma(n/2)}{\pi^{n/2}} v - \rho_{X_+}(-\omega)^n]^{1/n} & \text{for } \omega_n < 0. \end{cases}$$

Note that we do not need to specify the radial function on the set $\{\omega \in S^{n-1} : \omega_n = 0\}$ because this set has $\lambda_n^{n-1}-$measure zero.

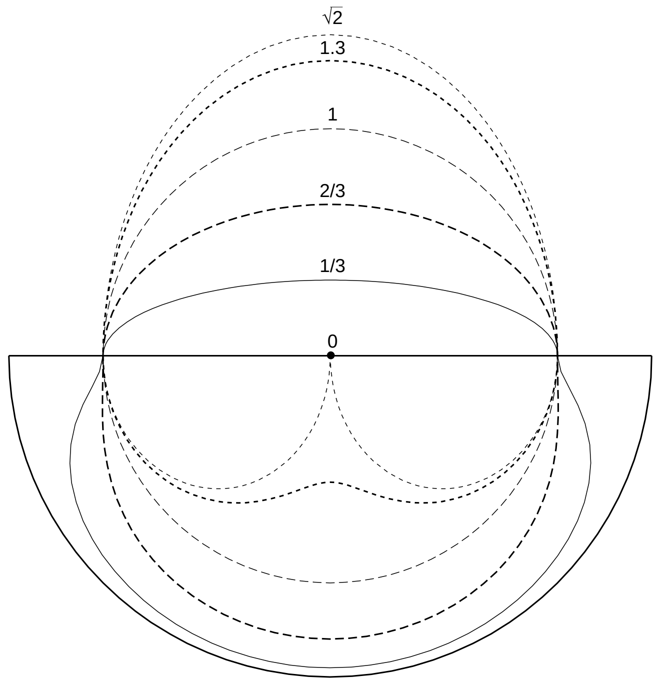

Figure 1. Five quasi-spherical planar bodies of order $(2,1,0)$, constructed from their ’upper halfs’ which are half-ellipses with horizontal semi-axis of length $a = 1$ and vertical semi-axes of lengths $b = 1/3, 2/3, 1, 1.3, \sqrt{2}$, respectively. The limiting body, for $b \to 0$, is also shown.

In Figure 1, a collection of six star-shaped quasi-spherical planar bodies are shown. They have all area (2–dimensional volume v) equal to π. Five of them have been constructed from their ’upper halfs’ X_+ which are half-ellipses with horizontal semi-axis of length $a = 1$ and vertical semi-axes of lengths $b = 1/3, 2/3, 1, 1.3, \sqrt{2}$, respectively. The value $b = \sqrt{2}$ is an upper bound for the length of the vertical semi-axis, according to (10). For $b \to 0$, X will approach a half-circle of radius $\sqrt{2}$, shown as the sixth planar body in Figure 1. Note that although X_+ is convex, X may well be non-convex.

In Figure 2, another example of a star-shaped quasi-spherical planar body is shown. It has been obtained by adding a systematic set of sine waves to a circle and modifying them such that the 2–chord function remains constant.

In Gardner (1995, Theorem 6.3.2), a more sophisticated method of constructing non-spherical convex bodies in $\mathcal{X}_0$ of arbitrary dimension is described. The idea is to start with a non-spherical compact convex set X' in the $\{x_1, x_n\}-$plane of R^n such that $O \in X'$, X' has constant $n-$chord function and X' is symmetric with respect to the x_n-axis. Let X be the convex body obtained by rotating X' about the x_n-axis. Then, X will also have constant $n-$chord function. To see this, let $\omega \in S^{n-1}$. Then, there is a unique plane containing ω and the x_n-axis and $-\omega$ also lies in this plane. The intersection of X with this vertical plane will be a copy of X' and therefore the $n-$chord function of X is constant.

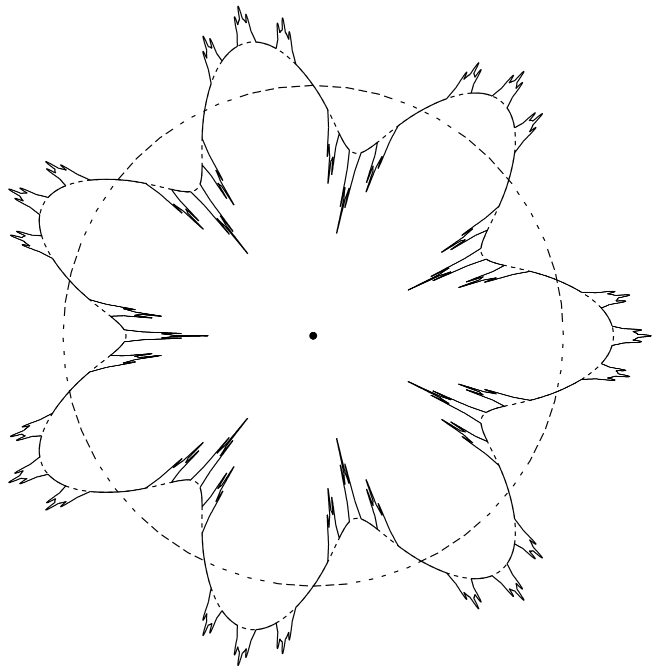

Figure 2. Quasi-spherical body of order (2,1,0).
The boundary of the body is the full-drawn curve.

In Figure 3, an example of this construction is shown for $n = 3$. The planar body X', shown to the left in the figure, has in the $\{x_1, x_3\}-$plane a radial function with polar representation

$$\rho_{X'}(\theta) = (\sin\theta + \frac{3}{2})^{1/3}, \theta \in [0, 2\pi).$$

It is easy to check that X' is convex, $O \in X'$, X' has constant 3–chord function and is symmetric about the x_3-axis.

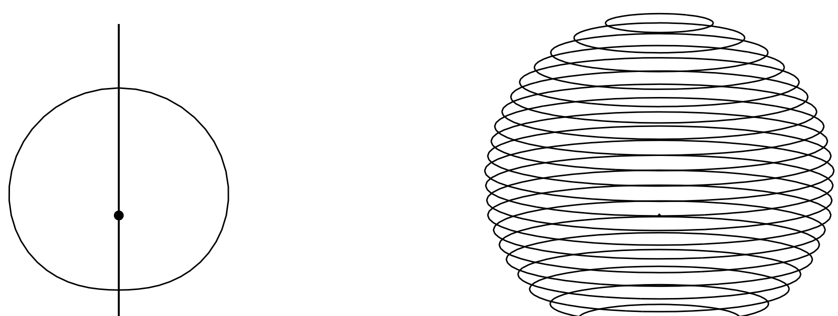

Figure 3. Quasi-spherical body of order (3,2,0) and (3,1,0) (right), obtained by rotating the planar body shown to the left about the vertical axis. The planar body is shown at a smaller scale.

If X' is non-convex, but the other assumptions are still satisfied, then X becomes a non-convex element of $\mathcal{X}_0$. Two examples are given in Figure 4. Here, X' is constructed from its upper half which is a half-ellipse with ratios $b/a = 2^{1/3}, 1/3$, respectively, between the lengths of the vertical semi-axis to the horizontal semi-axis. Note that X' is here constructed such that the 3–chord function is constant.

In Gardner (1995, Section 6.3), matters concerning the existence of quasi-spherical bodies are called equichordal problems. In particular, if X is a star-shaped body which contains O in its interiour and satisfies (9) for all $\omega \in S^{n-1}$, then O is called an $n-$equichordal point of X.

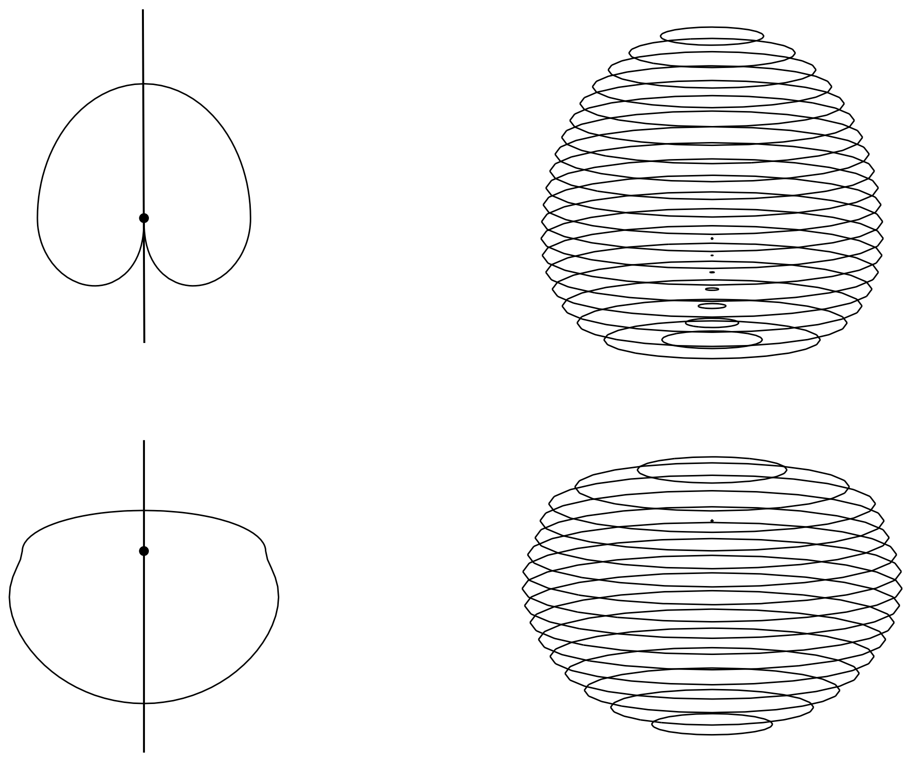

Figure 4. Two quasi-spherical bodies (right), both of order (3,2,0) and (3,1,0), obtained by rotating the planar bodies shown to the left about their vertical axis. The planar bodies are shown at a smaller scale.

5. A variance formula

The above considerations may be used to give a qualitative idea about the kind of bodies for which the volume can be estimated accurately by a local stereological estimator. In this section, we will derive an expression for the variance which gives some more insight into the problem.

First, we need the following lemma.

Lemma 5. Let $L_r \in \mathcal{L}_r^n, r < n$, and let X be a body in R^n. Let

$$g : S^{n-1} \cap L_r^{\perp} \to S^{n-1} \cap L_r^{\perp}$$

be a non-negative, even Borel function. Then,

$$\int_X g\left(\frac{\pi_{L_r^\perp} x}{\|\pi_{L_r^\perp} x\|}\right) dx^n = V(X) \int_{S^{n-1}\cap L_r^\perp} g(\omega) f_{X,L_r}(\omega) d\omega^{n-r-1}, \tag{11}$$

where f_{X,L_r} is the probability density with respect to λ_n^{n-r-1} of the form

$$f_{X,L_r}(\omega) = \frac{1}{\sigma_{n-r} V(X)} \widehat{V}_{n,r+1,r}(X \cap [L_r \oplus \operatorname{span}\{\omega\}]; L_r), \omega \in S^{n-1} \cap L_r^\perp.$$

Proof. Let us first notice that $X \cap L_r$ is a set of λ_n-measure zero and the integral on the left-hand side of (11) is therefore well-defined. Using translative decomposition of Lebesgue measure, we find,

$$\int_X g\left(\frac{\pi_{L_r^\perp} x}{\|\pi_{L_r^\perp} x\|}\right) dx^n = \int_{L_r} \int_{(X-y)\cap L_r^\perp} g\left(\frac{z}{\|z\|}\right) dz^{n-r} dy^r. \tag{12}$$

Next, we concentrate on the inner integral and use polar decomposition of the Lebesgue on $L_r^\perp$

$$\begin{aligned}
&\int_{(X-y)\cap L_r^\perp} g\left(\frac{z}{\|z\|}\right) dz^{n-r} \\
&= \int_{L_r^\perp} \mathbf{1}\{z \in X - y\} g\left(\frac{z}{\|z\|}\right) dz^{n-r} \\
&= \int_{S^{n-1}\cap L_r^\perp} \int_{\{u\omega : u>0\}} \mathbf{1}\{u\omega \in X - y\} g(\omega) u^{n-r-1} du d\omega^{n-r-1} \\
&= \frac{1}{2} \int_{S^{n-1}\cap L_r^\perp} \int_{\operatorname{span}\{\omega\}} \mathbf{1}\{z \in X - y\} g(\omega) \|z\|^{n-r-1} dz^1 d\omega^{n-r-1}.
\end{aligned} \tag{13}$$

Here, $\mathbf{1}\{\cdot\}$ is the notation used for the indicator function. At the last equality sign, we have used that g is an even function. Inserting (13) into (12) and changing the order of integration, we get, using (1) and (3),

$$\begin{aligned}
&\int_X g\left(\frac{\pi_{L_r^\perp} x}{\|\pi_{L_r^\perp} x\|}\right) dx^n \\
&= \frac{1}{2} \int_{S^{n-1}\cap L_r^\perp} g(\omega) \int_{L_r} \frac{\sigma_1}{\sigma_{n-r}} \widehat{V}_{n-r,1,0}((X-y) \cap \operatorname{span}\{\omega\}; O) dy^r d\omega^{n-r-1} \\
&= \frac{1}{\sigma_{n-r}} \int_{S^{n-1}\cap L_r^\perp} g(\omega) \widehat{V}_{n,r+1,r}(X \cap [L_r \oplus \operatorname{span}\{\omega\}]; L_r) d\omega^{n-r-1},
\end{aligned}$$

and the result now follows immediately. $\square$

Notice that the probability density f_{X,L_r} is constant almost surely if and only if X is quasi-spherical of order $(n, r+1, r)$ or according to Proposition 4 quasi-spherical of order (n, p, r) for some p.

We are now ready to present the general formula for the variance. Note that also in the case $p > r+1$ it is possible to express $\mathrm{Var}(\widehat{V}_{n,p,r}(X \cap L_p; L_r))$ in terms of the density f_{X,L_r}.

Proposition 6. $L_r \in \mathcal{L}_r^n$ and let X be a body in R^n. Then, if $p = r+1$

$$\mathrm{Var}(\widehat{V}_{n,p,r}(X \cap L_p; L_r)) = \sigma_{n-r} V(X)^2 \int\limits_{S^{n-1} \cap L_r^\perp} [f_{X,L_r}(\omega)^2 - \frac{1}{\sigma_{n-r}^2}] d\omega^{n-r-1},$$

while for $p > r+1$

$$\begin{aligned} &\mathrm{Var}(\widehat{V}_{n,p,r}(X \cap L_p; L_r)) \\ &= \frac{\sigma_{n-r}}{\sigma_{n-r-1}} \frac{\sigma_{p-r-1}}{\sigma_{p-r}} V(X)^2 \int\limits_{S^{n-1} \cap L_r^\perp} \int\limits_{S^{n-1} \cap L_r^\perp} (1 - \langle \omega_1, \omega_2 \rangle^2)^{-(n-p)/2} \\ &\qquad \times [f_{X,L_r}(\omega_1) f_{X,L_r}(\omega_2) - \frac{1}{\sigma_{n-r}^2}] \prod_{i=1}^{2} d\omega_i^{n-r-1}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

Proof. The result for $p = r+1$ follows immediately from the fact that

$$\widehat{V}_{n,r+1,r}(X \cap L_{r+1}; L_r)$$

is distributed as

$$\widehat{V}_{n,r+1,r}(X \cap [L_r \oplus \mathrm{span}\{\omega\}]; L_r),$$

where ω is uniform random on $S^{n-1} \cap L_r^\perp$.

Next, let us consider the case $p > r+1$. The proof is based on the classical Blaschke-Petkantschin formula for two sets. This version of the Blaschke-Petkantschin formula may be formulated as follows, cf. Jensen (1998, p. 104),

$$\begin{aligned} &c(n-2-r, p-2-r) \int\limits_{X_1} \int\limits_{X_2} g(x_1, x_2) \prod_{i=1}^{2} dx_i^n \\ &= \int\limits_{\mathcal{L}_{p(r)}^n} \int\limits_{X_1 \cap L_p} \int\limits_{X_2 \cap L_p} g(x_1, x_2) \nabla_2(\pi_{L_r^\perp} x_1, \pi_{L_r^\perp} x_2)^{n-p} \prod_{i=1}^{2} dx_i^p dL_{p(r)}^n, \end{aligned}$$

where $\nabla_2(y_1, y_2)$ is equal to 2 times the area of the triangle spanned by O, y_1 and y_2, i.e.

$$\nabla_2(y_1, y_2) = [\|y_1\|^2 \|y_2\|^2 - \langle y_1, y_2 \rangle^2]^{1/2}.$$

Using this formula and (1), we find

$$
\begin{aligned}
&E(\widehat{V}_{n,p,r}(X \cap L_p; L_r)^2) \\
&= \int_{\mathcal{L}^n_{p(r)}} \frac{\sigma^2_{n-r}}{\sigma^2_{p-r}} [\int_{X \cap L_p} \|\pi_{L_r^\perp} x\|^{n-p} dx^p]^2 \frac{dL^n_{p(r)}}{c(n-r, p-r)} \\
&= \frac{\sigma_{n-r}}{\sigma_{n-r-1}} \frac{\sigma_{p-r-1}}{\sigma_{p-r}} \int_X \int_X (1 - \langle \frac{\pi_{L_r^\perp} x_1}{\|\pi_{L_r^\perp} x_1\|}, \frac{\pi_{L_r^\perp} x_2}{\|\pi_{L_r^\perp} x_2\|} \rangle^2)^{-(n-p)/2} \prod_{i=1}^2 dx_i^n \\
&= \frac{\sigma_{n-r}}{\sigma_{n-r-1}} \frac{\sigma_{p-r-1}}{\sigma_{p-r}} V(X)^2 \\
&\quad \times \int_{S^{n-1} \cap L_r^\perp} \int_{S^{n-1} \cap L_r^\perp} (1 - \langle \omega_1, \omega_2 \rangle^2)^{-(n-p)/2} f_{X,L_r}(\omega_1) f_{X,L_r}(\omega_2) \prod_{i=1}^2 d\omega_i^{n-r-1}, \quad (14)
\end{aligned}
$$

where we at the last equality sign have used Lemma 5 two times. In particular, if X_0 is a ball of volume $V(X)$, then X_0 is quasi-spherical of order (n, p, r) and $(n, r+1, r)$ and it follows from (14) that

$$
\begin{aligned}
V(X)^2 &= V(X_0)^2 \\
&= E(\widehat{V}_{n,p,r}(X_0 \cap L_p; L_r)^2) \\
&= \frac{\sigma_{n-r}}{\sigma_{n-r-1}} \frac{\sigma_{p-r-1}}{\sigma_{p-r}} V(X)^2 \\
&\quad \times \int_{S^{n-1} \cap L_r^\perp} \int_{S^{n-1} \cap L_r^\perp} (1 - \langle \omega_1, \omega_2 \rangle^2)^{-(n-p)/2} \frac{1}{\sigma^2_{n-r}} \prod_{i=1}^2 d\omega_i^{n-r-1}. \qquad (15)
\end{aligned}
$$

Combining (14) and (15), the variance formula for $p > r + 1$ follows immediately. $\square$

If X is star-shaped and $O \in X$, then it follows from (4) that

$$
\widehat{V}_{n,1,0}(X \cap \operatorname{span}\{\omega\}; O) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} r_X(\omega)^n = V(B(r_X(\omega))),
$$

where

$$
r_X(\omega) = [\frac{1}{2}\big(\rho_X(\omega)^n + \rho_X(-\omega)^n\big)]^{1/n},
$$

and $B(r)$ is the ball in R^n with centre O and radius r. Therefore, the formulae for $\operatorname{Var}(\widehat{V}_{n,p,0}(X \cap L_p; O))$ have for star-shaped X with $O \in X$ the following nice formulations

$$p = 1 : \mathrm{Var}(\widehat{V}_{n,p,0}(X \cap L_p; O)) = \frac{1}{\sigma_n} V(X)^2 \int_{S^{n-1}} [\frac{V(B(r_X(\omega)))^2}{V(X)^2} - 1] d\omega^{n-1}.$$

$$p > 1 : \mathrm{Var}(\widehat{V}_{n,p,0}(X \cap L_p; O)) = \frac{1}{\sigma_n \sigma_{n-1}} \frac{\sigma_{p-1}}{\sigma_p} V(X)^2 \int_{S^{n-1}} \int_{S^{n-1}} (1 - \langle \omega_1, \omega_2 \rangle^2)^{-(n-p)/2} \times [\frac{V(B(r_X(\omega_1)))}{V(X)} \frac{V(B(r_X(\omega_2)))}{V(X)} - 1] \prod_{i=1}^{2} d\omega_i^{n-1}.$$

6. Further research

One of the most obvious topics for further research is a study of the classes of bodies $\mathcal{X}_r, r = 1, \ldots, n-2$. Recall that $\mathcal{X}_r$ consists of bodies which are quasi-spherical of order $(n, p, r), p = r+1, \ldots, n-1$. The formula (3) may be useful in such a study.

Another interesting topic is the variance of local stereological estimators of lower-dimensional properties such as surface area and length.

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