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On the ruin problem for some adapted premium rules

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Abstract

We consider risk processes where the premium rate $p(t)$ at time t is calculated according to past claims statistics, for example $p(t) = (1+\eta)A_{t-}/t$ or $p(t) = (1+\eta)(A_{t-} - A_{t-s})/s$ where η is the safety loading and A_t the total compound Poisson claims in $[0,t]$. We perform a comparison of the ruin probabilities with those of the Cramér–Lundberg model, and characterize the claims experience leading to ruin. With heavy tails, the controlled risk process has typically at least as large a ruin probability as the Cramér–Lundberg model. With light tails, the adjustment coefficient is typically larger so that the ruin probability is smaller; a key tool is the Gärtner–Ellis theorem from large deviations theory. We also consider similar problems for diffusion approximations.

Keywords Adjustment coefficient, Brownian motion, Brownian bridge, change of measure, conditioned limit theorems, diffusion approximation, exponential decay, Gärtner–Ellis theorem, insurance risk models, large deviations, subexponential distribution.

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1 Introduction

The classical model for the reserve R_t^* of an insurance corporation at time t is

$$R_t^* \;=\; u + pt - \sum_{i=1}^{N_t} U_i \tag{1.1}$$

where $\{N_t\}$ is a Poisson process with rate β and the claims $U_1, U_2, \ldots$ are i.i.d. with common distribution B and independent of $\{N_t\}$. The premium rate p is calculated according to the expected value principle with safety loading $\eta > 0$ (e.g. [14]), i.e.

$$p \;=\; (1+\eta)\beta m$$

where m is the mean of B.

Given the safety loading η, the model (1.1) implicitly assumes that the Poisson intensity β and the claim size distribution B (or at least its mean m) are known. Of course, this is not realistic. An apparent solution to this problem is to calculate the premium rate $p = p(t)$ at time t based upon claims statistics. Most obviously, the best estimator of βm based upon $\mathcal{F}_{t-}$, where $\mathcal{F}_t = \sigma(A_s : 0 \le s \le t)$, $A_t = \sum_{i=1}^{N_t} U_i$, is A_{t-}/t. Thus, one would take $p(t) = (1+\eta)A_{t-}/t$ where , and instead of (1.1) one would consider

$$R_t \;=\; u + (1+\eta)\int_0^t \frac{A_s}{s} ds - A_t \tag{1.2}$$

The purpose of the present paper is the investigation of the effect of such adapted premium rules on the ruin probability. Formally, we require that $p(t)$ is measurable w.r.t. $\mathcal{G}_{t-}$ where the filtration $\{\mathcal{G}_t\}_{t\ge 0}$ satisfies

$$\mathcal{F}_t \quad \subseteq \quad \mathcal{G}_t \text{ for all } t \tag{1.3}$$

$$\{A_{t+s} - A_{t-}\}_{s\ge 0} \quad \text{is} \quad \text{independent of } \mathcal{G}_{t-} \text{ for all } t \tag{1.4}$$

$$\frac{1}{t}\int_0^t p(s)\,ds \quad \stackrel{\text{a.s.}}{\to} \quad (1+\eta)\beta m. \tag{1.5}$$

Here (1.4) expresses that the premium rule is non–anticipative; $\mathcal{F}_t \subset \mathcal{G}_t$ could occur, e.g., if $\mathcal{F}_0$ contains some claims history prior to $t = 0$ (for an example, see Section 3). The content of (1.5) is that no unfair loading is charged; except for Theorem 1, we will in fact have the stronger $\mathbb{E}p(t) = (1+\eta)\beta m$ in the examples we study. The controlled risk process is

$$R_t \;=\; u + \int_0^t p(s)\,ds - A_t\,. \tag{1.6}$$

Let

$$T^*(u) = \inf\{t > 0 : R_t^* \le 0\}, \quad T(u) = \inf\{t > 0 : R_t \le 0\}$$

be the ruin times for the two processes and

$$\Psi^*(u) = \mathbb{P}(T^*(u) < \infty), \quad \Psi(u) = \mathbb{P}(T(u) < \infty)$$

the corresponding ruin probabilities. An obvious question is then how $\Psi^*(u)$ and $\Psi(u)$ compare, and what is the claims experience given ruin. That is, given the unlikely event of ruin: what was the atypical behaviour of the claims process $\{A_t\}$ that caused ruin?

Consider first the case of light–tailed claims. With $\gamma^* > 0$ given as solution of

$$\kappa^*(\gamma^*) \equiv \beta(\phi(\gamma^*) - 1) - (1+\eta)\gamma^*\beta m = 0, \tag{1.7}$$

where $\phi(\alpha) = \mathbb{E}e^{\alpha U} = \int_0^\infty e^{\alpha s} B(dx)$, the classical Cramér–Lundberg approximation then states that $\Psi^*(u) \sim C^* e^{-\gamma^* u}$ for some suitable constant C^* as $u \to \infty$. In Asmussen [1], it is shown that ruin is caused by β and the distribution B of the claims to be changed initially in an exponentially twisted way, so that in particular the linear drift of $\{R_t^*\}$ is changed from positive to negative.

It is notable that such a behaviour is taken care of by the adaptive premium rule in (1.2). Namely, if the drift βm of $\{A_t\}$ is replaced by $\alpha > \beta m$, then the drift of $\{R_t\}$ changes from $\eta\beta m$ to $\eta\alpha > \eta\beta m$. This might lead to expecting $\Psi(u)$ to be substantially smaller than $\Psi^*(u)$. However, one could also argue that the opposite should be true because $\{R_t\}$ is more variable then $\{R_t^*\}$ and because there are new ways in which ruin could occur: initially, the claims could be atypically small so that one would charge a too low premium and when the typical behaviour sets in (or when claims grow atypically large), the adapted premium rule could be too slow in compensating for this. In Section 2, we will in fact see that $\Psi(u)$ is smaller than $\Psi^*(u)$ in the sense that the adjustment coefficient γ (the exponential decay parameter of the ruin probability) for the model (1.2) satisfies $\gamma \ge \gamma^*$, with strict equality except for a degenerate claim size distribution. The analysis also shows that indeed ruin will occur as consequence of atypically small and few early claims followed by atypically large and many claims. We will next see in Section 3 that there are different adaptive premium rules such that γ is not only substantially larger than γ^* but in fact in a certain sense the largest possible.

With heavy–tailed claims, there is no hope for such a reduction of the ruin probability:

Theorem 1 *Assume that both the claim size distribution* $B(x) = \mathbb{P}(U_i \le x)$ *and its integrated tail* $\int_x^\infty (1 - B(y))\,dy$ *are subexponential. Then for any adapted premium rule satisfying* (1.3)*,* (1.4)*,* (1.5) *one has*

$$\liminf_{u\to\infty} \frac{\Psi(u)}{\Psi^*(u)} \;\ge\; 1.$$

[for subexponential distributions, see e.g. [6]]. The heuristics behind this result is quite simple: with heavy tails, ruin for $\{R_t^*\}$ occurs as consequence of one large claim occuring rather late (Asmussen & Klüppelberg [3]) and because of (1.5), this behaviour is just as dangerous for $\{R_t\}$ as for $\{R_t^*\}$. The rigorous proof is given in Section 4. We do not consider it a straightforward matter to prove that the limit actually is 1.

Finally, in Section 5 we consider similar issues when $\{R_t^*\}$, $\{R_t\}$ are replaced by processes $\{r_t^*\}$, $\{r_t\}$ driven by SDE's; this replacement may either be motivated by a heavy–traffic approximation ($\eta \downarrow 0$), or one may apriori postulate the SDE model, as is a current trend in much current insurance risk literature. In this setting, we obtain somewhat sharper conclusions.

2 The rule (1.2) with light–tailed claims

With light–tailed claims, a standard measure of the risk inherent in a given model is the adjustment coefficient γ. We will work with techniques from large deviations, so we define γ in the logarithmic sense, i.e. by

$$\gamma \;=\; -\lim_{u\to\infty} \frac{1}{u} \log \Psi(u) \tag{2.1}$$

To show that the limit exists and to identify γ, a convenient tool is the following consequence of the Gärtner–Ellis theorem (e.g. [4] p. 14) given in Glynn & Whitt [8], Theorem 2; see also Nyrhinen [13] for closely related discussion. Note that no independence or stationarity is assumed.

Theorem 2 *Let* $\{X_n\}_{n=1,2,\ldots}$ *be a sequence of real r.v.'s,* $S_n = X_1 + \ldots + X_n$*,* $\Psi(u) = \mathbb{P}(S_n > u \text{ for some } n)$*. Assume that there exist a finite function* κ *and positive constants* γ, ϵ *such that*
(i) $n^{-1} \log \mathbb{E}e^{\theta S_n} \;\to\; \kappa(\theta)$ *for* $|\theta - \gamma| < \epsilon$*,*
(ii) $\kappa(\gamma) = 0$*,* $\kappa'(\gamma) > 0$*,*
(iii) $\mathbb{E}e^{\gamma S_n} < \infty$ *for all* n*.*
Then $-\frac{1}{u} \log \Psi(u) \;\to\; \gamma$*,* $u \to \infty$*.*

From this result, we easily get:

Theorem 3 *Consider the model* (1.2). *Then* $\frac{1}{u}\log\Psi(u) \;\to\; -\gamma$ *where* γ *is the solution of the equation* $\kappa(\gamma)=0$ *with*

$$\kappa(\alpha) \;=\; \beta\int_0^1 \phi(\alpha(1+(1+\eta)\log u)\,du \;-\; \beta \tag{2.2}$$

$$\;=\; \beta\mathbb{E}\left[\frac{e^{\alpha U}}{1+(1+\eta)\alpha U}\right] \;-\; \beta \tag{2.3}$$

Proof The result is a variant of one given recently in Example 2 of Nyrhinen [13]. For the verification via Theorem 2, note first that if $\{A'_t\}$ is a time inhomogeneous compound Poisson process with arrival intensity $\beta(t)$ and jump size distribution with m.g.f. $\phi_t(\alpha)$ at time t, then (derive, e.g., a differential equation in t)

$$\log\mathbb{E}e^{\alpha A'_t} \;=\; \int_0^t \beta(s)(\phi_s(\alpha)-1)\,ds\,. \tag{2.4}$$

Write

$$S_t \;=\; \sum_{i=1}^{N_t} U_i \;-\; (1+\eta)\int_0^t \frac{\sum_{i=1}^{N_s} U_i}{s}\,ds \;=\; \sum_{i=1}^{N_t} U_i\left(1-(1+\eta)\log\frac{t}{T_i}\right) \tag{2.5}$$

so that $R_t = u - S_t$, and let $\kappa_t(\alpha) = \log\mathbb{E}e^{\alpha S_t}$. It is then follows from (2.4) that

$$\kappa_t(\alpha) \;=\; \beta\int_0^t \phi\left(\alpha\left[1-(1+\eta)\log\frac{t}{s}\right]\,ds\right) - \beta t \;=\; t\kappa(\alpha) \tag{2.6}$$

where κ is given by the first expression in (2.2). For the second, note the probabilistic interpretation $S_1 \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_1} Y_i$ where

$$Y_i \;=\; U_i(1+(1+\eta)\log\Theta_i) \;=\; U_i(1-(1+\eta)V_i)$$

where the Θ_i are i.i.d. uniform$(0,1)$ or, equivalently, the $V_i = -\log\Theta_i$ are i.i.d. standard exponential. Then

$$\mathbb{E}e^{\alpha Y} \;=\; \mathbb{E}\left[\Theta^{(1+\eta)\alpha U}e^{\alpha U}\right] \;=\; \mathbb{E}\left[e^{\alpha U}\int_0^1 t^{(1+\eta)\alpha U}\,dt\right] \;=\; \mathbb{E}\left[\frac{e^{\alpha U}}{1+(1+\eta)\alpha U}\right].$$

Considering a discrete skeleton $\{S_{nh}\}_{n=0,1,\ldots}$, (2.6) implies that $\kappa_{nh}(\alpha)/n$ has a limit (in fact, is constant), and so by Theorem 2 (the remaining regularity conditions are straightforward to verify) we conclude that $\mathbb{P}(\max_n S_{nh} > u) \sim e^{-\gamma u}$ in the logarithmic sense. An easy argument that we omit shows that one can replace the maximum over nh by the continuous time maximum over t, and since $\Psi(u) = \mathbb{P}(\max_{t\ge 0} S_t > u)$, we obtain the desired conclusion. $\square$

We proceed to some aspects of Theorem 3 not discussed in [13].

2.1 Comparison of the adjustment coefficients γ, γ^*

The defining equations $\kappa(\gamma) = 0$, $\kappa^*(\gamma^*) = 0$ mean

$$\mathbb{E}\left[\frac{e^{\gamma U}}{1+(1+\eta)\gamma U}\right] \;=\; 1, \quad \text{resp.} \quad \mathbb{E}e^{\gamma^* U} \;=\; \mathbb{E}\left[1+(1+\eta)\gamma^* U\right] .$$

From this is follows at once that if U is degenerate, $U \equiv m$, then $\gamma = \gamma^*$.

Theorem 4 *$\gamma \geq \gamma^*$, with equality if and only if U is degenerate.*

Proof The function $k(x) = e^{\gamma^* x} - 1 - (1+\eta)\gamma^* x$ is convex with $k(\infty) = \infty$, $k(0) = 0$, $k'(0) < 0$, so there exists a unique zero $x_0 = x_0(\eta) > 0$ such that $k(x) > 0$, $x > x_0$, and $k(x) < 0$, $0 < x < x_0$. Therefore

$$\begin{aligned}
\mathbb{E}\left[\frac{e^{\gamma^* U}}{1+(1+\eta)\gamma^* U}\right] - 1 \;&=\; \mathbb{E}\left[\frac{k(U)}{1+(1+\eta)\gamma^* U}\right] \\
&=\; \int_0^{x_0} \frac{k(y)}{1+(1+\eta)\gamma^* y}\, B(dy) \;+\; \int_{x_0}^{\infty} \frac{k(y)}{1+(1+\eta)\gamma^* y}\, B(dy) \\
&\leq\; \frac{1}{1+(1+\eta)\gamma^* x_0}\left\{\int_0^{x_0} k(y)\, B(dy) \;+\; \int_{x_0}^{\infty} k(y)\, B(dy)\right\} \;=\; 0, \qquad (2.7)
\end{aligned}$$

using $\mathbb{E}k(U) = 0$. This implies $\kappa(\gamma^*) \leq 0$, and since $\kappa(s)$, $\kappa^*(s)$ are convex with $\kappa'(0) < 0$, ${\kappa^*}'(0) < 0$, this in turn yields $\gamma \geq \gamma^*$. Further, equality in (2.7) can only occur if $U \equiv x_0$. □

In Section 5, we consider the heavy traffic limit $\eta \downarrow 0$. The detailed properties of γ, γ^* in this limit are given by the following result. Write m_2, m_3 for the second, resp. third moment of B, $m_1 = m$.

Proposition 1 *As $\eta \downarrow 0$,*

$$\gamma^* \;=\; \gamma^*(\eta) \;=\; \frac{2m_1}{m_2}\eta - \frac{4m_1^2 m_3}{3m_2^3}\eta^2 + O(\eta^3), \qquad (2.8)$$

$$\gamma \;=\; \gamma(\eta) \;=\; \frac{2m_1}{m_2}\eta + \frac{8m_1^2 m_3 - 12 m_1 m_2^2}{3m_2^3}\eta^2 + O(\eta^3)\,. \qquad (2.9)$$

Proof By Taylor expansion, we have up to $O(\eta^3)$ terms that

$$\begin{aligned}
&\frac{e^{\gamma x}}{1+(1+\eta)\gamma x} \\
&=\; \left(1 + \gamma x + \frac{\gamma^2 x^2}{2} + \frac{\gamma^3 x^3}{6}\right)\left(1 - \gamma(1+\eta)x + \gamma^2(1+\eta)^2 x^2 - \gamma^3(1+\eta)^3 x^3\right) \\
&=\; 1 - \eta\gamma x + (\gamma^2/2 + \eta\gamma^2)x^2 - \gamma^3/3\, x^3\,.
\end{aligned}$$

Taking expectations, equating to 1 and dividing by γ yields

$$\eta m_1 \;=\; (1+2\eta)\frac{\gamma}{2}m_2 - \frac{\gamma^2}{3}m_3.$$

Thus, $\gamma = 2m_1/m_2\,\eta + O(\eta^2)$. Substituting the trial solution $\gamma = 2m_1/m_2(\eta + c\eta^2) + O(\eta^3)$ yields

$$\eta m_1 \;=\; (1+2\eta)(\eta+c\eta^2)m_1 - \frac{4m_1^2m_3}{3m_2^2}\eta^2, \quad 2+c-\frac{4m_1m_3}{3m_2^2}=0$$

and the desired expansion for γ follows easily. The one for γ^* is obtained similarly, though slightly easier. □

Remark 1 It follows from Proposition 1 that

$$\gamma'(\eta)|_{\eta=0} = \gamma^{*\prime}(\eta)|_{\eta=0} = \frac{2m_1}{m_2}$$

whereas

$$\gamma''(\eta)|_{\eta=0} = 2\frac{8m_1^2m_3 - 12m_1m_2^2}{3m_2^3} \geq \frac{8m_1^2m_3 - 12m_1m_2^2}{3m_2^3} = \gamma^{*\prime\prime}(\eta)|_{\eta=0}$$

That $\geq$ holds is immediate from Theorem 4 but follows also from the moment inequality $(\mathbb{E}U^2)^2 \leq \mathbb{E}U \cdot \mathbb{E}U^3$ which is well-known and may easily be derived from the log convexity of the m.g.f. of $\log U$.

2.2 The large deviations path

We next consider the question of how ruin occurs. One would expect that there is again an initial change of linear drift of $\{R_t\}$ from positive to negative. However, this issue is not addressed in [8], and it is also not clear what are the implications for the more interesting question of how $\{A_t\}$ itself behaved given ruin.

The answer is roughly: with initial reserve u, ruin occurs by $\{A_t\}$ changing distribution from a time–homogeneous Poisson process with parameters β, B to a time–inhomogeneous one with arrival intensity $\beta_s^{(u)}$ and jump size distribution $B_s^{(u)}$ with m.g.f. $\phi_s(\alpha)$ at time s given by

$$\beta_s^{(u)} \;=\; \beta\phi(\gamma + (1+\eta)\gamma\log\{s\kappa'(\gamma)/u\}), \tag{2.10}$$

$$\phi_s^{(u)}(\alpha) \;=\; \frac{\phi([\alpha+\gamma][1+(1+\eta)\log\{s\kappa'(\gamma)/u\}])}{\phi(\gamma+(1+\eta)\gamma\log\{s\kappa'(\gamma)/u\})} \tag{2.11}$$

This follows since according to [8], ruin occurs roughly at time $u/\kappa'(\gamma)$ and as if the cumulant g.f. of $S_{u/\kappa'(\gamma)}$ was changed from $\kappa_{u/\kappa'(\gamma)}(\alpha)$ (cf. (2.6)) to

$$
\begin{aligned}
&\kappa_{u/\kappa'(\gamma)}(\alpha+\gamma) - \kappa_{u/\kappa'(\gamma)}(\gamma) \\
= \quad & \beta \int_0^{u/\kappa'(\gamma)} \phi\left((\alpha+\gamma)\left[1-(1+\eta)\log\frac{u/\kappa'(\gamma)}{s}\right] ds\right) \\
- \quad & \beta \int_0^{u/\kappa'(\gamma)} \phi\left(\gamma\left[1-(1+\eta)\log\frac{u/\kappa'(\gamma)}{s}\right] ds\right) \\
= \quad & \int_0^{u/\kappa'(\gamma)} \beta_s^{(u)}\left(\phi_s^{(u)}(\alpha)-1\right) ds
\end{aligned}
$$

The desired conclusion thus follows by appealing to (2.4).

The implications of (2.10), (2.11) are:

1. $\beta_s^{(u)}$ increases monotonically from 0 to $\beta\hat{B}[\gamma]$ as s increases from 0 to the time $u/\kappa'(\gamma)$ of ruin. On the time interval $[0, s_0)$, $\beta^{(u)}(s)$ is smaller than the typical value β whereas $\beta_s^{(u)} > \beta$ for $s \in (s_0, u/\kappa'(\gamma)]$; here the switchover point is $s_0 = ue^{-1/(1+\eta)}/\kappa'(\gamma)$.

2. $B_s^{(u)}$ increases in stochastic ordering as s increases from 0 to $u/\kappa'(\gamma)$, such that $B_s^{(u)} <_{\mathrm{st}} B$ for $s < s_0$ and $B_s^{(u)} >_{\mathrm{st}} B$ for $s > s_0$.

3. For simulation, the implication is that the natural algorithm for simulating $\Psi(u)$ is to simulate not the given time–homogeneous compound Poisson model with parameters β, B but a time–inhomogeneous one with parameters $\beta_s^{(u)}$, $B_s^{(u)}$. The simulation estimator is then the likelihood ratio at time $T(u)$. See for example Bucklew, Ney & Sadowsky [5] and Glynn & Glynn [7] for related discussion.

3 Some premium rules with short memory

The analysis of Section 2.2 seems to indicate that a main problem with the premium rule (1.2) is its long memory. Motivated from this, we now investigate alternatives which put more weight on recent claims statistics.

For mathematical convenience, we will carry out the analysis in a discrete time setting. Thus, let U_n be the claims encountered in year n and assume that the premium $p(n)$ charged in year n is a weighted average of $U_{n-1}, U_{n-2}, \ldots$:

$$
p(n) \;=\; (1+\eta)\left\{f_0 U_{n-1} + f_1 U_{n-2} + \cdots\right\} \qquad (3.1)
$$

where $\{f_n\}$ is a probability mass function (we assume a complete history of claims so that $\mathcal{G}_n = \sigma(U_k : k = n, n-1, n-2, \ldots, -\infty))$. The corresponding risk process is $R_n = u - S_n$ where

$$S_n = \sum_{i=1}^{n} U_i - (1+\eta) \sum_{i=1}^{n} \sum_{j=-\infty}^{i-1} U_j f_{i-1-j}.$$

For example, if $f_0 = 1$, $f_1 = f_2 = \ldots = 0$, then $p(n) = (1+\eta)U_{n-1}$ and $S_n = U_n - \eta \sum_0^{n-1} U_i$.

We solely consider the case where the radius θ^* of convergence of $\phi(\theta) = \mathbb{E}e^{\theta U}$ is finite and where $\mathbb{E}e^{\theta U} \uparrow \infty$, $\theta \uparrow \theta^*$ (much of the analysis carries over in a straightforward way to the case $\theta^* = \infty$ but the results need some reformulation).

It is clear that in the model formulation of Section 1, one will always have $\gamma \leq \theta^*$ (one can never avoid the effect of one early large claim). However, indeed the model (3.1) attains the optimal rate of decay θ^* of $\psi(u)$:

Theorem 5 *Assume* $0 < \theta^* < \infty$. *Then for any premium rule of the form* (3.1) *with* $f_n \geq 0$, $\sum_0^\infty f_n = 1$, $\sum_0^\infty n f_n < \infty$, *one has* $\gamma = \theta^*$.

Proof Let $F_n = f_0 + \cdots + f_n$, $\omega(\alpha) = \log \phi(\alpha)$, $\kappa_n(\alpha) = \log \mathbb{E}e^{\alpha S_n}$. Then

$$\begin{aligned}
S_n &= U_n - (1+\eta) \sum_{j=-\infty}^{n-1} U_j \sum_{i=1\wedge(j+1)}^{n} f_{i-1-j} \\
&= U_n - \sum_{j=1}^{n-1} U_j((1+\eta)F_{n-1-j} - 1) - (1+\eta) \sum_{j=-\infty}^{0} U_j(F_{n-1-j} - F_{-j-1}), \\
\kappa_n(\alpha) &= \omega(\alpha) + \sum_{j=1}^{n-1} \omega(-\alpha[(1+\eta)F_{n-1-j} - 1]) \\
&\quad + \sum_{j=-\infty}^{0} \omega(-\alpha[1+\eta][F_{n-1-j} - F_{-j-1}])
\end{aligned}$$

Taylor expanding ω around 0 and using $\sum_0^\infty n f_n < \infty$, it follows easily that the last sum has the finite limit

$$\sum_{j=-\infty}^{0} \omega(-\alpha[1+\eta][f_{-j} + f_{1-j} + \cdots])$$

Since each term in the first sum has limit $\omega(-\eta\alpha)$, it follows that

$$\frac{1}{n}\kappa_n(\alpha) \to \omega(-\eta\alpha)$$

Note that this limit is the cumulant g.f. of a negative r.v. so we cannot directly apply Theorem 2 (the root in (ii) fails to exist). However, let $\tilde{S_n} = S_n + n\epsilon$. Then in obvious notation, $\tilde{\kappa}_n(\alpha)/n \to \epsilon\alpha + \omega(-\eta\alpha)$. For $0 < \epsilon < \eta m$, there exists a unique root $\tilde{\gamma} > 0$ of $\epsilon\tilde{\gamma} + \omega(-\eta\tilde{\gamma}) = 0$ and $\tilde{\gamma} \to 0$ as $\epsilon \uparrow \eta m$, $\tilde{\gamma} \to \infty$ as $\epsilon \downarrow 0$. Hence we can choose ϵ such that $\tilde{\gamma}$ is smaller than but arbitrarily close to θ^*. Then Condition (iii) of Theorem 2 holds, and since the rest are trivial, we get $\tilde{\Psi}(u) \sim e^{-\tilde{\gamma}u}$ in the logarithmic sense. Since obviously $\Psi(u) \le \tilde{\Psi}(u)$, we get

$$\limsup_{u\to\infty} \frac{\log \Psi(u)}{u} \le -\tilde{\gamma}, \quad \limsup_{u\to\infty} \frac{\log \Psi(u)}{u} \le -\theta^*.$$

As noted above, $\liminf \ge$ is trivial, so the proof is complete. □

4 Heavy–tailed claims: proof of Theorem 2

From $A_t/t \stackrel{\text{a.s.}}{\to} \beta m$, (1.2) and (1.5) we get $S_t/t \stackrel{\text{a.s.}}{\to} -\eta\beta m$ where $S_t = u - R_t$. Hence given $\epsilon > 0$, we can choose K such that $\mathbb{P}(F) > 1 - \epsilon$ where

$$F_t \;=\; \{-K - (\eta\beta m + \epsilon)v \;< S_v \;< K - (\eta\beta m - \epsilon)v \text{ for all } v \le t\}\,.$$

Now the predictable intensity for the event of ruin is $I(T(u) \ge t)\beta\overline{B}(x - S_{t-})$ where $\overline{B}(x) = 1 - B(x)$, so

$$\begin{aligned}\Psi(u) \;&=\; \beta\int_0^\infty \mathbb{E}\left[\overline{B}(u - S_{t-};\, T(u) \ge t\right]\,dt\\ &\ge\; \beta\int_0^\infty \mathbb{E}\left[\overline{B}(u - S_{t-};\, T(u) \ge t, F_t\right]\,dt\end{aligned}$$

Let $u \ge K$. Then $F_t \subseteq \{T(u) \ge t\}$ and hence

$$\begin{aligned}\Psi(u) \;&\ge\; \beta\int_0^\infty \overline{B}(u + (\eta\beta m + \epsilon)t)\mathbb{P}(F_t)\,dt\\ &\ge\; (1-\epsilon)\beta\int_0^\infty \overline{B}(u + (\eta\mu + \epsilon)t)\,dt\\ &\ge\; \frac{1-\epsilon}{\eta m}\int_u^\infty \overline{B}(y)\,dy\end{aligned}$$

Combining with the well–known asymptotics

$$\Psi^*(u) \;\sim\; \frac{1}{\eta m}\int_u^\infty \overline{B}(y)\,dy$$

(e.g. [6]) and letting $\epsilon \downarrow 0$, the proof is complete. □

5 Models governed by SDE's

Write $\mu = \beta m$, $\sigma^2 = \beta^{(2)}$, $D_t = A_t - \mu t$. Assume in the following that $\sigma^2 = 1$ which can be achieved by a change of time scale, and that $\mu = 1$ which can be achieved by a change of scale. Then it is standard (Donsker's theorem in continuous time) that

$$\left\{\eta D_{t/\eta^2}\right\}_{t\geq 0} \xrightarrow{\mathcal{D}} \{w_t\}_{t\geq 0}$$

(in the sense of weak convergence in the Skorokhod space $D[0,\infty)$), where $\{w_t\}$ is standard Brownian motion; the interpretation of η as the safety loading is not important for this, it suffices that $\eta \downarrow 0$. Writing

$$\eta R^*(t/\eta^2) \;=\; \eta u + \eta(1+\eta)t/\eta^2 - \eta D_{t/\eta^2} - \eta t/\eta^2$$

and assuming $\eta u \to x$, we thus obtain the well–established diffusion approximation

$$\left\{\eta R^*_{t/\eta^2}\right\}_{t\geq 0} \xrightarrow{\mathcal{D}} \{x + t - w_t\}_{t\geq 0} \tag{5.1}$$

cf. Iglehart [12] and Grandell [9], [10], [11]. In the same way, we can write

$$\begin{aligned}
\eta R(t/\eta^2) \;&=\; \eta u + \eta(1+\eta)\int_0^{t/\eta^2} \frac{D_s}{s} ds + \eta(1+\eta)t/\eta^2 - \eta D_{t/\eta^2} - \eta t/\eta^2 \\
&=\; \eta u + t + (1+\eta)\int_0^t \frac{\eta D_{s/\eta^2}}{s} ds - \eta D_{t/\eta^2}
\end{aligned} \tag{5.2}$$

and are lead to:

Proposition 2 *Consider the process* (1.2)*. Then if* $\eta \downarrow 0$*,* $\eta u \to x$*, then*

$$\left\{\eta R_{t/\eta^2}\right\}_{t\geq 0} \xrightarrow{\mathcal{D}} \{x + t + z_t\}_{t\geq 0}\ , \tag{5.3}$$

where $z_t \;=\; \int_0^t \frac{w_s}{s} ds - w_t$*.*

The proof is given at the end of this section. In fact, the limits in (5.1) and (5.3) have the same distribution:

Proposition 3 $\{z_t\}_{t\geq 0}$ *is standard Brownian motion.*

In particular, the limiting diffusions $\{r^*_t\}$, $\{r_t\}$ for $\{R^*_t\}$, $\{R_t\}$ given by

$$dr^*_t = 1 - dw_t, \quad dr_t = 1 + dz_t, \quad r^*(0) = r(0) = x,$$

have the same ruin probabilities

$$\psi^*(x) \;=\; \mathbb{P}(\tau^*(x) < \infty) \;=\; \psi(x) = \mathbb{P}(\tau(x) < \infty) \;=\; e^{-2x}$$

where

$$\tau^*(x) \;=\; \inf\{t > 0 : \, r^*(t) = 0\}\,, \quad \tau(x) \;=\; \inf\{t > 0 : \, r(t) = 0\}$$

are the corresponding ruin times. On one hand, this is not surprising in view of the asymptotic properties of γ, γ^* for small η given in Proposition 1 and Remark 1. On the other, it is not straightforward to recognize $\{z_t\}$ as standard Brownian motion. However, Proposition 3 can be found in Ch. 1 of Yor [15]. For a direct proof, note that $\{z_t\}$ is obviously Gaussian with mean zero so that it suffices to show $Cov(z_t, z_u) = t$ for $t \le u$. To this end, let $w'_t = tw_{1/t}$ and note that $\{w'_t\}$ is again standard Brownian motion. Thus

$$z_t \;=\; \int_0^t w'_{1/s} ds - tw'_{1/t} \;=\; -\int_{1/t}^\infty \frac{1}{s^2} w'_s \, ds - tw'_{1/t} \;=\; \int_{1/t}^\infty \frac{1}{s} dw'_s \tag{5.4}$$

which yields

$$Cov(z_t, z_u) \;=\; \int_{[1/t,\infty)\cap[1/u,\infty)} \frac{1}{s^2}\, ds \;=\; \int_{1/t}^\infty \frac{1}{s^2}\, ds \;=\; t.$$

It is standard how ruin occurs for $\{r^*_t\}$, $\{r_t\}$, namely as if the drifts for the governing Brownian motions $\{w_t\}$, resp. $\{z_t\}$, were changed from 0 to 2, resp. from 0 to -2. For example,

$$\mathbb{P}^x\left(\{z_t\}_{0\le t\le \tau(x)} \in \cdot\right) \;=\; \mathbb{P}\left(\{z_t - 2t\}_{0\le t\le \tau_1(x)} \in \cdot\right) \tag{5.5}$$

where $\tau_1(x) = \inf\{t : \, z_t - t = -x\}$ and similarly for $\{r^*_t\}$. However, in the case of $\{r_t\}$ the more interesting question is what this means for $\{w_t\}$ and thereby the claims process $\{A_t\}$ (cf. the similar discussion in Section 2.2). Our first result gives an exact description; one possible interpretation is as a simulation algorithm for generating a sample path of $\{w_t\}_{0\le t\le \tau(x)}$ with distribution $\mathbb{P}^x$, the conditional distribution of $\{w_t\}_{t\ge 0}$ given $\tau(x) < \infty$.

Theorem 6 *Let $\tilde z$ be a standard Brownian motion, $\tilde\tau_1(x) = \inf\{t : \, \tilde z_t - 2t = -x\}$ and let V be a standard normal variable independent of $\tilde z$,*

$$\tilde w_t \;=\; t\int_t^{\tilde\tau_1(x)} \frac{1}{s} d\tilde z_s \;+\; 2t \log \frac{t}{\tilde\tau_1(x)} \;+\; t\sqrt{\tilde\tau_1(x)}\, V\,.$$

Then the distribution of $\{\tilde w_t\}_{0\le t\le \tilde\tau_1(x)}$ is the same as $\mathbb{P}^x$–distribution of $\{w_t\}_{0\le t\le \tau(x)}$.

Let $\mathcal{F}^w = \{\mathcal{F}^w_t\}_{t\ge 0}$ denote the filtration generated by w, i.e. $\mathcal{F}^w_t = \sigma(w_s : 0 \le s \le t)$ (similar notation is used for $\mathcal{F}^z$, $\mathcal{F}^z_t$ etc.)

For the proof of Theorem 6, we first note that the representation (5.4) also shows that $z_v = \int_{1/v}^{\infty} s^{-1} dw'_s$ and $w_u = u \int_0^{1/u} dw'_s$ are uncorrelated for for $v \le t \le u$ so that

Lemma 1 *For any t, $\mathcal{F}^z_t$ and $\{w_s\}_{s\ge t}$ are independent. In particular, if $\tau < \infty$ is a stopping time w.r.t. $\mathcal{F}^z$, then the conditional distribution of w_τ given $\mathcal{F}^z_\tau$ is normal $(0,\tau)$.*

(the last statement can be shown, e.g., by first considering the case where τ has a discrete support and next considering an approximation of τ from above with such stopping times).

Let $b^t = \{b^t_s\}_{0\le s\le t}$ be the Brownian bridge up to time t, $b^t_s = w_s - \dfrac{s}{t} w_t$, and $\mathcal{F}^b_t = \sigma(b^t) = \sigma(b^t_s : 0 \le s \le t)$.

Lemma 2 *For $v \le t \le u$, $z_t = \displaystyle\int_0^t \frac{b^u_s}{s}\, ds - b^u_t$, $\quad b^t_v = v \displaystyle\int_v^t \frac{1}{s}\, dz_s$. In particular, $\mathcal{F}^z_t = \mathcal{F}^b_t$.*

Proof The expression for z_t in terms of b^u is obvious from the definition of the Brownian bridge, which also yields

$$\frac{d}{dt} b^t_v \;=\; -\frac{v}{t} dw_t + \frac{v}{t^2} w_t \;=\; \frac{v}{t} dz_t$$

and hence the expression for b^t in terms of $\{z_s\}_{s\le t}$. $\square$

Proof of Theorem 6. Analogously to (5.5), the $\mathbb{P}^x$–distribution of $\{z_t\}_{t\le\tau(x)}$ is the same as the distribution of $\{\tilde z_t - 2t\}_{t\le\tilde\tau_1(x)}$. Hence by Lemma 2, the $\mathbb{P}^x$–distribution of $\left\{b^{\tau(x)}_t\right\}_{t\le\tau(x)}$ is the same as the distribution of

$$\left\{ t\int_t^{\tilde\tau_1(x)} \frac{1}{s}(d\tilde z_t - 2dt)\right\}_{t\le\tilde\tau_1(x)} \;=\; \left\{ \tilde b^{\tilde\tau_1(x)}_t + 2t\log\frac{t}{\tilde\tau_1(x)}\right\}_{t\le\tilde\tau_1(x)}$$

Writing

$$w_t \;=\; b^{\tau(x)}_t + \frac{t}{\tau(x)} W_{\tau(x)}, \quad t \le \tau(x),$$

and appealing to Lemma 1, the proof is complete. $\square$

Corollary 1 *For any $p > 1/2$, $\displaystyle\sup_{0\le t\le\tau(x)} \left|\frac{1}{x^p}\left(w_t - 2t\log\frac{t}{x}\right)\right| \;\to\; 0$ in $\mathbb{P}^x$–probability as $x \to \infty$.*

Proof By Theorem 6 and its proof,

$$\left\{t\int_t^{\tilde{\tau}_1(x)} + t\sqrt{\tilde{\tau}_1(x)}\,V\right\}_{0\le t\le\tau(x)} \ \stackrel{\mathcal{D}}{=}\ \left\{b_t^{\tau^*(x)}\right\}_{0\le t\le\tau^*(x)}\ \Big|\ \tau^*(x)<\infty \qquad (5.6)$$

in $\mathbb{P}^x$ distribution uiformly in $0\le t\le\tau(x)$. Further $x^{-p}\sup_{0\le t\le ax} w_t \xrightarrow{\mathbb{P}} 0$ for any a implies $x^{-p}\sup_{0\le t\le v\le ax} b_t^v \xrightarrow{\mathbb{P}} 0$ and since

$$\lim_{x\to\infty}\mathbb{P}(\tau^*(x)>ax\,|\,\tau^*(x)<\infty) \ =\ 0 \qquad (5.7)$$

(use, e.g., that $\tau^*(x) = x+\sqrt{x}V(x)$ where the limiting $\mathbb{P}^x$–distribution of $V(x)$ is standard normal), it follows that (5.6) is negligible after division by x^p. Thus

$$\begin{aligned}
&\sup_{0\le t\le\tau(x)}\left|\frac{1}{x^p}\left(w_t - 2t\log\frac{t}{x}\right)\right| \\
&\stackrel{\mathcal{D}}{=}\ \sup_{0\le t\le\tau(x)}\left|\frac{1}{x^p}\left(2t\log\frac{t}{\tilde{\tau}_1(x)} - 2t\log\frac{t}{x}\right)\right| \ +\ o(1) \\
&=\ \sup_{0\le t\le\tau(x)}\frac{1}{x^p}2t\log(1+V(x)/\sqrt{x}) \ +\ o(1) \\
&=\ \frac{1}{x^p}O(x)O\left(\frac{1}{x^{1/2}}\right) \ +\ o(1) \ =\ o(1)
\end{aligned}$$

(using once more (5.7) for the $O(x)$ term). $\square$

The implication of the above results is roughly a change of drift at time t of w from 0 to

$$\frac{d}{dt}\,2t\log(t/x) \ =\ 2+2\log t - 2\log x$$

given $\tau(x)<\infty$. However, this is in an asymptotic sense, not exact as in (5.5), and the $\mathbb{P}^x$–distribution of $\{w_t\}_{0\le t\le\tau(x)}$ cannot be viewed as the distribution of a stopped diffusion.

At the end, we briefly consider the limits as $\eta\downarrow 0$ of the models in Section 3 with short memory. For a simple case, consider

$$v_t \ =\ x+t+w_t-w_{t-1}$$

corresponding to $f_{\lfloor 1/\eta^2\rfloor}=1$. Then obviously the ruin probability satisfies

$$\begin{aligned}
\mathbb{P}\left(\inf_{t\ge 0} v_t\le 0\right) \ &\ge\ \sup_{t\ge 0}\mathbb{P}(w_{t-1}-w_t > x+t) \\
&=\ \mathbb{P}(N(0,1)>x) \ =\ 1-\Phi(x) \ \sim\ \frac{e^{-x^2/2}}{x\sqrt{2\pi}}.
\end{aligned}$$

There are many results in the literature which indicate that a lower bound obtained in this way is often close to the correct asymptotics but we will not go into a further discussion of this.

We finally turn to the proof of Proposition 2. The problem is to control the behaviour at $t = 0$.

Lemma 3 *Define* $Z_t^a = \int_0^t \frac{D_s}{s+a}\, ds$. *Then there exists* t_0 *such that* $\mathbb{P}(Z_t^a \geq 0) \geq 1/4$ *for all* $t \geq t_0$ *and all* $a \leq t$.

Proof In the same way as in the proof of Theorem 3, we can write

$$Z_t^a \ \stackrel{\mathcal{D}}{=}\ \sum_{i=1}^{N_t} U_i W_i - \int_0^t \frac{s}{s+a}\, ds$$

where $W_i^a = \log((t+a)/(t\Theta_i + a))$: Note by explicit calculation that

$$\mathbb{E}W_i^a \ = \ 1 - \frac{a}{t}\log\frac{t+a}{t} \ = \ \frac{1}{t}\int_0^t \frac{s}{s+a}\, ds$$

and that

$$\inf_{a\leq t} \mathbb{E}W_i^{a^2} > 0, \quad \sup_{a\leq t} \mathbb{E}W_i^{a^3} < \infty$$

Hence by the Berry–Esseen theorem, there exists a constant C such that

$$\left| \mathbb{P}\left(Z_t^a \geq 0 | \, N_t = n\right) - \frac{1}{2}\right| \ \leq \ \frac{C}{n^{1/2}}$$

for all t and for all $a \leq t$. Using the LLN to bound N_t below, the result follows. □

Lemma 4 *For any* $\delta > 0$, $\mathbb{P}\left(\sup_{0\leq v\leq t} \left|Z_v^0\right| > \delta\right) \ \leq \ 4\mathbb{P}\left(\left|Z_{2t}^0\right| > \delta\right)$ *for all* $t \geq t_0$.

Proof Define $\sigma = \inf\{v > 0 : \ |Z_v^0| > \delta\}$. Writing

$$Z_{2t}^0 \ = \ Z_\sigma^0 + D_\sigma \log\frac{2t}{\sigma} + \int_\sigma^{2t} \frac{D_s - D_\sigma}{s}\, ds$$

and noting that $D_\sigma \geq 0$ on $\{Z_\sigma^0 > \delta\}$, we obtain

$$\begin{aligned}
\mathbb{P}\left(Z_{2t}^0 > \delta\right) \ &\geq \ \mathbb{P}\left(\sigma \leq t, Z_\sigma^0 > \delta, \int_\sigma^{2t} \frac{D_s - D_\sigma}{s}\, ds \geq 0\right) \\
&\geq \ \frac{1}{4}\mathbb{P}\left(\sigma \leq t, Z_\sigma^0 > \delta\right),
\end{aligned}$$

using Lemma 3. Adding with a similar bound at $-\delta$ yields

$$4\mathbb{P}\left(\left|Z^0_{2t}\right| > \delta\right) \;\geq\; \mathbb{P}\left(\sigma \leq t, \left|Z^0_\sigma\right| > \delta\right) \;=\; \mathbb{P}\left(\sup_{0\leq v\leq t}\left|Z^0_v\right| > \delta\right).$$

$\square$

Proof of Proposition 2. Clearly for any $\epsilon > 0$,

$$\left\{\int_\epsilon^{t+\epsilon} \frac{\eta D_{s/\eta^2}}{s} ds\right\}_{t\geq 0} \xrightarrow{\mathcal{D}} \left\{\int_\epsilon^{t+\epsilon} \frac{w_s}{s}\, ds\right\}_{t\geq 0} \tag{5.8}$$

Hence the desired conclusion follows immediately from (5.2) and

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq t\leq\epsilon}\left|\int_0^t \frac{\eta D_{s/\eta^2}}{s}\, ds\right|\right) \;&=\; \mathbb{P}\left(\sup_{0\leq t\leq \epsilon/\eta^2} \eta\left|Z^0_t\right| > \delta\right) \;\leq\; 4\mathbb{P}\left(\eta\left|Z^0_{2\epsilon/\eta^2}\right| > \delta\right)\\ &\leq\; \frac{4O(\epsilon/\eta^2)}{\delta^2/\eta^2} \;\to 0\end{aligned}$$

as $\epsilon \downarrow 0$ with δ fixed. Here the last inequality follows from $Var(Z^0_t) = O(t)$ (explicit calculation). $\square$

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