The quantum stochastic evolution of an open system under continuous in time nondemolition measurement

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We propose a new approach to the description in the general case of continuous in time indirect measurement of an open system. Our approach is based not on the concept of a generating map of an instrument as the way for the description of an indirect measurement process and not on quantum stochastic calculus as a tool of consideration but on the methods of quantum theory and the Schrödinger equation.

Our approach is valid for a broad class of quantum measurement models and quantum input processes but not only in the case of the Markovian approximation.

In the general case we introduce the operator describing the evolution of an open system under the condition that the output process was continuously observed until the moment t and found to have the definite trajectory. We derive the integral equation describing the quantum stochastic evolution of an open system in the general case of nondemolition observation.

As an example of application of our results to concrete measurement models we consider the special measurement model which is the extended variant (including the gauge term) usually considered in the frame of so called quantum stochastic mechanics. We get the new equation describing the quantum stochastic evolution of an open system under continuous in time diffusion observation.

This equation can be rewritten in the stochastic form which in case of the vacuum initial state of a reservoir and the absence of the gauge term coincides with the well known quantum filtering equation in quantum stochastic mechanics introduced by V.P.Belavkin.

Keywords and phrases: Quantum theory, continuous in time measurements, nondemolition observation of an open system.

1. Introduction.

Different aspects of description of continuous in time observation of an open system were considered in the well known papers of A.Barchielli, V.P.Belavkin, E.B. Davies, L.Diosi, A.S.Holevo, M.Ozawa, A.Peres and others [3-22, 26, 27].

The usual approach to the description of continuous in time indirect observation of an open system is based on the use of the dynamic measurement model of the so called stochastic Schrödinger equation -- the particular case of quantum stochastic differential equation introduced by R.L.Hudson et al. [24].

For a description of the process of indirect measurement the concept of a generating map of an instrument of indirect observation introduced by A.Barchielli [3-7] is used.
The well known quantum filtering equation introduced first in papers of V.P.Belavkin [9-15] and describing the posterior quantum stochastic evolution of an open system subjected to continuous in time nondemolotion measurement is a quantum stochastic one. The mathematical properties of solutions of such stochastic equations were studied in [8,23].

But as it was shown by us in [1] even for this simple quantum measurement model the quantum stochastic calculus is not an obligatory technique for the description of that continuous in time observation and the derivation of the quantum filtering equation. The standard methods of quantum theory can be used as well and give the same results in this special model.
In fact the stochastic Schrödinger equation corresponds to only one very special kind of quantum theory model -- the model which describes the behaviour of an open system plus an extremely idealized reservoir. Despite of its name this equation does not include the stochasticity caused by the measurement process. As well as the "usual" Schrödinger equation it describes the unitary time evolution of an open system plus the extremely idealized reservoir but only in the new type of calculus.
In [25] it was shown that in the sense of weak solutions the more general quantum stochastic differential equation of R.L.Hudson and K.R.Parthasarathy (including the gauge term) is equivalent to the "usual" Schrödinger equation with the special kind of Hamiltonian.
The situation in the case of a general quantum theory model was considered in papers of L.Accardi [2]. L. Accardi showed that in the general case the quantum stochastic differential equations appear in quantum theory only as a result of some limit procedure corresponding to the Markovian approximation.

In the present paper we would like to consider in the general case the problem of correct description of the quantum stochastic evolution of an open system under continuous in time indirect observation. We do not concretize the type of interaction between an open system and a reservoir. Our approach is based not on quantum stochastic calculus and not on the concept of a generating map of an instrument but on standard methods of quantum theory. That is why it is valid for a broad class of quantum measurement models and quantum input processes and not only in the case of the Markovian approximation.

It is well known [17, 26] that from the mathematical point of view the POV measure of any indirect measurement can be regarded as the result of averaging some projection-valued measure given on a larger Hilbert space over auxiliary degrees of freedom. This mathematical fact is in perfect correlation with the real physical situation where any indirect observation of an open system is performed by means of direct observation of some von Neumann observable of the reservoir modelling the measuring device. Due to the interaction with an open system the direct measurement of an observable of the reservoir gives us indirect information and a reduced description of an open system.
The case of continuous in time measurement cannot be an exclusion for such a situation since in the Heisenberg picture we can describe the whole process of continuous in time indirect observation of an open system with the help of the POV measure on the product space of all sets of possible outcomes of continuous observation at all moments of time until t. This POV measure can be also regarded [1] as the result of averaging some

projection-valued measure given on a larger Hilbert space and describing the process of continuous in time direct observation of an extended system.
That is why for the description of continuous in time indirect observation of an open system we use the scheme where indirect observation is performed by means of continuous in time direct observation of an observable of the reservoir modelling the measuring device. We call such indirect measurements R-indirect.

The basic ideas of our approach to the description of continuous in time measurements (direct and indirect) and the derivation of the equation for the posterior state of an open system were first formulated in cooperation with C.Aberg in [1].

2. The description of continuous in time direct observation of a quantum system

In this section we first consider the problem of description of continuous in time direct measurement of any quantum system.
We use the clear and consistent approach based on the main principles of quantum theory and developed in [1].

It is well known that simultaneous direct measurement of distinct observables of a quantum system is possible if and only if these observables commute [17]. The result of a direct measurement of each of this observable does not depend on its context, that is it does not depend on whether we measure every observable alone or together with some, or all, other observables.
In the Heisenberg picture the concept of an observable wholly incorporates the unitary time evolution of the system, that is why the problem of description of continuous in time direct measurement of some observable of a system is equivalent to the problem of description of simultaneous direct measurements of several distinct system observables.
Consequently, *continuous in time direct measurement of a quantum observable is possible if and only if in the Heisenberg picture this observable satisfies the conditions [1]:*

(1)
$$\begin{gathered}\hat{A}_H(t) = \hat{A}_H^+(t),\\ \left[\hat{A}_H(t),\ \hat{A}_H(t_1)\right] = 0, \quad \text{for} \forall t, t_1.\end{gathered}$$

In this case the repeated direct measurements of this kind of observable at different moments of time do not interfere and one can perform a continuous in time direct measurement of a system with arbitrary precision without disturbing the results of consecutive observations.
Observables of a quantum system satisfying (1) are usually called quantum nondemolition observables.
It was shown in [1] that the process of continuous in time direct observation of a nondemolition system observable $\hat{A}_H(s)$ until the moment t is described by a projection-valued measure $\hat{P}^{(0,t]}_{\hat{A}_H}(\cdot)$ on the product space $\Omega^{(0,t]}$ of all sets of possible outcomes of continuous observation of $\hat{A}_H(s)$ at all moments of time $0 < s \le t$.
Denote by $\Omega^{(t_1,t_2]}$ the product space of all sets of possible outcomes of continuous observation of $\hat{A}_H(t)$ for all moments $t_1 < s \le t_2$.

Then

$$\Omega^{(0,t_2]} = \Omega^{(0,t_1]} \times \Omega^{(t_1,t_2]}.$$

For a self-adjoint observable $\hat{A}_H(t)$ the set of possible outcomes of measurement at any moment of time coincides with the spectrum of this operator which is a subset of $\boldsymbol{R}$.

Let $\hat{P}^{(t_1,t_2]}_{\hat{A}_H}(\cdot)$ be a projection valued measure on $\Omega^{(t_1,t_2]}$ describing the process of continuous in time direct measurement of $\hat{A}_H(s)$ for all moments $t_1 < s \le t_2$.
The introduced measures must satisfy the following relations:

(2)
$$\begin{aligned}
&\hat{P}^{(0,t_2]}_{\hat{A}_H}(E^{(0,t_2]}) = \hat{P}^{(0,t_1]}_{\hat{A}_H}(E^{(0,t_1]})\,\hat{P}^{(t_1,t_2]}_{\hat{A}_H}(E^{(t_1,t_2]}),\\
&E^{(0,t_2]} = E^{(0,t_1]} \times E^{(t_1,t_2]},\\
&\mathit{for} \quad \forall\, E^{(0,t_1]} \subseteq \Omega^{(0,t_1]}, \quad \forall\, E^{(t_1,t_2]} \subseteq \Omega^{(t_1,t_2]}.
\end{aligned}$$

For all moments of time the introduced measures commute and are compatible in the sense that

(3)
$$\begin{aligned}
&\hat{P}^{(0,t_1]}_{\hat{A}_H}(E^{(0,t_1]}) = \hat{P}^{(0,t_2]}_{\hat{A}_H}(E^{(0,t_1]} \times \Omega^{(t_1,t_2]}), \quad \mathit{for} \quad \forall E^{(0,t_1]} \subseteq \Omega^{(0,t_1]},\\
&\hat{P}^{(t_1,t_2]}_{\hat{A}_H}(E^{(t_1,t_2]}) = \hat{P}^{(0,t_2]}_{\hat{A}_H}(\Omega^{(0,t_1]} \times E^{(t_1,t_2]}), \quad \mathit{for} \quad \forall E^{(t_1,t_2]} \subseteq \Omega^{(t_1,t_2]}.
\end{aligned}$$

The first relation in (3) corresponds to the causality principle and expresses the fact that future measurements cannot influence measurements in the past. The second relation in (3) corresponds to the fact that a continuous in time direct measurement until the moment t of a nondemolition observable (1) does not disturb the results of subsequent measurements.
The probability that a continuous direct measurement of the nondemolition observable $\hat{A}_H(t)$ until the moment t on a system being initially in a pure state Φ gives a result in the subset $E^{(0,t]} \subseteq \Omega^{(0,t]}$ is given by:

(4a)
$$\mu^{(0,t]}(E^{(0,t]}) = \left\langle \Phi, \hat{P}^{(0,t]}_{\hat{A}_H}(E^{(0,t]})\Phi \right\rangle.$$

The process of description of continuous in time measurement of a *nondemolition* observable $\hat{A}_H(t)$ is similar to the description of a *classic stochastic process* with a probability measure $\mu^{(0,t]}(\cdot)$ on $\Omega^{(0,t]}$ being induced by an initial state of a system.

From (3) and (4a) it follows that for scalar probability measures $\mu^{(0,t]}(\cdot)$ we have the relations of compatibility similar to (3):

(4b)
$$\mu^{(0,t_1]}(E^{(0,t_1]}) = \mu^{(0,t_2]}(E^{(0,t_1]} \times \Omega^{(t_1,t_2]}), \quad for \quad \forall E^{(0,t_1]} \subseteq \Omega^{(0,t_1]};$$
$$\mu^{(t_1,t_2]}(E^{(t_1,t_2]}) = \mu^{(0,t_2]}(\Omega^{(0,t_1]} \times E^{(t_1,t_2]}), \quad for \quad \forall E^{(t_1,t_2]} \subseteq \Omega^{(t_1,t_2]}.$$

3.. The description of continuous in time R-indirect measurement of an open system

Consider now the interaction of an open system described by the complex separable Hilbert space $\boldsymbol{H}_S$ and being initially in the pure state ψ_0 with reservoir, described by the complex Hilbert space $\boldsymbol{H}_R$ and being initially in the pure state φ_R. The case when an open system and a reservoir are described initially by density operators can be easily considered as well..
The extended system, i.e. the system plus reservoir, is described by the self-adjoint Hamiltonian

(5)
$$\hat{H}_{S+R} = \hat{H}_S \otimes \hat{I} + \hat{I} \otimes \hat{H}_R + \hat{H}_{\text{int}}$$

on the Hilbert space $\boldsymbol{H}_S \otimes \boldsymbol{H}_R$ of the extended system. In (5) $\hat{H}_{\text{int}}$ is the Hamiltonian of interaction of an open system with the reservoir.
The time evolution of the extended system in the interaction picture generated by $\hat{H}_R$ and corresponding to the free dynamics of a reservoir is described by the unitary operator
(6a)
$$\hat{U}(t,s) = \exp\left(\frac{i}{\hbar}(\hat{I} \otimes \hat{H}_R)\, t\right) \exp\left(-\frac{i}{\hbar}\left(\hat{H}_{S+R}(t-s)\right)\right) \exp\left(-\frac{i}{\hbar}(\hat{I} \otimes \hat{H}_R)\, s\right),$$

satisfying the Cauchy problem:

(6b)
$$i\hbar \frac{\partial\, \hat{U}(t,s)}{\partial\, t} = (\hat{H}_S \otimes \hat{I} + \hat{H}_{\text{int}}(t))\, \hat{U}(t,s),$$
$$\hat{U}(t,s)\Big|_{t=s} = \hat{I},$$

here $\hat{H}_{\text{int.}}(t)$ is the interaction Hamiltonian in the interaction picture:
(7)
$$\hat{H}_{\text{int}}(t) = \exp\left(\frac{i}{\hbar}(\hat{I} \otimes \hat{H}_R)t\right) \hat{H}_{\text{int}} \exp\left(-\frac{i}{\hbar}(\hat{I} \otimes \hat{H}_R)t\right).$$

The operator $\hat{U}(t,s)$ represents the two-parameter family of unitary operators strongly continuous with respect to s and t and being a cocycle:

(8) $$\hat{U}(t,s) = \hat{U}(t,t_2)\hat{U}(t_2,s).$$

We use sometimes the special notation for

$$\hat{U}(t,0) \equiv \hat{U}(t).$$

Consider now the process of description of continuous in time R-indirect measurement of an open system which is performed through continuous in time direct measurement of a nondemolition observable $\hat{Q}_H(t)$ corresponding in the Heisenberg picture to some free dynamics observable $\hat{Q}(t)$ of the reservoir:

(9) $$\hat{Q}_H(t) = \hat{U}^+(t,0)(\hat{I} \otimes \hat{Q}(t))\hat{U}(t,0),$$

The POV measure $\hat{M}^{(0,t]}(\cdot)$ on the product space $\Omega^{(0,t]}$ describing the process of continuous in time R-indirect measurement of an open system until the moment t is a family of linear, self-adjoint, positive operators on the Hilbert space $\boldsymbol{H}_S$ of an open system and is defined [1] by a projection-valued measure $\hat{P}^{(0,t]}_{\hat{Q}_H}(\cdot)$ of the process of continuous in time direct measurement of a nondemolition observable $\hat{Q}_H(t)$:

(10) $$\hat{M}^{(0,t]}(E^{(0,t]}) = <\varphi_R, \hat{P}^{(0,t]}_{\hat{Q}_H}(E^{(0,t]})\,\varphi_R>_{H_R}, \quad \forall E^{(0,t]} \subseteq \Omega^{(0,t]},$$

here $<\cdot,\cdot>_{H_R}$ is a scalar product in the Hilbert space $\boldsymbol{H}_R$ of a reservoir.

The probability of getting a result of indirect measurement of an open system in $E^{(0,t]} \subseteq \Omega^{(0,t]}$ is given by

(11) $$\mu^{(0,t]}(E^{(0,t]}) = \left\langle \psi_0, \hat{M}^{(0,t]}(E^{(0,t]})\psi_0 \right\rangle_{H_S},$$

where $\langle \cdot,\cdot \rangle_{H_S}$ is a scalar product in the Hilbert space $\boldsymbol{H}_S$ of an open system**.** The relation (11) defines a scalar probability measure on $\Omega^{(0,t]}$ corresponding to the process of continuous in time indirect measurement.

The family of POV measures (10) (and the family of scalar probability measures (11) as well) are compatible in the sense that

(12)

$$\hat{M}^{(0,t_1]}(E^{(0,t_1]}) = \hat{M}^{(0,t_2]}(E^{(0,t_1]} \times \Omega^{(t_1,t_2]}), \quad for \quad \forall E^{(0,t_1]} \subseteq \Omega^{(0,t_1]},$$

$$\hat{M}^{(t_1,t_2]}(E^{(t_1,t_2]}) = \hat{M}^{(0,t_2]}(\Omega^{(0,t_1]} \times E^{(t_1,t_2]}), \quad for \quad \forall E^{(t_1,t_2]} \subseteq \Omega^{(t_1,t_2]}.$$

4. The principles of nondemolition observation

We are interested now in a special case of continuous in time R-indirect measurement called a nondemolition case.

The main principles of continuous in time nondemolition measurement were introduced by V.P. Belavkin [9-11] and imply that in the Heisenberg picture there must exist an observable of the extended system corresponding to some free dynamics observable $\hat{Q}(t)$ of a reservoir

$$\hat{Q}_H(t) = \hat{U}^+(t,0)(\hat{I} \otimes \hat{Q}(t))\hat{U}(t,0),$$

satisfying the conditions

(13a)
$$\hat{Q}_H(t) = \hat{Q}_H^+(t),$$
$$\left[\hat{Q}_H(t), \hat{Q}_H(t_1)\right] = 0, \quad \text{for} \forall t, t_1.$$

and commuting with all observables of an open system under $s \le t$

(13b)
$$\left[\hat{Z}_H(t), \hat{Q}_H(s)\right] = 0 \,.$$

The condition (13a) is similar to (1) and corresponds simply to the fact that this kind of measurement is performed through continuous in time direct observation of a reservoir and consequently, according to our considerations in section 2, there must exist a nondemolition observable of a reservoir which can be continuously directly observed.

The condition (13b) corresponds to a further restriction inside the considered class of continuous in time R-indirect measurements and expresses the fact that under such kind of measurement the open system is in fact not demolished as a quantum object and that it is in principle possible to make at any moment of time t measurements on any observable of the open system simultaneously with the continuous observation until t of $\hat{Q}_H(s), s \le t$.

Thus in this case it is possible to introduce the notion of a posterior state of the continuously observed open system.

Sufficient conditions (14) (considered below) for a continuous in time R-indirect measurement to be a nondemolition one (due to Belavkin's definition) correspond to such demands on a reservoir and it's interaction with an open system which are quite natural from a physical point of view ----

in order that this reservoir could pretend to be a measuring device.

Particularly, for a measuring device:

a) It is quite natural to demand that without an interaction with an open system, that is under its free dynamics, a reservoir observable $\hat{Q}(t)$ could be continuously directly observed as well and that is why it must be a nondemolition observable as well:

(14a)
$$\hat{Q}(t) = \hat{Q}^{+}(t),$$
$$\left[\hat{Q}(t),\ \hat{Q}(t_1)\right] = 0, \quad \text{for} \forall t, t_1.$$

The families of operators $\{\hat{Q}(s), 0 < s \le t\}$ and $\{\hat{Q}_H(s),\ 0 < s \le t\}$ are usually called input and output operator-valued processes, respectively.

b) For a measuring device it is natural to demand as well that the interaction with an open system should be of such kind that the results of indirect measurement of the open system would not be dependent on the meanings of the input observable $\hat{Q}(t)$ before the beginning of the process of measurement

(14b)
$$[\hat{U}(t,s), (\hat{I} \otimes \hat{Q}(r))] = 0, \quad \forall\ r < s \le t\ ,$$

Due to (6) the relation

(14c)
$$[\hat{H}_{\text{int}}(t), (\hat{I} \otimes \hat{Q}(s))] = 0, \qquad \forall s < t$$

is a necessary and sufficient condition for (14b) to be valid.

It is easy to show that the conditions (14) are sufficient for the conditions (13) to be true, that is for a continuous in time indirect measurement to be a nondemolition one according to Belavkin's definition.
In the present paper we consider the conditions (14) to be valid and we shall identify *a continuous in time nondemolition R- indirect measurement as one corresponding to (14).*

6. The POV measure in case of continuous in time nondemolition measurement of an open system

Let us now construct the POV measure defined by (10) in case of continuous in time nondemolition measurement.

From (14) it follows

(15)
$$\hat{Q}_H(s) = \hat{U}^{+}(t,0)(\hat{I} \otimes \hat{Q}(s))\hat{U}(t,0), \qquad s \le t\ .$$

Let $\hat{P}_{\hat{Q}}^{(0,\,t]}(\cdot)$ be a projection -valued measure describing the process of continuous direct measurement of the input nondemolition observable $\hat{Q}(t)$ until the moment t (in case of the free dynamics of reservoir), and let $\hat{P}^{(0,t]}_{\hat{Q}_H}(\cdot)$ be a projection -valued measure for the output nondemolition observable $\hat{Q}_H(t)$.

From (15) and from the definition (given in the section 2) of projection-valued measures corresponding to nondemolition observables, *we derive the following important relation:*

(16)
$$\hat{P}^{(0,t]}_{\hat{Q}_H}(E^{(0,t]}) = \hat{U}^+(t_1,0)(\hat{I} \otimes \hat{P}^{(0,t]}_{\hat{Q}}(E^{(0,t]}))\hat{U}(t_1,0),$$
$$for \quad \forall E^{(0,t]} \subseteq \Omega^{(0,t]}, \forall t_1 \geq t.$$

Consequently, in the case of nondemolition measurement the POV measure defined by (10) can be represented in the form

(17)
$$\hat{M}^{(0,t]}(E^{(0,t]}) = \left\langle \varphi_R, \hat{U}^+(t)(\hat{I} \otimes \hat{P}^{(0,t]}_{\hat{Q}}(E^{(0,t]}))\hat{U}(t)\varphi_R \right\rangle_{H_R},$$
$$\forall E^{(0,t]} \subseteq \Omega^{(0,t]}.$$

Consider a nondemolition observable $\hat{Q}(t)$ describing the free dynamics of a reservoir. For simplicity suppose that $\hat{Q}(t)$ is continuous with respect to t in the strong operator topology and that for any moment t the family of commuting operators $\{\ \hat{Q}(s)\ \}_{s\in(0,t]}$ on $\boldsymbol{H}_R$ is full.

The spectrum of the self-adjoint output operator $\hat{Q}_H(t)$, defined by (15), at any moment of time, coincides with the spectrum of the input observable $\hat{Q}(t)$ and is a subset of $\boldsymbol{R}$. Denote by $Q(s) \in R$ a possible outcome of a direct measurement of $\hat{Q}_H(s)$ at the moment s. A time-ordered sequence of possible outcomes of continuous measurement of $\hat{Q}_H(s)$ at all moments of time $t_1 < s \leq t_2$, where $t_1 \geq 0$,

(18)
$$Q^{(t_1,t_2]} = \{Q(s)\}_{s\in(t_1,t_2]}$$

is an element of $\Omega^{(t_1,t_2]}$. We shall use the special notation

$$Q^t = \{Q(s)\}_{s\in(0,\,t]}.$$

From the general representation of an observable through its projection-valued measure (due to the the spectral theorem [28]), it is clear that all information about the kind of spectrum is contained in a projection-valued measure corresponding to that operator. Having that in mind, without loss of generality, we identify a space $\Omega(s)$ of possible outcomes of measurement at the moment s with $\boldsymbol{R}$. Consequently, $\Omega^{(0,t]}$ -- the product space of all sets of possible outcomes of continuous observation of $\hat{Q}_H(s)$ (or $\hat{Q}(t)$) until the moment t -- can be identified with a space of trajectories. The family of projection-valued measures $\hat{P}^{(0,t]}_{\hat{Q}_H}(\cdot)$ on $\Omega^{(0,t]}$ gives us then automatically the probability distributions of possible results of observation.

The projection-valued measure $\hat{P}_{\hat{Q}}^{(0,\,t]}(\cdot)$ describing the process of continuous in time measurement until the moment t of the input nondemolition observable $\hat{Q}(s)$ of a reservoir being initially in the state φ_R induces a scalar probability measure on $\Omega^{(0,t]}$ (see section 2) by

(19)
$$\nu_{\hat{Q}}^{(0,\,t]}(E^{(0,\,t]}) = <\varphi_R,\, \hat{P}_{\hat{Q}}^{(0,\,t]}(E^{(0,t]})\,\varphi_R>_{H_R} \quad ,$$

satisfying the relations of compatibility (4b).
In the considered case, due to our assumption of the continuity with respect to t of the operator $\hat{Q}(t)$ in the strong operator topology , $\Omega^{(0,t]}$ is a space of trajectories "continuous" with respect to t in the sense of the probability measure (19).
At any moment of t the following resolution of identity on the Hilbert space $\boldsymbol{H}_R$ of a reservoir is valid

(21)
$$\hat{I} = \int\limits_{Q^t\in\Omega^{(0,\,t]}} \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t).$$

Let us introduce a linear operator $\hat{V}[Q^t, t; \varphi_R]$ *on the Hilbert space* $\boldsymbol{H}_S$ *of an open system by the following relation*

(22)
$$(\hat{V}[Q^t, t; \varphi_R] \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t))(\psi \otimes \varphi_R) = (\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t))\,\hat{U}(t)(\psi \otimes \varphi_R)$$
$$\textit{for} \quad \forall \psi \in H_S,$$

understood in the infinitesimal sense.

Using the definition (22) of the operator $\hat{V}[Q^t, t; \varphi_R]$ and the relation (19) for a scalar probability measure $\nu_{\hat{Q}}^{(0,t]}(\cdot)$ of the input process, we can rewrite the POV measure represented by (17) in the form

(23)
$$\hat{M}^{(0,\,t]}(E^{(0,t]}) = \int\limits_{Q^t\in E^{(o.t]}} \hat{V}^{+}[Q^t, t; \varphi_R]\; \hat{V}[Q^t, t; \varphi_R]\,\nu_{\hat{Q}}^{(0,t]}(dQ^t),$$
$$\textit{for} \quad \forall E^{(0,\,t]} \subseteq \Omega^{(0,t]}.$$

The probability of getting a result in a subset $E^{(0,t]} \subseteq \Omega^{(0,t]}$ under continuous in time R-indirect measurement of an open system is given by

(24) $$\mu^{(0,t]}(E^{(0,t]}) = \int_{Q^t \in E^{(0,t]}} \left\langle \psi_0, \hat{V}^+[Q^t, t; \varphi_R]\,\hat{V}[Q^t, t; \varphi_R]\,\psi_0 \right\rangle_{H_S} \nu_{\hat{Q}}^{(0,t]}(dQ^t)\,.$$

(24) defines the scalar probability measure on $\Omega^{(0,\,t]}$ describing the process of indirect measurement through a scalar probability measure $\nu_{\hat{Q}}^{(0,\,t]}(\cdot)$ of the input process.

From the definition (22) *the following representation follows*

(25) $$\hat{U}(t)(\psi \otimes \varphi_R) = \int_{Q^t \in \Omega^{(0,t]}} (\hat{V}[Q^t, t; \varphi_R] \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t))\,(\psi \otimes \varphi_R),$$

for $\forall \psi \in H_S$.

Consequently, the linear operator $\hat{V}[Q^t, t, \varphi_R]$ on the Hilbert space $\boldsymbol{H}_S$ of an open system introduced by (22) represents the operator-valued distribution in resolution (25) standing under the projection onto the subspace of $\boldsymbol{H}_R$ corresponding to the definite observed trajectory Q^t in the case when the initial state of the reservoir is φ_R.

From (25) and (19) we get the *following resolution for the reduced evolution for an open system in the case of continuous in time R- indirect measurement*

(26) $$< \varphi_R, \hat{U}(t)\varphi_R >_{H_R} = \int_{Q^t \in \Omega^{(0,t]}} \hat{V}[Q^t, t; \varphi_R]\,\nu_{\hat{Q}}^{(0,t]}(dQ^t)\,.$$

Rewriting the Cauchy problem (6b) for the unitary operator $\hat{U}(t)$ describing the time evolution of the extended system in the form of an integral equation and substituting it into (22), we derive the following *integral equation for the operator* $\hat{V}[Q^t, t; \varphi_R]$***:***

(27)
$$\hat{V}[Q^t, t; \varphi_R] = \hat{I} + \left(-\frac{i}{\hbar}\right)\int_0^t \hat{H}_S\,\hat{V}[Q^\tau, \tau; \varphi_R]\,d\tau +$$
$$+\left(-\frac{i}{\hbar}\right)\int_0^t \left(\int_{Q_1^\tau \in \Omega^{(0,t]}} \hat{W}[Q^\tau, Q_1^\tau, \tau; \varphi_R]\,\hat{V}[Q_1^\tau, \tau; \varphi_R]\,\nu_{\hat{Q}}^{(0,\tau]}(dQ_1^\tau) \right) d\tau\,,$$

In (27) we introduced the linear operator $\hat{W}[Q^t, Q_1^t, t; \varphi_R]$ on the Hilbert space $\boldsymbol{H}_S$ by:

(28)

$$(\hat{W}[Q^t, Q_1^t, t; \varphi_R]\, \nu_{\hat{Q}}^{(0,\, t]}(dQ_1^t)) \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t)(\psi \otimes \varphi_R) =$$
$$= (\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,\, t]}(dQ^t))\hat{H}_{\mathrm{int}}(t)(\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,\, t]}(dQ_1^t))(\psi \otimes \varphi_R),$$

for $\forall \psi \in H_S$.

The relation (28) must be understood in the infinitesimal form.
We show in the section 7 that, due to (14c), the third term in (27) corresponding to the "memory" effects can be essentially simplified.

The notion of quantum stochastic evolution operator as well as the notion of the posterior wave function under continuous in time nondemolition measurement were first introduced in papers of V.P.Belavkin [9-15] for the case of a special measurement model of quantum stochastic calculus. His definition is introduced through the method of a generating map of an instrument in the case of classic input process and is given in the integral form.
Our definition (22) of the operator $\hat{V}(Q^t, t, \varphi_R)$ is new, it is more detailed since this operator is defined in the infinitesimal form directly through the unitary time evolution operator $\hat{U}(t)$ of the extended system, the projection-valued measure corresponding to the observed trajectory Q^t and the initial state of the reservoir φ_R.
Due to our definition we could derive the new representations (25), (26) as well as the new integral equation (27).

From the mathematical point of view it is clear, that relations (22) - (26) and the integral eqution (27) are valid not only in the case of nondemolition measurement but for any continuous in time R-indirect measurement as well - in this case one should only understand the introduced measures $\hat{P}_{\hat{Q}}^{(0,t]}(\cdot)$ and $\nu_{\hat{Q}}^{(0,t]}(\cdot)$ of the input process in the different context.
In the general case of continuous in time R-indirect measurement of an open system the introduction of the operator $\hat{V}[Q^t, t, \varphi_R]$ is formal though very convenient since through (23), (24) this operator describes the indirect measurement process.

In the next section we show that from the point of view of quantum theory the operator $\hat{V}[Q^t, t, \varphi_R]$ can be interpreted as the operator describing the quantum stochastic evolution of an open system subjected to continuous in time observation only in the case of nondemolition measurement. We show that only in the case of such measurement process there is a sense to introduce the notion of the posterior state of the continuously observed open system.

6. The quantum stochastic evolution of an open system under continuous in time nondemolition measurement

The mean value at the moment t of any observable $\hat{Z}$ of an open system in the initial state $\psi_0 \otimes \varphi_R$ is given by

(29) $$\left\langle \hat{Z}_H(t) \right\rangle = \left\langle \psi_0 \otimes \varphi_R, \hat{U}^+(t)(\hat{Z} \otimes \hat{I})\hat{U}(t)(\psi_0 \otimes \varphi_R \right\rangle,$$

Considering (22) and (19), we can rewrite (29) in the form

(30) $$\left\langle \hat{Z}_H(t) \right\rangle = \int_{Q^t \in \Omega^{(0,t]}} < \psi_{0,} \ \hat{V}^+[Q^t, t; \varphi_R] \ \hat{Z} \ \hat{V}[Q^t, t; \varphi_R] \psi_0 >_{H_S} \ \nu_{\hat{Q}}^{(0,t]}(dQ^t).$$

Consider now a probability of a possible joint direct measurement on an extended system being in the initial state $\psi_0 \otimes \varphi_R$. This joint measurement corresponds to continuous in
time until the moment t measurement of the observable $\hat{Q}_H(s)$ and measurement at the moment t of any system observable $\hat{Z}_H(t)$.
Since we consider a nondemolition measurement when these observables commute

$$\left[\hat{Z}_H(t), \hat{Q}_H(s) \right] = 0, \qquad s \le t,$$

such joint direct measurement is possible and it is described by a projection-valued measure

(31) $$\hat{P}_{\hat{Z}_H}(F)\hat{P}_{\hat{Q}_H}^{(0,t]}(E^{(0,t]}) = \hat{U}^+(t)(\hat{P}_{\hat{Z}}(F) \otimes \hat{I})\,(\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,t]}(E^{(0,t]}))\,\hat{U}(t)$$

on the space $R \times \Omega^{(0,t]}$ of possible outcomes of such joint measurement. Here $\hat{P}_{\hat{Z}}(F)$ is a projection-valued measure for a system observable $\hat{Z}$.
The probability of getting a result of such measurement in the subset $F \times E^{(0,t]}$ is given by

(32)
$$\omega(F \times E^{(0,t]}) = \int_{Q^t \in E^{(o,t]}} < \psi_0, \hat{V}^+[Q^t, t; \varphi_R] \, \hat{P}_{\hat{Z}}(F) \, \hat{V}[Q^t, t; \varphi_R] \psi_0 > \nu_{\hat{Q}}^{(0,t]}(dQ^t)$$

$$for \quad \forall \ F \times E^{(0,t]} \subseteq R \times \Omega^{(0,t]}$$

From (32) and the expression (24) for a scalar probability measure $\mu^{(0,t]}(\cdot)$ describing the process of continuous in time indirect observation of an open system it follows that the conditional probability of finding a result of measurement of a system observable $\hat{Z}_H(t)$ at the moment t in the subset F, under the condition that the output process

$\hat{Q}_H(s)$ has been continuously observed until the moment t and found to have a trajectory Q^t is given by

(33)

$$<\psi_0, \hat{V}^+[Q^t, t; \varphi_R]\, \hat{P}_{\hat{Z}}(F)\, \hat{V}[Q^t, t; \varphi_R]\, \psi_0 >_{H_S} \quad / \quad <\psi_0, \hat{V}^+[Q^t, t; \varphi_R]\, \hat{V}[Q^t, t; \varphi_R]\, \psi_0 >_{H_S}$$

Consequently, due to (24), (30) and (33), in the case of nondemolition measurement the introduced operator $\hat{V}\left[Q^t, t; \varphi_R\right]$ *describes the irreversible in time stochastic evolution of an open system under the condition that the output process* $\hat{Q}_H(s)$ *was continuously observed until the moment t and found to have the trajectory* Q^t.
The state

(34)
$$\chi[Q^t, t; \varphi_R] = \hat{V}[Q^t, t; \varphi_R]\, \psi_0\ , \quad for \quad \forall \psi_0 \in H_S$$

should be interpreted as the posterior state of an open system. It satisfies the linear integral equation

(35)

$$\chi[Q^t, t; \varphi_R] = \psi_0 + \left(-\frac{i}{\hbar}\right)\int_0^t \hat{H}_S\, \chi[Q^\tau, \tau; \varphi_R]\, d\tau +$$
$$+\left(-\frac{i}{\hbar}\right)\int_0^t \left(\int_{Q_1^\tau \in \Omega^{(0,t]}} \hat{W}[Q^\tau, Q_1{}^\tau, \tau; \varphi_R]\, \chi[Q_1{}^\tau, \tau; \varphi_R]\, \nu_{\hat{Q}}^{(0,\tau]}(dQ_1{}^\tau) \right) d\tau.$$

Consider now what simplifications we can make in the "memory" terms of the equations (27) and (35) if we take into account the conditions (14).

For the most general nondemolition case defined by (14) we have the following relation

(36)
$$(Q_1(s) - Q(s))(\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t))\hat{H}_{\text{int}}(t)(\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ_1^t))(\psi \otimes \varphi_R) = 0, \qquad \forall s < t,$$
$$for \quad \forall \psi \in H_S\,,$$

understood in the infinitesimal sense.
Consequently, from (28) and (36) we conclude that in the nondemolition case the "memory" operator $\hat{W}[\cdot]$ in (27) and in (35) must have the following construction:

(37)
$$\hat{W}[Q^t, Q_1^t, t; \varphi_R] = \hat{W}_1[Q^t, Q_1^t, t; \varphi_R]\, \delta_\nu(Q_1^{(0,t)} - Q^{(0,t)})\ ,$$

where $\delta_\nu(Q_1^{(0,t)} - Q^{(0,t)})$ is a functional $\delta(\cdot)$-function with respect to the scalar measure $\nu_{\hat{Q}}^{(0,t]}(\cdot)$ of the input process. $\hat{W}_1[\cdot]$ in (37) is a linear operator on the Hilbert space $\boldsymbol{H}_S$, defined in any concrete nondemolition measurement model by $\hat{H}_{\text{int}}(t)$ and, in particular, by its commutation relation with $\hat{Q}(t)$ at the moment t.

Thus, in the most general case of nondemolition measurement the "memory" term in (27) and (35) can be simplified and is given by:

(38)

$$\int_{Q^t \in \Omega^{(0,t]}} \hat{W}[Q^t, Q_1^t, t; \varphi_R]\, \hat{V}[Q_1^t, t; \varphi_R] \nu_{\hat{Q}}^{(0,t]}(dQ^{(0,t]}) =$$

$$= \int_{Q_1(t)\in\Omega(t)} \left\{ \hat{W}_1[Q^t, Q_1^t, t, \varphi_R] \hat{V}[Q_1^t, t, \varphi_R] \right\}\Big|_{\substack{Q_1(s)=Q(s),\\ s\in(0,t)}} \tilde{\nu}_{\hat{Q}}(dQ_1(t)),$$

where $\Omega(t)$ is a space of possible outcomes of measurement at the moment t and

$$\tilde{\nu}_{\hat{Q}}(dQ_1(t)) = \int_{\Omega^{(0,t)}} \delta_\nu (Q^{(0,t)} - Q_1^{(0,t)}) \nu_{\hat{Q}}^{(0,t]}(dQ_1^t)$$

is a scalar measure on this space.

Finally, from (38) it follows, that in the most general case of nondemolition measurement the integral equation for the posterior state $\chi[Q^t, t, \varphi_R]$ of an open system subjected to continuous in time nondemolition measurement has the form:

(39a)

$$\chi[Q^t, t; \varphi_R] = \psi_0 + \left(-\frac{i}{\hbar}\right) \int_0^t \hat{H}_S\, \chi[Q^\tau, \tau; \varphi_R] d\tau +$$

$$+ \left(-\frac{i}{\hbar}\right) \int_0^t \left(\int_{Q_1(\tau)\in\Omega(\tau)} \left\{ \hat{W}_1[Q^\tau, Q_1^\tau, \tau, \varphi_R] \chi[Q_1^\tau, \tau, \varphi_R] \right\}\Big|_{\substack{Q_1(s)=Q(s),\\ s\in(0,\tau)}} \tilde{\nu}_{\hat{Q}}(dQ_1(\tau)) \right) d\tau\,,$$

We can also rewrite (39a) in a differential form in the sense of generalized distributions calculus:

(39b)

$$i\hbar \frac{d}{dt} \chi[Q^t, t; \varphi_R] = \hat{H}_S \chi[Q^t, t; \varphi_R] + \int_{Q_1(t)\in\Omega(t)} \left\{ \hat{W}_1[Q^t, Q_1^t, t, \varphi_R] \chi[Q_1^t, t, \varphi_R] \right\}\Big|_{\substack{Q_1(s)=Q(s),\\ s\in(0,\, t)}} \tilde{\nu}_{\hat{Q}}(dQ_1(t))$$

$$\chi|_{t=0} = \psi_0$$

The equations (39) describe in the most general nondemolition case the quantum stochastic evolution of an open system under the condition that the output process was continuously observed until the moment t and found to have the trajectory Q^t.
The "memory" term in (39b) points out that in the most general nondemolition case the posterior state of the continuously observed open system can depend not only on the meanings of possible outcomes of measurement for all moments of time up to the moment t but it can also depend on the posterior states corresponding to trajectories different from Q^t by jumps at the moment t.

We would like also to mention that from our definition (22) of the quantum stochastic evolution operator it follows that $\hat{V}\left[Q^t, t; \varphi_R\right]$ and $\chi[Q^t, t; \varphi_R]$ depend not only on the meanings of the outcomes of measurement of the output process until the moment t but on the initial state φ_R of a reservoir as well. This can be easily understood since *different initial states of reservoir induce different probability measures on $\Omega^{(0,t]}$ and consequently, different quantum stochastic evolution of an open system.*

The notion of quantum stochastic evolution operator as well as the notion of the posterior wave function were first introduced in papers of V.P.Belavkin [9-15] in the special nondemolition measurement model of quantum stochastic calculus. His definition is based on the concept of a generating map of an instrument and is given in the integral form.The equation for the posterior state derived by V.P.Belavkin [9-11] is valid only for the case of the measurement model of quantum stochastic calculus which implies the Markovian approximation.
Our approach to the description of continuous in time indirect measurement as well as our definition (22) of the quantum stochastic evolution operator $\hat{V}[Q^t, t, \varphi_R]$ allow us to introduce the new representations (24), (25) and to derive the new equations (27), (35), (39) describing the quantum stochastic evolution of an open system under continuous in time observation in the most general case of nondemolition measurement.

7.The special nondemolition measurement model of quantum stochastic calculus, the extended variant

In this section we apply our results to the concrete special measurement model of quantum stochastic mechanics -- the only dynamic model which is considered in the theory of continuous in time indirect measurements. This measurement model satisfies the principles of nondemolition observation [9-15].
In the frame of quantum stochastic mechanics the unitary time evolution of an extended system in the interaction picture is supposed to be described by the quantum stochastic differential equation of Ito's type introduced by Hudson R.L. and Parthasarathy K.R. in [24]. In our paper we take the general variant of this equation:

(40a)
$$d\hat{U}(t) = \left((-\hat{K} \otimes \hat{I})\, dt - (\hat{L}^{+} \hat{R}) \otimes d\hat{A}(t) + \hat{L} \otimes d\hat{A}^{+}(t) + (\hat{R} - \hat{I}) \otimes d\hat{\Lambda}(t)\right)\hat{U}(t),$$

including the gauge term which is usually omited under the consideration of continuous in time measurement.

In (40a) $\hat{L}$; $\hat{R}$; $\hat{K}+\hat{K}^{+}=\hat{L}^{+}\hat{L}$ are operators on the Hilbert space $\boldsymbol{H}_S$ of an open system, the operator $\hat{R}$ is unitary .

The exact relation between the operator $\hat{K}$ in (40a) and the Hamiltonian $\hat{H}_S$ of an open system in the case when $\hat{R}\neq-\hat{I}$ is given [25] by:

(40b) $$\hat{K}=\frac{i}{\hbar}H_S+\hat{L}^{+}\hat{R}(\hat{R}+\hat{I})^{-1}\hat{L}\,.$$

The reservoir, modelling the measuring device, is described as a Bose field and is represented by a symmetric Fock space $\Gamma(Z))$ over a single particle Hilbert space $\boldsymbol{Z}$ which in applications in quantum stochastic mechanics is taken to be $L^2(R)$ **.**

The annihilation, creation and gauge (or number) operators $\hat{A}(t),\hat{A}^{+}(t),$ $\hat{\Lambda}(t)=\hat{\Lambda}^{+}(t)$ describing the free dynamics of the Bose field form the annihilation, creation and gauge processes, respectively. These operators are continuous with respect to t in the strong operator topology , hence locally square integrable [24].

Denote by $e(f),\, f\in L^2(R)$ -- a normalized coherent vector for the Bose field:

(41) $$\hat{A}(t)e(f)=(\int_0^t\tilde{f}(\tau)\,d\tau)e(f),$$

where $\tilde{f}(t)$ is a Fourier transform of a function $f\in L^2(R)$.

Let us consider now the nondemolition observables in this model.

The input operator

(42) $$\begin{aligned}&\hat{Q}(t)=\hat{A}^{+}(t)+\hat{A}(t),\\&[\hat{Q}(t),\hat{Q}(t_1)]=0,\qquad \forall t,t_1\end{aligned}$$

represents a nondemolition observable and the Heisenberg observable $\hat{Q}_H(t)$, corresponding to $\hat{Q}(t)$ in the Heisenberg picture defined by the operator $\hat{U}(t)$ is also nondemolition and satisfies the principles of nondemolition observation. As it was shown in [1], the process of continuous in time direct observation of $\hat{Q}_H(t)$ corresponds to a classic stochastic process of diffusion type on $\Omega^{(0,t]}$.

The input operator

(43)
$$\hat{\Pi}_l(t)=\hat{\Lambda}(t)+\sqrt{l}(\hat{A}^{+}(t)+\hat{A}(t))+l\,t\hat{I},\qquad l\geq 0.$$

$$[\hat{\Pi}(t_1),\hat{\Pi}(t_2)]=0,\qquad \forall t_1,t_2\ ,$$

represents a nondemolition observable as well and the output operator corresponding to $\hat{\Pi}_l(t)$ in the Heisenberg picture defined by $\hat{U}(t)$, satisfies the principles of nondemolition measurement.

The family of self adjoint, commutative operators $\{\hat{\Pi}_l(t), t \geq 0\}$ corresponds to an operator-valued realization of the Poisson process of intensity l [24]. The gauge process $\{\hat{\Lambda}(t), t \geq 0\}$ can be considered as the Poisson one with itensity $l = 0$.
We would like to mention that no two of the processes in (43) commute. Consequently, as it follows from our consideration in the previous sections, no two of these quantum processes can be directly observed simultaneously.

In the present paper we consider only the description of continuous in time R-indirect measurement of an open system performed by means of continuous in time direct observation of the observable $\hat{Q}_H(t)$ of diffusion type. The special case, when $\hat{R} = \hat{I}$, was considered by us on the basis of the presented approach in [1].
Now we would like to consider the more general case of indirect diffusion observation. The description of the processes of continuous in time indirect observations of an open system due to continuous in time measurements of observables $\hat{\Pi}_l(t)$ or $\hat{\Lambda}(\mathrm{t})$ will be presented in a forthcoming paper.

The equation for the posterior state $\chi[Q^t, t, \varphi_R]$ of an open system can be easily derived from (35) and (28).
Take the most general case when the initial state of the reservoir $\varphi_R = e(f)$.
Considering the Hamiltonian [25] of the Schrödinger equation corresponding (when $\hat{R} \neq -\hat{I}$) to the quantum stochastic differential equation (40), we derive the following expression for the "memory" term in (35) (and (39)) in case $\hat{R} \neq -\hat{I}$:

(44)

$$\int_{Q_1^t \in \Omega^{(0,t]}} \hat{W}[Q^t, Q_1^t, t; e(f)]\chi[Q_1^t, t; e(f)]\nu_{\hat{Q}}^{(0,t]}(dQ^t) =$$
$$= -i\hbar\left\{\hat{G}(t) - q(t)(\hat{L} + \tilde{f}(t)(\hat{R} - \hat{I}))\right\}\chi[Q^t, t; e(f)],$$

where we introduced the operator on the Hilbert space of an open system
(45)

$$\hat{G}(t) = \frac{1}{2}\hat{L}^2 + \hat{L}^+\hat{R}(\hat{R}+\hat{I})^{-1}\hat{L} + \frac{1}{2}\tilde{f}(t)(\hat{R}-\hat{I})\hat{L} +$$
$$+\frac{1}{2}\tilde{f}(t)\hat{L}(\hat{R}+\hat{I}) + \tilde{f}(t)\hat{L}^+\hat{R} + \frac{1}{2}\tilde{f}^2(t)(\hat{R}^2 - \hat{I}),$$

and the notation $q(t) = \dfrac{dQ(t)}{dt}$ of the generalized derivative of the function $Q(t)$. In the considered case the observed trajectories of the quantum process are continuous and consequently, $q(t)$ is a piecewise continuous function.
In particular, the relation (44) shows that *in the case of diffusion observation -- when the trajectories Q^t are continuous -- the "memory" term in (39) is simplified at the most.*

Substituting (44) into (39,) we get the integral equation for the posterior state of an open system subjected to continuous in time indirect observation of diffusion type:
(46)

$$\chi[Q^t,t;e(f)]=\psi_0-\int_0^t(\frac{i}{\hbar}\hat{H}_S+\hat{G}(\tau))\chi[Q^\tau,\tau;e(f)]\,d\tau+$$

$$+\int_0^t q(\tau)\{\hat{L}+\tilde{f}(\tau)(\hat{R}-\hat{I})\}\,\chi[Q^\tau,\tau;e(f)]\,d\tau,\qquad \forall\psi_0\in H_S,$$

We can rewrite the integral equation (46) for the posterior state $\chi[Q^t,t,\varphi_R]$ in the stochastic form using the methods of classic stochastic calculus:
(47a)

$$\chi[Q^t,t;e(f)]=\psi_0-$$

$$-\int_0^t\{\frac{i}{\hbar}\hat{H}_S+\hat{L}^+\hat{R}(\hat{R}+\hat{I})^{-1}\hat{L}+\tilde{f}(\tau)(\hat{L}^+\hat{R}+\hat{L})+\tilde{f}^2(\tau)(\hat{R}-\hat{I})\}\chi[Q^\tau,\tau;e(f)]\,d\tau+$$

$$+\int_0^t\{\hat{L}+\tilde{f}(\tau)(\hat{R}-\hat{I})\}\,\chi[Q^\tau,\tau;e(f)]dQ(\tau),\qquad \forall\psi_0\in H_S.$$

The stochastic integral in (47a) is understood in Ito's sense.
The stochastic differential equation for the posterior state $\chi[Q^t,t,\varphi_R]$ of an open system is given by

(47b)

$$d\chi[Q^t,t,e(f)]+\left\{\frac{i}{\hbar}\hat{H}_S+\hat{L}^+\hat{R}(\hat{R}+\hat{I})^{-1}\hat{L}+\tilde{f}(t)(\hat{L}^+\hat{R}+\hat{L})+\tilde{f}^2(t)(\hat{R}-\hat{I})\right\}\chi[Q^t,t;e(f)]=$$

$$=\left\{\hat{L}+\tilde{f}(t)(\hat{R}-\hat{I})\right\}\chi[Q^t,t;e(f)]dQ(t),$$

$$\chi|_{t=0}=\psi_0.$$

In (47) $dQ(t)$ is a stochastic differential of the classic diffusion process induced by continuous in time observation of the input observable (42) in the initial state $\varphi_R=e(f)$. The scalar probability measure of this input process is defined by (19).

We would like to note that we could derive the stochastic equations (47) in another way --substituting the unitary operator $\hat{U}(t)$ describing the time evolution of the extended system in this model and given by the quantum stochastic differential equation (40) directly to the relation (22) defining the quantum stochastic evolution operator $\hat{V}[Q^t,t;\varphi_R]$.

The stochastic equations (47) for the posterior state of an open system are more general than the quantum filtering equation in quantum stochastic mechanics first introduced by V.P.Belavkin [12-13] and describing the posterior evolution of an open system under continuous in time observation of diffusion type. The equations (47) coincide with the stochastic equation given in [12-13] only if the initial state of a reservoir is vacuum and $\hat{R} = \hat{I}$. The exact correlation between the operators in the equations (46), (47) for the posterior state of an open system and the operators in the quantum stochastic differential equation (40) is very important for further concrete calculations.

8. Concluding remarks

In this paper we present a clear and consistent approach to the description of continuous in time measurements.
Our approach allows us to derive the most general results valid not only in the case of the Markovian approximation.
As an example of application of our results to concrete quantum measurement models we consider the special case of continuous in time indirect observation of diffusion type in quantum stochastic mechanics.
In a forthcoming paper we shall consider the further application of our approach to the description of other types of continuous in time indirect observations of an open system.

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REFERENCES

[1] Aberg, C., Loubenets, E.R. (1994). *Quantum stochastic calculus and nondemolition measurements. Notes on a recent approach to quantum measurements.* ITP, report 94-28, 48 pp.

[2] Accardi, L., Lu, Y.G. (1992). *From Markovian approximation to a new type of quantum stochastic calculus.* Quantum Probability and Related Topics **V11**, 1-14.

[3] Barchielli, A., Lanz, L., Prosperi, G.M. (1983). *Statistic of continuous trajectories in quantum mechanics: operation valued stochastic processes*.Found.Phys, **13**, 79-812.

[4] Barchielli, A. (1985).*Quantum stochastic calculus, operation-valued stochastic processes and continual measurements in quantum mechanics*. J.Math.Phys. **26**, 2222-2230**.**

[5] Barchielli, A. (1986). *Measurement theory and stochastic differential equations in quantum mechanics*. Phys. Rev. A, **34**, 1642-1648.

[6] Barchielli, A.. Belavkin, V.P. (1991).*Measurements continuous in time and a posteriori states in quantum mechanics*. J.Phys.A; Math.Gen. **24**,1495-1514.

[7] Barchielli, A., Holevo, A.S. (1995). *Constructing quantum measurement processes via classical stochastic calculus*. Stochastic Processes Appl. **58**, 293-317

[8] Barchielli, A. (1998). *On stochastic differential equations and semigroups of probability operators in quantum probability*. Stoch. Proc. and their Applications **73**, 69-86.

[9] Belavkin, V.P. (1988). *Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic processes*. In: Blaquiere, A. (Ed.), Lecture Notes in Control and Information Sciences **121**, 245-265, Springer, Berlin.

[10] Belavkin, V.P. (1989). *Stochastic calculus of input-output processes and nondemolition filtering*. VINITI **36**, 29-67, Moscow.

[11] Belavkin, V.P. (1990). *Stochastic equations of quantum filtering*. Proceedings of the Fifth Int.Conference "Probab.Theory and Math.Statistics", Vilnius.

[12] Belavkin, V.P. (1989). *A new wave equation for a continuous nondemolition measurement.* Phys. Lett. A **140**, 355-358 .

[13] Belavkin, V.P. and Staszewski, P. (1992). *Nondemolition observation of a free quantum particle* Phys. Rev. A, **45**, 1347-1356.

[14] Belavkin, V.P. (1989). *A continuous counting observation and posterior quantum dynamics.* J. Phys.A **22**, L1109-L1114.

[15] Belavkin, V.P. (1992). *Quantum continual measurements and a posteriori collapse on CCR.* Commun. Math.Phys.**146**, 611-635.

[16] Davies, E.B., Levis, J.T.(1970). *An operational approach to quantum probability.* Commun.Math.Phys.**17**, 239-260.

[17] Davies, E. B. (1976). *Quantum Theory of Open Systems*. Academic Press, London.

[18] Diosi, L.(1988). *Continuous quantum measurement and Ito formalism*. Phys.Lett.A **129**, 419-423.

[19] Ghirardi, G.C., Pearle, P.,Rimini, A. (1990). *Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles.* Phys.Rev. A **42**, 78-79.

[20] Gisin, N. (1984). *Quantum measurement and stochastic processes*. Phys.Rev.Lett. **52**, 1657-1660.

[21] Gisin, N., Percival, I.C. (1992). *The quantum-state diffusion model applied to open systems.* J. Phys. A **25**, 5677-5691.

[22] Holevo, A.S. (1987), *Quantum estimation.* Advances in Statistical Signal Processing **1**, 157-202 , JAI Press Inc.

[23] Holevo, A.S. (1996). *On dissipative stochastic equations in a Hilbert space.* Probab. Theory Rel. Fields **104**, 483-500.

[24] Hudson, R.L., Parthasarathy, K.R. (1984). *Quantum Ito's formula and stochastic evolutions.* Comm. in Math. Phys., **93**, 301-323.

[25] Loubenets, E.R. (1999). *Quantum stochastic evolutions and the Schrödinger equation.* Russian Journal of Mathematical Physics. (To appear).

[26] Ozawa, M.(1884). *Quantum measuring processes of continuous observables.* J.Math.Phys.**25**, 79-87.

[27] Peres, A. (1993). *Quantum Theory: Concepts and Methods*. Kluwer, Dordrecht.

[28] Reed, M., Simon, B.(1972). *Methods of Modern Mathematical Physics,* Academic Press, New York, London.

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