Markov jump processes with a singularity

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Abstract

Certain types of Markov jump processes $x(t)$ with continuous state space and one or more absorbing states are studied. Cases where the transition rate in state x is of the form $\lambda(x) = |x|^\delta$ in a neighbourhood of the origin in $\mathbf{R^d}$ are considered, in particular. This type of problem arises from quantum physics in the study of laser cooling of atoms, and the present paper connects to recent work in the physics literature. The main question addressed is that of the asymptotic behaviour of $x(t)$ near the origin for large t. The study involves solution of a renewal equation problem in continuous state space.

Keywords: confluent hypergeometric function; laser cooling; renewal theory.

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1 Introduction

Laser cooling of atoms is a recently developed field in physics of great current interest, and the 1997 Nobel price in Physics was awarded for pathbreaking work in this field. The most effective cooling techniques are capable of bringing down the temperature to the astonishingly low order of about one nano-Kelvin. Stochastic reasoning has played a significant role in understanding and improving the effectiveness of the cooling procedures. We refer to Barndorff-Nielsen and Benth (1999) for an account of this.

The quantum mechanical processes involved in the cooling are of great complexity but, as explained in Bardou et al. (1999) (cf. also Bardou et al., 1994, Bardou and Castin, 1998, and Barndorff-Nielsen and Benth (1999)), a key aspect for the understanding can be formulated in classical probabilistic terms as follows. Consider a Markov jump process $x(t)$ in a region B of $\mathbf{R^d}$ containing the origin, let $\lambda(x)$ denote the jump intensity in state $x \in B$ and suppose that λ is a smooth function of x and such that $\lambda(0) = 0$ while $\lambda(x) > 0$ for $x \neq 0$. Her $x(t)$ represents the momentum of the atom at time t. The problem consists in mathematically describing the behaviour of $x(t)$ near the origin, as t tends to ∞, two main questions being how much of the total experimental time does an atom spend in a small neighbourhood - the 'trap' - of 0 and, given that the atom is in the trap, what is the distribution of its momentum. In analysing this problem the physicists made several simplifying Ansätze. An aim of the work reported here has been to give an analysis of the situation, in order to see whether the conclusions about the (first order) asymptotics stand without these Ansätze. However, quite apart from the physical motivation, it would seem of interest to study such Markov jump processes with a singularity.

In the present paper, we focus on the cases where λ is of the form $\lambda(x) = |x|^\delta$, at least near the origin. Specifications of this type, in particular the cases $\delta = 2$ or 4, rest on the basic (quantum) physical nature of the cooling methodology (see, for instance, Barndorff-Nielsen and Benth (1999) and references given there).

Section 2 sets up the relevant Markov process model in its general form and gives the solution to Kolmogorov's forward equation in terms of the solution to a generalized renewal equation. We have not, so far, been able to give a complete rigorous treatment of the asymptotic behaviour for arbitrary jump transition laws, but in Section 3 such a treatment is provided for the case where the transition law does not depend on present position. The main results are contained in Proposition 3.2 which specifies the asymptotic behaviour of the law of $x(t)$, in terms of a confluent hypergeometric function. The asymptotic behaviour of the time spent in the present state since the last jump to that state is also derived, and several examples are given. It seems likely that the results obtained will hold for general transition laws, as we argue in Section 5. However, since a complete resolution is not presently available, to throw added light on the situation we have, in Section 4, considered a discretized version of the model for which we are able to treat state-dependent transitions. The concluding Section 6 briefly discusses some related work, including a comparison with the results obtained in Bardou et al. (1999), and indicates several possibilities for extensions of the theory.

2 General model

We will consider a particle moving around in the region B in $\mathbf{R}^d$. When the particle is at position x it stays there for a period that is exponentially distributed with mean $\lambda(x)^{-1}$. When the particle jumps the distribution of the new point has density $p(\cdot|x)$. A special case of some interest is $p(y|x) = q(x)1(y \in B_x)$, where $B_x = B(x,r) \cap B$ with $B(x,r)$ a ball with center x and radius r and $q(x) = 1/|B_x|$, that is $q(x)$ is one divided by the volume of B_x. We assume that the initial position $x(0)$ of the particle is distributed according to the density $a(x)$ with respect to Lebesgue measure. We will make the assumptions that

$$p(y|x) \leq k_1 \quad \forall\ x, y \in B, \quad \text{and} \quad \lambda(x) \leq \Lambda \quad \forall\ x \in B, \tag{1}$$

for some constants k_1 and Λ. From these assumptions we show in Appendix A that $x(t)$ has a density $p(x,t)$ with respect to Lebesgue measure and that this density satisfies Kolmogorov's forward equation

$$p_t(x,t) = -\lambda(x)p(x,t) + \int_B \lambda(y)p(x|yp(y,t)dy, \tag{2}$$

where $p_t(x,t)$ is the derivative of $p(x,t)$ with respect to t. Define

$$h(x,t) = \int_B \lambda(y)p(x|y)p(y,t)dy, \tag{3}$$

so that Kolmogorov's forward equation (2) can be written

$$p_t(x,t) = -\lambda(x)p(x,t) + h(x,t). \tag{4}$$

If $h(x,\cdot)$ is known the unique solution to (4) is given as

$$p(x,t) = a(x)\mathrm{e}^{-t\lambda(x)} + \int_0^t h(x,\tau)\mathrm{e}^{-(t-\tau)\lambda(x)}d\tau. \tag{5}$$

Direct checking shows that (5) is in fact a solution to (4). For given $h(x,\cdot)$ it is also the unique solution with $p(x,0) = a(x)$: if p_1 and p_2 are solutions we have that $m(x,t) = p_1(x,t) - p_2(x,t)$ satisfies the equation $m_t(x,t) = -\lambda(x)m(x,t)$, or $m(x,t) = c(x)\exp(-t\lambda(x))$ for some function $c(x)$, but since $m(x,0) = 0$ we have $c(x) = 0$.

We will use (5) to establish the asymptotic form of $p(x,t)$ for $t \to \infty$ from the asymptotic form of $h(x,t)$. Note that

$$h(x,t) \le k_1\Lambda \quad \text{and} \quad \int_B h(x,t)dx \le \Lambda.$$

To study $h(x,t)$ we derive a renewal type equation. Define

$$f(x,y,s) = p(x|y)\lambda(y)\mathrm{e}^{-s\lambda(y)}$$

and

$$v(x,t) = \int_B a(y)f(x,y,t)dy.$$

From the definition (3) we get

$$\begin{aligned} h(x,t) &= \int_B \left\{p(x|y)\lambda(y)p(y,t) - p(x|y)\lambda(y)a(y)\mathrm{e}^{-t\lambda(y)}\right\} dy + v(x,t) \\ &= \int_B \left\{\left[p(x|y)\lambda(y)p(y,t)\mathrm{e}^{-(t-\tau)\lambda(y)}\right]_{\tau=0}^{\tau=t}\right\} dy + v(x,t) \\ &= \int_B \int_0^t \left\{p(x|y)\lambda(y)p_t(y,\tau)\mathrm{e}^{-(t-\tau)\lambda(y)} + p(x|y)\lambda(y)^2 p(y,\tau)\mathrm{e}^{-(t-\tau)\lambda(y)}\right\} d\tau dy \\ &\quad +v(x,t) \end{aligned}$$

Using (4) we find that

$$h(x,t) = v(x,t) + \int_B \int_0^t h(y,t-w)f(x,y,w)dwdy, \tag{6}$$

As for the ordinary renewal equation we can write the solution to (6) explicitly in terms of f. Define recursively, for $x, y \in B$,

$$f^{n*}(x,y,t) = \int_B \int_0^t f^{(n-1)*}(z,y,t-w) f(x,z,w) dw dz \tag{7}$$

with $f^{1*}(x,y,t) = f(x,y,t)$. Next define the (generalized) renewal density

$$u(x,y,t) = \sum_{n=1}^{\infty} f^{n*}(x,y,t). \tag{8}$$

Proposition 2.1 *We have that the renewal density $u(x,y,t)$ is bounded on $B \times B \times [0,t]$ for any $t > 0$, and*

$$h(x,t) = v(x,t) + \int_B \int_0^t v(y,t-w) u(x,y,w) dw dy \tag{9}$$

exists and solves (6). The solution is unique among all functions $\tilde{h}$ satisfying $\tilde{h} \geq 0$ and $\sup_{s \leq t} \int_B \tilde{h}(x,s) dx < \infty$.

Proof. We first obtain a bound on $f^{n*}(x,y,t)$. We have

$$\begin{aligned} & f^{n*}(z_0, z_n, w_n) \\ &= \int_B \cdots \int_B \int_0^{w_n} \cdots \int_0^{w_2} \prod_{i=1}^n f(z_{i-1}, z_i, w_i - w_{i-1}) dz_{n-1} \cdots dz_1 dw_{n-1} \cdots dw_1 \\ &= \int_B \cdots \int_B \prod_{i=1}^n p(z_i | z_{i-1}) \\ &\quad \times \left(\int_0^{w_n} \cdots \int_0^{w_2} \prod_{i=1}^n \lambda(z_i) \mathrm{e}^{-\lambda(z_i)(w_i - w_{i-1})} dw_{n-1} \cdots dw_1 \right) dz_{n-1} \cdots dz_1. \end{aligned} \tag{10}$$

For the inner integrals in (10) we have the bound

$$\Lambda^n \int_0^{w_n} \cdots \int_0^{w_2} dw_{n-1} \cdots dw_1 = \frac{\Lambda^n w_n^{n-1}}{(n-1)!} \leq \frac{\Lambda^n t^{n-1}}{(n-1)!}$$

for $w_n \leq t$. We then obtain for (10) the bound

$$f^{n*}(z_0, z_n, w_n) \leq c_t (\frac{1}{2})^n \int_B \cdots \int_B \prod_{i=1}^n p(z_{i-1} | z_i) dz_{n-1} \cdots dz_1 = \frac{\Lambda^n t^{n-1}}{(n-1)!} p^n(z_0 | z_n), \tag{11}$$

where $p^n(x|y)$ is the n-step transition density for a Markov chain with transition density $p(x|y)$. From (1) we have $p^n(x|y) \leq k_1$ and we therefore obtain

$$u(x,y,s) \leq \sum_{n=1}^{\infty} \frac{\Lambda^n t^{n-1}}{(n-1)!} p^n(x|y) = k_1 \Lambda \mathrm{e}^{\Lambda t} \quad \text{for } s \leq t, x, y \in B.$$

This establishes the boundedness of u on $B \times B \times [0,t]$ for any $t > 0$. Since $\int_B v(y,s)dy$ is bounded by Λ the integral in (9) is less than $(\Lambda t)k_1\Lambda \mathrm{e}^{\Lambda t}$ and so the right hand side of (9) is well defined. Furthermore, we find

$$\begin{aligned}\int_B\int_B\int_0^t v(y,t-w)u(x,y,w)dwdydx &\leq \int_B\int_0^t \Lambda \mathrm{e}^{\Lambda w} v(y,t-w)dwdy \\ &\leq \Lambda^2\int_0^t \mathrm{e}^{\Lambda w}dw \\ &\leq \Lambda \mathrm{e}^{\Lambda t}\end{aligned}$$

so that $\sup_{s\leq t}\int_B h(x,s)dx \leq \Lambda + \Lambda \mathrm{e}^{\Lambda t}$ for $h(x,t)$ defined in (9.

Let us denote the right hand side of (9) by $\tilde{h}(x,t)$. That this is a solution to (6) follows from

$$\begin{aligned}&\int_B\int_0^t \tilde{h}(y,t-w)f(x,y,w)dwdy + v(x,t) \\ &= \int_B\int_0^t \left(v(y,t-w) + \int_B\int_0^{t-w} v(z,t-w-\xi)u(y,z,\xi)d\xi dz\right) f(x,y,w)dwdy \\ &\quad +v(x,t) \\ &= \int_B\int_0^t v(y,t-w)\sum_{n=1}^{\infty} f^{n*}(x,y,w)dwdy + v(x,t) \\ &= \tilde{h}(x,t).\end{aligned}$$

If h_1 and h_2 are two solutions to (6) we have

$$(h_1-h_2)(x,t) = \int_B\int_0^t (h_1-h_2)(y,t-w)f(x,y,w)dwdy.$$

Iteration of this equation gives for any n

$$(h_1-h_2)(x,t) = \int_B\int_0^t (h_1-h_2)(y,t-w)f^{n*}(x,y,w)dwdy,$$

and from (11) we therefore find

$$|h_1(x,t)-h_2(x,t)| \leq k_1\frac{\Lambda^n t^{n-1}}{(n-1)!}tk_t, \tag{12}$$

where

$$k_t = \sup_{s\leq t}\int_B h_1(x,s)dx + \sup_{s\leq t}\int_B h_2(x,s)dx.$$

Letting n tend to infinity in (12) we find $h_1 = h_2$ provided that k_t is finite for every t. □

Thus, in summary, the density $p(x,t)$ is given as the solution (5) to Kolmogorov's forward equation and with the function $h(x,t)$ given through (9). We return to the general setup of this section in Section 5 making a conjecture as to the asymptotic behaviour of the solution to the equation (6). First, however, we turn to the special case where $p(y|x)$ does not depend on x. In that case we are able to give a complete rigorous discussion of the (first order) asymptotics for the physically motivated forms of $\lambda(x)$ mentioned in the Introduction.

3 Simple model

In this section we consider the special case of the general model where a jump is independent of the present position, that is, we assume that $p(y|x)$ is a function $b(y)$ independent of x. In this situation the renewal equation (6) reduces to an ordinary renewal equation for the function $g(t)$ defined in (14) below and this allows us to derive the asymptotic form of the function. From this we next establish the asymptotic form of the density $p(x,t)$.

We will assume that the density b is a bounded and continuous function. The initial density $a(\cdot)$ is also assumed bounded and continuous. The Kolmogorov forward equation (2) becomes for the special case considered here

$$p_t(x,t) = -\lambda(x)p(x,t) + b(x)\int_B \lambda(y)p(y,t)dy. \tag{13}$$

Define

$$g(t) = \int_B \lambda(y)p(y,t)dy. \tag{14}$$

Then the function $h(x,t)$ from the general case is given by $h(x,t) = b(x)g(t)$.

3.1 Properties of g

We start by defining

$$v(t) = \int_B a(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}dy \quad \text{and} \quad u(t) = \int_B b(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}dy.$$

Similarly to the derivation of (6) we find

$$\begin{aligned}
g(t) &= \int_B \left\{\lambda(y)p(y,t) - a(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}\right\} dy + v(t) \\
&= \int_B \left\{\left[\lambda(y)p(y,\tau)\mathrm{e}^{-(t-\tau)\lambda(y)}\right]_{\tau=0}^{\tau=t}\right\} dy + v(t) \\
&= \int_B \int_0^t \left\{\lambda(y)p_t(y,\tau)\mathrm{e}^{-(t-\tau)\lambda(y)} + \lambda(y)^2 p(y,\tau)\mathrm{e}^{-(t-\tau)\lambda(y)}\right\} d\tau dy + v(t) \\
&= \int_B \int_0^t b(y)g(\tau)\lambda(y)\mathrm{e}^{-(t-\tau)\lambda(y)} d\tau dy + v(t) \\
&= v(t) + \int_0^t g(t-s)u(s)ds,
\end{aligned}$$

where in the fourth line we have used (13). Thus g satisfies the renewal equation

$$g(t) = v(t) + \int_0^t g(t-s)u(s)ds. \tag{15}$$

According to Feller (1966, VI.6) the solution to the equation (15) is given by

$$g(t) = \int_0^t v(t-s)Q(ds), \tag{16}$$

where $Q(s) = \sum_{n=0}^{\infty} U^{n*}(s)$ with $U(s) = \int_0^s u(\tau)d\tau$. The solution is unique among functions that are bounded on bounded intervals.

We will study the asymptotic behaviour of $g(t)$ as $t \to \infty$ from the properties of u and v for large values of t. In Section 3.4 we show that for the physically motivated choices of $\lambda(x)$ we will have that u and v are asymptotically power functions of t. In the next proposition we consider different cases according to the value of $\alpha > 0$ in $u(t) \sim ct^{-(1+\alpha)}$ for $t \to \infty$. We then have that

$$1 - U(t) = \int_t^\infty u(\tau)d\tau \sim \frac{c}{\alpha} t^{-\alpha}$$

and

$$m(t) = \int_0^t (1 - U(\tau))d\tau \sim \begin{cases} \frac{c}{\alpha(1-\alpha)} t^{1-\alpha} & \alpha < 1 \\ c\log(t) & \alpha = 1. \end{cases} \tag{17}$$

Furthermore, if $\alpha > 1$ we have

$$\mu = m(\infty) = \int_0^\infty tu(t)dt = \int_0^\infty \int_B b(y)t\lambda(y)\mathrm{e}^{-t\lambda(y)}dydt = \int_B \frac{b(y)}{\lambda(y)}dy < \infty \tag{18}$$

Proposition 3.1 *Assume that $u(t) \sim ct^{-(1+\alpha)}$ and $v(t) \sim c_1 t^{-(1+\alpha)}$ for $t \to \infty$. Then*

(i) if $\alpha > 1$ we have $g(t) \to \mu^{-1}$ for $t \to \infty$, where $\mu = \int_0^\infty \tau u(\tau)d\tau$;

(ii) if $\alpha = 1$ we have $g(t) \sim \{c\log t\}^{-1}$ for $t \to \infty$;

(iii) if $\frac{1}{2} < \alpha < 1$ we have $g(t) \sim \alpha\{c\Gamma(\alpha)\Gamma(1-\alpha)\}^{-1} t^{-(1-\alpha)}$ for $t \to \infty$;

(iv) if $0 < \alpha \leq \frac{1}{2}$ we have $G(t) = \int_0^t g(\tau)d\tau \sim \{c\Gamma(\alpha)\Gamma(1-\alpha)\}^{-1} t^{\alpha}$ for $t \to \infty$.

Proof. According to Feller (1966, XI.1, Theorem 2) we have in the case $\mu = \int_0^\infty \tau u(\tau)d\tau < \infty$ that $g(t) \to \frac{1}{\mu}\int_0^\infty v(t)dt$, which proves (i).

The cases (ii) and (iii) are obtained from Erickson (1970, Theorem 3):

$$g(t) \sim \{\Gamma(\alpha)\Gamma(2-\alpha)\}^{-1} \left(\int_0^\infty v(t)dt\right) m(t)^{-1}$$

with $m(t)$ given in (17) above.

For case (iv) define $\hat{G}(s) = \int_0^\infty e^{-st} g(t)dt$, $\hat{V}(s) = \int_0^\infty e^{-st} v(t)dt$, and $\hat{U}(s) = \int_0^\infty e^{-st} u(t)dt$. From the renewal equation (15) we have

$$\hat{G}(s) = \hat{V}(s) + \hat{G}(s)\hat{U}(s),$$

or

$$\hat{G}(s) = \frac{\hat{V}(s)}{1 - \hat{U}(s)}. \tag{19}$$

From the assumption on $u(\cdot)$ we have that $1 - U(t) \sim \frac{c}{\alpha} t^{-\alpha}$. From Bingham, Goldie and Teugels (1987, Corollary 8.1.7) we find that $1 - \hat{U}(s) \sim s^\alpha \frac{c\Gamma(1-\alpha)}{\alpha}$ for $s \to 0$, and from (19) we see that

$$\hat{G}(s) \sim s^{-\alpha} \frac{\alpha}{c\Gamma(1-\alpha)} \quad \text{for } s \to 0.$$

Karamata's Tauberian theorem (Bingham et.al., Theorem 1.7.1) finally gives

$$G(t) \sim t^\alpha \frac{\alpha}{c\Gamma(1-\alpha)\Gamma(1+\alpha)}.$$

□

3.2 Properties of $p(x,t)$

Noting that $h(x,t) = b(x)g(t)$ we have from (5) that the solution to Kolmogorov's equation (13) is given by

$$p(x,t) = e^{-t\lambda(x)} \left\{ a(x) + b(x) \int_0^t g(\tau) e^{\tau\lambda(x)} d\tau \right\}. \tag{20}$$

Define the function $\Psi(z,\beta)$ to be

$$\Psi(z,\beta) = \int_0^1 (1-s)^{\beta-1} e^{-sz} ds \tag{21}$$

for $z \geq 0$ and $0 < \beta \leq 1$. Apart from a norming constant this is a type I confluent hypergeometric function. Note that for $\beta = 1$ we have

$$\Psi(z,1) = \frac{1 - e^{-z}}{z}.$$

Proposition 3.2 *Assume that b is bounded. If*

(i) $g(t) \to c$, $0 < c < \infty$, *then*

$$p(x,t) \frac{1 + t\lambda(x)}{t} = cb(x)(1 + t\lambda(x))\Psi(t\lambda(x), 1) + o(1)$$

and the $o(1)$ term is uniform in x;

(ii) $g(t) \sim c/\log t$ *then*

$$p(x,t)\frac{1+t\lambda(x)}{t}\log t = cb(x)(1+t\lambda(x))\Psi(t\lambda(x),1)+o(1)$$

and the $o(1)$ *term is uniform in* x*;*

(iii) $g(t) \sim ct^{-\gamma}$, $0<\gamma<1$, *then*

$$p(x,t)\frac{1+t\lambda(x)}{t^{1-\gamma}} = cb(x)(1+t\lambda(x))\Psi(t\lambda(x),1-\gamma)+o(1)$$

and the $o(1)$ *term is uniform in* x*;*

(iv) $G(t)=\int_0^t g(\tau)d\tau \sim ct^{1-\gamma}/(1-\gamma)$, $0<\gamma<1$, *then*

$$p(x,t)t^{-(1-\gamma)} = cb(x)\Psi(t\lambda(x),1-\gamma)+o(1)$$

and the $o(1)$ *term is uniform in* x *(note that in this case the error* $o(1)$ *is not a relative error for* $t\lambda(x)$ *large).*

Proof. We use formula (20) and write λ for $\lambda(x)$. In case (i) we have

$$a(x)\mathrm{e}^{-t\lambda}\frac{1+t\lambda}{t} = O(t^{-1}) = o(1).$$

For the integral in (20) we consider the cases $t\lambda \geq 1$ and $t\lambda < 1$ separately. In the former case we write

$$\begin{aligned}
&\frac{1+t\lambda}{t}\int_0^t g(\tau)\mathrm{e}^{-(t-\tau)\lambda}d\tau \\
&= \frac{1+t\lambda}{t\lambda}\int_0^{t\lambda} g(t-\frac{y}{\lambda})\mathrm{e}^{-y}dy \\
&= \frac{1+t\lambda}{t\lambda}\left(\int_0^{t\lambda(1-\epsilon)}+\int_{t\lambda(1-\epsilon)}^{t}\right) \\
&= \frac{1+t\lambda}{t\lambda}\left(\int_0^{t\lambda(1-\epsilon)} c(1+o(1))\mathrm{e}^{-y}dy + O(\epsilon t\lambda \mathrm{e}^{-t\lambda(1-\epsilon)})\right) \\
&= c\frac{1+t\lambda}{t\lambda}\int_0^{t\lambda}\mathrm{e}^{-y}dy + o(1),
\end{aligned}$$

where ϵ is chosen so that $\epsilon \to 0$ and $t\epsilon \to \infty$. For the case $t\lambda < 1$ we write

$$\begin{aligned}
&\frac{1+t\lambda}{t}\int_0^t g(\tau)\mathrm{e}^{-(t-\tau)\lambda}d\tau \\
&= (1+t\lambda)\int_0^1 g(tw)\mathrm{e}^{-(1-w)t\lambda}dw \\
&= (1+t\lambda)\left(\int_0^{\epsilon}+\int_{\epsilon}^1\right) \\
&= (1+t\lambda)\left(O(\epsilon)+\int_{\epsilon}^1 c(1+o(1))\mathrm{e}^{-(1-w)t\lambda}dw\right) \\
&= c(1+t\lambda)\int_0^1 \mathrm{e}^{-(1-w)t\lambda}dw + o(1),
\end{aligned}$$

with ϵ chosen as before.

In case (ii) we proceed exactly as above. In the situation with $t\lambda \geq 1$ we get a term $O(\log(t)\epsilon t\lambda \mathrm{e}^{-t\lambda(1-\epsilon)})$ and therefore need that $\epsilon \log(t) \to 0$. We can take $\epsilon = (\log t)^{-2}$ to achieve this. Furthermore, we use that

$$g(t - \frac{y}{\lambda}) \log t = \frac{c \log t}{\log t + \log(1 - \frac{y}{t\lambda})}(1 + o(1)) = c + o(1)$$

for $y < t\lambda(1 - \epsilon)$.

In case (iii) the proof is also as before. For $t\lambda \geq 1$ we find

$$\frac{1 + t\lambda}{t^{1-\gamma}} \int_0^t g(\tau)\mathrm{e}^{-(t-\tau)\lambda} d\tau = \frac{1 + t\lambda}{t\lambda} \left(\int_0^{t\lambda} c(1 - \frac{y}{t\lambda})^{-\gamma} \mathrm{e}^{-y} dy + O(t^\gamma \epsilon t\lambda \mathrm{e}^{-t\lambda(1-\epsilon)}) \right)$$

and we must require that $t^\gamma \epsilon \to 0$.

For case (iv) the proof is slightly different. We rewrite the integral as

$$\begin{aligned} & t^{-(1-\gamma)} \int_0^t g(\tau)\mathrm{e}^{-(t-\tau)\lambda} d\tau \\ = \;& t^{-(1-\gamma)} \left\{ G(t) - \int_0^t G(\tau)\lambda \mathrm{e}^{-(t-\tau)\lambda} d\tau \right\} \\ = \;& \frac{G(t)}{t^{1-\gamma}} \left\{ 1 - \int_0^{t\lambda} \frac{G(t - \frac{y}{\lambda})}{G(t)} \mathrm{e}^{-y} dy \right\}, \end{aligned}$$

and as before the integral is split into the two parts with $0 < y < t\lambda(1 - \epsilon)$ and $t\lambda(1 - \epsilon) < y < t$. We then end up with

$$\frac{G(t)}{t^{1-\gamma}} \left\{ (1 - \gamma) \int_0^1 (1 - s)^{-\gamma} \mathrm{e}^{-st\lambda} ds + o(1) \right\}.$$

Since the integral tends to zero for $t\lambda$ large the error is not relative in this limit. □

Let us conclude this section with a discussion of case (i) in Proposition 3.2, where the Markov process $x(t)$ actually admits an invariant distribution. If we let $t \to \infty$ in $\int_B p(x, t)dx = 1$ we see from (i) in Proposition 3.2 that the constant c is μ^{-1}, in accordance with (i) in Proposition 3.1. In this case it follows from (18) that

$$\omega(x) = \frac{b(x)}{\mu\lambda(x)}.$$

is a probability density. If we take the initial density $a(x) = \omega(x)$ we see that $v(t) = \mu^{-1} \int_B b(x) \exp(-t\lambda(x))dx$ and $U(t) = \int_B b(x)[1 - \exp(-t\lambda(x))]dx = 1 - \mu v(t)$. It therefore follows that $g(t) = \mu^{-1}$ solves the renewal equation (15). From (20) we then have

$$p(x, t) = \frac{b(x)}{\mu\lambda(x)} \mathrm{e}^{-t\lambda(x)} + b(x)\mu^{-1}(1 - \mathrm{e}^{-t\lambda(x)})\lambda(x)^{-1} = \omega(x),$$

that is, $\omega(x)$ is an invariant density. We can rewrite (i) in Proposition 3.2 in the form

$$p(x,t) = \omega(x)(1 - \mathrm{e}^{-t\lambda(x)}) + \frac{t}{1+t\lambda(x)} o(1),$$

which shows that $p(x,t)$ converge to $\omega(x)$, but the convergence is not uniform near $\lambda(x) = 0$.

3.3 Time spent in present state

It is of some interest to consider, together with the momentum at time t, that is $x(t)$, also the time spent in state $x(t)$ since the last transition to that state. Thus consider the pair of variables $(x(t), u(t))$, where $u(t)$ is the time spent in the present state $x(t)$. Since the event $\{u(t) > u, x(t) = x\}$ is the same as $x(t-u) = x$ and no jumps having taken place in the time interval from $t-u$ to t, we have that the density of $x(t)$ constrained to the event $u(t) > u$ is $p(x, t-u)\exp(-\lambda(x)u)$. This shows that the conditional probability of $u(t) > u$ given that $x(t) = x$ is

$$P(u(t) > u|x(t) = x) = \frac{p(x,t-u)}{p(x,t)} \mathrm{e}^{-\lambda(x)u}. \tag{22}$$

We will now consider the case where both $t \to \infty$ and $t - u \to \infty$. We then have from Proposition 3.2

$$P(u(t) > u|x(t) = x) \sim \begin{cases} \frac{1-\mathrm{e}^{-(t-u)\lambda(x)}}{1-\mathrm{e}^{-t\lambda(x)}} \mathrm{e}^{-\lambda(x)u} & \text{case (i)} \\ \frac{(1-\mathrm{e}^{-(t-u)\lambda(x)})\log(t)}{(1-\mathrm{e}^{-t\lambda(x)})\log(t-u)} \mathrm{e}^{-\lambda(x)u} & \text{case (ii)} \\ (1-\frac{u}{t})^{1-\gamma} \frac{\Psi((t-u)\lambda(x),1-\gamma)}{\Psi(t\lambda(x),1-\gamma)} \mathrm{e}^{-\lambda(x)u} & \text{case (iii).} \end{cases}$$

Letting u and x be fixed we get in all three cases the limit

$$\lim P(u(t) > u|x(t) = x) = \mathrm{e}^{-\lambda(x)u},$$

that is, we get an exponential distribution in the limit. However, if instead we fix $w = t\lambda(x)$, corresponding typically to x tending to zero, and at the same time scale u by considering the new variable $z = u/t$ we find

$$P(u(t) > tz|t\lambda(x(t)) = w) \sim \begin{cases} \frac{1-\mathrm{e}^{-(1-z)w}}{1-\mathrm{e}^{-w}} \mathrm{e}^{-wz} & \text{case (i)} \\ \frac{(1-\mathrm{e}^{-(1-z)w})}{(1-\mathrm{e}^{-w})} \mathrm{e}^{-wz} & \text{case (ii)} \\ (1-z)^{1-\gamma} \frac{\Psi((1-z)w,1-\gamma)}{\Psi(w,1-\gamma)} \mathrm{e}^{-wz} & \text{case (iii).} \end{cases}$$

Note that in case (ii) there is a non-uniformity in z since we have replaced $(\log(t) + \log(1-z))/\log t$ by one.

3.4 Examples

Example 3.3 Assume that $\lambda(x) = |x|^\delta$ for $|x| \leq c_1$ and $\lambda(x) \geq c_2$ for $|x| > c_1$ for some positive constants c_1 and c_2. Assume also that $b(0) > 0$. Then we have

$$\begin{aligned} u(t) &= \int_B b(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}dx \\ &= \int_{y\in B(0,c_1)} b(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}dx + O(\Lambda \mathrm{e}^{-tc_2}) \\ &= t^{-(1+d/\delta)} \int_{t^{1/\delta}B(0,c_1)} b(wt^{-1/\delta})|w|^\delta \mathrm{e}^{-|w|^\delta} dw + O(\Lambda \mathrm{e}^{-tc_2}) \\ &\sim b(0)\frac{C_d}{t^{1+d/\delta}} \int_0^{c_1 t^{1/\delta}} r^{d-1+\delta}\mathrm{e}^{-r^\delta}dr \\ &\sim b(0)\frac{C_d\Gamma(1+d/\delta)}{\delta t^{1+d/\delta}}, \end{aligned} \tag{23}$$

where $C_1 = 2$, $C_2 = 2\pi$, and $C_3 = 4\pi$. The same holds with u replaced by v and $b(0)$ replaced by $a(0)$, assuming that $a(0) > 0$. This justifies the assumption made in Proposition 3.1. □

Example 3.4 Let $\lambda(x) = |x|^\delta$ and let $b(0) > 0$ and $a(0) > 0$. From (23) we have

$$u(t) \sim b(0)\frac{C_d\Gamma(1+d/\delta)}{\delta t^{1+d/\delta}}$$

Define $w = xt^{1/\delta}$. If $d/\delta < 1$ we have from (23) and Proposition 3.1 that $G(t) \sim c_1 t^{d/\delta}$. We then obtain from Proposition 3.2 that the density $p^w(w,t)$ of w is

$$p^w(w,t) = p(\frac{w}{t^{1/\delta}}, t)t^{-d/\delta} \to c_2\Psi(|w|^\delta, d/\delta),$$

where

$$c_2 = \delta\{C_d\Gamma(\frac{d}{\delta})^2\Gamma(1-\frac{d}{\delta})\}^{-1}.$$

If $d/\delta = 1$ the situation is somewhat different. For illustration let us take $d = 1$ and $\lambda(x) = |x|$. If we define

$$w = \ \mathrm{sign}(x)\frac{\log(1+t|x|)}{\log(t)}$$

the density of w is from (ii) in Proposition 3.2

$$p^w(w,t) = p(x,t)\frac{1+t|x|}{t}\log(t) = cb(x)t^{|w|}\Psi(t^{|w|}-1,1) + o(1) \to cb(0)$$

for $|w| < 1$.

Finally, if $d/\delta > 1$ we have the invariant density $\omega(x) = b(x)/(\mu|x|^\delta)$, and the convergence of $p(x,t)$ to the stationary density is given in (i) of Proposition 3.2. □

Example 3.5 Consider $d=1$ and

$$\lambda(x)=\begin{cases} |x|^\alpha & |x|<1/4 \\ (1-|x|)^\beta & 3/4<|x|<1, \end{cases}$$

with a smooth transition in between the two regions and with $\alpha, \beta \geq 2$. Then

$$u(t) \sim c_1 t^{-(1+1/\alpha)} + c_2 t^{-(1+1/\beta)} \sim c_3 t^{-(1+1/\gamma)},$$

with $\gamma = \max\{\alpha, \beta\}$. From Proposition 3.1(iv) we obtain

$$G(t) \sim c_4 t^{1/\gamma}.$$

Letting $w = t^{1/\alpha}x$ we find from Propostion 3.2(iv) that the density of w is

$$\begin{aligned} p^w(w,t) &= p(\frac{w}{t^{1/\alpha}},t)t^{-1/\alpha} \\ &= t^{1/\gamma-1/\alpha}p(\frac{w}{t^{1/\alpha}},t)t^{-1/\gamma} \\ &= t^{1/\gamma-1/\alpha}\left\{c_5 b(\frac{w}{t^{1/\alpha}})\Psi(w^\alpha,\frac{1}{\gamma})+o(1)\right\} \quad \text{for } |w|<\sqrt{t}/4. \end{aligned}$$

This shows that, conditionally on being in the trap $|x|<1/4$, the density of w converges to

$$c_6 \Psi(w^\alpha, 1/\beta).$$

In case $\beta > \alpha$ we have $\gamma = \beta$. Then the probability of being in the trap $|x|<1/4$ is asymptotically $c_7 t^{-(\beta-\alpha)/(\alpha\beta)}$. When $\beta = \alpha$ a finite fraction of the atoms will be in the trap as $t \to \infty$ and when $\beta < \alpha$ the probability of being in the trap will tend to one. □

4 Discretized model

In this section we consider another special case of the general model where the jump density $p(y|x)$ depends on x only through which 'box' x belongs to. The general renewal equation (6) is then turned into a finite set of coupled renewal equations for the functions g_i defined in (25) below. For the equations that we obtain some results are already known in the literature. We first find the asymptotic form of the functions g_i and this can then be combined with the analysis in Proposition 3.2 to obtain the asymptotic form of the density $p(x,t)$.

Let the box B_i have side length Δ and center at $i\Delta$, $i \in \mathbf{Z}^d$. Let $S = \{i \in \mathbf{Z}^d | B_i \cap B \neq \emptyset\}$. We assume that $p(y|x) = p(y;i)$ for $x \in B_i$ and that the support of $p(y;i)$ is in $\cup_{j \in J_i} B_i$, where J_i consists of i and its neighbours, and where $|J_i| \leq k_0$ for some constant k_0. Define $S_m = \{i \in S | m \in J_i\}$. A special case is $J_i = \{j||j-i|<\kappa, j \in S\}$ and $p(y;i) = q_i/\Delta^d$ with $q_i = 1/|J_i|$.

The Kolmogorov forward equation is, for the case with $x \in B_m$,

$$\begin{aligned} p_t(x,t) &= -\lambda(x)p(x,t) + \sum_{j \in S_m} p(x;j) \int_{B_j} \lambda(y)p(y,t)dy \\ &= -\lambda(x)p(x,t) + \sum_{j \in S_m} p(x;j)g_j(t), \end{aligned} \tag{24}$$

where

$$g_j(t) = \int_{B_j} \lambda(y)p(y,t)dy. \tag{25}$$

We can now proceed as in the proof of (15) and get a set of coupled renewal equations for g_i. Define

$$v_i(t) = \int_{B_i} a(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}dy \quad \text{and} \quad u_{ij}(t) = \int_{B_i} p(y;j)\lambda(y)\mathrm{e}^{-t\lambda(y)}dy,$$

where $a(y)$ is the initial density. We then find that

$$\begin{aligned} g_i(t) &= \int_{B_i} \left\{\lambda(y)p(y,t) - a(y)\lambda(y)\mathrm{e}^{-t\lambda(y)}\right\} + v_i(t) \\ &= \int_{B_i} \int_0^t \frac{d}{d\tau} \left\{\lambda(y)p(y,\tau)\mathrm{e}^{-(t-\tau)\lambda(y)}\right\} d\tau dy + v_i(t) \\ &= \sum_{j \in S_i} \int_{B_i} \int_0^t g_j(\tau)p(y;j)\lambda(y)\mathrm{e}^{-(t-\tau)\lambda(y)} d\tau dy + v_i(t) \\ &= \sum_{j \in S_i} \int_0^t g_j(t-\tau)u_{ij}(\tau)d\tau + v_i(t). \end{aligned} \tag{26}$$

We can write the equations in vector form as

$$g(t) = v(t) + (M * g)(t), \tag{27}$$

where M is the matrix with (i,j)'th entry given by $1(j \in S_i)U_{ij}$, $U_{ij}(t) = \int_0^t u_{ij}(s)ds$, and $(M * g)_i = \sum_j M_{ij} * g_j$ with $*$ being ordinary convolution.

Our problem is now to analyze (26) in order to etablish the asymptotic form of $g_i(t)$ for $t \to \infty$. We first divide the discussion into the two cases of a bounded region B and an unbounded region B. For the bounded case we make a further division into the case where all the functions u_{ij} admit a finite mean and the case where at least one function u_{ij} has an infinite mean. For the case of a finite mean we can use results from the literature.

Case I: B bounded. We start with the case where the region B is bounded so that S is finite. According to Crump (1970a, Theorem 2.1) the solution to (27) is

$$g(t) = (Q * v)(t),$$

where $Q(t) = \sum_{n=0}^{\infty} M^{n*}(t)$ with $M^{1*} = M$ and $M^{n*}(t)_{ij} = \sum_l \int_0^t M^{(n-1)*}(t-s)_{lj} M_{il}(s)ds$. From Crump (1970a, Theorem 3.1, 1970b) we can also get information on the limiting behaviour of $g_i(t)$ in certain cases. Since $\int_0^\infty u_{ij}(t)dt = p(i|j)$, with $p(i|j) = \int_{B_i} p(y;j)dy$, we have that $M_{ij}(\infty) = 1(j \in S_i)p(i|j)$. Thus the column sums are equal to one and the largest eigenvalue of $M(\infty)$ is one, which allow us to use Theorem 3.1 in Crump (1970a). The theorem states that

$$g_i(t) \to \sum_{j\in S} \alpha_{ij} \int_0^\infty v_j(s)ds = \sum_{j\in S} \alpha_{ij} \int_{B_j} a(y)dy, \tag{28}$$

where the α_{ij}'s will be non-zero only if all the U_{ij}'s have a finite mean, that is, if $\int_0^\infty tu_{ij}(t)dt = \int_{B_i} p(y;j)\lambda(y)^{-1}dy < \infty$ for all i, j, or $\int_B p(y;j)\lambda(y)^{-1}dy < \infty$ for all j.

Case IA: $\int_0^\infty tu_{ij}(t)dt < \infty$ for all i, j. We now assume that $\int_B p(y;j)\lambda(y)^{-1}dy < \infty$ for all j so that $g_i(t) \to c_i > 0$ for all i. Our problem is now to determine the constant c_i. Letting t tend to infinity in (26) we get

$$c_i = \sum_{j\in S_i} c_j \int_{B_i} p(y;j)dy = \sum_{j\in S_i} c_j p(i|j).$$

We therefore conclude that $c_j = c\pi_j$ for some constant c, where π is the invariant distribution corresponding to the transition probabilities $p(i|j)$. For $x \in B_i$ the solution to (24) is

$$p(x,t) = a(x)\mathrm{e}^{-t\lambda(x)} + \sum_{j\in S_i} p(x,j) \int_0^t g_j(t-\tau)\mathrm{e}^{-\tau\lambda(x)}d\tau, \tag{29}$$

Integrating with respect to x and letting $t \to \infty$ we obtain

$$1 = \sum_{j\in S} c_j \int_B \frac{p(x;j)}{\lambda(x)}dx,$$

which shows that

$$c = \left(\sum_{j\in S} \pi_j \int_B \frac{p(x;j)}{\lambda(x)}dx\right)^{-1}.$$

Case IB: $\int_0^\infty tu_{ij}(t)dt = \infty$ for some i, j. When some of the U_{ij}'s have an infinite mean, that is $\int_B p(y;j)\lambda(y)^{-1}dy = \infty$ for some j, we proceed as in case (iv) of Proposition 3.1. Let $\hat{G}_i$, $\hat{V}_i$, and $\hat{U}_{ij}$ be the respective Laplace transforms. Then from (26)

$$\hat{G}_i(\theta) = \hat{V}_i(\theta) + \sum_{j\in S_i} \hat{G}_j(\theta)\hat{U}_{ij}(\theta), \tag{30}$$

or in matrix form (using column vectors)

$$\hat{G}(\theta) = \hat{V}(\theta) + D(\theta)A\hat{G}(\theta),$$

where A is the matrix with (i,j)'th entry $1(j \in S_i)$ and D is the matrix with elements $\hat{U}_{ij}(\theta)$. We thus have

$$\hat{G}(\theta) = (I - D(\theta)A)^{-1}\hat{V}(\theta). \tag{31}$$

Summing (30) we get

$$\sum_i \hat{G}_i(\theta) = \sum_i \hat{V}_i(\theta) + \sum_{i,j} 1(i \in S_j)\hat{U}_{ij}(\theta)\hat{G}_j(\theta),$$

or

$$0 = \sum_i \hat{V}_i(\theta) + \sum_{i,j} 1(i \in S_j)(\hat{U}_{ij}(\theta) - p(i|j))\hat{G}_j(\theta). \tag{32}$$

Since

$$\hat{U}_{ij}(\theta) = \int_{B_i} p(y;j)\frac{\lambda(y)}{\lambda(y)+\theta}dy \to p(i|j) \quad \text{for } \theta \to 0$$

and

$$\sum_i \hat{V}_i(\theta) = \int_B a(y)\frac{\lambda(y)}{\lambda(y)+\theta}dy \to 1$$

we see from equation (32) that there must be an i, j, $i \in S_j$ so that $\hat{G}_j(\theta)$ tends to infinity with the rate $(\hat{U}_{ij}(\theta) - p(i|j))^{-1}$. However, (30) shows that all the $\hat{G}_j$ have to tend to infinity at the same rate. Returning then to (32) we must have that none of the terms $1(i \in S_j)(\hat{U}_{ij}(\theta) - p(i|j))\hat{G}_j(\theta)$ must tend to infinity, and therefore the rate at which $\hat{G}_j(\theta)$ tends to infinity is given as the inverse of the maximum of $\hat{U}_{ij}(\theta) - p(i|j)$. Let us therefore consider the case

$$\hat{U}_{ij}(\theta) = \int_{B_i} p(y;j)\frac{\lambda(y)}{\lambda(y)+\theta}dy = p(i|j) - c_{ij}\theta^{\beta_{ij}} + o(\theta^{\beta_{ij}}), \tag{33}$$

where c_{ij} and β_{ij} are positive constants and at least one β_{ij} is less than one (since we have assumed that some of the U_{ij}'s have infinite mean). Let $\beta = \min_{ij} \beta_{ij}$. The argument above shows that

$$\hat{G}_i(\theta) \sim \tilde{c}_i\theta^{-\beta}, \tag{34}$$

and Karamata's Tauberian theorem (Bingham et.al., Theorem 1.7.1) finally gives

$$G_i(t) \sim \tilde{c}_i t^{\beta}/\Gamma(1+\beta). \tag{35}$$

We can find $\tilde{c}_i$ from (30). Multiplying both sides by θ^{β} and letting $\theta \to 0$ we get from (34)

$$\tilde{c}_i = \sum_{j \in S_i} p(i|j)\tilde{c}_j.$$

This shows that $\tilde{c}_i = c\pi_i$ for some constant c. To find c we return to (32). Letting $\theta \to 0$ in this equation we get

$$1 = c \sum_{i,j:\beta_{ij}=\beta} 1(i \in S_j) c_{ij} \pi_i,$$

with the c_{ij}'s defined in (33).

Combining (29) with (35) we get as in Proposition 3.2, case (iv), the asymptotic form of $p(x,t)$.

When instead of (33) we have that $p(i|j) - \hat{U}_{ij}(\theta)$ is of order $O(\theta)$ for $i \neq 0$, say, and $p(i|j) - \hat{U}_{0j}(\theta) \sim c_j\theta/\log(\theta)$ we get $\hat{G}_i(\theta) \sim \tilde{c}_i \log(\theta)/\theta$. Karamata's Tauberian theorem then gives that $G(t) \sim \tilde{c}_i t/\log(t)$

Case II: B unbounded. To illustrate what can be done when B is not bounded we turn to the case $d = 1$ and $B = \mathbf{R}$. We will consider the special case with $J_i = \{j||j-i| < \kappa\}$ and $p(y;i) = q/\Delta^d$ with $q = 1/|J_0|$. We will assume that for some R we have

$$\lambda(x) = \lambda^0 \ \text{ and } \ a(x) = 0 \ \text{ for } |x| > R. \tag{36}$$

For $|i| > R + \kappa$ the equation (30) reads

$$\hat{G}_i(\theta) = \frac{\lambda^0}{\lambda^0 + \theta} q \sum_{j \in S_i} \hat{G}_j(\theta), \tag{37}$$

For $i > R + \kappa$ we have a finite difference equation. The solution to this is determined by the roots of the equation

$$\frac{\lambda^0}{\lambda^0 + \theta} \frac{1}{2\kappa + 1} \sum_{s=-\kappa}^{\kappa} z^s - 1 = 0. \tag{38}$$

Since G_i is real we are only interested in the real roots of (38). A root has to be different from 1 when $\theta > 0$ and therefore a root cannot be a multiple root since this would imply

$$\sum_{s=-\kappa}^{\kappa} s z^s = 0.$$

Since G_i is bounded we look for the roots that are less than 1. Write $z = 1 - \epsilon$. In the limit $\theta \to 0$ we have $\epsilon \to 0$ and (38) becomes

$$1 + \frac{\theta}{\lambda^0} = \frac{1}{2\kappa + 1} \sum_{s=-\kappa}^{\kappa} (1 - s\epsilon + \frac{1}{2} s(s-1)\epsilon^2 + (\epsilon^3)).$$

This shows that

$$\epsilon = c\sqrt{\theta} + O(\theta).$$

Putting together our observations we have that

$$\hat{G}_i(\theta) = (1 - \epsilon(\theta))^{i-(R+2\kappa)} A(\theta), \quad \text{for } i \geq R + 2\kappa. \tag{39}$$

For $i < -(R+2\kappa)$ we have a similar statement with the same $\epsilon(\theta)$. We now use (39) together with (32). This gives

$$
\begin{aligned}
0 &= \sum_i \hat{V}_i(\theta) + \frac{\lambda^0}{\lambda^0+\theta} \sum_{|j|\geq R+2\kappa} \hat{G}_j(\theta) + \sum_{|j|<R+2\kappa} \hat{G}_j(\theta) \left(q \sum_{i\in S_j} (\hat{U}_i(\theta) - 1) \right) \\
&= \sum_i \hat{V}_i(\theta) + \frac{\lambda^0}{\lambda^0+\theta} \frac{2}{\epsilon(\theta)} A(\theta) + \sum_{|j|<R+2\kappa} \hat{G}_j(\theta) \left(q \sum_{i\in S_j} (\hat{U}_i(\theta) - 1) \right) \qquad (40)
\end{aligned}
$$

where $\hat{U}_i(\theta) = \hat{U}_{ii}(\theta)/q$ and we have used that $\hat{U}_{ij}(\theta) = 1(j \in S_i)\hat{U}_{ii}(\theta)$. From (30) we have that all the $\hat{G}_j(\theta)$ appearing in (40) have the same asymptotic form for $\theta \to 0$. Also from (39) we see that the asymptotic form of $A(\theta)$ is the same as that of $\hat{G}_j(\theta)$. Let us now assume that

$$
\begin{cases} \lambda(x) = |x|^\delta & \text{for } |x| < r \\ \lambda(x) > c > 0 & \text{for } |x| > r. \end{cases} \qquad (41)
$$

Then we have

$$
1 - \hat{U}_i(\theta) = \begin{cases} c_1\theta^{d/\delta} + O(\theta) & \text{for } i = 0 \\ \theta + O(\theta^2) & \text{for } i \neq 0. \end{cases}
$$

Returning to (40) we have terms of the order $\theta A(\theta)/\epsilon(\theta)$, $\theta A(\theta)$ and $\theta^{d/\delta}A(\theta)$. The conclusion is therefore that

$$
\hat{G}_j(\theta) \sim \begin{cases} c_2\theta^{-1/2} & \text{if } d/\delta \geq \frac{1}{2} \\ c_2\theta^{-d/\delta} & \text{if } d/\delta < \frac{1}{2} \end{cases} \qquad (42)
$$

From here we proceed as before to get the asymptotic form of $p(x,t)$, that is, Karamata's Tauberian theorem gives the asymptotic form of $G_j(t)$ for large t and then Proposition 3.2(iv) gives the asymptotic form of $p(x,t)$.

In summary, the conclusion of this section parallels the conclusions from the simple model in Section 3. Since our analysis shows that all the functions g_i appearing in (24) have the same asymptotic form they will give rise to the term $\Psi(t\lambda(x), \beta)$ in the asymptotic form of the density $p(x,t)$. The difference is that the function $b(x)$ appearing in the results of Proposition 3.2 is replaced by

$$
\sum_{j\in S_i} p(x,j)\pi_j, \quad \text{for } x \in B_i.
$$

5 General case revisited

We now return to the general setup of Section 2. Our problem is to find the asymptotic form of $h(x,t)$ for $t \to \infty$. To this end we transform (6) into an equation for the Laplace transform of $h(x,t)$. We are not able at present to give a completely rigorous analysis of this equation. Instead we indicate, non-rigorously, what kind of asymptotic behaviour to expect for the Laplace transform, and then by Karamata's Tauberian theorem, what to expect for $h(x,t)$ itself.

Let $\hat{H}(x,\theta) = \int_0^\infty \mathrm{e}^{-\theta t} h(x,t)dt$ be the Lapace transform of $h(x,t)$. Since, by (1), $h(x,t)$ is clearly bounded we have that $\hat{H}(x,\theta)$ is finite for $\theta > 0$. From (6) we find

$$\hat{H}(x,\theta) = \int_B a(y)p(x|y)\frac{\lambda(y)}{\lambda(y)+\theta}dy + \int_B \hat{H}(y,\theta)p(x|y)\frac{\lambda(y)}{\lambda(y)+\theta}dy. \tag{43}$$

Define $\phi(\theta) = \sup_{x\in B}\hat{H}(x,\theta)$. Since $\phi(\theta)\to\infty$ for $\theta\to 0$ we obtain from (43) that

$$\begin{aligned}\frac{\hat{H}(x,\theta)}{\phi(\theta)} &= \frac{\hat{V}(x,\theta)}{\phi(\theta)} + \int_B \frac{\hat{H}(y,\theta)}{\phi(\theta)}p(x|y)\frac{\lambda(y)}{\theta+\lambda(y)}dy \\ &\sim \int_B \frac{\hat{H}(y,\theta)}{\phi(\theta)}p(x|y)dy,\end{aligned}$$

for $\theta\to 0$. This shows that as θ tends to zero the function $\hat{H}(x,\theta)/\phi(\theta)$ approaches an invariant function for the transition density $p(x|y)$. Thus let $\kappa(x)$ be the solution to

$$\kappa(x) = \int_B \kappa(y)p(x|y)dy \;\; \forall\, x \in B,$$

normalized so that $\sup_x \kappa(x) = 1$. The above argument indicates that

$$\hat{H}(x,\theta) \sim \kappa(y)\phi(\theta) \quad \text{for } \theta\to 0. \tag{44}$$

In the special case where $p(x|y) = 1(x\in B_y)q(y)$ with $B_y = B\cap B(y,r)$ and $q(y) = 1/|B_y|$ we get $\kappa(y) = cq(y)^{-1}$.

To find the asymptotic form of $\phi(\theta)$ we will assume that B is bounded, in which case it seems plausible that the convergence in (44) is uniform in x. Integrating (43) with respect to x we get

$$\int_B \frac{\hat{H}(y,\theta)}{\phi(\theta)}\frac{\theta}{\theta+\lambda(y)}dy = \frac{1}{\phi(\theta)}\int_B a(y)\frac{\lambda(y)}{\theta+\lambda(y)}dy.$$

In the limit $\theta\to 0$ this gives us the relation

$$\int_B \kappa(y)\frac{\theta}{\theta+\lambda(y)}dy \sim \frac{1}{\phi(\theta)},$$

or

$$\theta\phi(\theta) \sim \left(\int_B \kappa(y)\frac{1}{\theta+\lambda(y)}dy\right)^{-1}. \tag{45}$$

If $\lambda(y)^{-1}$ is integrable we immediately get that the right hand side of (45) converges to $(\int_B \kappa(y)\lambda(y)dy)^{-1}$. To study the remaining cases we need to specify $\lambda(y)$. We will assume that $\lambda(y) = |y|^\delta$ for $|y| \le c_1$ and $\lambda(y) \ge c_2$ for $|y| > c_1$. The non-integrability of $\lambda(y)^{-1}$ implies that $d \le \delta$. We first note that

$$\begin{aligned}
\int_{B(0,c_1)} \frac{|y|^k}{\theta + |y|^\delta} dy &= C_d \int_0^{c_1} \frac{r^{k+d-1}}{\theta + r^\delta} dr \\
&= C_d \theta^{(k+d)/\delta - 1} \int_0^{c_1 \theta^{-1/\delta}} \frac{z^{k+d-1}}{1+z^\delta} dz \\
&\sim C_d \theta^{(k+d)/\delta - 1} \begin{cases} \int_0^\infty \frac{z^{k+d-1}}{1+z^\delta} dz & k+d-\delta < 0 \\ \log(\theta^{-1})/\delta & k+d-\delta = 0 \\ (\theta^{-1/\delta} c_1)^{k+d-\delta} & k+d-\delta > 0. \end{cases}
\end{aligned}$$

Using this together with

$$\int_{B(0,c_1)} \kappa(y) \frac{1}{\theta + \lambda(y)} dy = \kappa(0) \int_{B(0,c_1)} \frac{1}{\theta + \lambda(y)} dy + O\left(\int_{B(0,c_1)} \frac{|y|}{\theta + \lambda(y)} dy \right)$$

we find that the first term here dominates as θ tend to zero. Furthermore, the integral for $|y| > c_1$ is bounded, and we therfore end up with

$$\phi(\theta) \sim \begin{cases} \theta^{-d/\delta} (C_d \kappa(0) \int_0^\infty \frac{z^{k+d-1}}{1+z^\delta} dz)^{-1} & d < \delta \\ \delta / (C_d \kappa(0) \theta \log(\theta^{-1})) & d = \delta \\ \left(\theta \int_B \frac{\kappa(y)}{\lambda(y)} dy \right)^{-1} & d > \delta. \end{cases} \tag{46}$$

Using now Karamata's Tauberian theorem we find

$$H(x,t) \sim \begin{cases} \frac{t^{d/\delta} \kappa(x)}{\Gamma(1+d/\delta) C_d \kappa(0) \int_0^\infty \frac{z^{k+d-1}}{1+z^\delta} dz} & d < \delta \\ \frac{t \log(t) \delta \kappa(x)}{\Gamma(2) C_d \kappa(0)} & d = \delta \\ \frac{t \kappa(x)}{\Gamma(2) \int_B \frac{\kappa(y)}{\lambda(y)} dy} & d > \delta. \end{cases} \tag{47}$$

With this result we can find the asymptotic form of $p(x,t)$ as in Propostion 3.2. When $d > \delta$ we have an invariant distribution with density $\omega(x) = c\kappa(x)/\lambda(x)$.

6 Concluding remarks

An equation similar to (6) has been considered in Mode (1971,1972). Using L_2 theory the existence and the uniqueness of the solution is proved under conditions that are satisfied in the set up here as long as the region B is bounded and $\lambda(\cdot)$ is bounded. Mode (1971) study the equation (43) for the Laplace transform via Fredholm theory in order to obtain an asymptotic form for $\theta \to 0$. An equation for the Laplace transform of the renewal density $u(x,y,t)$ is established which involves the Fredholm determinant. To study the latter Mode (1972) makes an assumption that in our case translates into

$$\int_B \int_B \frac{p(x|y)^2}{\lambda(y)^2} dxdy < \infty.$$

This condition will typically not be satisfied in the cases of interest to us. It is an interesting question, though, whether the Fredholm theory used by Mode can still be used, via some other approach, to study the Fredholm determinant.

Schumitzky and Wenska (1975) have also studied an equation similar to (43). They require that the integral operator in (43) is analytic in θ. Seemingly, to apply this theory we need that the integral operator in (43) can be expanded around $\theta = 0$, which is not possible in our setup.

For the case of a finite set of coupled renewal equations, as in Section 4, we have used results (Crump, 1970 a,b) for the case of a finite mean. We have not been able to find similar results for the case of an infinite mean, that is it seems to be an open question whether the result of Erickson (1970) for the ordinary renewal equation can be generalized to the case of coupled renewal equations.

A number of generalizations to the model discussed here can be physically motivated. For example it is of interest to consider a modification of λ to the form $\lambda(x) = c + |x|^\delta$ with c small. If c is kept fixed this is a trivial model, instead we can imagine that $c = c(t)$ with $c(t) \to 0$ for $t \to \infty$. Another generalization concerns the waiting time distribution. For laser cooling the exponential distribution is an approximation. A better approximation is obtained by letting the waiting time be a sum, with a small number of terms, of exponentially distributed variables. This can be incorporated into the model by including an extra discrete state variable. These generalizations are discussed in Barndorff-Nielsen, Benth, and Jensen (2000).

As demonstrated in Examples 3.4 and 3.5 when rescaling x by an appropriate power of t we get a limiting distribution with a density proportional to $\Psi(|w|^\delta, \beta)$. This is in accordance with the findings in Bardou et al. (1999) and has furthermore being experimentally verified in Saubaméa, Leduc and Cohen-Tannoudji (1999) for the case $\delta = 2$ and $\beta = 1/2$. A difference between the treatment here and the one given in Bardou et al. (1999) is that we are able to treat the time spent in the present state and we are able to give a more complete description of the case where an invariant distribution exists.

7 Appendix

In this appendix we verify that $x(t)$ has a density with respect to Lebesgue measure and we derive Kolmogorov's forward equation for the density.

Let $p(A, t, y)$ be the probability that the process is in the set A at time t given that it starts at y at time zero. We can write this probability as follows

$$p(A,t,y) = 1(y \in A)\mathrm{e}^{-t\lambda(y)} + \sum_{i=1}^{\infty} \int_A p(x,i,t|y)dx, \tag{48}$$

where

$$\begin{aligned} p(x,i,t|y) &= \int_B \cdots \int_B \int_0^t \int_{w_1}^t \cdots \int_{w_{i-1}} \prod_{j=1}^{i} p(z_i|z_{i-1})\mathrm{e}^{-\lambda(x)(t-w_i)} \\ &\quad \times \prod_{j=1}^{i} \lambda(z_{j-1})\mathrm{e}^{-\lambda(z_{j-1})(w_j-w_{j-1})} dz_{i-1}\cdots dz_1 dw_1 \cdots dw_i, \end{aligned} \tag{49}$$

with $z_i = x$, $z_0 = y$ and $w_0 = 0$. Since $x(0)$ has a density $a(y)$ we see from (48) that $x(t)$ has a density $p(x,t)$ given by

$$p(x,t) = \mathrm{e}^{-t\lambda(x)}a(x) + \sum_{i=1}^{\infty} \int_B p(x,i,t|y)a(y)dy.$$

We can use (48) also to establish Kolmogorov's forward equation. We have that

$$p(x,t+\delta) = \mathrm{e}^{-\delta\lambda(x)}p(x,t) + \sum_{i=1}^{\infty} \int_B p(x,i,\delta|y)p(y,t)dy. \tag{50}$$

From the assumption (1) we see that

$$p(x,i,\delta|y) \le k_1 \lambda^i \frac{\delta^i}{i!},$$

which implies the bound

$$\sum_{i=2}^{\infty} p(x,i,\delta|y) \le k_1(1 - \mathrm{e}^{\delta\lambda} - \delta\lambda\mathrm{e}^{\delta\lambda}) \le k_2\delta^2.$$

Furthermore,

$$p(x,1,\delta|y) = \int_0^t p(x|y)\mathrm{e}^{-\lambda(x)(\delta-w_1)}\lambda(y)\mathrm{e}^{-\lambda(y)w_1}dw_1 = p(x|y)\lambda(y)\delta + O(\delta^2).$$

Returning to (50) we find

$$p(x,t+\delta) = \mathrm{e}^{-\delta\lambda(x)}p(x,t) + \delta \int_B \lambda(y)p(x|y)p(y,t)dy + \omega k_3\delta^2,$$

where $|\omega| \le 1$ and k_3 is a constant. Rearranging and letting δ tend to zero we obtain

$$p_t(x,t) = -\lambda(x)p(x,t) + \int_B \lambda(y)p(x|y)p(y,t)dy,$$

where $p_t(x,t)$ is the derivative of $p(x,t)$ with respect to t.

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