ISSN 1398-599X MPS-RR 1999-35 www.maphysto.dk

Atiyah-Patodi-Singer Type Index Theorems for Manifolds with Corners and Splitting of η-Invariants II.*

Gorm Salomonsen.

Contents

Abstract

We extend results of [Sa1] and prove an index theorem for manifolds with corners of codimension 3. A splitting formula for η-invariants of Dirac operators on manifolds with a restricted class of wedge singularities into η-invariants of manifolds with corners of codimension 2 is proved. The method of proof is by applying the index theorem to a special index problem and combining with a simpler splitting formula for η-invariants.

Ams subject classification: 35F15, 58A14, 58G10, 58G11, 58G20.

Keywords: Manifolds with Corners, Manifolds with Wedges, Index Theory, Boundary value problems, Eta-Invariants.

*This work was supported by MaPhySto – Centre for Mathematical Physics and Stochastics, funded by a grant from the Danish National Research Foundation. Parts of this work was done at the University of Bonn, funded by the SFB 256.

0 Introduction

Let M be a manifold with corners and product structure around the boundary and corners. The purpose of this paper and its predesessor [Sa1] is to define and study index invariants for Dirac type operators on M. The ansatz was that we wanted to generalize the Atiyah-Patodi-Singer boundary conditions. Because of some incompatibility problems in the corners, in [Sa1] we were forced to include some sections not in the standard Sobolev spaces to the domain of a Dirac operator in order to get a self-adjoint extension generalizing the Atiyah-Patodi-Singer boundary conditions. Further we noticed that there are several ways to do that, some of which are canonical inside the class of all generalized Dirac operators. In [Sa1] only the case of corners of codimension 2 was considered. The purpose of this paper is to extend the results of [Sa1] to higher codimension. Much of what we will do applies to arbitrary codimension, whereas the final results are only worked out in codimension 3 and the methods of this paper and [Sa1] together suffice to prove a particular index theorem (without unreasonable assumptions) in codimension 4, but not in codimension 5. The self-adjoint extensions of the Dirac operator, which we can construct, are related to geometric constructions, the simplest of which we explain here.

It was observed in [Sa1] that if M has corners of codimension 2, a canonical smooth structure is induced on ∂M by the geometry of M. If M has corners of codimension higher than 2, $\partial M \setminus \{\text{corners of } M \text{ of codimension } 3\}$ can still be smoothened in the same way. In this case the completion Z of the smoothened boundary is a manifold with stratifications.

A manifold $\tilde{M}$ with stratifications and a stratified boundary is given by

$$\tilde{M} := M \cup_Z (Z \times [0,1]). \tag{0.0.1}$$

If $E \mapsto M$ is a Dirac bundle over M respecting the product structure, E can be extended to a bundle $\tilde{E} \mapsto \tilde{M}$ respecting the product structure on the cylindrical piece. There may be more than one way to extend E. In [Sa1, Lemma 1.1.14] it was proved that there is a canonical choice of extension, which is well-defined and canonical inside the class of all Dirac bundles.

Let $\tilde{D}$ be the associated Dirac type operator on $\tilde{M}$ and let A_Z be the induced Dirac type operator on $\{1\} \times Z$. Then A_Z is a Dirac type operator defined on the space of smooth sections of $\tilde{E}_{|Z}$ and we have the same decomposition of $\tilde{D}$ on the cylinder as for manifolds with boundary

$$\tilde{D} = \gamma(\frac{\partial}{\partial u} + A_Z).$$

For good choices of self-adjoint ideal boundary conditions for A_Z, the commutation relation $\gamma A_Z = -A_Z \gamma$ holds in the strong sense. That means that γ preserves the domain of A_Z and thus that on the cylindrical piece, $\tilde{D}$ can be handled exactly as if Z had been closed. Like for a cylinder over a closed manifold, Atiyah-Patodi-Singer boundary conditions can be defined at $Z \times \{1\}$. If $\tilde{M}$ has no stratifications, a Lagrangian subspace of $\ker(A_Z)$ has to be chosen in order to get self-adjointness of $\tilde{D}$. In our case in addition ideal boundary conditions have to be defined at the stratifications of $\tilde{M}$

corresponding to corners of M. If M has corners of codimension 3 and $\tilde{E}$ is glued using the canonical gluing, we can construct such ideal boundary conditions up to a symplectic space W, arising from the singularities of $\tilde{M}$, and a Lagrangian subspace of $\ker(A_Z) \oplus W$ has to be chosen. Like for a manifold with cylindrical ends there is a canonical choice given by $\ker(S-1)$, where S corresponds to the scattering matrix in 0 for a manifold with cylindrical ends.

If the dimension of M is even an index theorem, Theorem 6.1, can be proved. Like for an Atiyah-Patodi-Singer boundary, the contribution to the index from the boundary is given by $-\frac{1}{2}\eta(A_{Z+}, 0)$ together with a term depending on the scattering matrix. This form of the boundary contribution is however not always the most desirable one. For example in order to be able to prove theorems like the Wall non-additivity theorem [Wa] for manifolds with corners of higher codimension using this index theorem, the η-invariant of A_{Z+} has to be split into η-invariants of operators on the smooth boundary components of M. The main and final result of this paper will be this splitting formula, given in Theorem 6.4.

For manifolds with corners of codimension 2, index theorems comparable to Theorem 6.1 are well-known. In [HMM] and [Mü2], Hassel-Mazzeo-Melrose and Werner Müller, respectively, prove equivalent index theorems in different, overlapping generalities and different setups. The index theorems from [Sa1] and this paper are related but different. Apparently there are no comparable index theorems for manifolds with corners of codimension 3 or higher except from under assumptions, which course all contributions to the index from higher codimensional corners to vanish.

In order to make constructions like (0.0.1) rigorous we have to construct a greater class of model corners than the ones used by most authors. Further we have to study the spaces arising by gluings of manifolds with corners along common boundary components. Such questions are addressed in Section 1. Our definition of a manifold with corners has been chosen among numerous other possible definitions, such that the class of manifolds with corners is local, contains the examples of smooth Riemannian manifolds and convex polytopes in $\mathbb{R}^n$, and such that it is closed under a restricted class of gluings along common boundary components, taking (intrinsic) boundary components and taking products. With those constraints on the class additional requirements can be made in order to avoid corners with very pathetic behaviour. We require that model corners have trivial holonomy and that none of their faces have stratifications. The condition that model corners have trivial holonomy is pleasant to work with and implies that trivial Dirac bundles exist over model corners.

We emphasize that our definition of a manifold with corners and product structure around the corners has been chosen such that it accommodates the methods used in this paper. This is also the motivation for the rather long Section 1. Other methods lead to other definitions. For example the definition in [Me, Page 26] is designed such that global boundary defining functions exist. Yet another definition, which is designed to give the concept of a manifold with corners (of codimension 3) with minimal structure will be given in [Sch2]. Given that corners have product structure our class contains the corners in the sense of [Me] and has a significant overlap with the corners in the sense of [Sch2].

The splitting formula for η-invariants of closed manifolds into η-invariants of manifolds with boundary and product structure around the boundary is well-known. Different proofs have been given by Bunke [Bu], Brüning & Lesch [BL1], Wojciechowski [DW], [Wo1], [Wo2], Mazzeo & Melrose [MM1], [MM2] and Müller [Mü2]. It is also well-known how the η-invariant behaves under variation of the boundary conditions. The case, where only the augmentation is varied, is handled in [LW] and [Mü1]. A case of very general pseudo-differential boundary conditions is handled in [Wo3].

In Section 5 we prove a slight generalization of the splitting formula for η-invariants using yet another proof from [Sa2]. This generalization is very minor, and it seems that each of the other proofs could also easily be adapted to cover this generalization.

In Section 6 we take a manifold M_0 with corners of codimension 3 and consider the smoothened boundary Z of M_0. Further we let E_Z be the restriction of $\tilde{E}$ to Z, where $\tilde{E}$ is glued using the canonical gluing. A manifold M with corners, a stratification and a stratified boundary is given by

$$M := (Z \times [-1,0]) \cup_{Z_1 \sqcup \ldots \sqcup Z_k} (Z_1 \times [0,1] \sqcup \cdots \sqcup Z_k \times [0,1]),$$

where $Z_1, \ldots, Z_k$ are the intrinsic (see Definition 1.22) boundary components of M_0. Using the index theorem for manifolds with corners of codimension 3 on a Dirac operator on M we relate the η-invariant of A_{Z+} to the η-invariant of a Dirac type operator $A_{\tilde{Z}+}$ on another manifold $\tilde{Z}$. $\tilde{Z}$ has more singularities than Z, but it has collar neighborhoods of the separating sub-manifolds, where it has to be split. The splitting formula into η-invariants of manifolds with corners of codimension 2 is then proved by splitting the η-invariant of $A_{\tilde{Z}+}$.

Finally we remark that if M is an odd dimensional manifold with corners, the definition of the η-invariant, we are led to by the proof of the splitting formula, is $\eta(D_M, 0) = \eta(D_{\tilde{M}}, 0)$, where $\tilde{M}$ is defined by (0.0.1). This is a reasonable definition of the η-invariant of D_M since it is independent of the length of the attached cylinder. See [Mü1, Section 2].

acknowledgement: We will like to thank B.W. Schulze for giving useful comments and suggestions.

1 The Definition of a Corner.

In this section we define a model corner, a stratified space, a stratification and so on. Much of the terminology is used in index theory in a more restricted sense. Other parts of the terminology has been taken from the theory of convex polytopes. See [Brø]. Except for Definition 1.9 we will make no claims.

Definition 1.1. A *simple model* corner C of dimension n is an intersection of closed half-planes $H_1, \ldots, H_k$ in $\mathbb{R}^n$, which all contain 0 in their boundary, such that the interior C° of C is non-empty. C is given the structure of a complete metric space induced by the pullback of the Riemannian metric on $\mathbb{R}^n$ to the interior of C.

We will always make the assumption that the intersection is non-redundant, i.e. that the intersection of any proper subset of $\{H_1, \ldots, H_k\}$ gives a space containing C as a proper subset.

For a non-redundant intersection we may to each H_i associate the $i'th$ $(n-1)$-face F_i of C, given by $C \cap \partial H_i$. F_i is a simple model corner, arising as the intersection of the half-planes $\partial H_i \cap H_j$ for $j \neq i$ in $\partial H_i \cong \mathbb{R}^{n-1}$.

The $(n-q)$-faces of C are defined inductively to be the $(n-q)$-faces of the $(n-(q-1))$-faces.

The codimension of C is the smallest number k such that C is isometric to $C' \times \mathbb{R}^{n-k}$, where C' is a simple model corner in $\mathbb{R}^k$.

Notice that every simple model corner is a convex cone.

Lemma 1.2. *The codimension of an $(n-1)$-face F of a simple model corner C of codimension k is $k-1$.*

Proof: Because $\{0\} \times \mathbb{R}^{n-k} \subseteq \partial H_i$ for each H_i, where $\{H_i\}$ is as above. □

Lemma 1.3. *The codimension of a simple model corner C of dimension n is k if and only if $n-k$ is the highest number, such that C contains a subspace of $\mathbb{R}^n$ of dimension $n-k$.*

Proof: Given a subspace $L \subseteq \mathbb{R}^n$. C contains L if and only if each defining half-plane H_i of C contains L. This is further equivalent to that each H_i decomposes $H_i = H_i' \times L$, where H_i' is a half-plane in $L^\perp$. But this is true if and only if $C = C' \times L$, where C' is a simple model corner in $L^\perp$. □

Definition 1.4. A *triangulized generalized model corner* C is a space arising in the following way:

1) Let I be a finite index set and let $\{C_i\}_{i\in I}$ be a set of simple model corners of dimension n. Further let J be an index set and let $\{(i_j, k_j, F_{i_j j}, F_{k_j j}, \varphi_j)\}_{j\in J}$ be a set of vectors, where $i_j \neq k_j \in I$, $F_{i_j j}$ is an $(n-1)$-face of C_{i_j}, $F_{k_j j}$ is an $(n-1)$-face of C_{k_j} and $\varphi_j : F_{i_j j} \mapsto F_{k_j j}$ is an isometric homeomorphism between $F_{i_j j}$ and $F_{k_j j}$. We assume:

   a) Each $(n-1)$-face of each C_i is present at most once as an $F_{i_j j}$ or an $F_{k_j j}$.

   b) The undirected graph $\mathcal{G}$ with vertices C_i and edges φ_j between C_{i_j} and C_{k_j} is connected.

2) Let $\sim$ be the equivalence relation generated by the relation $F_{i_j j} \ni x \sim \varphi_j(x) \in F_{k_j j}$ and let $C = (\sqcup_{i\in I} C_i)/\!\sim$ as a topological space. Then we assume that each C_i is topologically imbedded in C with the quotient map as imbedding.

   We call an $(n-1)$-face F of a C_i a *glued* respectively *non-glued simple boundary face* depending on, whether F arises as an $F_{i_j j}$ or $F_{k_j j}$ or not.

The *triangulized codimension* of a triangulized model corner C is the smallest number k such that each C_i is isomorphic to $\mathbb{R}^{n-k} \times C_i'$, where C_i' is a simple model corner in $\mathbb{R}^k$.

The *origin* 0 of C is the image of $0 \in C_i$ for each $i \in I$.

Definition 1.5. Let C be a triangulized generalized model corner and let the notation be like in Definition 1.4.

1) Let , be the set of connected subgraphs γ of $\mathcal{G}$, for which it holds that if we define C_γ in the same way as C with γ in place of $\mathcal{G}$, then up to composition by an element of the orthogonal group $O(n)$ there is a unique map $f_\gamma : C_\gamma \mapsto \mathbb{R}^n$ satisfying that f_γ is continuous and that each C_i in C_γ is mapped homeomorphically and isometrically to its image in $\mathbb{R}^n$ by f_γ. Further if C_i and C_j are connected in γ by a φ_j, their images only intersect in the image of $F_{i_j j}$ and $F_{k_j j}$.

   The interior C_γ° of C_γ is the set of points $c \in C_\gamma$, for which an open neighborhood of c in C_γ is mapped homeomorphically to an open subset of $\mathbb{R}^n$ by f_γ. The interior of each C_γ is supplied with the pullback of the Riemannian metric on $\mathbb{R}^n$. There is a canonical continuous function $p_\gamma : C_\gamma \mapsto C$ given by that it is the identity on each C_i. Notice that , contains all non-cyclic connected subgraphs of $\mathcal{G}$.

2) The triangulized interior C^Δ of C is the union of the images of C_γ° for each γ. The set of triangulized interior points contains the sets of interior points for each C_i together with the interior points of glued $(n-1)$-faces. In particular C^Δ is connected and dense in C. In each of the cases mentioned one realizes that a neighborhood U_c of an interior point c can be found together with realizations of the f_γ, such that for each γ with $c \in f_\gamma(C_\gamma^\circ)$, there exists an open neighborhood U_c of c such that $f_\gamma \circ p_\gamma^{-1}$ is defined on U_c and is independent of , up to composition by an element of $O(n)$. This shows that C^Δ is a smooth Riemannian manifold. The metric on C^Δ defined above is called the canonical metric on C^Δ.

3) We notice that C is the completion of C^Δ with respect to the metric defined in 4). The interior C° of C is the set of points in C for which a neighborhood is isometrically homeomorphic to an open subset of $\mathbb{R}^n$. Also C° is a smooth Riemannian manifold with the Riemannian metric locally inherited from $\mathbb{R}^n$.

Considered as the completion of C°, C is called a *generalized model corner.*

The *boundary* ∂C of C is the complement $C \setminus C^\circ$ of the interior of C.

The *codimension* of a generalized model corner C is the smallest number k such that C is isometric to $\mathbb{R}^{n-k} \times C'$, where C' is a generalized model corner of dimension k.

Remark 1. The interior of C may differ from the topological interior.

Lemma 1.6. *The triangulized codimension of a triangulized generalized model corner C is also the codimension of each C_i.*

Proof: Take an $(n-1)$-face F of a simple model corner C_i. By Lemma 1.2 the codimension of F is $k-1$. Thus it is only isometric to simple model corners of codimension $k-1$, so if it occurs as a F_{i_j} also the codimension of F_{k_j} is $k-1$ and thus the codimension of C_{k_j} is k. Since $\mathcal{G}$ is connected this proves the lemma.

□

Definition 1.7. A *model stratified space* is a generalized model corner, for which every $(n-1)$-face F of every C_i is glued.

Definition 1.8. The set of *extrinsic* $(n-1)$-faces of a generalized model corner C is the set of closures in C of maximal open connected subsets of ∂C, for which the inward-pointing normal vector is well-defined.

The set of *intrinsic* $(n-1)$-faces of a generalized model corner C is the set of completions of maximal open subsets of ∂C, for which the inward-pointing normal is well-defined, with respect to the Riemannian metric induced on smooth subsets of ∂C from C.

Each intrinsic $(n-1)$-face is mapped continuously to an extrinsic $(n-1)$-face by the extension of the identity map in C by continuity. We say that the $(n-1)$-faces of C are *simple* if each of those maps are homeomorphisms.

The only n-face of C is C itself.

The intrinsic $(n-k)$-faces of C are defined iteratively as the intrinsic $(n-k)$-faces of the $((n-(k-1))$-faces.

We say that C has *simple faces* if all $(n-k)$-faces of C are simple for $k=1,\dots,n$, and the simpleness of the $(n-k)$-faces is defined with respect to the $(n-(k-1))$-faces.

If C has simple faces we drop the words intrinsic/extrinsic.

Remark 2. For a generalized model corner both the union of the extrinsic $(n-1)$-faces and the union of all the intrinsic faces may be properly contained in ∂C.

Definition 1.9. A model corner C is a generalized model corner for which the holonomy group of C° is trivial, all faces are simple, and for which the boundary of every face F of C is of dimension $\dim(F)-1$.

Lemma 1.10. *The definitions of a model stratified space and a model corner are independent of the triangulation.*

Proof: A reformulation of the definition of a model stratified space is, that a model stratified space is a generalized model corner, for which no open subset of a point of its boundary is isometric to a neighborhood of 0 in $\mathbb{R}^{n-1}\times[0,\infty)$. The definition of a model corner does not refer to any triangulation. □

Definition 1.11. A stratification Y of dimension $k\geq 0$ and codimension $n-k$ of a generalized model corner C is the closure of a maximal connected smooth Riemannian manifold Y° imbedded locally isometrically in ∂C, for which some connected neighborhood is isometrically homeomorphic to a neighborhood of $Y^\circ\times\{0\}$ in a fibre bundle $N\mapsto S\mapsto Y^\circ$, with a model stratified space N of dimension $(n-k)$ as fibre, and every $y\in Y^\circ$ is sent into $0\in S_{|y}$. S is given the Riemannian metric, which is locally the product metric of the metric on Y° and the canonical metric on N.

A triangulized stratification is defined like a stratification, but with ∂C replaced by $C\setminus C^\Delta$.

A triangulized stratification Y is said to be removable if it is not a stratification.

Lemma 1.12. *Assume F is a face of dimension $k\geq 0$ of some C_i for a triangulized generalized model corner C, that $F\subseteq\partial C$ and that all $(n-1)$-faces containing F are glued. Then F is a subset of a stratification.*

Proof: We have to show that the image of F° in C is subset of a stratification. Let W be the $(n-k)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $F \subset C_i \subseteq \mathbb{R}^n$. Let $H_1, H_2, \ldots, H_q$ be those of the defining half-planes of C_i, which contain F in their boundary. Set $V_i := W \cap H_1 \cap \ldots \cap H_q$. Then $C_i \subseteq V_i + F$, where i runs over the indices from I, for which the image of C_i in C contains the image of F. A neighborhood of F° in C can be mapped continuously injectively and isometrically to a neighborhood of $F^\circ \times 0$ in $S \mapsto F^\circ$, where $\mathcal{S}$ is the fibre bundle with basis F° and fibre V given by the gluing of the V_i arising from $(i_j, k_j, G_{i_j j}, G_{k_j j}, \psi_j)$ given in the following way:

1) With notation like in Definition 1.5, (i_j, k_j) runs over the $j \in J$ for which both the image of $F_{i_j j}$ and the image of $F_{k_j j}$ in C contain interior points of the images of the boundary components $G_{i_j j}$ of V_{i_j} and $G_{k_j j}$ of V_{k_j}, respectively.

2) $\psi_j : G_{i_j j} \mapsto G_{k_j j}$ is the function arising by first restricting φ_j to $G_{i_j j} \cap F_{i_j}$ and then extending it to $G_{i_j j}$ as an isometry.

The map is given by $x + c \mapsto (x, c)$ for $x \in F, c \in W$. The full triangulized stratification Y arises, when also possible gluings of the various k-faces are taken into account. $\square$

Corollary 1.13. *The boundary of a model stratified space is the union of its stratifications.*

Proof: Let $\{C_i\}$ be a triangulation of C. Since no F is unglued the boundary of C is contained in the images of ∂F in C, where F runs over the boundary components of the C_i. By Lemma 1.12 every $(n-k)$-face for $k \geq 2$ of every C_i in every triangulation for C is a part of a triangulized stratification. Further it is contained in a stratification if and only if it is contained in ∂C. $\square$

Lemma 1.14. *Assume that C is a generalized model corner and that C' is the completion of a Riemannian manifold such that C is isometric to $C' \times \mathbb{R}$. Then C' is a generalized model corner.*

Proof: Take a triangulation $\{C_i\}$ of C. We fix an imbedding of C_i in $\mathbb{R}^n$ such that $\{0\} \times \mathbb{R} \subset C' \times \mathbb{R}$ is mapped to $\{0\} \times \mathbb{R} \in \mathbb{R}^{n-1} \times \mathbb{R} \cong \mathbb{R}^n$. Let C_i' be the projection of the imbedding of C_i onto $\mathbb{R}^{n-1}$.

The product structure of $C' \times \mathbb{R}$ gives that the imbedding $C_i \mapsto C$ extends to an injective isometry $C_i' \times \mathbb{R} \mapsto C$. Further C is covered by sets of the type $C_i' \times \mathbb{R}$.

The set of minimal intersections of $C_i' \times \mathbb{R}$ of full dimension now is another triangulation of $C \times \mathbb{R}$. Consequently the set of minimal intersections of $C_i' \times \{0\}$ is a triangulation of C'. $\square$

Lemma 1.15. *Let C be a model corner. Then*

1) The codimension of C is smaller than or equal to the triangulized codimension of C for any triangulation.

2) Whenever C has a decomposition $C = C' \times \mathbb{R}^{n-k}$, where C' is the completion of a Riemannian manifold, also C' is a model corner.

3) The $(n-1)$-faces of C are model corners of dimension $(n-1)$.

Proof: Let $\{C_i\}$ be a triangulation of C. Then by Lemma 1.6 the codimension of each C_i coincides with the triangulized codimension of C. Write $C_i = C_i' \times \mathbb{R}^{n-k}$. The $\mathbb{R}^{n-k}$ factors are sent into each other by the gluing, such that C° is a vector bundle over some k-dimensional $(C')^\circ$. By the trivial holonomy of C° it follows that this is the trivial vector-bundle. This proves 1).

We prove 2). By Lemma 1.14 C' is a generalized model corner. That C' has trivial holonomy follows by the product structure and the trivial holonomy of C. The faces F' of C' stand in bijective correspondence to the faces of C by $F' \leftrightarrow F' \times \mathbb{R}^{n-k}$. The simpleness of faces of C' easily follows from that. In the same way the maximal dimension of $\partial F'$ for all faces F' of C' follows.

For 3) first notice that the local product structure implies that also the boundary components of C have trivial holonomy. Next notice that the faces of faces of C are also faces of C, so that the simpleness and maximal dimension on the faces of C is inherited. In the same way the condition that the boundary of all faces has maximal dimension is inherited.

This proves the lemma. □

Example 1.16. *Let $C \subseteq \mathbb{R}^3$ be a simple model corner constructed as the intersections of the half-planes $x \geq 0$, $y \geq 0$, $z \geq 0$ and the half-plane $H = \{(x,y,z) \mid -x-y+z \geq 0\}$. Take 8 copies of C and glue them together on a chain by iteratively rotating the (x,y)-plane into the (y,z)-plane and finally gluing the (y,z)-plane of the last one to the (x,y)-plane of the first one. The resulting generalized model corner C' has trivial holonomy, simple faces and boundary of full dimension. The boundary component corresponding to the (x,z)-plane however has a stratification. Thus the condition that the boundaries of* all *faces have to have full dimension is necessary in order to get 3) of Lemma 1.15.*

If C is replaced with the union of C and the reflection of C in the (x,y)-plane, the above construction gives a model corner, the topological boundary of which is not a topological manifold.

Lemma 1.17. *For every generalized model corner there exists a triangulation such that the triangulized codimension is smaller than or equal to the codimension.*

In particular every model corner has a triangulation for which the triangulized codimension coincides with the codimension.

Proof: If C has codimension k, C is isometrically homeomorphic to $C' \times \mathbb{R}^{n-k}$ for some generalized model corner C'. Any triangulation of C' gives a triangulation of C with triangulized codimension smaller than or equal to k.

The second part follows by combining the first with 1) of Lemma 1.15. □

Remark 3. For a generalized model corner the triangulized codimension can be strictly smaller than the codimension. The interiors of such generalized model corners have the structure of non-trivial vector-bundles.

Definition 1.18. The gluing $C\#_{N_1,\dots,N_k}C'$ of two triangulized generalized model corners C and C' of the same dimension n along $(n-1)$-dimensional intrinsic $(n-1)$-faces $N_1,\dots,N_k$, of which no two are the same, which appear in both C and C', and for which the triangulations of C and C' induce the same triangulation on each N_i is given by gluing $(n-1)$-faces of simple model corners in the triangulation $\{C_i\}_{i\in I}$ to the $(n-1)$-faces of simple model corners in the triangulation $\{C_i'\}_{i\in I'}$ of C' giving the same image in some N_j.

If $N_{11}, N_{12}, N_{21}, N_{22},\dots,N_{k1},N_{k2}$ are instead different boundary components of the same triangulized generalized model corner C and for each i there is an isometric homeomorphism $N_{i1}\mapsto N_{i2}$ preserving the induced triangulation, and such that no $(n-1)$-face of some C_i is sent into an $(n-1)$-face of the same C_i, we may define $C_{N_{11}=N_{12},\dots,N_{k1}=N_{k2}}$ to be the generalized model corner arising by identifying $(n-1)$-faces in N_{i1} and N_{i2}, which are mapped to each other.

Remark 4. In our applications it will be obvious that there are triangulations inducing the same triangulations on the boundary.

Definition 1.19. Let $C\cong\mathbb{R}^{n-k}\times C'$ be a generalized model corner of codimension k. A *product metric* on C° is a metric of the form $g+h$, where g is a complete metric on $\mathbb{R}^{n-k}$ and h is the canonical metric on C'. We write $\mathbb{R}^{n-k}_g$ for the Riemannian manifold $(\mathbb{R}^{n-k},g)$.

Lemma 1.20. *The product $C\times C'$ of two triangulized generalized model corners of dimensions n, n', triangulized codimensions k_t, k_t', codimensions k and k, respectively and product metrics is a generalized model corner of dimension $n+n'$, triangulized codimension k_t+k_t', codimension $k+k'$ and a product metric. Further*

a) If C and C' are both model corners, $C\times C'$ is a model corner.

b) If C and C' are both model stratified spaces, $C\times C'$ is a model stratified space.

Proof: First we realize that the product of two simple model corners is a simple model corner. This follows since $H_i\times H_j=H_i\times\mathbb{R}^m\cap\mathbb{R}^n\times H_j$ for half-planes H_i and H_j of dimension n and m respectively. Next we see that if C and C' are simple model corners and $C\cong\hat{C}\times\mathbb{R}^{n-k}_g$, $C'\cong\hat{C}'\times\mathbb{R}^{n'-k'}_{g'}$ then $C\times C'\cong(\hat{C}\times\hat{C}')\times\mathbb{R}^{n+n'-k-k'}_{g+g'}$. The metric given is a product metric with respect to this decomposition. We prove that the codimension of $\hat{C}\times\hat{C}'$ is $k+k'$ if $\hat{C}$ has codimension k and $\hat{C}'$ has codimension k'. Clearly the codimenison of $\hat{C}\times\hat{C}'$ is at most $k+k'$. Now assume that the codimension is smaller than $k+k'$. Assume that $L^\times$ is a subspace contained in $\hat{C}\times\hat{C}'$. Taking the projections we get subspaces L, L', where $L\subseteq C$ and $L'\subseteq C'$. Thus, by Lemma 1.3, $\dim(L^\times)\leq\dim(L)+\dim(L')=n-k+n'-k'$. This proves that the codimension of $\hat{C}\times\hat{C}'$ is at least $k+k'$, so it must be $k+k'$.

Now assume that C and C' are generalized model corners. We recover $C\times C'$ as a generalized model corner as follows: Set $(C\times C')_{ij}:=C_i\times C_j'$. The $(n-1)$-faces of $(C\times C')_{ij}$ are the simple model corners of the form $C_i\times F'$ and $F\times C_j$. The φ_l can thus be constructed as the identity on one coordinate and φ_j or φ_j', respectively, on the other.

By the first part of the lemma the dimension and the triangulized codimension is $n+n'$ and k_t+k_t', respectively. For the codimension q we first realize, that $q \leq k+k'$. By Lemma 1.17 we may change the triangulations, if necessary, such that the codimensions are at least as high as the triangulized codimensions. Assume that $C \times C' \cong C'' \times \mathbb{R}_{g''}^{n+n'-k-k'+j}$ for some generalized model corner C'' of dimension and codimension $k+k'+j$. Assume that $C \cong C_1 \times \mathbb{R}_g^{n-k}$. Vectors $0 \times w \times \{0\} \in C \times \mathbb{R}^{n-k_g} \times C'$ are necessarily mapped to $0 \times w' \in C'' \times \mathbb{R}^{n+n'-k-k'+j}$ since the geodesic flow in directions tangential to $\mathbb{R}^{n-k}$ can be continued to all of $\mathbb{R}$. Therefore we may divide out the $\mathbb{R}_g^{n-k}$-factor. In the same way we may divide out the $\mathbb{R}_{g'}^{n-k'}$-factor of C'. Thus we only have to consider the case, where the codimension of each of C and C' is maximal and C and C' are supplied with the canonical metric.

If now $C \times C' \overset{\psi}{\cong} C'' \times \mathbb{R}$, the geodesic flow in directions tangential to the $\mathbb{R}$-factor gives rise to a faithful $\mathbb{R}$-action by isometries on at least one of C and C'. This follows by considering the inverse image of ψ of the orbits of the $\mathbb{R}$-action on $C'' \times \mathbb{R}$, which are geodesics. Assume that C has such a faithful $\mathbb{R}$-action. Then the inverse image of $C'' \times \{0\}$ by the map $\varphi : C^\circ \mapsto C'' \times \mathbb{R}$ given by $\varphi(c) = \psi((c,0))$ is isomorphic to $\mathbb{R} \backslash C^\circ$ and C° thus has a structure as a vector-bundle $C^\circ \cong V \mapsto C_1^\circ$, where C_1° is a flat Riemannian manifold. We let C_1 be the completion of C_1°. If the $\mathbb{R}$-action on C is trivial we set $C_1 = C$ and identify C with the trivial 0-dimensional bundle $\mathbb{R}^0 \mapsto C_1$. In the same way we define C_1'. We may form the direct sum of vector bundles $C^\circ \boxplus (C')^\circ \mapsto C_1 \times C_1'$.

Since $C^\circ \times (C')^\circ \cong C'' \times \mathbb{R}$, $C^\circ \boxplus (C')^\circ$ has a trivial flat factor L. That means that $C^\circ \boxplus (C')^\circ$ comes from a representation $\rho : \pi_1(C_1^\circ \times (C_1')^\circ) \mapsto O(\mathbb{R}^p)$, for $p \in \{1,2\}$ with a trivial term in its decomposition into irreducible representations. On the other hand $\pi_1(C_1^\circ \times (C_1')^\circ) \cong \pi_1(C_1^\circ) \times \pi_1((C_1')^\circ)$ and correspondingly $\rho = \rho_1 \times \rho_2$, where neither ρ_1 nor ρ_2 has a trivial term in its decomposition into irreducible representations by assumption. Since the decomposition into irreducible representations is unique up to equivalent representations, we are done.

We prove a): Since C° and $(C')^\circ$ are flat the holonomy of a loop is given by its class in the first fundamental group. Since $\pi_1(C \times C') = \pi_1(C) \times \pi_1(C')$, the holonomy has a similar decomposition, and we conclude that the holonomy group is trivial. The properties that the faces are simple and the boundaries of all faces have maximal dimension are easily checked.

Points b) is an immediate consequence of Definition 1.7. □

Definition 1.21. A manifold M with simple corners, generalized corners, corners or stratifications, respectively, and product structure near the corners is the completion of an open Riemannian manifold M° such that each point $x \in M$ is isometrically homeomorphic to a neighborhood of 0 in a simple model corner, generalized model corner, a model corner or a model stratified space, respectively, supplied with a product metric.

For $x \in M$ denote by C_x the generalized model corner, for which a neighborhood of 0 is isometrically homeomorphic to a neighborhood of x in M, such that 0 is sent into x by the isometry.

An *atlas* associated to M is a set of local isometric homeomorphisms sending open subsets of M into open neighborhoods of 0 in generalized model corners, such that every $x \in M$ is mapped to 0 for at least one of the homeomorphisms.

Remark 5. Since isometries of $\mathbb{R}^n$ and more generally of Riemannian manifolds are automatically smooth, smoothness follows automatically above.

Definition 1.22. A geometrically open face N° of codimension k of a manifold M with corners is a maximal connected set of points $x \in V$, for which there is a neighborhood in M isometrically homeomorphic to a neighborhood of 0 in a generalized model corner of codimension k, such that x is sent to 0 by the isometric homeomorphism.

An extrinsic face N of codimension k of a manifold M with corners is the closure of an open face N° of codimension k in M.

An intrinsic face N' of codimension k of a manifold M with corners is the *completion* of an open face N° of codimension k with respect to the induced Riemannian metric.

Denote by $\mathcal{S}_k$ the k-skeleton of ∂M, i.e. $\mathcal{S}_k$ is the union in M of all extrinsic faces of codimension k of M.

Now let V be a union of $(n-k)$-faces of M.

The topological interior $V^\bullet$ of V is given as $V^\bullet = \mathcal{S}_k \setminus \overline{\mathcal{S}_k \setminus V}$.

The geometric interior V° of V is the set of points $x \in V$ contained in open faces of codimension k.

Remark 6. The geometric and topological interiors need not coincide, but we always have the inclusion $V^\circ \subseteq V^\bullet$. For model corners there is only one concept of interior of a k-face.

Remark 7. Let Y be an extrinsic face of codimension $k > 1$ of a manifold M with corners. The local product structure in the definition of a manifold with corners does *not* imply, that there is an isometry of a neighborhood of $\{0\} \times Y^\circ \in C \times Y^\circ$ into a neighborhood of Y° in M for some model corner C (unless the isometry group of C is trivial). Instead what we have is that a neighborhood of Y in M is isometric to a neighborhood of $Y \times \{0\}$ in a fibre bundle $C \mapsto S \mapsto Y$, where the metric on S is such that local trivializations have product structure.

Theorem 1.23. *The class of manifolds with corners and product structure near the corners contains the following examples:*

a) Any convex polytope K in $\mathbb{R}^n$.

b) Any complete Riemannian manifold.

c) Any convex polytope K in $\mathbb{R}^n$ supplied with a metric, which is equal to the metric induced from $\mathbb{R}^n$ in a neighborhood of ∂K.

d) Any finite product of manifolds with corners and product structure near the corners.

e) Any intrinsic boundary component of a manifold with corners and product structure near the corners.

f) Assume M_1 and M_2 are manifolds with corners, $N_1 \subset M_1$ and $N_2 \subset M_2$ is a union of boundary components $N_1 = N_{11} \cup \cdots \cup N_{1k}$, $N_2 = N_{21} \cup \ldots \cup N_{2k}$ satisfying the following:

1) Each connected component of M_2 contains at most one N_{2i} and $N_{2i}^{\bullet} = N_{2i}^{\circ}$ for all i.

2) $N_{1i} \neq N_{1j}$ for $i \neq j$ and there is an isometric homeomorphism $h_i : N_{2i}^{\circ} \mapsto N_{1i}^{\circ}$. Denote the extension by continuity of h_i to N_{2i} by h_i also.

3) Given $x \in N_1$. Let C_x be the model corner at x. For every $y \in N_{2i}$ with $h_i(y) = x$ denote by C_y the model corner at y. Then we assume that there exist a triangulation of C_x such that for each y there is a triangulation of C_y, which induces the same triangulation on $h_i(F_y)$ as the triangulation induced by h_x.

Denote by $M_1 \#_N M_2$ the gluing of M_1 and M_2 as metric spaces, where for each i and $y \in N_i$, y and $h_i(y)$ are identified.

Then $M_1 \#_N M_2$ is a manifold with corners.

Proof: We leave a), b) and c) to the reader. It is enough to prove d) for a product of two manifolds with corners. Given M, M'. For $x \in M$, $x' \in M'$ there are neighborhoods U and U' and isometric homeomorphisms φ_1, φ_2 to open neighborhoods V and V' of $\{0\}$ in model corners C and C' with $\varphi_1 \circ h_1(x) = 0$, $\varphi_2 \circ h_2(x') = 0$. The product of those isometries gives an isometry $U \times U' \mapsto V \times V'$, and $V \times V'$ is an open neighborhood of 0 in a model corner by Lemma 1.20. Finally $\varphi_1 \circ h_1 \times \varphi_2 \circ h_2$ sends (x, x') to $0 = (0, 0)$.

e) follows by 4) of Lemma 1.15 since the atlas on M induces an atlas on each of its boundary components by restriction.

Finally we prove f). By 1), 2) and the hard demand on the triangulation of C_x in 3), it suffices to consider the case, where M_2 and N_2 have only one component. Further, by 3), the problem reduces to the case of model corners $C_1 \#_N C_2$, which induce the same triangulation over the boundary component N to be glued. Since only one boundary component is glued it follows that also $C_1 \#_N C_2$ has trivial holonomy. Further triangulized stratifications of $C_1 \#_N C_2$ are easily seen to be removable. The faces F of $C_1 \#_N C_2$ are faces of either C_1 or C_2, or gluings of a face of C_1 and a face of C_2. In the first two cases F is a model corner and thus satisfies the necessary assumptions. In the second case we realize that $F = F_1 \cup_G F_2$, where G is a connected boundary component of $F_1 \subseteq C_1$ and $F_2 \subseteq C_2$. In particular the boundary of F is of full dimension. That the same holds for the faces of F follows by iterating the above argumentation. □

2 Self-Adjointness Properties of Dirac Operators on Stratified Spaces.

In this section we briefly review some theory regarding self-adjoint extensions of Dirac operators on wedges and make some modifications for the more general stratified spaces.

The theory will be developed up to the point, where we can handle Dirac operators on $\tilde{M}$ defined in (0.0.1) in the case where M is a manifold with corners of codimension 3 and $\tilde{E}$ is glued using the canonical gluing. Some relevant references are [Bä], [Ch1], [Ch2], [Chou], [Le] and [Sch1]. Most theory has though been sufficiently adapted such that it can not be referenced away directly. For example the restriction to the submanifold N below is covered in [Bä] for the spin bundle, whereas we need the trivial generalization to the local twisted spin bundle.

Let in the following N and Y be open Riemannian manifolds with Riemannian metrics g_N and g_Y. We form the wedge

$$M := \mathbb{R}_+ \times N \times Y \tag{2.0.1}$$

supplied with the wedge metric

$$g = dr^2 + r^2 g_N + g_Y. \tag{2.0.2}$$

Here $r \in \mathbb{R}_+$ denotes the first coordinate.

Remark 8. A simple model corner with a product metric is a wedge. If $C = C' \times \mathbb{R}_g^{n-k}$ we take $Y = \mathbb{R}_g^{n-k}$ and $N = S^{k-1} \cap C'$. In particular N is given a Riemannian metric with constant curvature 1. Gluing together we see that also a generalized model corner is a wedge and that N is a manifold with generalized model corners supplied with a metric of curvature 1. A corner of a manifold with corners or a neighborhood of a subset of a stratification of a manifold with stratifications is however only a generalized wedge. Generalized wedges will be defined below.

Let $E \mapsto M$ be a Dirac bundle with Hermitian structure g^E, connection ∇^E and structure of Clifford multiplication $\mathbf{c} \in C^\infty(M, End(TM \otimes E, E))$. With those conventions the Dirac operator D_M on M is given by $D_M := \mathbf{c} \circ g^{-1} \circ \nabla^E$.

Let ∇ denote the Levi-Civita connection on M. Then we have the following basic facts.

- Let $e \in T_n N$, $f \in T_y Y$, $\frac{\partial}{\partial r} \in C^\infty(\mathbb{R}_+, T\mathbb{R}_+)$. Then $\frac{1}{r}e$, f and $\frac{\partial}{\partial r}$ are parallel in the direction $\frac{\partial}{\partial r}$.
- Let $e_1, \dots, e_{\dim(N)}$ be locally defined orthonormal vector-fields on N. Then the Levi-Civita connection on N is given by the formula
$$\nabla^N = \nabla - \sum_{j,k} g(\nabla_{e_j} e_k, \frac{\partial}{\partial r}) g(e_j) \otimes \frac{\partial}{\partial r} \otimes g(e_k).$$
- We have the identity
$$\nabla_{\frac{\partial}{\partial r}} e_j = \nabla_{e_j} \frac{\partial}{\partial r} = \frac{1}{r} e_j.$$

Using parallel transport in the direction $\frac{\partial}{\partial r}$, E can be trivialized with respect to the first coordinate. The connection ∇^E splits canonically into $\nabla^{\mathbb{R}_+ \times N, E} + \nabla^{Y,E}$, where $\nabla^{\mathbb{R}_+ \times N, E} : C^\infty(M, E) \mapsto C^\infty(M, (T^*\mathbb{R}_+ \times N) \otimes E)$ and $\nabla^{Y,E} : C^\infty(M, E) \mapsto C^\infty(M, T^*Y \otimes E)$.

We assume that E respects the wedge structure in the following sense:

Assumption 2.1.

a) With the trivialization given above, ∇^E does not depend on r.

b) Given local coordinates on $\mathbb{R}_+ \times N$ and on Y, let V be a coordinate vector-field on $\mathbb{R}_+ \times N$ and let W be a coordinate vector field on Y. Then clearly $[V, W] = 0$. We assume that also

$$\nabla^E_V \nabla^E_W - \nabla^E_W \nabla^E_V = 0,$$

i.e. the mixed curvatures vanish. With other words, covariant differentiation in directions tangential to Y commutes with covariant differentiation in directions tangential to $\mathbb{R}_+ \times N$.

Under this assumption the Dirac operator D_M takes the form:

$$D_M = \nu \left(\frac{\partial}{\partial r} + \frac{1}{r} B_{N,0} + \frac{\dim(N)}{2r} + B_{Y,0} \right). \tag{2.0.3}$$

Here ν denotes the operator of Clifford multiplication by $\frac{\partial}{\partial r}$. The operators $B_{N,0}$ and $B_{Y,0}$ are Dirac type operators on N and Y, respectively, with respect to the operator of Clifford multiplication $\mathbf{b}$ given by

$$\mathbf{b}(\gamma) = -\nu \mathbf{c}(\gamma)$$

and the connections

$$\nabla^{N,E} := \imath_y^*(\nabla^E) - \frac{1}{2} \sum_{j,k} g(\nabla_{e_j} e_k, \frac{\partial}{\partial r}) g(e_j) \otimes \mathbf{c}(e_k)\nu, \tag{2.0.4}$$

and $\nabla^{Y,E}$. Here $\imath_y : N \mapsto M$ is given by $\imath_y(n) = (1, n, y)$ and $\{e_1, \dots, e_{\dim(N)}\}$ is a moving orthonormal frame of vector fields, which are normal with respect to ∇^N.

The operators ν, $B_{N,0}$, $B_{Y,0}$ satisfy the following formal commutation relations

$$\nu B_{N,0} = -B_{N,0}\nu \quad ; \quad \nu B_{Y,0} = -B_{Y,0}\nu, \tag{2.0.5}$$

$$B_{N,0} B_{Y,0} = -B_{Y,0} B_{N,0}. \tag{2.0.6}$$

Definition 2.2. Given an N-bundle $N \mapsto S \mapsto Y$. A *generalized wedge* is the product $\mathbb{R}_+ \times S$, supplied with a Riemannian metric, which is a wedge metric on local trivializations of S.

All of the above is completely local and thus also holds for generalized wedges.

For fixed $y \in Y$, $\imath_y^*(E)$ is a Dirac bundle over N. Further parallel transport along curves of the form $(1, n, f(t))$, where $f : (0,1) \mapsto Y$ is smooth, gives isomorphisms of Dirac bundles $\imath_y^*(E) \cong \imath_{y'}^*(E)$ for $y, y' \in Y$.

Now assume that we have a self-adjoint extension B_N of $B_{N,0}$ satisfying the following

- B_N is canonically determined by the isomorphism type of $\imath_y^*(E) \mapsto N$.
- $\nu B_N = -B_N \nu$ in the strong sense. That means that ν preserves the domain of B_N.
- B_N has pure point spectrum with eigenvalues of finite multiplicity.

If the above assumptions hold, for each λ^2 eigenvalue of B_N^2 the corresponding eigenspace E_{λ^2} is a Dirac bundle over Y. We now also assume that we have a canonical self-adjoint extension B_{Y,λ^2} of $B_{Y,0}$, defined in each eigenbundle E_{λ^2} of B_N^2 and that further B_Y satisfies:

- $\nu B_Y = -B_Y \nu$ in the strong sense.
- For each λ^2, B_{Y,λ^2} has pure point spectrum with eigenvalues of finite multiplicity.
- $B_N B_{Y,\lambda^2} = -B_{Y,\lambda^2} B_N$.

In this case it follows that D_M has a Direct sum decomposition

$$D_M = \bigoplus_{\substack{\lambda^2 \in \operatorname{spec}(B_N^2) \\ \mu^2 \in \operatorname{spec}(B_Y^2)}} \nu \left(\frac{\partial}{\partial r} + \frac{\dim(N)}{2r} + \frac{1}{r} B_{N,\lambda^2,\mu^2} + B_{Y,\lambda^2,\mu^2} \right), \tag{2.0.7}$$

where each term in the sum is an operator in $L^2(\mathbb{R}_+, E_{\lambda^2,\mu^2}, r^{\dim(N)} dr)$ and E_{λ^2,μ^2} denotes the eigenspace of B_Y^2 to the eigenvalue μ^2 inside E_{λ^2}. If we further conjugate by multiplication by $r^{-\frac{\dim(N)-1}{2}}$ we get

$$D_M \sim \bigoplus_{\substack{\lambda^2 \in \operatorname{spec}(B_N^2) \\ \mu^2 \in \operatorname{spec}(B_Y^2)}} \nu \left(\frac{\partial}{\partial r} + \frac{1}{2r} + \frac{1}{r} B_{N,\lambda^2,\mu^2} + B_{Y,\lambda^2,\mu^2} \right) =: \bigoplus D_{\lambda^2,\mu^2}, \tag{2.0.8}$$

where each term in the sum is an operator in $L^2(\mathbb{R}_+, E_{\lambda^2,\mu^2}, r dr)$.

Now let X be a compact manifold with generalized wedge singularities and let $E \mapsto X$ be a Dirac bundle over X respecting the wedge singularities. We will define canonical self-adjoint ideal boundary conditions on X. This can be done by defining canonical self-adjoint extensions of each D_{λ^2,μ^2} for $\lambda^2 \neq 0$ or $\mu^2 \neq 0$, and finally defining a self-adjoint extension of the direct sum of the $D_{0,0}$ for all wedges. In the following let $\rho \in C^\infty(\mathbb{R}_+)$ be a function with $\rho(r) = r^{-1}$ for $r \in (0,1)$ and $\rho(r) = 0$ for $r > 2$.

Depending on λ^2 and μ^2 our prefered self-adjoint extension $\mathcal{D}_{\lambda^2,\mu^2}$ of D_{λ^2,μ^2} is given by (see [Ch1], [Chou], [Sa1]):

1) $\lambda^2 \geq \frac{1}{4}$. In this case D_{λ^2,μ^2} is essentially self-adjoint on C_0^∞. We set $\mathcal{D}(\mathcal{D}_{\lambda^2,\mu^2}) = \mathcal{D}(\bar{D}_{\lambda^2,\mu^2})$, where $\bar{D}_{\lambda^2,\mu^2}$ denotes the closure of D_{λ^2,μ^2}.

2) $\lambda^2 \neq 0$ and $\lambda^2 < \frac{1}{4}$. In this case $\mathcal{D}(D^*_{\lambda^2,\mu^2})$ is of the form

$$\mathcal{D}(D^*_{\lambda^2,\mu^2}) = \mathcal{D}(\bar{D}_{\lambda^2,\mu^2}) \oplus \operatorname{span}\left\{\rho^{(\frac{1}{2}-|\lambda|)}\varphi_j, \rho^{(|\lambda|+\frac{1}{2})}\psi_j\right\}_{j\in J},$$

where J is some finite index set and φ_j, ψ_j are independent of r for $j \in J$. A canonical self-adjoint extension $\mathcal{D}_{\lambda^2,\mu^2}$ of D_{λ^2,μ^2} is given by including the slowest growing sections of $\mathcal{D}(D^*_{\lambda^2,\mu^2})$ to $\mathcal{D}(\bar{D}_{\lambda^2,\mu^2})$. That means

$$\mathcal{D}(\mathcal{D}_{\lambda^2,\mu^2}) := \mathcal{D}(\bar{D}_{\lambda^2,\mu^2}) \oplus \operatorname{span}\left\{\rho^{(\frac{1}{2}-|\lambda|)}\varphi_j\right\}_{j\in J}$$

for $\mathrm{Re}(s) \gg 0$.

3) $\lambda^2 = 0$ and $\mu^2 \neq 0$. In this case a conjugation by the operator of multiplication by $r^{-\frac{1}{2}}$ conjugates D_{μ^2,λ^2} to an operator of the form $\nu(\frac{\partial}{\partial r} + B_Y)$ on $L^2((0,\infty), E_{\lambda^2,\mu^2}, dr)$. The Atiyah-Patodi-Singer boundary conditions gives a canonical choice of boundary conditions. That means that we define

$$\mathcal{D}(\mathcal{D}_{\lambda^2,\mu^2}) := \mathcal{D}(\bar{D}_{\lambda^2,\mu^2}) \oplus \operatorname{span}\{\rho^{\frac{1}{2}}\varphi\}_{B_{Y,0,\mu^2}\varphi=\mu\varphi\,;\,\mu<0}.$$

Also in this case $\mathcal{D}_{\lambda^2,\mu^2}$ is self-adjoint and canonically defined.

4) $\lambda^2 = \mu^2 = 0$. In this case

$$\mathcal{D}(D^*_{\lambda^2,\mu^2})/\mathcal{D}(\bar{D}_{\lambda^2,\mu^2}) \sim \operatorname{span}\{\rho^{\frac{1}{2}}\varphi\}_{\varphi\in E_{\lambda^2,\mu^2}}. \tag{2.0.9}$$

There is no canonical choice of self-adjoint extension of $D_{\lambda^2,\mu^2,0}$ and in some cases there is no self-adjoint extension at all. Like for manifolds with boundary this problem can be solved by a global construction.

Let D_X be the closed realization of the Dirac operator on X given by imposing the above defined boundary conditions for $(\lambda^2, \mu^2) \neq 0$ and imposing Dirichlet boundary conditions on each $D_{0,0}$.

Then $D^*_X D_X$ is a Fredholm operator (see for example [Sa1, Lemma 1.3.6]) and $\mathcal{D}(D^*_X)/\mathcal{D}(D_X)$ is the direct sum of a finite number of spaces like (2.0.9) coming from the generalized wedges. The construction in [Sa1, Section 2.2] gives a scattering matrix S exactly like in the case of wedges of codimension 2.

Thus a self-adjoint extension $\mathcal{D}_X$ of D_X is given by that sections of the type in (2.0.9), for which $\varphi \in \ker(S-1)$ are added to $\mathcal{D}(\mathcal{D}_X)$.

Definition 2.3. Let X be a manifold with generalized wedge singularities and let $E \mapsto X$ be a Dirac bundle respecting the structure of the wedge singularities. We call the (ideal) boundary conditions defined above *slow growing ideal Atiyah-Patodi-Singer boundary conditions*.

For manifolds with conical singularities the slow growing ideal boundary conditions were introduced in [Chou]. In connection with boundary value problems for manifolds with corners, slow growing ideal boundary conditions play a special role because they give the best possible restriction to $\mathbb{R}_+ \times \{n\} \times Y$. See [Sa1, Section 1.4].

Notice that slow growing Atiyah-Patodi-Singer boundary conditions have the following properties.

- They are canonical.
- They respect every super-structure on E.
- Assume that E is the restriction of a Clifford bundle $\tilde{E} \mapsto N \times X$ to $\{n\} \times X$, where $N \times X$ is supplied with a product metric and product connection. Then Clifford multiplication by vectors in TN preserves $\mathcal{D}(\mathcal{D}_X)$ if $\mathcal{D}_X$ is given slow growing ideal boundary conditions.
- Let B_N be like in (2.0.8) and assume the restriction of B_N to each fixed $y \in Y$ is given slow growing Atiyah-Patodi-Singer boundary conditions. Then the operator ν preserves the domain of B_N.

The above properties are absolutely vital for the iteration of the theory of this section. On the other hand if N and Y are manifolds with generalized wedge singularities and B_N and B_Y are given slow growing ideal Atiyah-Patodi-Singer boundary conditions, all of the assumptions on B_N and B_Y hold, such that this section can be iterated.

3 About Gluings of Dirac Bundles.

In this section we only consider conical singularities. Everything however also holds for wedge singularities.

In [Sa1, Lemma 1.1.14] we introduced a canonical gluing of Dirac bundles. This gluing is given as $\sqrt{-\delta_1\delta_2}$, where δ_1 and δ_2 are the operators of Clifford multiplication by parallel vector fields, also denoted by δ_1 and δ_2, defined in a neighborhood of a corner of codimension 2. The sign conventions are such that $\sqrt{-\delta_1\delta_2} = 1$ for $\sigma = \pi$ and such that $\sqrt{-\delta_1\delta_2}$ is continuous with respect to σ.

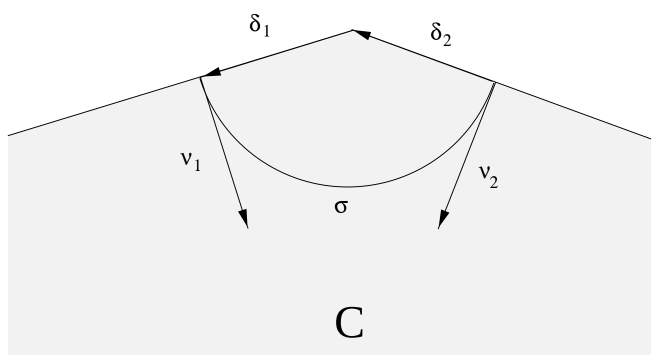

Fig3: A cone $C \subset \mathbb{R}^2$.

Only if C is oriented we can distinguish $\sqrt{-\delta_1\delta_2}$ from $\sqrt{-(-\delta_2)(-\delta_1)}$. However we have $\sqrt{-(-\delta_2)(-\delta_1)} = \left(\sqrt{-\delta_1\delta_2}\right)^{-1}$, so the glued vector bundle becomes the same if we change the orientation.

If in a neighborhood of the corner, E has a product decomposition $\mathcal{W} \otimes \$$, where $\$$ is a spin-bundle, this gluing corresponds to gluing $\mathcal{W}$ trivially.

Notice that the *cone* corresponding to the gluing $\sqrt{-\delta_1\delta_2}$ has as base a copy of S^1 supplied with a Riemannian metric such that the length of S^1 is $\pi + \sigma$.

For the more demanding applications of corners of higher codimension we need a different construction of this gluing operator.

Let $I = [a, b]$ be a closed interval parametrized by $\theta \in I$. Further let C be the cone over I. If we let γ be the curve $\gamma(\theta) = (1, \theta)$, we can consider two different types of parallel transport:

1) The parallel transport in TN, which we denote by $\gamma(\theta)_{*N}$.

2) The parallel transport in TC, which we denote by $\gamma(\theta)_{*C}$.

Notice that if we trivialize TC using the trivial holonomy of a corner, in fact $\gamma(\theta)_{*C}$ is tautologically the identity map for all θ.

Now start out with $\delta_1(a) = \gamma'(a)$ and $\delta_2(a) = \delta_1(a)$ and let

$$\delta_1(\theta) = \gamma(\theta)_{*C}\delta_1(a), \tag{3.0.1}$$

$$\delta_2(\theta) = \gamma(\theta)_{*N}\delta_1(a). \tag{3.0.2}$$

Let E be a Dirac bundle, which is trivial over C. We define $\sqrt{-\delta_1(\theta)\delta_2(\theta)} \in \mathrm{spin}(T_{\gamma(\theta)}C)$ by continuity and the convention $\sqrt{-\delta_1(a)\delta_2(a)} = 1$. It follows from Fig. 3 that when $I = [-\frac{\pi}{2}, \sigma + \frac{\pi}{2}]$, $\sqrt{-\delta_1(\sigma + \frac{\pi}{2})\delta_2(\sigma + \frac{\pi}{2})}$ coincides with the gluing operator from [Sa1, Lemma 1.1.14].

Lemma 3.1. $\left(\sqrt{-\delta_1(\theta)\delta_2(\theta)}\right)^{-1}$ *is the parallel transport in* $E_{|N}$ *with respect to* $\nabla^{N,E}$, *where* $\nabla^{N,E}$ *is defined by (2.0.4).*

Proof: By 6) of [Sa1, Lemma 1.1.14] it follows that

$$\left(\sqrt{-\delta_1(\theta)\delta_2(\theta)}\right)^{-1} = \left(e^{\frac{1}{2}\delta_1\nu_1(\pi-(\theta-\pi))}\right)^{-1} = -e^{\frac{1}{2}\nu(\theta)\delta_2(\theta)\theta}, \tag{3.0.3}$$

where $\nu(\theta)$ denotes Clifford multiplication by $\frac{\partial}{\partial r}$ in $(1, \theta)$. Here as usual, r denotes the distance to $0 \in C$. Since $\frac{1}{r}\nu(\theta)\delta_2(\theta)$ is a constant multiple of the image of the volume form in the Clifford bundle on C, it is parallel with respect to ∇^E. Consequently, if $\varphi \in C^\infty(E)$ is a section parallel with respect to ∇^E, we get

$$\nabla^E_{\frac{\partial}{\partial\theta}}\left(-e^{\frac{1}{2r}\nu(\theta)\delta_2(\theta)\theta}\varphi\right)_{|r=1} = -\frac{1}{2}\nu(\theta)\delta_2(\theta)e^{\frac{1}{2}\nu(\theta)\delta_2(\theta)}\varphi.$$

Therefore and by (2.0.4)

$$\nabla^{N,E}_{\frac{\partial}{\partial\theta}}\left(-e^{\frac{1}{2}\nu(\theta)\delta_2(\theta)\theta}\varphi\right) = -\frac{1}{2}\left(\nu(\theta)\delta_2(\theta) + \delta_2(\theta)\nu(\theta)\right)e^{\frac{1}{2}\nu(\theta)\delta_2(\theta)\theta}\varphi = 0.$$

This proves the lemma. □

Corollary 3.2. *If $\tilde{E}$ is glued using the canonical gluing, $\tilde{E}$ has trivial holonomy with respect to $\nabla^{N,E}$.*

Now consider a model corner M of dimension and codimension 3. Assume that $E_M \mapsto M$ is a trivial Clifford bundle over M. (A trivial Clifford bundle exists since M° has trivial holonomy). Let Z° be a topological boundary component of $M \setminus \{0\}$ (there may be more than one, see Example 1.16), and let Z be the closure of Z° in M. Then for a triangulation $\{M_i\}$ of M, Z° is the union of $\{F_j \setminus \{0\}\}$, where F_j runs over a subset of the unglued 2-faces of M_i.

All M_i have codimension 3 by 1) of Lemma 1.15 and thus by Lemma 1.2, all F_j have codimension 2. Consequently each F_j has two 1-faces. It follows that the F_j are glued on a closed cycle. Let $Y_0^\circ, \ldots, Y_{k-1}^\circ$ be the interiors of the 1-faces of M, ordered such that for all i, Y_i and $Y_{(i+1) \operatorname{mod} k}$ are 1-faces of the same $F_{j(i)}$. Further let δ_{i1} and δ_{i2} be the normals of $F_{j((i-1) \operatorname{mod} k)}$ and $F_{j(i)}$ at Y_i, respectively. The directions are taken such that $\delta_{iq} \wedge \frac{\partial}{\partial r} \wedge \nu_{j((i-1) \operatorname{mod} k)}$ is the same for all i and q. Here r is the distance to $0 \in M$ and $\nu_{j(i-1)}$ is the inward pointing normal at $F_{j(i)}$.

Lemma 3.3. *Let $\tilde{M} = M \cup_Z ([0,1] \times Z)$ and let $\tilde{E}$ be the extension of E_M to $\tilde{M}$ arising from the canonical gluing. Then $\tilde{E}_{|Z}$ has trivial holonomy with respect to the connection $\nabla^{N,E}$ defined by (2.0.4), where N is the circle of distance 1 to the origin in Z.*

Proof: With respect to the trivialization of E_M on M, the gluing of $\tilde{E}_{|Z}$, starting at $Y_1 \subseteq F_{j(0)}$ is given by

$$\sqrt{-\delta_{11}\delta_{12}}\sqrt{-\delta_{21}\delta_{22}}\cdots\sqrt{-\delta_{01}\delta_{02}}.$$

Using Lemma 3.1 we may compute the holonomy in $\tilde{E}_{|Z}$ with respect to $\nabla^{N,E}$ (in the direction $Y_1, F_{j(0)}, Y_0, F(j(k), \ldots)$. The result is

$$\sqrt{-\delta_{11}\delta_{12}}\sqrt{-\delta_{12}\delta_{21}}\sqrt{-\delta_{21}\delta_{22}}\sqrt{-\delta_{22}\delta_{31}}\cdots\sqrt{-\delta_{(k-1)2}\delta_{01}}\sqrt{-\delta_{01}\delta_{02}}\sqrt{-\delta_{02}\delta_{11}}.$$

Since the Clifford algebra is not commutative, we can not telescope this expression. From [Sa1, Lemma 1.1.14, 3),4)] and because $\sqrt{-\delta_1\delta_2} \in \operatorname{spin}(\operatorname{span}\{\delta_1, \delta_2\})$, we however have:

$$\delta_1\sqrt{-\delta_1\delta_2} = \sqrt{-\delta_1\delta_2}\,\delta_2 \tag{3.0.4}$$

$$\nu_1\sqrt{-\delta_1\delta_2} = \sqrt{-\delta_1\delta_2}\,\nu_2 \tag{3.0.5}$$

$$e\sqrt{-\delta_1\delta_2} = \sqrt{-\delta_1\delta_2}\,e \quad ; e \perp \operatorname{span}\{\delta_1, \delta_2\}. \tag{3.0.6}$$

Further, up to multiplication by a scalar, $\sqrt{-\delta_1\delta_2}$ is the only element of $\operatorname{spin}(TM)$ with this property. Using (3.0.4), (3.0.5) and (3.0.6) on δ_{11}, ν_{11} and $\frac{\partial}{\partial r}_{|Y_1}$, where ν_{11} is the inward pointing vector normal to δ_{11} and $\frac{\partial}{\partial r}_{|Y_1}$, it follows that the holonomy is a scalar. Since it is an element of the real spin group $\operatorname{spin}_{\mathbb{R}}(T_{\gamma(a)}C)$ and is unitary, this scalar is ± 1. Further, since it is a product of unitary operators with determinant 1 on the odd-dimensional space $T_{\gamma(a)}C$, it follows that this scalar is 1. $\square$

Remark 9. Lemma 3.3 is also a useful starting point when one wants to compute the holonomy of $\tilde{E}_{|Z}$ for other gluings. If namely $E_M = W \otimes \$$ for a twisting bundle W, any other gluing operator has to be of the form $U \otimes \sqrt{-\delta_1\delta_2}$, where U is a unitary operator on W. Thus the holonomy is obtained by composing the U's.

Corollary 3.4. *If M is a manifold with corners and product structure around the corners, $\tilde{M}$ is the extension of M and $\tilde{E}$ is a Dirac bundle of $\tilde{M}$, which is glued with respect to the canonical gluing. Then for every N associated to a codimension 2 wedge singularity of $\tilde{M}$ we have* $\ker(B_{N,\lambda}) = \{0\}$ *and* $\eta(B_{N,\lambda,+}, 0) = 0$.

Proof: By the above we have that all pullbacks $\tilde{E} \mapsto N$ are glued using the canonical gluing. By [Sa1, Section 6.1] we further have that the spectrum of $B_{N,\lambda,+}$ is symmetric and does not contain $\{0\}$. Finally $\text{spec} B_{N,\lambda,-} = -\text{spec}(B_{N,\lambda,+}) \not\ni 0$, so the corollary holds for all of B_N. □

4 Contributions to the Index.

In this section we first assume that X is a compact, even dimensional, oriented manifold with generalized wedges. Later on we make the modifications necessary in order to cover the restricted class of manifolds with stratifications, which we will need. $E \mapsto X$ will be a Dirac bundle over X respecting the wedge singularities.

Given a generalized wedge singularity of X we let B_N and B_Y be given like in (2.0.7). Let τ be the canonical involution in E given by the image of the volume form in the Clifford algebra. Then τ anti-commutes with D_X, and in the local picture, τ commutes with B_N and B_Y and anti-commutes with ν. For many of the cases we will moreover need the involutions τ_N and τ_Y, given by volume forms in the Clifford algebras on N and Y and the Clifford multiplication $\mathbf{b}(\gamma) = -\nu\gamma$. We use the convention $\tau = \tau_N\tau_Y$, but in general τ_N and τ_Y depend on an arbitrary orientation of either N or Y. Notice that commutation relations between τ_N, τ_Y and other operators depend on the dimensions of N and Y. Let E_{λ^2,μ^2} be a joint eigenspace of B_N^2 and B_Y^2. We say that the contribution $I(K)$ to the index from a compact subset K of X exists if there exists a sequence of open subsets K_j of X containing K such that

$$\lim_{j\to\infty}\lim_{t\to 0}\int_{K_j\setminus K}\left|\text{tr}\,\left(\tau e^{-tD^2}\right)(x,x)\right|dx = 0.$$

In this case we set

$$I(K) = \lim_{j\to\infty}\lim_{t\to 0}\int_{K_j}\text{tr}\,\left(\tau e^{-tD^2}\right)(x,x)dx.$$

By Duhamels principle or finite propagation speed methods it can in all cases of immediate interest be established that $I(K)$ only depends on the structure of E and the boundary conditions for D in an arbitrarily small neighborhood of K. In particular one can compute the index contributions by considering the model spaces. For the model spaces we can further decompose the index contributions into index contributions arising from various parts of the spectra of B_N and B_Y. The natural contributions from a generalized wedge singularity are for $\lambda^2 \in \text{spec}(B_N^2)$, $\mu^2 \in \text{spec}(B_Y^2)$:

1) $\lambda^2 \geq \frac{1}{4}$. If $\mu^2 \neq 0$ the operator $\frac{1}{|\mu|}\nu B_{Y,\lambda^2,\mu^2}$ interchanges $D_{M-}D_{M+}$ and $D_{M+}D_{M-}$. Consequently the contribution to the heat super-trace from such an eigenspace vanishes identically for the model operator D_M. If however $\mu^2 = 0$ the above trick does no longer work. If $\dim(N)$ is odd the method for computing this index contribution used in [Ch1], [Chou] gives that the joint contribution coming from all λ^2 is given by $-\frac{1}{2}\eta_{\text{big}}(B_{N+}; B_Y, 0)$, where the η-function η_{big} is given by

$$\eta_{\text{big}}(B_{N+}; B_Y, s) = \sum_{\substack{\lambda \in \operatorname{spec}(B_{N+}) \\ |\lambda| \geq \frac{1}{2}}} \operatorname{sign}(\lambda)|\lambda|^{-s}\operatorname{tr}\left(\tau_{|E_{\lambda^2,0}\cap\ker(\tau_N - 1)}\right)$$

for $\operatorname{Re}(s) \gg 0$. Here B_{N+} denotes the restriction of B_N to $\ker(\tau_N - 1)$.

If $\dim(N)$ is even, τ_N anti-commutes with B_N and commutes with τ. Consequently the spectrum of the restriction of B_N to the ± 1 eigenspaces of τ is symmetric. Further ν conjugates $(B_N)_{|\ker(\tau-1)}$ into $-(B_N)_{|\ker(\tau+1)}$ so it follows that $(B_N)_{|\ker(\tau-1)\cap\ker(B_Y)}$ is conjugate to $(B_N)_{|\ker(\tau+1)\cap\ker(B_Y)}$ and consequently the index contribution vanishes.

2) $0 < \lambda^2 < \frac{1}{4}$. This is exactly like in 1) though the self-adjoint extension is different. Together with the contribution from 1), for $\dim(N)$ odd we get the full η-invariant of B_{N+}, modified with respect to τ and for $\dim(N)$ even the contribution vanishes.

3) $\lambda^2 = 0$, $\mu^2 \neq 0$. This is conjugate to an Atiyah-Patodi-Singer boundary (with $\ker(B_Y) = \{0\}$). If $\dim(Y)$ is odd, $\frac{1}{2}\eta((B_Y)_{|\ker(\tau-1)}, 0)$ comes out as the joint contribution for all $\mu^2 \neq 0$. If $\dim(Y)$ is even the operator τ_Y anti-commutes with B_Y and it follows that the spectrum of $(B_Y)_{|\ker(\tau-1)}$ is symmetric. Since ν conjugates $(B_Y)_{|\ker(\tau-1)}$ into $-(B_Y)_{|\ker(\tau+1)}$ it follows that $(B_Y)_{|\ker(\tau-1)}$ is conjugate to $(B_Y)_{|\ker(\tau+1)}$ and consequently the contribution to the index vanishes.

4) $\lambda^2 = \mu^2 = 0$. In this case the contribution depends on the augmentation, which we take to be the scattering matrix S, defined on the direct sum of all $E_{0,0}$, coming from all generalized wedges of X. Further the direct sum of all $D_{0,0}$ is conjugate to a self-adjoint extension of $\nu\frac{\partial}{\partial r}$ defined on $L^2((0,\infty), V)$, where V is the direct sum of the $\ker(B_N) \cap \ker(B_Y)$. For the global augmentation S, the contribution is $\frac{1}{2}\operatorname{tr}(S_+)$.

In the general case of a manifold with stratifications the global arguments above, valid for generalized wedges, do not apply. The problem is that stratifications of low codimension meet in stratifications of higher codimension. Thus the contribution to the index coming from a neighborhood of the boundary of a stratification of non-maximal codimension has to be computed using the local structure of stratifications of higher codimension. Since however the boundary of a stratification contributes to the global index contributions given above, the global contribution may not be the correct contribution to the index. In order to avoid this problem we have to localize the computation of the index contribution. The tools available for such a localization (in the case where $\dim(Y)$ is even and $\dim(N)$ is odd) are the local index theorem [BGV, Theorem 4.1] together with an explicit analysis of the operator B_N defined on each F_{μ^2} eigenspace of B_Y.

If $\ker(B_N) \neq \{0\}$ even the definition of ideal boundary conditions courses trouble in the non-global setup. The problem can though be solved using the theory from [Sa1, Section 3.2]. Fortunately, because we make use of the canonical gluing only and restrict attention to manifolds with corners of codimension 3, the need for localization only exists for stratifications of codimension 2, for which $\ker(B_N) = \{0\}$.

If M is an oriented manifold with corners of codimension 3 and product structure, the only stratifications of $\tilde{M}$, which can not be handled globally, are the stratifications corresponding to corners of codimension 2 of M and the stratifications of $Z\times[0,1]$. Both of those types of stratifications are manifolds with boundary and product structure around the boundary. Further they are orientable and because also the direction $\frac{\partial}{\partial r}$, where r is the distance to the stratification Y, is globally defined on Y, also the direction $\frac{\partial}{\partial \theta}$ can be fixed globally. Thus in fact the cone-bundle over Y is trivial since corners of codimension 2 allow only one global self-isometry, which is orientation reversing.

Now, instead of fixing $y \in Y$, we fix $n \in N$ and $r \in R_+$ and impose Atiyah-Patodi-Singer boundary conditions on ∂Y for each fixed (n, r). The μ^2 eigenspaces, F_{μ^2}, of B_Y^2 are Dirac bundles over $\mathbb{R}_+ \times N$.

Lemma 4.1. *Each F_{μ^2} eigenspace of B_Y^2 is glued using the canonical gluing.*

Proof: Just like Clifford multiplication acts on F_{μ^2}, also the gluing operators splits into gluing operators for each F_{μ^2} and are given in the same way as the gluing operators for $\tilde{E}$. □

In particular Lemma 4.1 gives that the spectrum of B_N is independent of μ^2, and that we to each μ^2 can associate a multiplicity m_{μ^2} such that the multiplicity of $\lambda^2 \in \operatorname{spec}((B_N)_{|F_{\mu^2}})$ is m_{μ^2} independently of λ.

Let D be the Dirac operator on $\mathbb{R}_+ \times N \times Y$. The restriction of D to each F_{μ^2} is given slow-growing ideal boundary conditions. This gives a self-adjoint realization of D. The local contribution to the index from the stratification is given by

$$\lim_{\varepsilon\to 0}\lim_{t\to 0}\int_0^\varepsilon\int_N \operatorname{tr}(\tau e^{-tD^2})((r,n,y),(r,n,y))dnrdr.$$

Now take $\{\varphi_{\mu^2}\}_{\mu^2\in\operatorname{spec}(B_Y^2)}$ to be an orthonormal basis for $\ker(\tau_Y - 1)$ such that $B_Y^2\varphi_{\mu^2} = \mu^2\varphi$, $\tau_N\varphi_{\mu^2} = \pm\varphi_{\mu^2}$, $\tau\varphi_{\mu^2} = \pm\varphi_{\mu^2}$ and such that further φ_{μ^2} is an eigenvalue of the canonical gluing operator. Then B_N preserves the space $\operatorname{span}\{\varphi_{\mu^2}, \nu\varphi_{\mu^2}\}$.

We use that B_Y^2 commutes with the operator $A^2 := D^2 - B_Y^2$. Further, by the remarks above, A^2 is a direct sum of unitarily equivalent operators A_0^2 defined on each $L^2(\mathbb{R}_+ \times N, \operatorname{span}(\varphi_{\mu^2}, \nu\varphi_{\mu^2}))$. For $t > 0$ we have

$$\int_0^\varepsilon\int_N \operatorname{tr}(\tau e^{-tD^2})((r,n,y),(r,n,y))dnrdr =$$

$$\sum_{\mu^2} e^{-t\mu^2}\int_0^\varepsilon\int_N \operatorname{tr}\left(\tau e^{-t(A_0^2)_{|\ker(\tau_Y-1)}}\right)((r,n),(r,n))dnrdr\,|\varphi_{\mu^2}(y)|^2 +$$

$$\sum_{\mu^2} e^{-t\mu^2} \int_0^{\varepsilon} \int_N \operatorname{tr}\left(\tau e^{(-tA_0^2)_{|\ker(\tau_Y+1)}}\right)((r,n),(r,n))dnrdr\,|\nu\varphi_{\mu^2}(y)|^2$$

$$= \operatorname{tr}\left(\tau_Y e^{-tB_Y^2}\right)(y,y) \int_0^{\varepsilon} \int_N \operatorname{tr}\left(\tau_N e^{-t(A_0^2)_{|\ker(\tau_Y-1)}}\right)((r,n),(r,n))dnrdr.$$

By the local index theorem [BGV, Theorem 4.1] we know that there are expansions

$$\operatorname{tr}\left(\tau_Y e^{-t(B_Y^2)_{|\ker(\tau_N-1)}}\right)(y,y) \overset{t\to 0}{\sim} \sum_{k=0}^{\infty} b_k(y) t^{\frac{k}{2}}, \tag{4.0.1}$$

$$\operatorname{tr}\left(\tau_N e^{-t(A_0^2)_{|\ker(\tau_Y-1)}}\right)((r,n),(r,n)) \overset{t\to 0}{\sim} \sum_{k=0}^{\infty} c_k((r,n)) t^{\frac{k}{2}}. \tag{4.0.2}$$

The expansions are locally uniform in the interior of Y and $\mathbb{R}_+ \times N$, respectively. By the local product structure we further have that $c_0(r,n) = 0$ for all (r,n). Further, by scale invariance of $\mathcal{D}(A_0^2)$ we have the identity

$$e^{-tA_0^2}((r,n),(r,n)) = \frac{1}{r^2} e^{-\frac{t}{r^2}A_0^2}((1,n),(1,n)).$$

This gives

$$\int_0^{\varepsilon} \int_N \operatorname{tr}\left(\tau_N e^{-t(A_0^2)_{|\ker(\tau_Y-1)}}\right)((r,n),(r,n))dnrdr$$

$$= \int_0^{\varepsilon} \int_N \frac{1}{r^2} \operatorname{tr}\left(\tau_N e^{-\frac{t}{r^2}(A_0^2)_{|\ker(\tau_Y-1)}}\right)((1,n),(1,n))dnrdr$$

$$= \frac{1}{2} \int_{\frac{t}{\varepsilon^2}}^{\infty} \int_N u^{-1} \operatorname{tr}\left(\tau_N e^{-u(A_0^2)_{|\ker(\tau_y-1)}}\right)((1,n),(1,n))dndu. \tag{4.0.3}$$

The integral is convergent because $\operatorname{tr}\left(\tau_N e^{-u(A_0^2)_{|\ker(\tau_Y-1)}}\right)((1,n),(1,n)) \leq Cu^{-\frac{1}{2}}$ for $u \in (\frac{t}{\varepsilon^2}, \infty)$ (see [Ch1], [Chou]). By (4.0.2) and since $c_0 = 0$ the limit for $t \to 0$ exists. This limit is computed in [Chou]. The result is $-\frac{1}{2}\eta(B_{N+}, 0)$. Here B_{N+} is the restriction of B_N to the $+1$ eigenspace of τ_N inside the $+1$ eigenspace of τ_Y. But by Corollary 3.4, $\eta(B_{N+}, 0) = 0$. We state the result in a lemma:

Lemma 4.2. *Let M, $\tilde{M}$, $\tilde{E}$ and $\tilde{D}$ be like in the introduction. Then the local contribution from the corners of codimension 2 vanishes if $\tilde{E}$ is glued using the canonical gluing.*

In the end of this section we will justify that index contributions from the various strata of $\tilde{M}$ can be computed as we do it.

Let Y be a stratification of $\tilde{M}$ of codimension 2 and let $Y' \subseteq \partial Y$ be a corner of codimension 3 of M. Let v denote the distance to Y' in Y and let v_0 be small enough such that all corners of codimension 2 satisfy that the distance between different boundary components is at least $3v_0$ and all have product structure on the set where the distance to the boundary is at most v_0. Further assume that v_0 is small enough such that $\tilde{M}$ has product structure on a $2v_0$ neighborhood of each corner of codimension 3. Let Y_{v_0} be the subset of Y such that the distance to Y' is greater than v_0. Then we may split $\tilde{M}$ in the following way:

Let $\varepsilon < \frac{v_0}{2}$ be such that the ε-neighborhood of 0 in the wedge $\mathbb{R}_+ \times N \times Y_{v_0}$ is imbedded in $\tilde{M}$ for each Y_{v_0} and such that these neighborhoods are disjoint. On each of these neighborhoods we take the heat kernel of $\mathbb{R}_+ \times N \times Y$ as a parametrix for the heat kernel of $\tilde{M}$.

For each corner Y^3 of codimension 3 of M we take the v_0 neighborhood of 0 in the wedge $\mathbb{R}_+ \times N^3 \times Y^3$ as a neighborhood, on which we use the heat kernel of $\mathbb{R}_+ \times N^3 \times Y^3$ as a parametrix. (In order to cover $\ker(B_{N^3})$ we actually have to take all codimension 3 wedges simultaneously and simultaneously with the boundary).

Let $y = (v, y')$ denote the variables close to the boundary of a stratification Y of codimension 2. Further let (r, n) denote the variables in the fibre $\mathbb{R}_+ \times N$ and let $\delta = v_0 - \sqrt{v_0^2 - \varepsilon^2}$. If $Y' = \partial Y$ is a wedge singularity of codimension 3 of $\tilde{M}$ the neighborhood

$$\{(r, n, u, y') \mid \sqrt{v_0^2 - v^2} < r < \varepsilon, n \in N, v \in (v_0 - \delta, v_0)\}$$

can not be treated using an interior parametrix because it touches a stratification. Such neighborhoods do however not contribute to the index. This can be seen similarly like the computation of the local index density along Y:

$$\int_{v_0-\delta}^{v_0} \int_{\sqrt{v_0^2-v^2}}^{\varepsilon} \int_N \mathrm{tr}\, \left(\tau e^{-tD^2}\right) ((r, n, v, y'), (r, n, v, y')) dnrdrdvdy' =$$

$$\int_{v_0-\delta}^{v_0} \mathrm{tr}\, (\tau_Y e^{-tB_Y^2})(y, y) \int_{\frac{t}{\varepsilon^2}}^{\frac{t}{\sqrt{v_0^2-v^2}}} u^{-1} \int_N \mathrm{tr}\, \left(\tau_N e^{-u(A_0^2)_{|\ker(\tau-1)}}\right) ((1, n), (1, n)) dndudvdy'.$$

Here we notice that the integral over $dndu$ is uniformly bounded in all variables whereas, because of the product structure, $\mathrm{tr}\, (\tau_Y e^{-tB_Y^2})(y, y)$ tends uniformly towards 0 on $v \in (v_0 - \delta, v_0)$ for $t \to 0$. This proves that the contribution coming from the transition vanishes.

On the set $[1 - u_0, 1] \times Z$ a parametrix for the Atiyah-Patodi-Singer problem is constructed. (Again, $\ker(A_Z)$ can not be treated alone.)

On the rest of M interior parametrices for the heat super-trace are appropriate.

All in all we have split M into neighborhoods, for each of which we have a parametrix of the heat kernel, from which we can compute the index contribution.

5 The Splitting Formula for η-Invariants.

In this section we prove a slight generalization of the splitting formula for the η-invariant.

Let $Z = Z_1 \cup_Y Z_2$ be an oriented odd dimensional Riemannian manifold with wedges and product structure around the separating manifold Y. Further let E be a Dirac bundle over Z, which also has product structure around Y and respects the wedge structure over the wedges. We assume that for each wedge $\mathbb{R}_+ \times N \times Y$, we have that $\ker(B_N) = \{0\}$, where B_N is the closure of the operator $B_{N,0}$ given by (2.0.3), defined for each $y \in Y$.

Let

$$M := Z \times [-1, 0] \cup_{Z_1 \sqcup Z_2} (Z_1 \sqcup Z_2) \times [0, 1]. \tag{5.1.1}$$

The bundle E can be pulled back to M and gives a Hermitian vector-bundle over M, which we will also denote by E. It is supplied with the connection $\nabla = \nabla^Z + du\frac{\partial}{\partial u}$, where u is the second coordinate in the definition of M. In order to construct a Clifford module from E, we let $F = E \oplus E$, with the Clifford module structure from E on the first component and Cliff(-1) times Clifford multiplication on the second component. Let the Clifford multiplication in the direction of $\frac{\partial}{\partial u}$ be given by the element

$$\nu' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $C^\infty(M, End(E \oplus E))$. Clearly ν' is parallel, anti-selfadjoint, $(\nu')^2 = -1$ and ν' anti-commutes with Clifford multiplication in directions tangential to Z. Thus F is a Dirac bundle.

The canonical involution τ is given as $\nu'\tau_Z$, where τ_Z is the canonical involution on $F \mapsto Z$ associated to the orientation.

The operator, for which we will split the η-invariant, is the operator

$$A_+ := (\nu' D_Z)_{|\ker(\tau - 1)}.$$

The bundle $\ker(\tau - 1)_{|Z}$ is a Dirac bundle over Z with the Clifford multiplication $\mathbf{c}'$ given by $\mathbf{c}'(v) = \nu'\mathbf{c}(v)$ for $v \in T^*M$. Thus A_+ is a Dirac type operator. A self-adjoint realization of A_+ is given by imposing slow growing ideal boundary conditions.

The Dirac operator D on M has a self-adjoint extension given by imposing Atiyah-Patodi-Singer boundary conditions on each Z_j, augmented with respect to the scattering matrices S_j, and imposing Atiyah-Patodi-Singer boundary conditions at $Z \times \{-1\}$ and $Z_j \times \{1\}$, $j = 1, 2$ and slow growing ideal Atiyah-Patodi-Singer boundary conditions at Y. We augment with respect to the scattering matrix S, which mixes the boundaries and the wedge singularity at Y. By assumption there is no need to augment in the singularities of Z.

With respect to the eigenspaces of τ, D has the usual decomposition $D = D_+ + D_-$, where D_+ maps $\ker(\tau - 1)$ to $\ker(\tau + 1)$ and D_- is the adjoint of D_+.

The index of D_+ can be computed.

$$\mathrm{Index}(D_+) = \frac{1}{2}\eta(A_+, 0) - \frac{1}{2}\eta(A_{1+}, 0) - \frac{1}{2}\eta(A_{2+}, 0) + \frac{1}{2}\mathrm{tr}\,(S_+) + \mathbf{c}. \tag{5.1.2}$$

Here A_{1+} and A_{2+} are the operators $(\nu' D_{Z_1})_{|\ker(\tau-1)}$ and $(\nu' D_{Z_2})_{|\ker(\tau-1)}$. The η-invariants occur with different signs because the operator ν normally used to define the operators A, A_1 and A_2 gets different signs corresponding to the orientation of the cylinders. The term $\mathbf{c}$ is the contribution from Y coming from the orthogonal complement of $\ker(B_N) \cap \ker(B_Y)$. This contribution is computed in [Sa1]. The undesirable terms in (5.1.2) we collect in a term $\mathbf{J}$ given by

$$\mathbf{J} := \mathrm{Index}(D_+) - \frac{1}{2}\mathrm{tr}\,(S_+). \tag{5.1.3}$$

Theorem 5.1. *We have*

$$\eta(A_+, 0) = \eta(A_{1+}, 0) + \eta(A_{2+}, 0) - \sum_{\beta} \left(\frac{1}{\pi}[\beta + \pi]\right) + 2\mathbf{J}, \tag{5.1.4}$$

where $e^{i\beta}$ runs over the eigenvalues of $-S_{1++}S_{2++}$ and for $i = 1, 2$, S_{i++} is the restriction of S_i to the joint eigenspace $\ker(\tau - 1) \cap \ker(\nu\delta - i)$.

Remark 10. [Sa3, Lemma 2.1.3] and [Sa3, Corollary 2.1.4] contain serious mistakes.

6 η-Invariants for Manifolds with Corners.

First we will prove an index theorem for manifolds with corners of codimension 3. This both motivates the splitting formula, we are going to prove, and at the same time it gives the main tool for proving the splitting formula.

Let M_0 be a compact, oriented, even dimensional manifold with corners of codimension 3. The corners of M_0 of codimension 3 are isometrically homeomorphic to $N \mapsto S \mapsto Y$. Let $\tilde{M}_0$ be the extension of M_0 defined in Section 3.

Let $E_0 \mapsto M_0$ be a Dirac bundle. We extend E_0 to $\tilde{E}_0 \mapsto \tilde{M}_0$ using the canonical gluing. Further we let E_Z be the pull-back of $\tilde{E}_0$ to $Z \times \{1\}$. E_Z is a Dirac bundle over Z with the Clifford multiplication given by $b(\gamma) = -\nu\mathbf{c}(\gamma)$, where ν is Clifford multiplication by $\frac{\partial}{\partial u}$, and u is the distance to M_0 on the cylinder $Z \times [0, \infty) \subseteq \tilde{M}_0$. This Clifford bundle further has parallel sections of $End(Z, E_Z)$ given by ν and τ.

Take a wedge singularity of Z, i.e. an open subset of Z isometric to a neighbourhood of $0 \times N \times Y \subset [0, \infty) \times N \times Y$, where Y is a corner of codimension 3 of M_0. Around such a singularity $\ker(B_Y) \mapsto N$ is a Dirac bundle over N. Let $B_{N,0}$ be the associated

Dirac operator. Further let τ_N be the image of the volume form on N in the Clifford bundle for some orientation of N.

By Corollary 3.2, Lemma 3.3 and Corollary 3.4 we have that $\ker(B_{N,0}) = \{0\}$ and that the spectra $\operatorname{spec}((B_{N,0})_{|\ker(\tau_N - 1)})$ and $\operatorname{spec}((B_{N,0})_{|\ker(\tau_N+1)})$ are both symmetric.

This gives that the Dirac operator A_Z on Z can be given slow-growing ideal boundary conditions, and is self-adjoint with those boundary conditions. Further the domain of A_Z is preserved by τ and ν.

We now consider the singularities of $\tilde{M}_0$ corresponding to corners of codimension 3 of M_0. Again we locally can write

$$D = \nu(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{1}{r}B_N + B_Y).$$

Here N is a 2-dimensional manifold with asymptotically conical singularities. Since the singularities of N are asymptotic to the singularities of Z together with the singularities of $\tilde{M}_0$ arising from codimension 2 corners of M_0, the restriction of B_N to any eigenspace of B_Y^2 is self-adjoint when it is given slow-growing ideal boundary conditions and has a discrete point spectrum.

We now define boundary conditions for $\tilde{D}$. Let D_1 be the closed Dirac operator on $\tilde{M}_0$ arising by imposing slow-growing ideal boundary conditions along all wedges with $\dim(N) = 1$. This can be done without fixing a self-adjoint extension of B_Y since $\ker(B_N) = \{0\}$. For the wedges for which $\dim(N) = 2$ we notice that the corresponding Y is closed and that N is a manifold with conical singularities and an asymptotic product metric. The cones of B_N are over circles with the vector bundle glued using the canonical gluing, and B_N is self-adjoint with slow growing ideal boundary conditions, which match the slow growing ideal boundary conditions imposed along the codimension 2 stratifications. In the same way A_Z is a self-adjoint operator with slow growing boundary conditions and A_Z anti-commutes in the strong sense with Clifford multiplication in the normal direction at the boundary. Consequently the theory from Section 2 can be iterated. This gives a self-adjoint extension of $\tilde{D}$.

Item 3) and 4) in the list of global index contributions in Section 4 gives the index contributions from the corners of codimension 3 and the joint scattering matrix. These contributions are given by

$$-\frac{1}{2}\eta(B_{Y^3+}; \ker(B_{N^3+}), 0) + \frac{1}{2}\operatorname{tr}(S_+).$$

Here B_{Y^3+} is the Dirac operator acting on $\ker(B_{N^3}) \cap \ker(\tau - 1) \mapsto Y$ and S is the scattering matrix acting on $(\bigoplus \ker(B_N) \cap \ker(B_Y)) \oplus \ker(A_Z)$. Notice that we do not even need an asymptotic expansion of the heat kernel of $\tilde{D}_0$ in order to deal with these terms because the cancelation in 1) and 2) in the list of index contributions depends only on the fact that B_N has point spectrum and of some commutation relations.

The simple wedge singularities of Z course a logarithmic term in the expansions of $\operatorname{tr}(e^{-tA_{Z+}})$ and $\operatorname{tr}(A_{Z+}e^{-tA^2_{Z+}})$. Proceeding like in [APS] we get that there exist coefficients $\mathbf{d}_0$, $\mathbf{d}_{-1}$ and $\{d_j\}$ such that

$$\int_{1-\varepsilon}^{1}\int_Z \operatorname{tr}(\tau e^{-t\tilde{D}^2})((z,u),(z,u))dzdr \sim \mathbf{d}_0 \log t + \mathbf{d}_{-1} t \log t + \sum_{j=-\infty}^{\dim(M)} d_j t^{\frac{j}{2}}. \qquad (6.0.1)$$

The integral of a tubular neighborhood of a strata of $\tilde{M}$ corresponding to a corner of codimension 2 of M has a similar expansion. Further the integral of the heat super-trace over any compact subset of the smooth interior of $\tilde{M}$ has an asymptotic expansion without log terms. Since

$$\mathrm{Index}(\tilde{D}_+) = \int_{\tilde{M}} \mathrm{tr}\, \left(\tau e^{-t\tilde{D}^2}\right)(x,x)dx,$$

we conclude that the log terms must cansel with other log terms in local expansions. But from the way the complicated parts of the index contributions cansel in Section 4, it follows that none of the heat super-trace expansions from the stratifications of codimension 2 or the stratifications of $\tilde{M}_0$ of codimension 3 have log terms. Thus we conclude that $\mathbf{d}_0 = 0$. This gives that the η-invariant of A_{Z+} exists.

The various index contributions listed in the end of Section 4 suffice to prove an index theorem for manifolds with corners of codimension 3.

Theorem 6.1. *Let M_0 be an even dimensional manifold with corners of codimension 3 and product structure around the boundary and corners. Further let $E \mapsto M_0$ be a Dirac bundle over M_0, which respects the product structure. Let $\tilde{M}_0$ be the extension of M_0 defined by (0.0.1) and $\tilde{E} \mapsto \tilde{M}_0$ be the Dirac bundle over $\tilde{M}_0$ arising by extending E using the canonical gluing. Finally let $\tilde{D}$ be the self-adjoint Dirac operator on $\tilde{M}_0$ given slow growing ideal boundary conditions and Atiyah-Patodi-Singer boundary conditions, augmented jointly at the boundary and ideal boundary with respect to the scattering matrix S defined in Section 2. Then we have*

$$\mathrm{Index}(\tilde{D}_+) = \int_M a_D(x)dx - \frac{1}{2}\eta(A_{Z+},0) + \frac{1}{2}\mathrm{tr}\,(S_+) - \frac{1}{2}\sum_{Y^3} \eta(B_{Y^3+};\ker(B_{N^3}),0). \quad (6.0.2)$$

Here a_D is the standard local formula. The $+$ denotes restriction to $\ker(\tau-1)$, where τ is the image of the volume form in the Clifford bundle. Y^3 runs over the codimension 3 corners of M_0 and for each Y^3, N^3 denotes the base of the cone factor of the wedge singularity of $\tilde{M}_0$ in $Y_3 \times 0$. The notation $\eta(B_{Y^3+};\ker(B_{N^3}),0)$ means the following: The kernel of B_{N^3} can be considered as a Dirac bundle over Y^3 on which B_{Y^3} and τ are defined. $\eta(B_{Y^3+};\ker(B_{N^3}),0)$ thus means the η-invariant of the restriction of B_{Y^3} to $\ker(B_{N^3}) \cap \ker(\tau-1)$.

Proof: The list of global index contributions given in the end of Section 4 gives all the terms in (6.0.2). The stratifications of the type $\mathbb{R}_+ \times N \times Y$, where N is of dimension 1, do not contribute to the index by Lemma 4.2, and by the comments after Lemma 4.2, we are done. □

In order to get gluing formulas for the index invariant $\mathcal{I}(M_0) := \mathrm{Index}(\tilde{D}_+) - \frac{1}{2}\mathrm{tr}\,(S_+)$, $\eta(A_{Z+},0)$ has to be split. Like in Section 5 the index theorem itself provides some of the tools necessary for proving a splitting formula.

In the following let M_0 and Z be as above and let M be the manifold with corners of codimension 3 and wedge singularities given by

$$M = (Z \times [-1,0]) \cup_{Z_1 \sqcup \cdots \sqcup Z_k} (Z_1 \sqcup \cdots \sqcup Z_k) \times [0,1],$$

where $Z_1, \ldots, Z_k$ denote the intrinsic boundary components of M_0. The pullback to M of $\tilde{E}_{|Z}$ has a canonical structure as a Dirac bundle over M. The smoothened boundary of M of maximal codimension is the union of Z and another boundary component, which we denote by $\tilde{Z}$.

Again we form an extended manifold $\tilde{M}$ given by

$$\tilde{M} = (M \cup_Z (Z \times [-2,-1])) \cup_{\tilde{Z}} (\tilde{Z} \times [0,1]).$$

Further, using the canonical gluing we get a Dirac bundle $\hat{E} \mapsto \tilde{M}$.

The singularities of $\tilde{M}$ have the same structure as the singularities of the manifold $\tilde{M}_0$ treated above and again, since we have used the canonical gluing, slow growing ideal Atiyah-Patodi-Singer boundary conditions can be imposed. This gives a self-adjoint Dirac operator $\hat{D}$ on $\tilde{M}$.

Lemma 6.2. *We have the identity*

$$\eta(A_{Z+},0) = \eta(A_{\tilde{Z}+},0) - 2\left(\mathrm{Index}(\hat{D}_+) - \frac{1}{2}\mathrm{tr}\,(\hat{S}_+)\right) - \sum_{Y^3} \eta(B_{Y_3+}; \ker(B_{N_1^3}), 0). \tag{6.0.3}$$

Here Y_3 runs over the corners of codimension 3 of M_0 and N_1^3 denotes the base of the cone part of the wedge singularity of the type $\mathbb{R}_+ \times N_1^3 \times Y^3$ of $\tilde{M}$. $\hat{S}^1$ denotes the scattering matrix of $\hat{D}$.

Proof: This is an immediate consequence of the proof of Theorem 6.1 applied to $\hat{D}$. The local formulas vanish because $\tilde{M}$ has product structure everywhere. The stratification of M is such that it makes no difference in the analysis of $\tilde{M}$, so the proof of Theorem 6.1 goes through without change. □

In order to proceed we have to understand the structure of $\tilde{Z}$. Consider an intrinsic boundary component Z_j of M_0. The image of $Z_j \times \{1\}$ in $\tilde{Z}$ is the end of a cylinder over Z_j. The cylinder is attached to $Z \times [-2,0]$ by mapping $Z_j \times \{0\}$ into $Z \times \{0\}$. It follows that for each $\varepsilon > 0$, $\tilde{Z}$ has a decomposition

$$\tilde{Z} = (Z_{1\varepsilon} \sqcup \cdots \sqcup Z_{k\varepsilon}) \cup_{\partial X_{1-\varepsilon}} X_{1-\varepsilon}.$$

Here, for $j = 1, \ldots, k$, $Z_{j\varepsilon} = Z_j \cup_{\partial Z_j} ([1-\varepsilon, 1] \times \partial Z_j)$ and $X_{1-\varepsilon}$ is the manifold with singularities arising by taking cylinders of length $1-\varepsilon$ over each smoothened boundary ∂Z_j, and identifying the sets $\partial Z_j \times \{0\}$ in the same way as the boundaries of the Z_j's are identified in Z.

For any $\varepsilon > 0$ each of $Z_{j\varepsilon}$ and $X_{1-\varepsilon}$ are manifolds with boundaries and wedge singularities, whose cone factors are over manifolds of dimension 1. Further, by Lemma 3.3 the restriction of $\hat{E}$ to each of $Z_{j\varepsilon}$ and $X_{1-\varepsilon}$ is glued using the canonical gluing.

Lemma 6.3. *$\eta(A_{Z_{j\varepsilon}+}, 0)$ is independent of ε.*

Proof: The proof of [Mü1, Proposition 2.16] applies to this setup also. □

The standard splitting formula for η-invariants now gives

$$\eta(A_{\tilde{Z}+}, 0) = \eta(A_{X_{1-\varepsilon}+}, 0) + \sum_j \eta(A_{Z_{j\varepsilon}+}, 0) + \mathbf{m}(S_{1++}, S_{2++}) + \mathbf{J},$$

where S_{1++} is the restriction of the direct sum of the scattering matrices of $A_{Z_{j\varepsilon}}$ to $\ker(\tau - 1) \cap \ker(\tau_N - 1)$ and S_{2++} is the restriction of the scattering matrix of A_X to $\ker(\tau - 1) \cap \ker(\tau_N - 1)$. In combination with Lemma 6.2 this gives:

Theorem 6.4. *Let Z be the smoothened boundary of a manifold M_0 with corners of codimension 3 and product structure around the boundary. Further let $E \mapsto M$ be a Dirac bundle over M respecting the product structure. Let $\tilde{E} \mapsto \tilde{M}$ be the extension of $E \mapsto M$ using the canonical gluing and let A_Z denote the Dirac operator on Z with slow growing boundary conditions. Then the following splitting formula holds*

$$\eta(A_{Z+}, 0) = \eta(A_{X_{1-\varepsilon}+}, 0) + \sum_j \eta(A_{Z_{j\varepsilon}+}, 0) + \mathbf{m}(S_{1++}, S_{2++}) + \mathbf{J} - 2\left(\mathrm{Index}(\hat{D}_+) - \frac{1}{2}\mathrm{tr}\,(\hat{S}_+)\right) - \sum_{Y^3} \eta(B_{Y_3+}; \ker(B_{N_1^3}), 0). \quad (6.0.4)$$

Each of the terms in (6.0.4) is defined above.

References

[APS] Atiyah, M.F., Patodi, V.K., Singer, I.M.: "Spectral Asymmetry and Riemannian Geometry I.", Math. Proc. Cambridge Philos. Soc. **77** (1975), 43-69.

[Bä] Bär, C.: "Metrics with Harmonic Spinors." GAFA **6**, (1996), 899-942.

[BGV] Berline, N.,Getzler, E., Vergne, M.: "Heat Kernels and Dirac Operators", Second Edition, Springer-Verlag (1996).

[BL1] Brüning, J., Lesch, M.: "On the eta-invariant of certain non-local boundary value problems" Duke math. J. bf 96 (1999) 425-468.

[BL2] Brüning, J., Lesch, M.: "On the Spectral Geometry of Algebraic Curves" J. reine angew. math. **474** (1996), 25-66.

[Br1] Brüning, J.: "Heat Equation Asymptotics for Singular Sturm-Liouville Operators" Math. Ann. **268** (1984), 173-196.

[Br2] Brüning, J.: "Heat Kernel Asymptotics for Operator Valued Sturm-Liouville Equations" Analysis **8** (1988), 73-93.

[Brø] Brøndsted, A.: "An Introduction to Convex Polytopes" Graduate Texts in Mathematics **90**, Springer-Verlag (1983).

[BS] Brüning, J., Seeley, R.: "The Resolvent Expansion for Second order Regular Singular Operators" J. Funct. Anal. **73** (1987), 309-429.

[Bu] Bunke, U.: "On the Gluing Problem for the η-Invariant" J. Diff. Geo. **41** (1995), 397-448.

[BW] Booß-Bavnbek, B., Wojciechowski, K.P.:"Elliptic Boundary Problems for Dirac Operators" Birkhäuser, Boston (1993).

[Ch1] Cheeger, J.:"Spectral Geometry of Singular Riemannian Spaces", J. Diff. Geom. **18** (1983), 575-657.

[Ch2] Cheeger, J.:"η-Invariants, The Adiabatic Approximation and Conical Singularities" J. Diff. Geom. **26** (1987), 175-221.

[Cn] Chernoff, P.R.: "Essential Self-adjointness of Powers of Generators of Hyperbolic Equations." J. Funct. Anal. **12** (1973), 401-414.

[Chou] Chou, A. W..:"The Dirac Operator on Spaces with Conical Singularities and Positive Scalar Curvatures.", Trans. of the Amer. Math. Soc. **289** (1985), 1-40.

[doC] do Carmo, M.P.: "Differential geometry of curves and surfaces" Prentice-Hall Inc. (1976) Englewood Cliffs, New Jersey.

[DW] Douglas, R.G., Wojciechowski, K.P.:"Adiabatic Limits of the η-Invariants. The Odd-Dimensional Atiyah-Patodi-Singer Problem" Commun. MAth. Phys. **142** (1991), 139-168.

[Gi] Gilkey, P.:"Invariance Theory, the Heat Equation and the Atiyah-Patodi-Singer Index Theorem", 2nd edition, CRC Press (1995).

[HMM] Hassel, A., Mazzeo, R., Melrose, R.B.:"A Signature Formula for Manifolds with Corners of Codimension Two" Topology **36** (1997).

[Le] Lesch, M.:"Operators of Fuchs Type, Conical Singularities, and Asymptotic Methods", Teubner-Texte zur Mathematik, **136** (1997), B.G. Teubner Verlagsgesellschaft, Stuttgart - Leipzig.

[LW] Lesch, M., Wojciechowski, K. P.:"On the η-invariant of generalized Atiyah-Patodi-Singer boundary value problems." Ill. J. Math. **40** (1996) 30-46.

[MM1] Mazzeo, R.R., Melrose, R.B.:"Analytic Surgery and the Eta Invariant" GAFA **5**, No 1 (1995), 15-75.

[MM2] Mazzeo, R.R., Melrose, R.B.:"Analytic Surgery and the Accumulation of Eigenvalues" Comm. Anal. Geom. **3** No 1. (1995), 115-222.

[Me] Melrose, R.B.:"The Atiyah-Patodi-Singer Index Theorem", MIT, AK Peters, Wellesley, 1993.

[Mo] Mooers, E.:"Heat Kernel Asymptotics on Manifolds with Conic Singularities" Preprint, December 1997.

[Mü1] Müller, W.:"Eta Invariants and Manifolds with Boundary" J. Diff. Geo. **40** (1994), 311-377.

[Mü2] Müller, W.:"On the L^2-Index of Dirac Operators on Manifolds with Corners of Codimension Two. I" J. Diff. Geo. **44** (1996), 97-177.

[Sa1] Salomonsen, G.: "Atiyah-Patodi-Singer Type Index Theorems for Manifolds with Corners and Splitting of η-Invariants I." In preparation.

[Sa2] Salomonsen, G.:"The Atiyah-Patodi-Singer Index Theorem for Manifolds with Corners I." Preprint #514, SFB 256, University of Bonn.

[Sa3] Salomonsen, G.:"The Atiyah-Patodi-Singer Index Theorem for Manifolds with Corners II." SFB 256 preprint nr. 564 (1998), University of Bonn.

[Sch1] Schulze, B.-W.:"Pseudodifferential Operators on Manifolds with Singularities" North-Holland (1991).

[Sch2] Schulze, B.-W.:"Operators on Corner Manifolds." Book in Preparation.

[Wa] Wall, C.T.C.:"Non-additivity of the Signature" Inv. math. **7** (1969), 269-274.

[Wo1] Wojciechowski, K.P.:"The Additivity of the η-Invariant: The Case of an Invertible Tangential Operator" Houston J. Math. **20** (1994), 603–621.

[Wo2] Wojciechowski, K.P.:"The Additivity of the η-Invariant. The Case of a Singular Tangential Operator" Commun. Math. Phys. **169** (1995), 315-327.

[Wo3] Wojciechowski, K.P.:"The ζ-Determinant and the Additivity of the η-Invariant on the Smooth Self-Adjoint Grassmannian" Preprint, 1998.