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Atiyah-Patodi-Singer Type Index Theorems for Manifolds with Corners and Splitting of η-Invariants I.*

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Abstract

We construct self-adjoint extensions of Dirac operators on manifolds with corners of codimension 2, which generalize the Atiyah-Patodi-Singer boundary condition. The boundary conditions are related to geometric constructions, which convert problems on manifolds with corners into problems on manifolds with boundary and wedge singularities. In the case, where the Dirac bundle is a super-bundle, we prove two general index theorems, which differ by the splitting formula for η-invariants. Further we work out the de Rham, signature and twisted spin complex in closer detail. Finally we give a new proof of the splitting formula for the η-invariant.

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0 Introduction

Let M be a Riemannian manifold with boundary and product structure around the boundary. Further let $E \mapsto M$ be a Dirac bundle over M (Definition 1.1.13) respecting the product structure. In [2], Atiyah, Patodi and Singer introduced global boundary conditions. If the dimension of M is even and a superstructure on $E = E_+ \oplus E_-$ is fixed, these boundary conditions give rise to the Atiyah-Patodi-Singer index theorem:

$$\mathrm{Index}(D_+) = \int_M a_D - \frac{1}{2}\eta(A_+, 0) + \frac{1}{2}\mathrm{tr}\,(S_+). \tag{0.0.1}$$

Here D_+ is the part of Dirac operator D associated to E mapping sections of E_+ into sections of E_-. A is the induced Dirac operator on the boundary, A_+ is the part of A mapping sections of E_+ to sections of E_+ and a_D are local formulas defined in the interior of M. The η-function $\eta(A_+, s)$ is the analytic continuation of

$$\eta(A_+, s) = \sum_{\lambda \in \mathrm{spec}(A_+)\setminus\{0\}} \mathrm{sign}(\lambda)|\lambda|^{-s}$$

from the part of $\mathbb{C}$ where the sum is convergent to all of $\mathbb{C}$. It is regular in 0 for all Dirac type operators on closed manifolds [16, Section 3.8]. The last term, $\frac{1}{2}\mathrm{tr}\,(S_+)$, depends on the augmentation of D, i.e. on the choice of boundary conditions in $\ker(A)$. Here we have stated it for a canonical choice given by the scattering matrix, introduced by Werner Müller in [31]. The Atiyah-Patodi-Singer (from now on APS) index theorem distinguishes itself by giving the correct index formula for special cases like the signature complex and by providing an index invariant $\mathcal{I}(E) := \mathrm{Index}(D_+) - \frac{1}{2}\mathrm{tr}\,(S_+)$, which is additive under gluing of manifolds along common boundaries.

For manifolds with corners much less is known. In particular nobody has so far given boundary conditions for manifolds with corners, which generalize the APS boundary conditions. However, for manifolds with corners of codimension 2 a number of special index theorems are known. We mention the Gauß-Bonnét theorem for surfaces with corners [12]. Further, for the signature complex the Wall non-additivity formula [42], which is a gluing formula for the signature independent of analytic index theory, is known. More recently Hassel-Mazzeo-Melrose [22] and Werner Müller [32] have proved index theorems for manifolds with corners. Both of the approaches build on an extension of M to a complete Riemannian manifold without boundary. The technical difficulty with that approach is that the continuous spectrum of the involved operators close to 0 has infinite multiplicity and is very difficult to study. The index theorems in [22] and [32] are equivalent to special cases of the index theorems in this paper. The vanishing of a term in the splitting formula for the η-invariant for the signature complex though only follows by combining the results of this paper with those of [22].

In this paper we take a different approach to index theory for manifolds with corners. Let in the following M be manifold with corners of codimension 2 and product structure in a neighborhood of the boundary and corners.

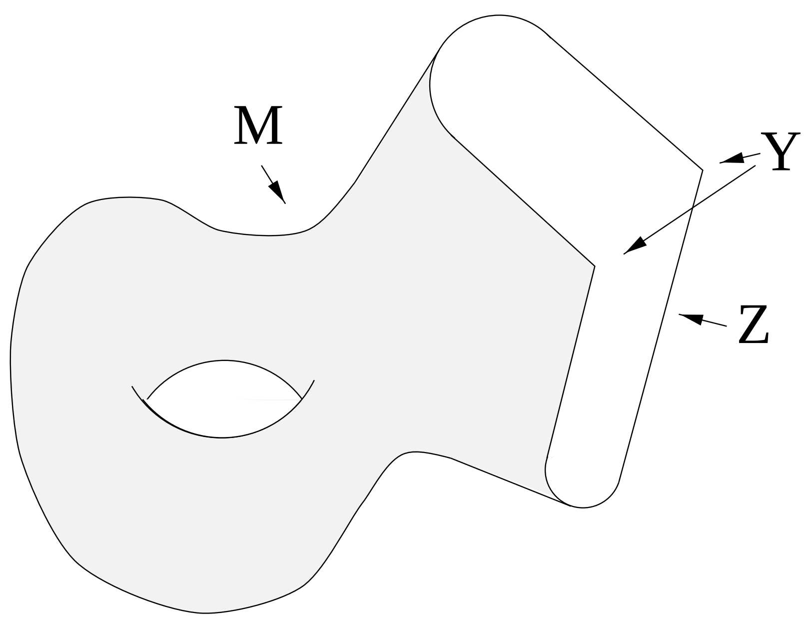

Fig1: A manifold M with boundary Z and corner Y.

A study of the structure of Z close to the corners shows that Z can be given a canonical smooth structure induced by the Riemannian metric. Next we form a cylinder $Z \times [0,1]$ and attach $Z \times \{0\}$ to M using the identity map on Z. This gives a Riemannian manifold $\tilde{M}$ with a smooth boundary and wedge singularities.

If $E \mapsto M$ is a Dirac bundle over M we prove in Lemma 1.1.14 that there always exists at least one extension of E to a Dirac bundle $\tilde{E} \mapsto \tilde{M}$. Let $\tilde{D}$ be the Dirac operator associated to $\tilde{E}$. Self-adjoint boundary conditions for D generalizing the APS boundary conditions can now be constructed in two steps. First we impose APS boundary conditions at $Z \times \{1\}$ and ideal boundary conditions in the wedge singularity of $\tilde{M}$ in order to get a self-adjoint extension of $\tilde{D}$. Next we use the self-adjointness of $\tilde{D}$ and an extension property of certain sections of E to construct a self-adjoint extension of D, which coincides with the APS extension if there are in fact no corners. The extension of D is further such that $\ker(D)$ is canonically isomorphic to $\ker(\tilde{D})$. We remark that this self-adjoint extension of D is by no means the only generalization of the APS boundary conditions. Different gluings of $\tilde{E}$ and different choices of ideal boundary conditions for $\tilde{D}$ give other extensions. In Section 5 we give a completely different extension based on a similar construction.

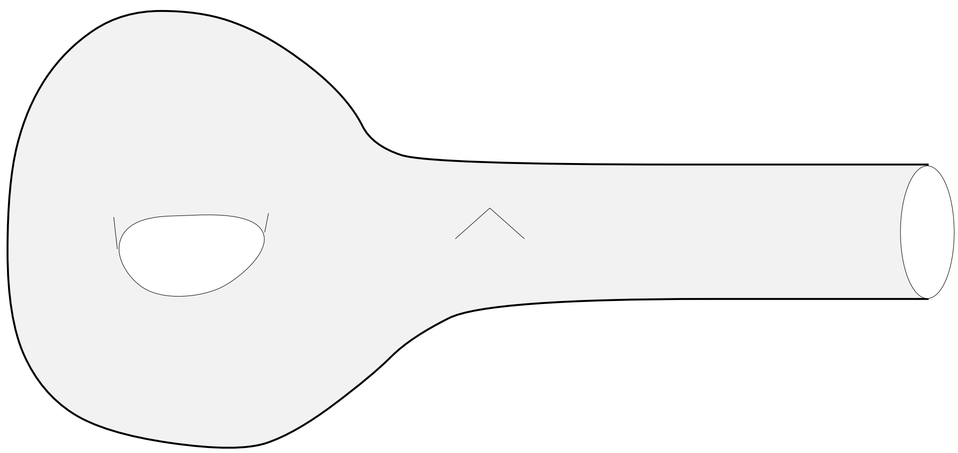

Fig2: The extension $\tilde{M}$ of M.

Since $\ker(D) \cong \ker(\tilde{D})$ we can proceed with the index theory by proving an index theorem for $\tilde{D}$. This can widely be done using standard methods and results from [2], [8], [9]. Working directly with D must be expected to be much harder since the complications from the boundary and the corner both appear in the same point, whereas they can be treated separately for $\tilde{D}$. The main results for general Dirac bundles are Theorem 4.2.3, Theorem 5.1.5 and its refinement Theorem 5.1.10. In Section 6.3 we further apply those theorems in order to give a new proof of the splitting formula for η-invariants.

In Section 3.3 we consider the de Rham and signature complexes. Like it is the case for manifolds with boundary, a well understood subspace $\ker_0(\tilde{D})$ of $\ker(\tilde{D})$ is isomorphic to the image of the relative cohomology in the absolute cohomology of M and the orthogonal complement of $\ker_0(\tilde{D})$ gives a vanishing contribution to the index. In particular the right hand side of the index theorem is the Euler characteristic and the signature, respectively. For those cases we can further work out some of the terms on the left hand side, and specializations of the index theorems give the well know

Gauß-Bonnet theorem (Theorem 6.2.2)

$$\chi(M) = \int_M e + \sum_Y \left(\frac{\pi - \sigma(Y)}{2\pi} \right) \chi(Y), \tag{0.0.2}$$

where Y runs over the corners and $\sigma(Y)$ denotes the interior angle at Y. Also the following formula for the signature for a manifold with corners, Theorem 6.2.1, is worked out

$$\operatorname{sign}(M) = \int_M L - \frac{1}{2}\eta(A_{Z+}, 0). \tag{0.0.3}$$

If all angles at the corners are $\frac{\pi}{2}$ this formula can be compared to the index formula of [22]. The only difference (except from a high number of different conventions) is that the η-invariant is split in [22] and together the theorems imply that the integer valued term in the splitting formula for the η-invariant vanishes in the case of the signature complex of an odd-dimensional boundary.

Another example, which covers all Dirac bundles, is a local twisted spin bundle. This example corresponds to the universal gluing of $\tilde{E}$ to be introduced in Lemma 1.1.14. For this example not much can be said about the right hand side, but the contribution to the index from the corner vanishes, and in this way Theorem 4.2.3 can be refined. In the case of the de Rham and signature complexes this index theorem differs from the Gauß-Bonnet and the signature theorems by a cut and paste formula for η-invariants.

Much of this work has appeared as preprints [37], [38]. In addition to the correction of some mistakes a number of changes in the theory have been done. The first and most noticeable change is that scattering theory on a manifold with cylindrical ends has been replaced by Section 2. In Section 2 what corresponds to the scattering matrix in 0 for a Dirac operator on a manifold with cylindrical ends is constructed for a manifold with boundary and wedge singularities. The advantage of this construction is that it does not use scattering theory and therefore the presentation becomes simpler. More important is however that it treats boundaries and wedge singularities equally. That means that we can make use of the trivial but important observation that *a boundary is the special case of a wedge singularity, where the conic part has a* 0*-dimensional base.* Where the ideal boundary conditions chosen in [37], [38] were somewhat arbitrary, we now have completely canonical boundary conditions determined by that they have to be a generalization of the APS boundary conditions as well as of *slow-growing ideal boundary conditions* (given by including the slowest growing local solutions of $\tilde{D}^2 f = -f$ in the domain of $\tilde{D}$) for a cone, and that the augmentation has to be given by the scattering matrix. This has tremendous advantages, the first of which is that the time-consuming process of considering different augmentations has now been made redundant. The most important is however that it gives a domain which is preserved by all operators, which satisfy commutation relations with certain underdefined realizations of $\tilde{D}$, and that it gives a self-adjoint extension at all in the case of odd-dimensional manifolds. In the generalization of APS boundary conditions to manifolds with corners of codimension 3 and 4, this gives a considerably increased performance, and already in this paper it leads to simplifications.

When we consider a wedge with product structure as a generalization of a boundary with product structure, it is natural to look for a generalization of the geometric operation of attaching a cylinder. In the case of a wedge it can not be done geometrically, but it can still be done operator theoretically. We do so in Section 3.2. In Section 3.3, where we work out the de Rham and signature complexes, the geometric extension still plays an important role. The significance of the operator theoretical extension will be clear when we consider manifolds with corners of codimension 3 and 4 and theory from this paper has to be iterated. What happens is that operator theoretical extensions are locally conjugate to Dirac operators on spaces with simpler singularity structure. Thus they can be used for specifying self-adjoint extensions similar to those appearing in this paper and for proving the corresponding index theorems. It turns out that the main approach to index theory in this paper and the approach given in Section 5 melt together in the sense that both are in use for specifying the same boundary conditions from a certain level of complication of corners and singularities.

Theorem 5.1.10 differs from Theorem 4.2.3 by that the η-invariant is split in Theorem 5.1.10. Depending on whether one holds the two index theorems together or not, the contribution from the corner in Theorem 5.1.10 can be considered either as the $\mathbf{m}$-term in the splitting formula for the η-invariant independent of the angle or as a term depending on the angle, which measures the dislocation of the scattering matrices from the various boundary components. The formula corresponds to that of [22] modulo $\mathbb{Z}$ in the case of the signature complex and angles $\frac{\pi}{2}$. The index term in Theorem 5.1.10 can not be interpreted cohomologically for the signature complex since it is not clear, whether harmonic sections are closed. Thus the comparison can only be carried out modulo $\mathbb{Z}$.

This approach generalizes to give index theorems for manifolds with corners of codimension 3 and 4. In codimension 4 we have though only worked it out in the case of the universal gluing of vector-bundles. Other gluings give rise to additional problems. The main result, which is proved so far, is an index theorem for manifolds with corners of codimension 3 and an associated splitting formula for η-invariants of odd-dimensional closed manifolds or manifolds with singularities into η-invariants of manifolds with corners. These results, both proved in [37], go further than the results obtained with other approaches.

1 A Boundary Value Problem.

1.1 Geometric Constructions Related to a Manifold with Corners.

First we will give our definition of a manifold with corners. The definition has been chosen such that it fits the methods used in this paper. More general definitions extending the definitions below to higher codimension have been given in [37]. Compared to [37] we have allowed ourselves to make more intensive use of group actions and covering spaces than it is possible in higher codimension. This leads to some simplifications. We will repeatedly use the fact that isometric homeomorphisms of Riemannian manifolds are automatically smooth [21, Theorem 11.1] and in this way reduce most proofs to proofs involving metric spaces only.

Let

$$\mathbb{R}^2_\times = \mathbb{R}^2 \setminus \{0\},$$

considered as a Riemannian manifold. We denote the universal covering by $\tilde{\mathbb{R}}^2_\times$. Then $\tilde{\mathbb{R}}^2_\times$ is a Riemannian manifold isometrically diffeomorphic to $(0,\infty) \times \mathbb{R}$ supplied with the Riemannian metric

$$g = dr^2 + r^2 d\theta^2.$$

There are two natural group actions on $\tilde{\mathbb{R}}^2_\times$. The cone structure and the lift of the group of rotations of $\mathbb{R}^2$ given by

$$\rho(s)(r,\theta) = (r, \theta + s) \quad ; \quad s \in \mathbb{R}.$$

The group action ρ acts by isometries.

Definition 1.1.1. A model corner of dimension 2, codimension 2 and product structure is the completion of a subset of $\tilde{\mathbb{R}}^2_\times$ of the form

$$C^\circ = \{(r,\theta) \in (0,\infty) \times \mathbb{R} \mid a < \theta < b\} \tag{1.1.1}$$

for some $a, b \in \mathbb{R}$ with $b > a$. The angle of C is the number $b - a \in \mathbb{R}_+$. The interior of C is the set C° defined in (1.1.1). A model corner $C \times Y$ of dimension n, codimension 2 and product structure is the completion of a Riemannian manifold of the form

$$C^\circ \times Y,$$

where C is a model corner of dimension 2, codimension 2 and product structure, and Y is a closed manifold. The angle of $C \times Y$ is the angle of C.

The category of model corners will be considered as a sub-category of the category of metric spaces with a cone structure. In particular isomorphisms of corners are isometric homeomorphisms.

Example 1.1.2. *An intersection of dimension 2 of two different half-planes in $\mathbb{R}^2$ is a model corner of codimension 2 and angle smaller than π. One half-plane in $\mathbb{R}^2$ is also a corner of codimension 2 and angle π. Notice that with Definition 1.1.1 there is no particular distinction between corners with angles smaller than π, equal to π or greater than π, whereas the spaces of smooth functions for the three cases are very different. This is our main reason for* not *working with smooth or C^k functions at all. Of the same reason we have chosen not to make use of the groupoid structure of a corner since it mainly captures phenomena, which will turn out to be irrelevant for our purposes.*

Definition 1.1.3. The boundary components of a model corner of codimension 2 are the completions of the subsets $\{\theta = a\}$ and $\{\theta = b\}$, where a and b are like in Definition 1.1.1.

Lemma 1.1.4. *Let $C_1 \times Y$ and $C_2 \times Y$ be model corners and let C be the space arising by identifying the image of a boundary component of C_1 with the image of a boundary component of C_2 (using the unique isometric homeomorphism). Then $C \times Y$ is a model corner.*

Proof: This is trivial with the definitions given. □

Definition 1.1.5. A closed model corner of codimension 1 is a space of the form

$$Z \times [0, \infty),$$

where Z is a closed Riemannian manifold.

Definition 1.1.6. A closed model cone C of dimension 2, angle σ and product structure is the completion of a Riemannian manifold of the form

$$C^\circ = \tilde{\mathbb{R}}^2_\times / \rho_\sigma,$$

where ρ_σ is the $\mathbb{Z}$-action on $\tilde{R}^2_\times$ given by

$$\rho_\sigma(k)(r, \theta) = (r, \theta + \sigma k) \quad ; \quad k \in \mathbb{Z}.$$

A closed model wedge of codimension 2 is a space of the form

$$C \times Y,$$

where C is a closed model cone of codimension 2 and Y is a closed Riemannian manifold.

Definition 1.1.7. A compact manifold M with corners and product structure around the boundary and corners is a compact connected metric space, such that for each $m \in M$, an open neighborhood U_m of m is isometrically homeomorphic to an open subset of a smooth Riemannian manifold, a model corner of codimension 1 or a model corner of codimension 2.

Remark 1.1.8. With Definition 1.1.7 the model space at a corner of a manifold with corners is a bundle of model corners over a closed manifold, supplied with a local product metric. Since, however, a model corner of codimension 2 allows only one self-isometry, which is orientation reversing, this bundle is trivial if M is orientable, such that in fact a neighborhood of a corner is isometric to a neighborhood of $\{0\} \times Y$ in a model corner $C \times Y$.

Definition 1.1.9. Let M be a compact manifold with corners of codimension 2 and product structure around the boundary and corners. Then:

An *open boundary component* N° of M is a maximal connected subset, such that a neighborhood of each $m \in N^\circ$ is isometrically homeomorphic to a neighborhood of $\{0\} \times \{z\}$ in a model corner $[0, \infty) \times Z$ of codimension 1.

An *intrinsic boundary component* is the completion of an open boundary component with respect to the induced Riemannian metric.

An *extrinsic boundary component* is the closure of an open boundary component in M.

The *boundary* ∂M of M is the union in M of the extrinsic boundary components.

Lemma 1.1.10. *Let M be a compact manifold with corners and product structure around the boundary and corners. Then the boundary Z of M has a canonical smooth structure such that Z is a smooth Riemannian manifold with the Riemannian metric given by the extension by continuity of the Riemannian metric induced on the open boundary components of M.*

Proof: An atlas is given on the open boundary components of M, so it suffices to consider the corners of codimension 2. If $C \times Y$ is a model corner, ∂C is the union of two half-lines with Riemannian metrics. Gluing those half-lines together gives that $\partial C \times Y$ is homeomorphic to $\mathbb{R} \times Y$, where $\mathbb{R}$ is considered as a Riemannian manifold. Further the homeomorphism is uniquely determined by the demand that its restriction to each half-line is an isometry. The smooth structure on $\mathbb{R}$ can now be pulled back to a smooth structure on ∂C and the pullback of the Riemannian metric on $\mathbb{R}$ is the extension by continuity of the Riemannian metrics on the boundary components of C.

Using the above, $\partial C \times Y$ is identified with the smooth manifold $\mathbb{R} \times Y$. Further these identifications extend the atlas on the open boundary components of M to all of ∂M. □

Definition 1.1.11. A compact manifold with boundary, closed wedge singularities and product structure around the boundary and the wedge singularities is a compact connected metric space such that some open neighborhood of each $m \in M$ is isometrically homeomorphic to an open subset of either a Riemannian manifold, a model corner of codimension 1 or a closed model wedge.

Lemma 1.1.12. *Let M be a compact manifold with corners of codimension 2. Then the space*

$$\tilde{M} := M \cup_Z Z \times [0,1] \tag{1.1.2}$$

is a compact manifold with boundary, closed wedge singularities of codimension 2 and product structure. The corners of M stand in bijective correspondence to the wedge singularities of $\tilde{M}$ and if a corner of M has angle σ, the corresponding wedge singularity has angle $\sigma + \pi$.

Proof: Clearly $Z \times \{1\}$ is a smooth boundary. Further the product structure gives that points in the open boundary components of M are mapped to interior points of $\tilde{M}$. Now consider a subset of M isometrically homeomorphic to $C_\varepsilon \times Y$, where $C_\varepsilon = \{(r,\theta) \in C \mid r < \varepsilon\}$ for a model corner $C = \overline{\{(r,\theta) \in \tilde{\mathbb{R}}^2_\times \mid a < \theta < b\}}$. The image of $\{0\} \times Y$ in $Z \times [0,1]$ has a neighborhood of the form $[0,\varepsilon) \times (-\varepsilon,\varepsilon) \times Y$. Further, a neighborhood of $\{0\} \times \{0\} \times Y \subset [0,\varepsilon) \times (-\varepsilon,\varepsilon) \times Y$ is isometrically homeomorphic to a neighborhood of $\{0\} \times Y$ in $C' \times Y$, where C' is the model corner

$$C' := \overline{\{(r,\theta) \in \tilde{\mathbb{R}}^2_\times \mid b < \theta < b + \pi\}}.$$

Further the identifications are such that $\{(r,\theta) \in C \mid \theta = b\}$ is identified with $\{(r,\theta) \in C' \mid \theta = b\}$ and $\{(r,\theta) \in C \mid \theta = a\}$ is identified with $\{(r,\theta) \in C' \mid \theta = b + \pi\}$. This space is isometrically homeomorphic to $\tilde{\mathbb{R}}^2_\times / \rho_{b-a+\pi}$, so $\{0\} \times Y$ is mapped to a closed wedge singularity. The remaining statements are clear from that. □

We notice that the interior of a model corner of dimension 2 and codimension 2 has trivial holonomy. In particular any tangent vector in a point can be extended to a globally defined parallel vector field on a model corner. If C is a model corner defined by (1.1.1) a number of tangent vectors are canonically given:

$$\delta_1 := \frac{\partial}{\partial r}_{|\theta=b,r=1} \quad ; \quad \delta_2 := -\frac{\partial}{\partial r}_{|\theta=a,r=1},$$
$$\nu_1 := \frac{\partial}{\partial \theta}_{|\theta=b,r=1} \quad ; \quad \nu_2 := -\frac{\partial}{\partial \theta}_{|\theta=a,r=1}.$$

All of those tangent vectors extend to globally defined parallel vector fields.

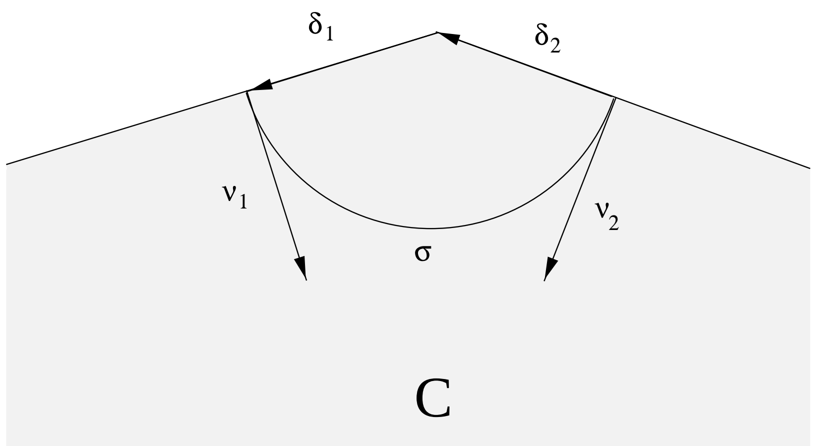

Fig3: A corner $C \subset \mathbb{R}^2$.

In the following we will assume that $E \mapsto M$ is a Dirac bundle over M. We recall the definition:

Definition 1.1.13. A vector bundle $E \mapsto M$ supplied with a Hermitian structure h and a Hermitian connection ∇ is a Dirac bundle if it is a module over the Clifford bundle Cliff(TM), such that if $\mathbf{c}$ denotes the structure of Clifford multiplication we have for all vector-fields X, Y and all smooth sections s_1 and s_2 of E

$$\nabla_Y \mathbf{c}(X) s_1 = \mathbf{c}(X) \nabla s_1 + \mathbf{c}(\nabla_Y X) s_1, \tag{1.1.3}$$
$$h(\mathbf{c}(X)s_1, s_2) = -h(s_1, \mathbf{c}(X)s_2). \tag{1.1.4}$$

If E is a Dirac bundle the associated Dirac operator is given by the composition

$$D := \mathbf{c} \circ g^{-1} \circ \nabla, \tag{1.1.5}$$

where $g \in C^\infty(End(TM, T^*M))$ is the Riemannian structure on M.

Further we will assume that E respects the structure near the boundary and the corners. That means that the local pullbacks of E to the model corners of codimension

1 and 2 are pullbacks of the restriction of $E_{|Z}$ and $E_{|Y'}$, respectively. And that the connection ∇ in E is of the form $du\frac{\partial}{\partial u} + \nabla^Z$ and $dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial \theta} + \nabla^Y$, respectively.

A bundle $TZ' = 1 \oplus TZ$ is given on Z. The trivial factor is mapped to the inward pointing normal on the open boundary components of M, such that over each open boundary component N° of M, $TZ'_{|N^\circ} \cong TM_{|N^\circ}$ canonically. In the corners TZ' is glued using the "gluing operator", which sends δ_2 into δ_1 and ν_2 into ν_1.

Since the gluing operators for TZ' are unitary, they induce canonical gluing operators for the Clifford bundle $\mathrm{Cliff}(TZ')$. We will construct gluing operators for $E_{|Z}$ at the corners, and in this way construct a vector-bundle $F \mapsto Z$, which is equal to E over each open boundary component, and which is a bundle of Clifford modules over $\mathrm{Cliff}(TZ')$. Such gluing operators are not necessarily uniquely determined. There is however a canonical choice.

Lemma 1.1.14. *Let V be a finite-dimensional real vector-space with an inner product $\langle .,. \rangle$. Let $U \subseteq V$ be a two-dimensional subspace, $\delta_1 \subseteq U$ be a unit vector and let*

$$W = \{\delta_2 \in U \mid |\delta_2| = 1 \text{ and } \langle \delta_1, \delta_2 \rangle > -1\}.$$

Then there exists a function, which we will denote by $\sqrt{-\delta_1\delta_2}$, defined on W and with values in $\mathrm{Spin}(V) \subseteq \mathrm{Cliff}(V)$, such that the following holds:

1) $\sqrt{-\delta_1\delta_2}$ is continuous with respect to δ_2 and $\sqrt{-\delta_1\delta_1} = 1$.

2) $\left(\sqrt{-\delta_1\delta_2}\right)^2 = -\delta_1\delta_2$ for all $\delta_2 \in W$.

3) $\delta_1\sqrt{-\delta_1\delta_2} = \sqrt{-\delta_1\delta_2}\,\delta_2$.

4) For e in the orthogonal complement of U in V we have

$$e\sqrt{-\delta_1\delta_2} = \sqrt{-\delta_1\delta_2}\,e.$$

The function $\sqrt{-\delta_1\delta_2}$ takes its values in $\mathrm{Spin}(U) \subseteq \mathrm{Spin}(V)$ and is uniquely determined by the properties 1), 2), 3) and 4). Further, it satisfies the following extra conditions:

5) $\sqrt{-\delta_1\delta_2}$ commutes with $\delta_1\nu_1$.

6) The eigenspaces of $\sqrt{-\delta_1\delta_2}$ coincide with the eigenspaces of $\delta_1\nu_1$. The eigenvalues of $\sqrt{-\delta_1\delta_2}$ are given by

$$\left\{ \begin{array}{ll} e^{i\frac{\pi-\sigma}{2}} & ; \text{on the } i \text{ eigenspace of } \delta_1\nu_1 \\ e^{-i\frac{\pi-\sigma}{2}} & ; \text{on the } -i \text{ eigenspace of } \delta_1\nu_1 \end{array} \right\},$$

where σ denotes the angle between δ_1 and $-\delta_2$.

We call $\sqrt{-\delta_1\delta_2}$ the universal gluing operator.

Proof: If we write

$$\delta_1 = \cos(\pi-\sigma)\delta_2 + \sin(\pi-\sigma)\nu_2,$$

it follows that

$$-\delta_1\delta_2 = \cos(\pi-\sigma) - \sin(\pi-\sigma)\nu_2\delta_2 = e^{-(\pi-\sigma)\nu_2\delta_2}.$$

Consequently, for $\pi-\sigma \in (-\pi,\pi)$ a canonical square root of $-\delta_1\delta_2$ is given by

$$\sqrt{-\delta_1\delta_2} := e^{-\frac{\pi-\sigma}{2}\nu_2\delta_2} = e^{\frac{\pi-\sigma}{2}\delta_1\nu_1}. \tag{1.1.6}$$

Computations similar to above give that

$$\sqrt{-\delta_1\delta_2} = -\left\{\cos(\frac{\pi-\sigma}{2})\delta_2 + \sin(\frac{\pi-\sigma}{2})\nu_2\right\}\delta_2 \in \mathrm{Spin}(U) \subseteq \mathrm{Spin}(V).$$

The properties 1), 2), 3) and 4) are now easily checked. It remains to prove the uniqueness part: Assume that $\omega \in \mathrm{Spin}(V)$ is another element satisfying 3) and 4). Then conjugation by $\omega^{-1}\sqrt{-\delta_1\delta_2}$ induces the identity on $U^\perp \oplus \mathrm{span}(\delta_2)$. Since conjugation by elements of $\mathrm{Spin}(V)$ give rise to unitary operators with determinant 1 it follows that conjugation by $\omega^{-1}\sqrt{-\delta_1\delta_2}$ induces the identity on V. But then $\omega^{-1}\sqrt{-\delta_1\delta_2}$ is in the centre of $\mathrm{Cliff}(V)$. Since $\omega^{-1}\sqrt{-\delta_1\delta_2}$ also belongs to $\mathrm{Spin}(U)$, which has centre ± 1, it follows that $\omega = \pm\sqrt{-\delta_1\delta_2}$. The uniqueness now follows from 1).

The claims 5) and 6) are satisfied by construction of $\sqrt{-\delta_1\delta_2}$. This proves the lemma. □

Remark 1.1.15. We define $\sqrt{-\delta_1\delta_2}$ by (1.1.6) for $\sigma \in (0,\infty)$. With this convention 1), 2), 3), 4), 5) and 6) remain valid. What is not true is that $\sqrt{-\delta_1\delta_2}$ is a globally defined function of δ_1 and δ_2. Instead it must be considered as a function of σ.

Denote by $R : \mathrm{Spin}(\mathbb{R}^n) \mapsto SO(\mathbb{R}^n)$ the covering homomorphism. We notice that $R(\sqrt{-\delta_1\delta_2})$ is the identification map for TZ' at the corner, which identifies δ_2 with δ_1 and ν_2 with ν_1. We can now in the same way construct the restriction of a bundle F over Z by at each corner identifying $\omega \in E_{|\{(0,a)\}\times Y}$ with $\sqrt{-\delta_1\delta_2}\,\omega \in E_{|\{(0,b)\times Y}$. Then the computation

$$R(\sqrt{-\delta_1\delta_2})(v)\sqrt{-\delta_1\delta_2}\omega = \sqrt{-\delta_1\delta_2}\,v\left(\sqrt{-\delta_1\delta_2}\right)^{-1}\sqrt{-\delta_1\delta_2}\omega \tag{1.1.7}$$

$$= \sqrt{-\delta_1\delta_2}v\omega \tag{1.1.8}$$

shows that F is a bundle of Clifford modules over TZ'. Since the connection commutes with Clifford multiplication, F can also be given the connection from E. Notice that when F is considered as a Clifford bundle over $\mathrm{Cliff}(TZ)$, rather than over $\mathrm{Cliff}(TZ')$, the structure $\mathbf{b}$ of Clifford multiplication is given by $\mathbf{b}(\gamma) = \nu\gamma$, where ν denotes the inward pointing normal at ∂M.

In neighborhoods of small open subsets of each open boundary component Z_i of Z, D takes the form

$$D = -\nu\frac{\partial}{\partial u} + D_{Z_i} = -\nu(\frac{\partial}{\partial u} + A).$$

Here $A = \nu D_{Z_i}$ is the Dirac operator on $F \mapsto Z$ coming from the structure of F as a Clifford module over TZ. There is no globally defined Dirac operator on $E \mapsto Z$, but by construction of F, the Dirac operators D_{Z_i} glue together to a Dirac operator D_Z on $F \mapsto Z$. By construction of F the operator of Clifford multiplication by ν is a smooth section in $End(F)$, which anti-commutes with D_Z. Thus also the operator $A := \nu D_Z$ is well defined as an operator on $F \mapsto Z$.

There is an extension of E to a Clifford-bundle $\tilde{E}$ on $\tilde{M}$ by letting $\tilde{E}_{|Z\times[0,1]}$ be the pullback of F. The connection and the Hermitian structure extend by the product structure. Finally, Clifford multiplication in the direction of the last variable is provided by the operator $-\nu$.

1.2 Analysis on a Cone.

In this section we will consider a number of different Sobolev spaces over two-dimensional cones.

Let $V \mapsto X$ be a Hermitian vector-bundle over a Riemannian manifold supplied with a connection ∇. Then we define

$$W^{2,k}(X,V) := \{f \in L^2(X,V) \mid \forall i = 0,\dots,k : \nabla^i f \in L^2(X,(T^*M)^{\otimes i} \otimes V)\},$$
$$W_0^{2,k}(X,V) := \text{Closure of } C_0^\infty(X,V) \text{ in } W^{2,k}(X,V).$$

If V is further a Dirac bundle and D is the associated Dirac operator, we may define

$$H^k(X,V) := \{f \in L^2(X,V) \mid \forall i = 0,\dots,k : D^i f \in L^2(X,V)\},$$
$$H_0^k(X,V) := \text{Closure of } C_0^\infty(X,V) \text{ in } H^k(X,V).$$

The following inclusions are standard:

Lemma 1.2.1. *We have*

$$\begin{aligned} W_0^{2,k}(X,V) &\subseteq W^{2,k}(X,V), \\ H_0^k(X,V) &\subseteq H^k(X,V), \\ W_0^{2,k}(X,V) &\subseteq H_0^k(X,V), \\ W^{2,k}(X,V) &\subseteq H^k(X,V). \end{aligned}$$

If the curvature term R occurring in the Weizenböck formula is bounded, we further have

$$W_0^{2,1}(X,V) = H_0^1(X,V).$$

Proof: The first two inclusions hold by definition. The next two by the expression $D = \mathbf{c}g^{-1}\nabla$, where $\mathbf{c}$ is the structure of Clifford multiplication and g is the metric. In the last equation the inclusion $\subseteq$ is already clear. The other follows by the Weizenböck formula in the following way: For $f, g \in C_0^\infty(X,V)$

$$\begin{aligned} |\langle \nabla f, \nabla g\rangle| &= |\langle \nabla^*\nabla f, g\rangle| \\ &= \left|\left\langle (D^2 - R)f, g\right\rangle\right| \\ &\le |\langle Df, Dg\rangle| + \|R\|_\infty \, |\langle f, g\rangle| \,. \end{aligned}$$

This implies equivalence of the norms on $W_0^{2,1}$ and H_0^1. □

Now let $\mathbb{R}_\times^{2,\sigma} = \tilde{\mathbb{R}}_\times^2/\rho_\sigma$, where ρ_σ is defined in Definition 1.1.6. Let $\mathcal{E}$ be a Clifford module over Cliff($\mathbb{R}^2$) and let $\mathcal{E}_1$ be the corresponding Clifford bundle over $\tilde{R}_\times^2$, arising by trivializing $T\tilde{R}_\times^2$ by the trivial holonomy of $\tilde{R}_\times^2$. The bundle $\mathcal{E}_1$ is a Dirac bundle over $\tilde{R}_\times^2$ if it is supplied with the pullback of the trivial connection $dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}$ to $\tilde{R}_\times^2$. Let U be a unitary operator on $\mathcal{E}$ such that $\mathcal{E}_1/(\rho_{\sigma,U})$ is a Dirac bundle over $\mathbb{R}_{\times,\sigma}^2$. Here $\rho_{\sigma,U}$ is given by

$$\rho_{\sigma,U}(k)((r,\theta,e)) = (r, \theta + k\sigma, U^k e).$$

We denote the resulting Dirac bundle $\mathcal{E}_1/(\rho_{\sigma,U})$ by E.

Lemma 1.2.2. *We have*

$$W_0^{2,1}(\mathbb{R}_\times^{2,\sigma}, E) = W^{2,1}(\mathbb{R}_\times^{2,\sigma}, E).$$

Proof: First consider E in the case $\sigma = 2\pi$. In this case Euclidean coordinates (x,y) are globally well-defined.

First we prove that $W^{2,1}(\mathbb{R}_\times^2, E) \cap L^\infty(\mathbb{R}_\times^2, E)$ is dense in $W^{2,1}(\mathbb{R}_\times^2, E)$. Let $v_1, \dots, v_m$ be an orthonormal basis of eigenvectors of U. Then an orthogonal basis of eigenvectors of the operator $-i\frac{\partial}{\partial\theta}$ on $L^2(S^1, E_{|S^1})$ is given by $\left\{e^{is_{k,q}\theta}v_q\right\}_{q=1,\dots,m;k\in\mathbb{Z}}$ for some discrete sequences $\{s_{k,q}\}_{k\in\mathbb{Z}}$ of eigenvalues. Every section $f \in W^{2,1}(\mathbb{R}_\times^2, E)$ can be split into a $W^{2,1}$-orthogonal sum

$$f(r,\theta) = \sum_{k,q} f_{k,q}(r)e^{is_{k,q}\theta}v_q.$$

By orthogonality each term $f_{k,q}(r)e^{is_{k,q}\theta}v_q$ belongs to $W^{2,1}$ and the sum is $W^{2,1}$-convergent. It thus suffices to prove that each section of the form $f_{k,q}(r)e^{is_{k,q}\theta}v_q$ can be approximated by bounded $W^{2,1}$-sections with respect to the $W^{2,1}$-norm.

The function $f_{k,q}$ belongs to $W^{2,1,loc}((0,\infty))$ and is therefore continuous, such that in particular it is everywhere defined. Now define for $n \in \mathbb{N}$:

$$f_{k,q,n}(r) := \max\{\min\{\mathrm{Re}(f_{k,q}(r)), n\}, -n\} + i\max\{\min\{\mathrm{Im}(f_{k,q}(r)), n\}, -n\}.$$

Then

$$f_{k,q,n}(r)e^{is_{k,q}\theta}v_q \overset{n\to\infty}{\longrightarrow} f_{k,q}(r)e^{is_{k,q}\theta}v_q$$

with respect to $\|\cdot\|_{W^{2,1}}$.

Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function, which is identically equal to 1 in a neighborhood of 0. Further, let $f \in W^{2,1}(\mathbb{R}_\times^2, E) \cap L^\infty(\mathbb{R}_\times^2, E)$. The estimate

$$\begin{aligned}\int_{\mathbb{R}^2} \left|\left(\frac{\partial}{\partial x}\varphi\left(n\sqrt{x^2+y^2}\right)\right) f(x,y)\right|^2 + \left|\left(\frac{\partial}{\partial y}\varphi\left(n\sqrt{x^2+y^2}\right)\right) f(x,y)\right|^2 dxdy \\ = n^2\int_{\mathbb{R}^2} \frac{x^2+y^2}{x^2+y^2}\left|\varphi'\left(n\sqrt{x^2+y^2}\right)\right|^2 |f(x,y)|^2\, dxdy \le C_{f,\varphi}\end{aligned}$$

and some trivial estimates shows that $\{\varphi(n\sqrt{x^2+y^2})f\}_{n\in\mathbb{N}}$ is bounded in $W^{2,1}$ for $n \to \infty$. Further, for $g \in W^{2,1}(\mathbb{R}^2_\times, E) \cap L^\infty(\mathbb{R}^2_\times, E)$ we get

$$\left| \int_{\mathbb{R}^2} \left(\frac{\partial}{\partial x} \varphi(n\sqrt{x^2+y^2}) \right) \langle f(x,y), g(x,y) \rangle \, dxdy \right| \leq$$

$$\int_{\mathbb{R}^2} \left| \frac{nx}{\sqrt{x^2+y^2}} \right| \left| \varphi'(n\sqrt{x^2+y^2}) \right| |f(x,y)| \, |g(x,y)| \, dxdy \to 0.$$

Thus $\{\varphi(n\sqrt{x^2+y^2})f\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{2,1}$ converging weakly towards zero with respect to a dense subset of $W^{2,1}$. It follows that it is in fact weakly convergent towards zero. Consequently

$$\left(1 - \varphi(n\sqrt{x^2+y^2})\right) f(x,y)$$

converges $W^{2,1}$-weakly towards f for $n \to \infty$. Now, sections with support away from 0 can be approximated in $W^{2,1}$ by sections in $C_0^\infty(\mathbb{R}^2_\times, E)$. It follows that the $W^{2,1}$-weak closure of $C_0^\infty(\mathbb{R}^2_\times, E)$ is all of $W^{2,1}(\mathbb{R}^2_\times, E)$. But the weak and the strong closure of a subspace always coincide. This proves the lemma in the special case.

Now, $\mathbb{R}^{2,\sigma}_\times$ is diffeomorphic to $\mathbb{R}^2_\times$ through a diffeomorphism, whose differential and inverse differential are bounded. From that it follows that the spaces $W^{2,1}$ and $W_0^{2,1}$ are preserved. Thus the lemma holds for all σ. □

In the special case $\sigma = 2\pi$, $U = 1$ we know that the Dirac operator is self-adjoint on the domain $W^{2,1}(\mathbb{R}^{2,\sigma}_\times, E) = W^{2,1}(\mathbb{R}^2, \mathbb{C}^m)$. This is not so in general. Instead we will have to introduce ideal boundary conditions in order to get a self-adjoint extension.

Consider the restriction E_r of E to the circle $N_r = r \cdot (\sigma\mathbb{Z} \setminus \mathbb{R})$. We write N for N_1. Let ν denote the operator of Clifford multiplication by $\frac{\partial}{\partial r}$ and let δ denote the operator of Clifford multiplication by $\frac{\partial}{\partial \theta}$. Operators

$$B_N := -\nu\delta \frac{\partial}{\partial \theta} - \frac{1}{2} \tag{1.2.1}$$

and

$$\tau_N := -\frac{i}{r}\nu\delta \tag{1.2.2}$$

are defined in $L^2(N_r, E_r)$. τ_N is the canonical involution on N_r given by a multiple of the image of the volume form in the Clifford bundle. It is at the same time equal to $-\tau_{\mathbb{R}^{2,\sigma}_\times}$, where the orientation on $\mathbb{R}^{2,\sigma}_\times$ has been taken such that $(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$ is an ordered frame of $T\mathbb{R}^{2,\sigma}_\times$. In particular it is parallel with respect to the connections on both N and $\mathbb{R}^{2,\sigma}_\times$, and the dimensions of the ± 1 eigenspaces of τ_N are both equal to $\frac{\dim(E)}{2}$.

The operators B_N, U and τ_N commute. Thus B_N and τ_N preserve the eigenspaces of U. Let $\alpha \in S^1$ vary over the eigenvalues of U. Then E_r splits into eigenbundles E_α to the α eigenvalues of U. Let $v_1, \ldots, v_p$ be an orthonormal basis of eigenvectors of τ_N in the α-eigenspace of U. Then

$$\left\{ e^{\nu\delta \frac{2\pi k + i\log(\alpha)}{\sigma}\theta} v_q \right\}_{q=1,\ldots,p;k\in\mathbb{Z}} \tag{1.2.3}$$

is an orthonormal basis of eigenvectors for B_N to the eigenvalues $\left\{\frac{2\pi k + i\log(\alpha)}{\sigma} - \frac{1}{2}\right\}$. This basis is independent on the branch of the logarithm used, though the indexing depends on the branch of the logarithm. Let

$$s_{k,\alpha} = \frac{2\pi k + i\log(\alpha)}{\sigma}. \tag{1.2.4}$$

Then $\{s_{k,\alpha}\}$ are the eigenvalues of $B_N + \frac{1}{2}$.

Remark: $\frac{1}{r}B_N$ is the induced Dirac operator on the sub-manifold N_r. It anti-commutes with ν. Since τ_N commutes with B_N and anti-commutes with ν we see that τ_N maps $\ker(B_N)$ to itself and that ν gives a symplectic structure on $\ker(B_N)$, for which $\ker(\tau_N - 1)$ is a Lagrangian subspace.

Definition 1.2.3. If $s_{k,\alpha} \neq \frac{1}{2}$ for all k and α, let $\sigma_1, \dots, \sigma_q$ be the values of $s_{k,\alpha}$ for which $s_{k,\alpha} \in (0, \frac{1}{2})$, counted with multiplicity and let $\varphi_1, \dots, \varphi_q$ be a corresponding orthonormal basis of eigensections. Then we set

$$D^{2,1}(\mathbb{R}^{2,\sigma}_{\times}, E) := W^{2,1}(\mathbb{R}^{2,\sigma}_{\times}, E) \oplus \operatorname{span}\left\{K_{\sigma_j}(r)\varphi_j \mid j = 1, \dots, q\right\}, \tag{1.2.5}$$

where K_s denotes the $s'th$ K-Bessel function.

If some $s_{k,\alpha}$ equals $\frac{1}{2}$, i.e. $\ker(B_N) \neq 0$, let W be a Lagrangian subspace of $\ker(B)$, which is a direct sum of subspaces of the eigenspaces of τ. Let $\varphi_1, \dots, \varphi_q$ and $\sigma_1, \dots, \sigma_q$ be like above, let $\varphi_{q+1}, \dots, \varphi_{q'}$ be a basis of W and let $\sigma_{q+1} = \cdots = \sigma_{q'} = \frac{1}{2}$. We set

$$D^{2,1}_W(\mathbb{R}^{2,\sigma}_{\times}, E) := W^{2,1}(\mathbb{R}^{2,\sigma}_{\times}, E) \oplus \operatorname{span}\{K_{\sigma_j}(r)\varphi_j \mid j = 1, \dots, q'\}. \tag{1.2.6}$$

We will use the terminology that we *augment* with respect to a Lagrangian subspace if we take that subspace as W and that we *augment* with respect to a self-adjoint involution ρ, if we take $W = \ker(\rho - 1)$.

It will often be convenient to write $D^{2,1}_W$ instead of $D^{2,1}$, also when $\ker(B_N) = \{0\}$. In this case $W = \{0\}$. We will prove that D is self-adjoint on $D^{2,1}_W(\mathbb{R}^{2,\sigma}_{\times}, E)$. In the case, where $\sigma = 2\pi$ and $U = 1$, $B_N + \frac{1}{2}$ has no eigenvalues in $[-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$, $D^{2,1} = W^{2,1}$ and we already know that D is self-adjoint on $W^{2,1}(\mathbb{R}^2, E)$. The general case requires that we compute the defect indices of $(D_\sigma, W^{2,1}(\mathbb{R}^{2,\sigma}_{\times}, E))$.

Assume that we have a solution $f \in L^2(\mathbb{R}^{2,\sigma}_{\times}, E)$ of the equation

$$(D^* \pm i)f = 0.$$

Then it follows

$$((D^*)^2 + 1)f = (D^* \mp i)(D^* \pm i)f = 0.$$

Thus we have the distributional equation

$$(\Delta + 1)f = 0,$$

where $\Delta = D^2$. In polar coordinates (r, θ), Δ takes the form $\Delta = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. Now assume f is a L^2-solution of the equation

$$(\Delta + \lambda^2)f = 0$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$. Since Δ commutes with U we may consider the components f_α of f, which take their values in $C^\infty(\mathbb{R}^{2,\sigma}_\times, E_\alpha)$, separately. It follows by local elliptic regularity that $f_\alpha \in C^\infty(\mathbb{R}^{2,\sigma}_\times, E_\alpha)$ and thus that f_α has an expansion of the form

$$f_\alpha(r,\theta) = \sum_{q=0}^{p} \sum_{k=-\infty}^{\infty} g_{k,q}(r) e^{\nu\delta s_{k,\alpha}\theta} v_q \, , \tag{1.2.7}$$

where the sum is convergent in the C^∞-topology and the coefficient functions $g_{k,q}$ are smooth. The coefficient functions are solutions of the ordinary differential equations

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s_{k,\alpha}^2}{r^2} + \lambda^2\right) g_{k,q}(r) = 0. \tag{1.2.8}$$

Let φ_s be a solution of the equation

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s^2}{r^2} + 1\right) \varphi_s(r) = 0. \tag{1.2.9}$$

Then we may compute

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s^2}{r^2} + \lambda^2\right) \varphi_s(\lambda r) = \lambda^2 \left(-\varphi_s''(\lambda r) - \frac{1}{\lambda r}\varphi_s'(\lambda r) + \left(\frac{s^2}{\lambda^2 r^2} + 1\right)\varphi_s(\lambda r)\right) = 0.$$

It follows that $\varphi_s(\lambda r)$ is a solution of (1.2.8) if $s^2 = s_{k,\alpha}^2$. The equation (1.2.9) is known to have the two-dimensional solution space spanned by the Bessel functions $I_s(r)$ and $K_s(r)$. In the following we will concentrate on the special case where $\lambda = 1$.

For $r \to \infty$, $I_s(r)$ has the following asymptotic expansion [14]

$$I_s(r) = e^{-i\frac{\pi}{2}s} J_s(ir) \sim e^{-i\frac{\pi}{2}s} \sqrt{\frac{2}{\pi r}} \cos(ir - \frac{\pi}{2}s - \frac{\pi}{4}).$$

The other solution, K_s, is known to have the asymptotic expansion [43, 7.23]

$$\begin{aligned} K_s(r) &\sim \sqrt{\tfrac{\pi}{2}} r^{-\frac{1}{2}} e^{-r} \quad ; r \to \infty, \\ K_s(r) &\sim 2^{|s|-1}, (|s|) r^{-|s|} \quad ; r \to 0 \quad \text{for } s \neq 0, \\ K_0(r) &\sim -\log(r) \quad ; r \to 0. \end{aligned} \tag{1.2.10}$$

By the asymptotic behaviour for $r \to \infty$ it follows that the I_s component of $g_{k,q}$ must vanish in order for f to be in $L^2(\mathbb{R}^{2,\sigma}_\times, E)$. Further, for $|s| \in [1,\infty)$ the integral $\int_0^1 r^{-2|s|} r dr = \int_0^1 r^{1-2|s|} dr$ is divergent. Thus $g_{k,q} = 0$ for all k such that $|s_{k,\alpha}| \in [1,\infty)$. In particular the sum (1.2.7) is finite.

By [43, 3.71] we have for all s:

$$\frac{d}{dr} K_s(r) = -\frac{s}{r} K_s(r) - K_{s-1}(r).$$

Thus we may compute

$$(D^* f_\alpha)(r,\theta) = \nu\left(\frac{\partial}{\partial r} - \frac{1}{r}\nu\delta\frac{\partial}{\partial\theta}\right)\left(\sum_k\sum_q c_{k,q}K_{s_{k,\alpha}}(r)e^{\nu\delta s_{k,\alpha}\theta}v_q\right)$$

$$= \nu\left\{\sum_k\sum_q c_{k,q}\left(\frac{\partial K_{s_{k,\alpha}}(r)}{\partial r} + \frac{s_{k,\alpha}}{r}K_{s_{k,\alpha}}(r)\right)e^{\nu\delta s_{k,\alpha}\theta}v_q\right\}$$

$$= -\nu\left\{\sum_k\sum_q c_{k,q}K_{s_{k,\alpha}-1}(r)e^{\nu\delta s_{k,\alpha}\theta}v_q\right\}.$$

By orthogonality and the identity $K_{-s}(r) = K_s(r)$ [43, 3.71] we see that for this to be a solution to $(D^* \pm i)f_\alpha = 0$ we must have that whenever

$$e^{\nu\delta s_{k,\alpha}\theta}v_q$$

is an eigensection of $B_N + \frac{1}{2}$ to the eigenvalue $s_{k,\alpha}$ then $-\nu e^{\nu\delta s_{k,\alpha}\theta}v_q$ is an eigensection of $B_N + \frac{1}{2}$ to the eigenvalue $1 - s_{k,\alpha}$. Since ν anti-commutes with B_N this is indeed so for all v_q. It follows by orthogonality that the solutions to $(D^* \pm i)g = 0$ are spanned by vectors of the form

$$K_{s_{k,\alpha}}(r)e^{\nu\delta s_{k,\alpha}\theta}v_q \pm iK_{1-s_{k,\alpha}}(r)e^{\nu\delta s_{k,\alpha}\theta}\nu v_q. \tag{1.2.11}$$

This excludes $s_{k,\alpha} \in (-1,0]$ since for $s_{k,\alpha} \in (-1,0]$, $1 - s_{k,\alpha} \geq 1$, so that $s_{k,\alpha}$ can not give rise to an L^2-solution of $(D^* \pm i)g = 0$ of the type (1.2.11). Thus we are left with $s_{k,\alpha} \in (0,1)$. In this case (1.2.11) is indeed a L^2-solution to $(D^* \pm i)g = 0$. Further we see that $\dim(\ker(D^* + i)) = \dim(\ker(D^* - i))$, so that D has self-adjoint extensions. Each self-adjoint extension is given by adding a Lagrangian subspaces of the symplectic form

$$\langle D^* f, g\rangle - \langle f, D^* g\rangle$$

on $\ker(D^* - i) \oplus \ker(D^* + i)$ to $\mathcal{D}(D)$. The identity

$$\langle D^* f, g\rangle = \langle f, D^* g\rangle - \lim_{r\to 0}\int_{N_r}\langle f, \nu g\rangle$$

gives that this is exactly the Lagrangian subspaces for the quadratic form $\langle \nu f, g\rangle$. We observe that the space

$$W_0 := \operatorname{span}\left\{K_{s_{k,\alpha}}(r)e^{\nu\delta s_{k,\alpha}\theta}v_q\right\}_{s_{k,\alpha}\in(0,\frac{1}{2})}$$

is a Lagrangian subspace for the restriction of ν to the space

$$\operatorname{span}\left\{K_{s_{k,\alpha}}(r)e^{\nu\delta s_{k,\alpha}\theta}v_q\right\}_{s_{k,\alpha}\in(0,\frac{1}{2})\cup(\frac{1}{2},1)}.$$

If a Lagrangian subspace W of $\ker(B_N)$ is added to W_0, $W \oplus W_0$ is a Lagrangian subspace of $\ker(D^* - i) \oplus \ker(D^* + i)$. This proves that D is self-adjoint on $D_W^{2,1}$.

We will end this discussion by noticing that since $\nu\delta$ commutes with any superstructure on E, *the domain* $D_W^{2,1}$ *is compatible with any superstructure on* E *if and only if* W *is compatible with the superstructure.*

We state the main results, which are well known from [8], [9], as a lemma:

Lemma 1.2.4. *Let* D *be the Dirac operator on a cone over* S^1. *Then* D *is self-adjoint on* $D_W^{2,1}(\mathbb{R}_\times^{2,\sigma}, E)$. *Further, for any superstructure* $E = E_+ \oplus E_-$ *into* ± 1 *eigenspaces of an involution* ρ *anti-commuting with Clifford multiplication,* $(D, D_W^{2,1})$ *anti-commutes with* ρ *if and only if* W *splits into* $W = W_+ \oplus W_-$, *where* $W_+ \subset C^\infty(E_+)$ *and* $W_- \subset C^\infty(E_-)$. □

We remark that Lemma 1.2.4 holds with minimal modifications on a cone over any closed manifold.

1.3 Analysis on a Wedge.

In the following Y will denote a closed Riemannian manifold supplied with a Hermitian vector-bundle $E_{|Y}$ and a Hermitian connection ∇^Y on $E_{|Y}$.

A vector bundle $\tilde{E}$ is defined on $\tilde{\mathbb{R}}_\times^2 \times Y$ by taking the pullback through the projection on the second component of the product.

The connection, metric and Hermitian structure on $\tilde{\mathbb{R}}_\times^2 \times Y$ is given by the product structure. For the structure of Clifford multiplication we will assume that there is a structure on $\tilde{E}$ as a Dirac bundle. The associated Dirac operator we will denote by $\tilde{D}$.

In polar coordinates (r, θ), $\tilde{D}$ can be written:

$$\tilde{D} = \nu\frac{\partial}{\partial r} + \frac{1}{r}\delta\frac{\partial}{\partial\theta} + D_Y = \nu\left(\frac{\partial}{\partial r} + \frac{1}{r}B_N + \frac{1}{2r} + B_Y\right), \tag{1.3.1}$$

where D_Y is a Dirac operator on $E_{|Y}$ defined with respect to the structure of Clifford multiplication on $\mathbb{R}_\times^{2,\sigma}$. The operators B_N and B_Y are the induced Dirac operators on N and Y with respect to the structure of Clifford multiplication induced on $N \times Y$ from the structure of Clifford multiplication on $\mathbb{R}_\times^{2,\sigma} \times Y$.

Let $U \in C^\infty(End(E_{|Y}))$ be a unitary section and let ρ_σ and $\rho_{U,\sigma}$ be given by:

$$\rho_\sigma(k)(r,\theta,y) = (r, \theta + \sigma k, y) \quad ; k \in \mathbb{Z},$$
$$\rho_{U,\sigma}(k)((r,\theta,y),e) = ((r,\theta+\sigma k, y), U^k e) \quad ; k \in \mathbb{Z}.$$

A vector bundle E over $\mathbb{R}_\times^{2,\sigma} \times Y$ is defined by $E := \rho_{U,\sigma} \setminus \tilde{E}$. We will assume that U is such that E is a Dirac bundle. Let D denote the Dirac operator on E.

Since δ, ν and U commute with B_Y^2, for each $\mu \in \operatorname{spec}(B_Y)$, a Dirac bundle F_{μ^2} over $\mathbb{R}_\times^{2,\sigma}$ is given as the direct sum of the eigenspaces

$$F_{\mu^2} := E_\mu \oplus E_{-\mu},$$

where E_μ is the eigenspace of B_Y to the eigenvalue μ, supplied with the Hermitian structure induced by the inner product on $L^2(Y, E_{|Y})$. Further, B_Y acts by an element B_μ of $End(F_{\mu^2})$ of operator norm μ.

The restriction D_{μ^2} of $\tilde{D}$ to F_{μ^2} is of the form

$$D_{\mu^2} = D_0 + B_{\mu^2},$$

where D_0 is an operator of the form from Lemma 1.2.4. Thus D_0 is self-adjoint on $D^{2,1}_W(\mathbb{R}^{2,\sigma}_\times, F_{\mu^2})$. Since B_{μ^2} is bounded and symmetric, also D_{μ^2} is self-adjoint on $D^{2,1}_W(\mathbb{R}^{2,\sigma}_\times, F_{\mu^2})$.

For $\mu^2 \neq 0$ the operator B_Y gives a canonical choice of augmentation for D_μ. We exploit this to define self-adjoint ideal boundary conditions for D up to the finite dimensional space $\ker(B_N) \cap \ker(B_Y)$.

Definition 1.3.1. Let ρ be a self-adjoint involution defined on the space $\ker(B_N) \cap \ker(B_Y)$, which anti-commutes with the restriction of ν to $\ker(B_N) \cap \ker(B_Y)$. Let $W_0 = \ker(\rho - 1)$. Further, for $\mu^2 \neq 0$ let $W_{\mu^2} := \ker(\frac{-1}{|\mu|} B_{\mu^2} - 1)$ and let

$$D^{2,1}_\rho(\mathbb{R}^{2,\sigma}_\times \times Y, E)$$

be the closure of

$$\bigoplus_{\mu^2 \in \operatorname{spec}(B_Y^2)} D^{2,1}_{W_{\mu^2}}(\mathbb{R}^{2,\sigma}_\times, E)$$

in $H^1(\mathbb{R}^{2,\sigma}_\times \times Y, E)$.

If an involution ρ is not given we let $W_0 = \{0\}$ and define $D^{2,1}_{\min}(\mathbb{R}^{2,\sigma}_\times, E)$ like above.

Proposition 1.3.2. *The Dirac operator D defined on $D^{2,1}_\rho(\mathbb{R}^{2,\sigma}_\times \times Y, E)$ is self-adjoint. If ρ is not given, the realization of D on the domain $D^{2,1}_{\min}(\mathbb{R}^{2,\sigma}_\times \times Y, E)$ is a closed symmetric operator with finite defect indices.*

Proof: $(D, D^{2,1}_\rho(\mathbb{R}^{2,\sigma}_\times \times Y, E))$ is by definition the closure of an orthogonal sum of self-adjoint operators, and is therefore self-adjoint. If ρ is not given $(D, D^{2,1}_{\min}(\mathbb{R}^{2,\sigma}_\times \times Y, E))$ is the closure of the Direct sum of symmetric operators with finite defect indices, of which only finitely many are not self-adjoint. □

Definition 1.3.3. If ρ is given and D is given the domain $D^{2,1}_\rho$ we say that *slow-growing ideal Atiyah-Patodi-Singer boundary conditions augmented with respect to ρ* are imposed on D.

If ρ is not given and D is given the domain $D^{2,1}_{\min}$ we say that *minimal slow-growing ideal Atiyah-Patodi-Singer boundary conditions* are imposed on D.

Remark 1.3.4. If the dimension of Y is odd it does occur that no ρ like in Definition 1.3.1 exists. This is part of our motivation for the construction of the scattering matrix in Section 2.

Lemma 1.3.5. *Assume $\mathcal{H}$ is a Hilbert space, $A \in B(\mathcal{H})$ and that A^*A is compact. Then A and A^* are compact.*

Proof: If A^*A is compact, for every bounded net $\{f_\lambda\}_{\lambda\in\Lambda}$ converging weakly towards 0, $\|A^*Af_\lambda\| \to 0$ for $\lambda \to \infty$. Thus

$$\|Af_\lambda\| = (\langle A^*Af_\lambda, f_\lambda\rangle)^{\frac{1}{2}} \to 0$$

for $\lambda \to \infty$. This implies that A is compact. Thus also A^* is compact, and the proof is complete. □

Lemma 1.3.6. *Let $\varphi \in C^\infty(\mathbb{R}^{2,\sigma}_\times)$ be a function depending only on r, such that $\sqrt{\varphi} \in C^\infty(\mathbb{R}^{2,\sigma}_\times)$, $\varphi(r,\theta) = 1$ for $r \le 1$ and $\varphi(r,\theta) = 0$ for $r \ge 2$. Let M_φ be the operator of multiplication by φ. Further, let D be the Dirac operator on $\mathbb{R}^{2,\sigma}_\times \times Y$ and let $(D^2-\lambda)^{-s}$ be the analytic continuation in λ of the operator $(D^2-\lambda)^{-s}$ defined for $\lambda \in (-\infty, 0)$ by the spectral theorem. Then, for $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$ and $s > 0$, the operator*

$$M_\varphi(D^2-\lambda)^{-s}$$

is compact.

Proof: First notice that by multiplicativity, Lemma 1.3.5 and the fact that the compact operators make up a closed $*$-ideal in the Banach algebra of bounded operators, we may take $s = 1$ without exception. Next notice that by the first resolvent equation, we may take $\lambda = -1$. First we consider the restriction of D to the orthogonal complement of $\ker(B_N)$. On this space we have the following two important properties of the domain:

- $\mathcal{D}(D)$ is independent of B_Y.
- For each μ^2 eigenspace of B_Y^2 we have $\mathcal{D}((D_0 + B_{\mu^2})^2) = \mathcal{D}(D_0^2)$.

Splitting $D_{|\ker(B_N^2)^\perp}$ into the eigenspaces of B_Y^2 gives that

$$D^2 = \bigoplus_\mu D_\mu^2 = \bigoplus_{\mu^2} D_0^2 + \mu^2$$

and thus that the sum

$$M_\varphi(D^2+1)^{-1} = \bigoplus_{\mu^2} M_\varphi(D_0^2+\mu^2+1)^{-1} \tag{1.3.2}$$

is convergent in operator norm (not necessarily absolutely convergent, but the orthogonality of the terms makes up for that). It thus suffices to prove that each term in (1.3.2) is compact. Again by the first resolvent equation we may assume $\mu = 0$ without loss of generality.

Let $\mathbb{R}^{2,\sigma}_{\times,2} = \{[(r,\theta)] \in \mathbb{R}^{2,\sigma}_\times \mid r < 2\}$ and let D_D^2 be the operator in $L^2(\mathbb{R}^{2,\sigma}_{\times,2}, E_\mu)$ given by imposing Dirichlet boundary conditions at $r = 2$. I.e. $D^2_{\sigma,D}$ is the Friedrich's extension of D_σ^2 restricted to the domain

$$\mathcal{D}_{0,D} = \{f \in D^{2,1}_\rho \mid \mathrm{supp}(f) \subseteq \mathbb{R}^{2,\sigma}_{\times,2} \text{ and } Df \in D^{2,1}_\rho\}.$$

Then since

$$M_\varphi(D^2+1)^{-1} = M_{\sqrt{\varphi}}(D_D^2+1)^{-1}\left((D^2+1)M_{\sqrt{\varphi}}(D^2+1)^{-1}\right)$$

and the operator

$$(D^2+1)M_{\sqrt{\varphi}}(D^2+1)^{-1}$$

is bounded, we may consider $M_{\sqrt{\varphi}}(D_D^2+1)^{-1}$ instead.

Also the operator $(D_D^2+1)^{-1}$ can be decomposed with respect to the eigenspaces of $B+\frac{1}{2}$. The operator estimate

$$-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s_{k,\alpha}^2}{r^2} + 1 \geq -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s_{k,\alpha}^2}{2^2} + 1,$$

which holds for $|s_{k,\alpha}| > \frac{1}{2}$, where both operators are the Friedrich's extensions from C_0^∞, implies

$$\left\| \left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s_{k,\alpha}^2}{r^2} + 1\right)^{-1} \right\| \leq \left\| \left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s_{k,\alpha}^2}{2^2} + 1\right)^{-1} \right\|.$$

From that it follows that the sum

$$M_{\sqrt{\varphi}}(D_D^2+1)^{-1} = \bigoplus_{k\in\mathbb{Z}} M_{\sqrt{\varphi}}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{s_{k,\alpha}^2}{r^2} + 1\right)^{-1} \tag{1.3.3}$$

is convergent in norm, and thus it again suffices to consider each term separately.

Now the domain of D_D^2 is such that each term in (1.3.3) is a bounded operator from $L^2((0,\infty), rdr)$ to $W^{2,1}((0,\infty), rdr) \oplus V$, where $\dim(V) < \infty$. By Rellichs lemma and since $\dim(V) < \infty$ it follows that each term is compact.

The restriction of D to $\ker(B_N)$ is conjugate to a Dirac operator on $(0,\infty)\times Y$ with Atiyah-Patodi-Singer boundary conditions (by the operator of multiplication by $r^{\frac{1}{2}}$). Thus it is well known that the compactness result also holds on this space.

This finishes the proof of the lemma. □

1.4 Characterization of $D^{2,1}$.

It will be convenient to have an abstract characterization of $D_\rho^{2,1}$ and $D_{\min}^{2,1}$, respectively. If $\ker(B_N) = 0$ or ρ is not given, set $W = \{0\}$. Otherwise let $W_0 := \ker(\rho - 1)$ and let $W \subseteq \ker(B_N)$ be the $H^{\frac{1}{2}}$-closure of the direct sum of W_0 and the negative spectral subspace of B_Y in $\ker(B_N)$. Let ξ be a smooth function on $C\times Y$ depending only on r, such that $\xi(r) = 0$ for $r > 2$ and $\xi(r) = r^{-\frac{1}{2}}$ for $0 < r < 1$. We realize that the closure in H^1 of the space

$$\mathbf{D}_W^{2,1} := \{f \in H^1 \mid \limsup_{r\to 0} r\int_0^{2\pi a} |f(r,\theta)|^2 d\theta = 0\} \oplus \xi W \tag{1.4.1}$$

contains $D_\rho^{2,1}$, and that D_μ is symmetric on $\mathbf{D}^{2,1}$. Since every self-adjoint operator is maximally symmetric it follows that $\mathbf{D}_W^{2,1} = D_W^{2,1}$. Next notice that $W^{2,1} = W_0^{2,1} = H_0^1$. Thus $W^{2,1}$ is a closed subspace of H^1 contained in $D^{2,1}$.

Let $L = (0,R)\times\{\theta_0\}\times Y$ be a “line segment” in $\mathbb{R}_\times^{2,\sigma}\times Y$.

Lemma 1.4.1. *If* $\ker(B_N)=\{0\}$, $D^{2,1}_{\min}(\mathbb{R}^{2,\sigma}\times Y,E)$ *is the only extension of* $W^{2,1}$ *such that* D *is self-adjoint on* $D^{2,1}_{\min}$ *and the restriction*

$$D^{2,1}_{\min}\mapsto L^2(L,E_{|L})$$

is well defined and bounded.

If $\ker(B_N)\neq 0$, *the domains* $D^{2,1}_W$ *satisfy that* D *is self-adjoint on* $D^{2,1}_\rho$ *and that the restriction*

$$D^{2,1}_\rho\mapsto W^{2,-\varepsilon}(L,E^{Y,a,U}_{|L}) \tag{1.4.2}$$

is well defined and bounded for all $\varepsilon>0$. *For* $\varepsilon=0$ *the restriction (1.4.2) is not well defined for any* ρ.

Proof: The space of sections in $\ker((D^2-D^2_Y)^*+1)$ is spanned by sections of the form $K_s(r)e^{\nu\delta s\theta}v$. If $s\neq\frac{1}{2}$ for all s, the restriction of a vector in this span to N is in L^2 if and only if only basis elements with $s<\frac{1}{2}$ occur. The norm on $D^{2,1}$ restricted to $H^1(\mathbb{R}^{2,a}_\times,F_{\mu^2})$ is given by

$$\|f\|^2_{D^{2,1}}=\langle(D_0+B_\mu)f,(D_0+B_\mu)f\rangle+\langle f,f\rangle\,.$$

Since D_0 and B_μ anti-commute and B_μ is bounded and preserves the domain of D_0, this can be rewritten

$$\langle D_0 f,D_0 f\rangle+\langle B_\mu f,B_\mu f\rangle+\langle f,f\rangle\,.$$

Thus convergence in $\|\cdot\|_{D^{2,1}}$ implies convergence in the norm $\|\cdot\|_\perp$ given by

$$\|f\|^2_\perp:=\langle D_0 f,D_0 f\rangle+\langle f,f\rangle\,.$$

Let $\bar{D}^{2,1}_\rho$ be the completion of $D^{2,1}_\rho(\mathbb{R}^{2,\sigma}_\times\times Y,E)$ with respect to $\|\cdot\|_\perp$. The orthogonal complement of $W^{2,1}$ in $\bar{D}^{2,1}$ is the closed span of the functions $K_{s_j}(r)e^{is_j\theta}\varphi_{\mu,j}$ in $D^{2,1}$. Since there are at most finitely many different s_j, the restriction from the orthogonal complement of $W^{2,1}$ to $L^2(L,E)$ is bounded. The completion of $W^{2,1}$ in $\bar{D}^{2,1}$ is contained in the space $W^{2,1}(\mathbb{R}^{2,\sigma}_\times,L^2(Y,E_{|Y}))$. The restriction to $L^2(L,E_{|L})$ from this space is bounded, as it can be seen by splitting

$$W^{2,1}(\mathbb{R}^{2,a}_\times,L^2(Y,E_{|Y}))=\bigoplus_\mu W^{2,1}(\mathbb{R}^{2,a}_\times,L^2(Y,E_\mu))$$

and using the standard restriction $W^{2,1}\mapsto W^{2,\frac{1}{2}}$.

For the second part, if $s=\frac{1}{2}$ occurs, we realize that the restriction to $W^{2,-\varepsilon}$ is well defined and bounded for $\varepsilon>0$ but a priori not for $\varepsilon=0$. The proof is like the proof of the first part. On the other hand a section of the form

$$\sum_j a_jK_{\frac{1}{2}}(r)e^{\frac{1}{2}\nu\delta\theta}v_j=K_{\frac{1}{2}}(r)e^{\frac{1}{2}\nu\delta\theta}\sum_j a_jv_j,$$

restricts to $L^2(L,E_{|L})$ if and only if it vanishes. This proves that the restriction to $L^2(L,E_{|L})$ is not well-defined. □

2 Globally defined Augmentations.

If M is a manifold with boundary and product structure around the boundary, D is a Dirac operator on a Dirac bundle $E \mapsto M$ respecting the product structure and A is the induced Dirac operator on ∂M, a canonical choice of augmentation for D is given by the scattering matrix in 0, denoted by S. See [30], where it is denoted by $C(0)$. In [30] the scattering matrix is constructed (or more precisely, its properties are deduced) using the spectral resolution of a Dirac operator on a manifold with cylindrical ends. The purpose of this section is to give an alternative construction, which works directly on compact manifolds. This construction further gives scattering matrices for manifolds with both wedge singularities and boundaries, which mix the various spaces, which have to be augmented.

2.1 The General Construction

In the following $E \mapsto X$ will denote a Dirac bundle over an open Riemannian manifold X° with completion X. Let D_0 be the associated Dirac operator, defined on some domain $\mathcal{D}(D_0)$ satisfying the following:

Assumption 2.1.1.

a) *D_0 is densely defined, closed and symmetric on $\mathcal{D}(D_0)$.*

b) *$\mathcal{D}(D_0^*)/\mathcal{D}(D_0)$ is of finite dimension and the restriction of the projection map $\ker(D_0^* D_0^*) \mapsto \mathcal{D}(D_0^*)/\mathcal{D}(D_0)$ is surjective.*

c) *There exists an exhaustion $\{X_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of X° such that each X_ε has a smooth boundary, $X_\varepsilon \subseteq X_{\varepsilon'}$ for $\varepsilon \geq \varepsilon'$ and for every $f \in \mathcal{D}(D_0^*)$, the limit*

$$\langle f, f\rangle_\partial := \lim_{\varepsilon\to 0} \int_{\partial X_\varepsilon} \langle f(x), f(x)\rangle\, dx$$

exists and satisfies that if $\langle f, f\rangle_\partial = 0$ then $f \in \mathcal{D}(D_0)$.

Remark 2.1.2. Often X_ε will be a manifold with corners rather than a manifold with smooth boundary. What matters is that (2.1.2) below is satisfied.

Let $\mathcal{L}$ be the orthogonal complement of $\ker(D_0^* D_0)$ in $\ker(D_0^* D_0^*)$ with respect to the inner product on $L^2(X, E)$. The restriction of $\langle\cdot,\cdot\rangle_\partial$ to $\mathcal{L}$ is an inner product by the assumptions above. We define an operator $\mathbf{c}_\partial : \mathcal{L} \mapsto \mathcal{L}$ by

$$\langle \mathbf{c}_\partial f, g\rangle_\partial := \lim_{\varepsilon\to 0} \int_{\partial X_\varepsilon} \langle \mathbf{c}(\nu_\varepsilon) f(x), g(x)\rangle\, dx, \tag{2.1.1}$$

where $\mathbf{c}(\nu_\varepsilon)$ denotes the inward pointing normal at ∂X_ε. The formula

$$\int_{X_\varepsilon} \langle D_0^* f, g\rangle_{|x} - \langle f, D_0^* g\rangle_{|x}\, dx = \int_{\partial X_\varepsilon} \langle \mathbf{c}(\nu_\varepsilon) f(x), g(x)\rangle_{|x}\, dx \tag{2.1.2}$$

proves that $\mathbf{c}_\partial$ is well defined and that

$$\langle \mathbf{c}_\partial f, g\rangle_\partial = \langle D_0^* f, g\rangle - \langle f, D_0^* g\rangle\,.$$

Using (2.1.1) it follows that $\mathbf{c}_\partial^* = -\mathbf{c}_\partial$. Further, if $\{e_j\}$ is an orthonormal basis for $\mathcal{L}$ with respect to $\langle\cdot,\cdot\rangle_\partial$ we may compute

$$\begin{aligned}\langle \mathbf{c}_\partial f, \mathbf{c}_\partial f\rangle_\partial &= \sum_j \left\langle \mathbf{c}_\partial f, \langle \mathbf{c}_\partial f, e_j\rangle_\partial\, e_j\right\rangle_\partial \\ &= \sum_j \langle \mathbf{c}_\partial f, e_j\rangle_\partial\, \overline{\langle \mathbf{c}_\partial f, e_j\rangle_\partial} \\ &= \sum_j \lim_{\varepsilon\to 0} \langle \mathbf{c}(\nu_\varepsilon) f, e_j\rangle_{L^2(\partial X_\varepsilon, E)}\, \overline{\langle \mathbf{c}(\nu_\varepsilon) f, e_j\rangle}_{L^2(\partial X_\varepsilon, E)} \\ &= \sum_j \lim_{\varepsilon\to 0} \left\langle \mathbf{c}(\nu_\varepsilon) f, \langle \mathbf{c}(\nu_\varepsilon) f, e_j\rangle_{L^2(\partial X_\varepsilon, E)}\, e_j\right\rangle_{L^2(\partial X_\varepsilon, E)}\,.\end{aligned}$$

Because $\langle\cdot,\cdot\rangle_{L^2(\partial X_\varepsilon, E)} \to \langle\cdot,\cdot\rangle_\partial$ as a family of quadratic forms on $\mathcal{L}$ this gives:

$$\begin{aligned}&= \lim_{\varepsilon\to 0} \langle \mathbf{c}(\nu_\varepsilon) f, \mathbf{c}(\nu_\varepsilon) f + o(1)\rangle_{L^2(\partial X_\varepsilon, E)} \\ &= \lim_{\varepsilon\to 0} \left\{ \langle f, f\rangle_{L^2(\partial X_\varepsilon, E)} + o(1)\right\} \\ &= \langle f, f\rangle_\partial\,.\end{aligned}$$

This proves that $\mathbf{c}_\partial$ is an isometry. We immediately conclude that in fact $\mathbf{c}_\partial$ is unitary and anti-self-adjoint.

Lemma 2.1.3. *Let* $\ker(D_0)^*/\ker(D_0) \overset{\imath}{\mapsto} \ker(D_0^* D_0^*)/\ker(D_0)$ *denote the map of inclusion. Then we have*

i) If Assumption 2.1.1 holds, the sequence

$$0 \overset{\imath}{\mapsto} \ker(D_0^*)/\ker(D_0) \mapsto \ker(D_0^* D_0^*)/\ker(D_0) \overset{D_0^*}{\mapsto} \ker(D_0^*)/\ker(D_0) \mapsto 0 \quad (2.1.3)$$

is exact.

ii) Let

$$\mathcal{L}_0 := \{f \in \mathcal{L} \mid D_0^* f = 0\}.$$

Then $\mathcal{L}_0$ *is a Lagrangian subspace of* $\mathcal{L}$ *with respect to* $\mathbf{c}_\partial$ *and* $\langle\cdot,\cdot\rangle$.

Proof: We identify $\ker(D_0^* D_0^*)/\ker(D_0)$ with $\mathcal{L}$ and $\ker(D_0^*)/\ker(D_0)$ with $\mathcal{L}_0$. Since for $g \in \ker(D_0)$, $f \in \ker(D_0^* D_0^*)$ we have

$$\langle D_0^* f, g\rangle = \langle f, D_0 g\rangle = 0,$$

we also have that D_0^* maps $\mathcal{L}$ to $\mathcal{L}_0$, so the sequence (2.1.3) is isomorphic to the sequence

$$0 \mapsto \mathcal{L}_0 \overset{\imath}{\mapsto} \mathcal{L} \overset{D_0^*}{\mapsto} \mathcal{L}_0 \mapsto 0.$$

That $\imath$ is injective and that $\mathrm{Im}(\imath) = \mathcal{L}_0 = \ker((D_0^*)_{|\mathcal{L}})$ are both obvious. We need to show that $D_0^* : \mathcal{L} \mapsto \mathcal{L}_0$ is surjective. We will use a dimension argument. First notice that the exactness of

$$0 \mapsto \mathcal{L}_0 \overset{\imath}{\mapsto} \mathcal{L} \overset{D_0^*}{\mapsto} \mathcal{L}_0$$

implies that we have the inequality

$$\dim(\mathcal{L}) \leq 2\dim(\mathcal{L}_0). \tag{2.1.4}$$

Next we notice that for $f, g \in \mathcal{L}_0$,

$$\langle \mathbf{c}_\partial f, g \rangle = \langle D_0^* f, g \rangle - \langle f, D_0^* g \rangle = 0.$$

Thus $\mathbf{c}_\partial$ is an injective operator mapping $\mathcal{L}_0$ to its orthogonal complement. It follows that

$$\dim(\mathcal{L}_0) \leq \frac{1}{2}\dim(\mathcal{L}).$$

Together with (2.1.4) this proves that $\dim(\mathcal{L}_0) = \frac{1}{2}\dim(\mathcal{L})$, so $D_0^* : \mathcal{L} \mapsto \mathcal{L}_0$ is surjective.

The same arguments also prove ii). □

Corollary 2.1.4. *Let D be the restriction of D_0^* to $\mathcal{D}(D_0) \oplus \mathcal{L}_0$. Then D is self-adjoint.*

Proof: Let $f_1, f_2 \in \mathcal{D}(D_0)$ and let $g_1, g_2 \in \mathcal{L}_0$. Then

$$\langle D(f_1 + g_1), f_2 + g_2 \rangle = \langle D_0 f_1, f_2 + g_2 \rangle = \langle f_1, D_0^*(f_2 + g_2) \rangle = \langle f_1, D_0 f_2 \rangle.$$

In the same way it follows

$$\langle f_1 + g_1, D(f_2 + g_2) \rangle = \langle D_0 f_1, f_2 \rangle = \langle f_1, D_0 f_2 \rangle.$$

Consequently D is symmetric. Since $D_0 \subseteq D$ and D is symmetric we have $D \subseteq D^* \subseteq D_0^*$. Now assume that there exists $f \in \mathcal{D}(D^*) \setminus \mathcal{D}(D)$. Then f is of the form $f_3 + f_4$, where $f_3 \in \mathcal{D}(D)$ and f_4 belongs to the orthogonal complement of $\mathcal{L}_0$ in $\mathcal{L}$. Consequently $D_0^* f_4 \in \mathcal{L}_0 \setminus \{0\} \subseteq \mathcal{D}(D)$. We check

$$\begin{aligned}\langle D_0^* f_4, D^*(f_3 + f_4) \rangle &= \langle D_0^* f_4, D f_3 \rangle + \langle D_0^* f_4, D_0^* f_4 \rangle \\ &= \langle D_0^* D_0^* f_4, f_3 \rangle + \langle D_0^* f_4, D_0^* f_4 \rangle \\ &= \langle D_0^* f_4, D_0^* f_4 \rangle \neq 0.\end{aligned}$$

On the other hand

$$\langle D(D_0^* f_4), f_3 + f_4 \rangle = \langle D_0^* D_0^* f_4, f_3 + f_4 \rangle = 0.$$

This is a contradiction against $f_3 + f_4 \in \mathcal{D}(D^*)$, so D is self-adjoint. □

Definition 2.1.5. We consider $\mathcal{L}$ as a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_\partial$. The *scattering matrix* $S:\mathcal{L}\mapsto\mathcal{L}$ is the operator $2P-1$, where P is the orthogonal projection on $\mathcal{L}_0$. We say that the operator D defined in Corollary 2.1.4 is *augmented with respect to* S.

Lemma 2.1.6. *Let T be a bounded normal operator on $L^2(X,E)$ such that*

1) T^* preserves $\mathcal{D}(D_0)$.

2) $T^*D_0 = D_0T^*$ or $T^*D_0 = -D_0T^*$.

Then T preserves $\mathcal{L}$ and $\mathcal{L}_0$ and in particular, T commutes with S.

Proof: Assume $f\in\mathcal{D}(D_0^*)$. Then for $g\in\mathcal{D}(D_0)$

$$|\langle D_0g,Tf\rangle| = |\langle T^*D_0g,f\rangle| = |\pm\langle D_0T^*g,f\rangle|.$$

Since D_0^*f is well defined and $T^*g\in\mathcal{D}(D_0)$ this is equal to

$$|\pm\langle T^*g,D_0^*f\rangle| \le \|T^*\|\cdot\|g\|\cdot\|D_0^*f\|.$$

Thus $Tf\in\mathcal{D}(D_0^*)$. Further $TD_0^* = \pm D_0^*T$ by calculations like above, so in particular T preserves $\ker(D_0^*)$. Further, for $f\in\mathcal{D}(D_0^*D_0^*)$ it follows $D_0^*D_0^*Tf = \pm D_0^*(TD_0^*f) = TD_0^*D_0^*f$, so T also preserves $\ker(D_0^*D_0^*)$. Since $\ker(D_0^*D_0^*)$ is finite dimensional, T is normal and T^* preserves $\ker(D_0)$ it follows that T preserves $\mathcal{L}$, $\mathcal{L}_0$ and the orthogonal complement of $\mathcal{L}_0$ in $\mathcal{L}$. This proves the lemma. □

Corollary 2.1.7. *We have*

- *S respects every superstructure on E respected by D_0.*
- *Assume that E is the restriction of a Clifford bundle $\tilde{E}\mapsto N\times X$ to $\{a\}\times X$, where $N\times X$ is supplied with a product metric and product connection. If Clifford multiplication by any tangent vector $\gamma\in T_aN$ preserves $\mathcal{D}(D_0)$, then γ sends $\mathcal{L}$ into $\mathcal{L}$ and $\gamma S = S\gamma$.*

2.2 Manifolds with Boundaries and Wedge Singularities.

We will now apply the above to the Dirac operator $\tilde{D}$ on the extended manifold $\tilde{M}$, where M is an oriented manifold with corners of codimension 2 and product structure around the corners. $\tilde{M}$ is a manifold with wedge singularities and a boundary. Let X be the sub-manifold of interior points of $\tilde{M}_0$. At the boundary Atiyah-Patodi-Singer boundary conditions can be imposed except for in $\ker(A)$, where A is the induced Dirac operator at the boundary. Thus we require sections to have restrictions to the boundary in the strictly positive spectral subspace for A. Similarly, in the wedge singularities minimal slow-growing ideal Atiyah-Patodi-Singer boundary conditions can be imposed. This gives a closed symmetric realization $\tilde{D}_0$ of $\tilde{D}$.

Lemma 2.2.1. *Assumption 2.1.1 is satisfied for $\tilde{D}_0$.*

Proof: a) is obvious. Further, $\mathcal{D}(\tilde{D}_0^*)/\mathcal{D}(\tilde{D}_0)$ is finite dimensional since

$$\mathcal{D}(\tilde{D}_0^*)/\mathcal{D}(\tilde{D}_0) \cong \ker(A) \oplus \bigoplus \ker(B_N) \cap \ker(B_Y) =: V,$$

where the direct sum is over the wedge singularities. Now consider the operator $\tilde{D}_0^*\tilde{D}_0$. Essentially by the proof of Lemma 1.3.6 it follows that $\tilde{D}_0^*\tilde{D}_0$ is a Fredholm operator. To each $\varphi \in \ker(A)$ and to each $\psi \in \ker(B_N) \cap \ker(B_Y)$, where N and Y are associated to a wedge singularity, we may associate an element $h \in \mathcal{D}(\tilde{D}_0^*) \setminus \mathcal{D}(\tilde{D}_0)$ with support in a small neighborhood of the boundary or singularity. Further h may be taken such that $\tilde{D}_0^* h \in \mathcal{D}(\tilde{D}_0^*)$. On the other hand each $f \in \mathcal{D}(\tilde{D}_0^*) \setminus \mathcal{D}(\tilde{D}_0)$ is asymptotic to a sum of such elements up to $\mathcal{D}(\tilde{D}_0)$. Further, $\tilde{D}_0^*\tilde{D}_0^* h$ is orthogonal to $\ker(\tilde{D}_0) = \ker(\tilde{D}_0^*\tilde{D}_0)$, so $h - (\tilde{D}_0^*\tilde{D}_0)^{-1}(\tilde{D}_0^*\tilde{D}_0^* h)$ is an element of $\ker(\tilde{D}_0^*\tilde{D}_0^*)$, which differs from h by an element in $\mathcal{D}(\tilde{D}_0)$. This gives that the map $\ker(\tilde{D}_0^*\tilde{D}_0^*)/\ker(\tilde{D}_0) \mapsto \mathcal{D}(\tilde{D}_0^*)/\mathcal{D}(\tilde{D}_0)$ is surjective. In order to prove c) take

$$X_\varepsilon := \{x \in \tilde{M} \mid \operatorname{dist}(x, \partial\tilde{M}) \geq \varepsilon\}.$$

For small ε this is a manifold with boundary and the family X_ε exhausts X. Further, by the asymptotics of elements of $\mathcal{D}(\tilde{D}_0^*)$, it follows that $\langle \cdot, \cdot \rangle$ is well defined and that it does not vanish on elements of $\mathcal{D}(\tilde{D}_0^*) \setminus \mathcal{D}(\tilde{D}_0)$.

This proves the lemma.

□

By Lemma 2.2.1 and Corollary 2.1.4 there is a canonical self-adjoint extension of $\tilde{D}_0$, which we (with a slight abuse of notation) denote by $\tilde{D}$.

Definition 2.2.2. When $\tilde{D}$ is given the domain defined above we say that $\tilde{D}$ is given slow-growing Atiyah-Patodi-Singer boundary conditions augmented with respect to the scattering matrix.

The significance of the slow-growing ideal Atiyah-Patodi-Singer boundary conditions is that

- They are always well-defined, also for odd dimensional manifolds, and they are canonical inside the class of all generalized compatible Dirac operators on manifolds with boundary and closed wedge singularities of codimension 2.
- They satisfy Lemma 2.1.6 and Corollary 2.1.7. This is crucial for the iteration of the theory to more complicated singularities. In Section 5 a simple example of this is given.
- As soon as we relate $\tilde{D}$ back to a self-adjoint realization of D in Section 3, the mixing of boundary conditions in the various boundaries occurs anyway. Consequently this mixing is not a particular draw-back by the boundary conditions, as it could appear by a first sight.
- They give rise to a canonical joint generalization of the Atiyah-Patodi-Singer boundary conditions on manifolds with boundary and slow-growing ideal boundary conditions on manifolds with cones.

We will end up by remarking that if augmentations are given in some singularities or boundary pieces of $\tilde{M}$, $\tilde{D}_0$ can be replaced by a corresponding symmetric extension of $\tilde{D}_0$ and we get a scattering matrix involving the remaining singularities. The extension of this theory to wedge singularities of higher codimension is also completely straightforward.

3 A Self-Adjoint Extension of D.

In this section we return to the original manifold M. The theory developed for $\tilde{M}$ turns out to be suitable for defining global boundary conditions on M, generalizing the Atiyah-Patodi-Singer boundary conditions. In all of this section M will thus be a manifold with corners of codimension 2 and product structure around the corners, $\tilde{M}$ will be the extension of M defined by (1.1.2) and Z will be the smoothened boundary of M. If $E \mapsto M$ is a Dirac bundle respecting the product structure, $\tilde{E} \mapsto \tilde{M}$ will be an extension of E respecting the product structure. The bundle $\tilde{E}$ need not be glued using the canonical gluing operator. In Section 3.3 the gluing will in fact be the gluing associated to the signature complex, which does not coincide with the canonical gluing operator.

Let $\tilde{D}$ be the Dirac operator associated to $\tilde{E} \mapsto \tilde{M}$. The operator $\tilde{D} \mapsto \tilde{M}$ will be given slow-growing Atiyah-Patodi-Singer boundary conditions augmented with respect to the scattering matrix, where the scattering matrix can possibly be defined relative to an augmentation of some of the wedge singularities and boundary components.

3.1 Self-Adjoint Boundary Conditions.

Let $\tilde{D}$ be the Dirac operator on $\tilde{M}$. A first naive attempt of constructing a Sobolev space on M associated to $\tilde{D}$ is to define:

Definition 3.1.1. Let

$$D^{2,1}(M,E) = \{f_{|M} \mid f \in \mathcal{D}(\tilde{D})\}.$$

The space $D^{2,1}(M,E)$ is not a Hilbert space under the H^1-norm. Let $\bar{D}^{2,1}(M,E)$ denote the completion of $D^{2,1}(M,E)$ with respect to the H^1-norm. For each $\varepsilon > 0$ there is an unbounded trace operator defined on all of $D^{2,1}(M,E)$, which we denote by $\mathcal{R}$:

$$\mathcal{R} : D^{2,1}(M,E) \mapsto W^{2,-\varepsilon}(Z,F) \oplus \bigoplus \ker(B_N) \cap \ker(B_Y),$$

where the direct sum is over the wedge singularities. The first component of R is restriction to ∂M. See Lemma 1.4.1. The second component is obtained by first taking the projection onto $\ker(B_Y)$, then taking the leading term with asymptotics like $r^{-\frac{1}{2}}$, where r is the distance to the corner, and finally exploiting that an element of $\ker(B_N)$ is uniquely determined by its restriction to M. The image of the trace $\mathcal{R}$ is not dense unless $\ker(B_N) \cap \ker(B_Y) = 0$ for all N, Y since the second component is a function of the first.

We recall that on $Z \times (0,1]$, $\tilde{D}$ has the decomposition

$$\tilde{D} = -\nu(\frac{\partial}{\partial u} + A),$$

where A is a self-adjoint Dirac operator on Z with a discrete point spectrum with eigenvalues of finite multiplicity. Let Π_+, Π_- and Π_0 denote the projections on the positive, negative and zero spectral subspaces for A. All of those operators are defined on $L^2(Z,F)$ and extend by continuity to $W^{2,-\varepsilon}(Z,F)$.

Definition 3.1.2. The domain of the Dirac operator D on M is given by

$$\mathcal{D}(D) = \{f \in D^{2,1}(M,E) \mid \Pi_- f_{|Z} = 0 \text{ and } (\Pi_0 f_{|Z}, \mathcal{R}_2 f) \in \ker(S-1)\}.$$

Here $\mathcal{R}_2$ denotes the second component of $\mathcal{R}$.

Lemma 3.1.3. *If $f \in H^1(\tilde{M},\tilde{E})$ vanishes identically on M then $\varphi f \in W_0^{2,1}(\tilde{M},\tilde{E})$ for every smooth function φ, which is constant in a neighborhood of each wedge singularity and vanishes on $Z \times (1-\varepsilon, 1]$ for some $\varepsilon > 0$.*

Proof: The orthogonal complement $\mathcal{H}$ of $W_0^{2,1}(\tilde{M},\tilde{E})$ in $H^1(\tilde{M},\tilde{E})$ consists of the distributional solutions of the equation $\tilde{D}^2 f = -f$, which are in L^2. Let $\{\varphi_{\mu,\alpha}\}$ be an orthogonal basis of common eigenvectors for B_Y^2 to the eigenvalues μ^2 and the identification operator U at the corner to the eigenvalue α. Developing the restriction of f to a neighborhood of Y with respect to this basis, we get a sum, which is orthogonal on the level of $L^2(Y, E_{|Y})$

$$f(r,\theta,y) = \sum_{\mu,\alpha} f_{\mu,\alpha}(r,\theta)\varphi_{\mu,\alpha}(y),$$

where $f_{\mu,\alpha}$ is a L^2-solution to the equation

$$(D_{a,\alpha}^2 + \mu^2) f_{\mu,\alpha} = -f_{\mu,\alpha}.$$

This gives that each $f_{\mu,\alpha}$ is of the form

$$f_{\mu,\alpha}(r,\theta) = f_{0,\mu,\alpha}(r,\theta) + \sum_{s_{k,\alpha} \in (-1,1)} a_{\mu,\alpha,k} K_{s_{k,\alpha}}(\sqrt{\mu^2+1}\,r) e^{\nu\delta s_{k,\alpha}\theta},$$

where $f_{0,\mu,\alpha}$ is the restriction of a section in $W_0^{2,1}(\tilde{M},\tilde{E})$. From that it is not difficult to see that all terms, which are not restrictions of sections in $W_0^{2,1}$, have to vanish identically if $f_{|M} = 0$. □

Lemma 3.1.4. *Every $\varphi \in \mathcal{D}(D)$ has an extension $\tilde{\varphi} \in \mathcal{D}(\tilde{D})$ such that $\tilde{D}\tilde{\varphi}$ vanishes on $Z \times (0,1]$.*

The restriction of $\tilde{\varphi}$ to $Z \times (0,1]$ only depends on the restriction of φ to Z. Further, for all $\varepsilon > 0$ the operator $\mathcal{R}_{\mathrm{cyl}} \sim : W^{2,-\varepsilon}(Z,F) \mapsto H^1(Z \times (0,1], \tilde{E})$ is a compact operator. Here $\mathcal{R}_{\mathrm{cyl}}$ denotes the operator of restriction to $Z \times (0,1]$ and $\sim$ is the operator $\varphi \mapsto \tilde{\varphi}$.

Proof: The restriction of φ to Z is a $W^{2,-\varepsilon}$-convergent sum

$$\varphi(z,0) = \sum_{i=1}^{\dim(\ker(A)\cap\ker(\tau-1))} a_i\varphi_i(z) + \sum_{\lambda>0} a_\lambda\varphi_\lambda(z)$$

for some orthonormal basis $\{\varphi_\lambda\}_{\lambda\in\sigma(A)}$ of eigensections for A, eigenvalues counted with multiplicity. It follows that φ can be continued to a solution $\tilde\varphi$ of $D\tilde\varphi = 0$ on $Z\times[0,1]$ by

$$\tilde\varphi(z,u) = \sum_{i=1}^{\dim(\ker(A)\cap\ker(\tau-1))} a_i\varphi_i(z) + \sum_{\lambda>0} a_\lambda e^{-\lambda u}\varphi_\lambda(z). \tag{3.1.1}$$

For $u>0$ this sum is convergent in $W^{2,1}(Z,F)$. Further, the condition $\varphi(z,0)\in W^{2,-\varepsilon}(Z,F)$ suffices to ensure that $\tilde\varphi\in L^2$.

By definition of $D^{2,1}(M,E)$, φ also has an extension $f\in\mathcal{D}(\tilde D)$ and by Lemma 3.1.3, $f-\tilde\varphi$ is locally in $W_0^{2,1}$ close to at wedge singularities. This proves that $\tilde\varphi\in D_W^{2,1}(\tilde M,\tilde E)$. That $\mathcal{R}\sim$ only depends on $\varphi_{|Z}$ and is compact follows immediately by construction. □

Lemma 3.1.5. *D is closed and symmetric on $\mathcal{D}(D)$.*

Proof: By Lemma 1.4.1 there exists a constant C depending on $\varepsilon>0$ such that for all $f\in\mathcal{D}(D)$,

$$\|f_{|Z}\|_{W^{2,-\varepsilon}(Z,E)} \le C(\|f\|_{H^1(M,E)} + \|\tilde f\|_{H^1(Z\times(0,1],\tilde E)}).$$

This follows since there is a continuous restriction $H^1(\tilde M,\tilde E)\mapsto W^{2,-\varepsilon}(Z,F)$. Since $\mathcal{R}_{\text{cyl}}\sim$ is compact, for $f_{|Z}$ in some subspace with finite dimensional complement, there is an estimate

$$\|\tilde f\|_{H^1(Z\times(0,1],\tilde E)} \le \frac{1}{2C}\|f_{|Z}\|_{W^{2,-\varepsilon}(Z,F)}.$$

These two estimates together give

$$\|f_{|Z}\|_{W^{2,-\varepsilon}(Z,F)} \le 2C\|f\|_{H^1(M,E)}. \tag{3.1.2}$$

This proves that restriction to Z is continuous. Thus also $\varphi\mapsto\tilde\varphi$ is H^1-H^1 continuous, and it follows from the closedness of $\tilde D$ that D is closed.

That D is symmetric follows by applying Greens formula to the extensions of sections defined in Lemma 3.1.4 on $M\cup_Z(Z\times[0,\delta])$ and letting $\delta\to 0$. □

Theorem 3.1.6. *There are maps, given by extension and restriction, respectively*

$$\Phi_1 \;:\; \ker(D)\mapsto\ker(\tilde D), \tag{3.1.3}$$
$$\Phi_2 \;:\; \ker(\tilde D)\mapsto\ker(D). \tag{3.1.4}$$

The maps Φ_1 and Φ_2 are inverse of each other.

Proof: Like in the start of the proof of Lemma 3.1.4 we see that elements of $\ker(D)$ can be extended as claimed.

On the other hand, if $\varphi \in \ker(\tilde{D})$, expanding it on $Z \times (0,1]$ gives that it has an expansion like (3.1.1). Further the restriction of every element of $\ker(\tilde{D})$ satisfies the condition $(\Pi_0 f_{|Z}, \mathcal{R}_2 f) \in \ker(S-1)$. Consequently the restriction belongs to $\mathcal{D}(D)$ and thus to $\ker(D)$.

That Φ_1 and Φ_2 are inverse of each other is clear. □

Lemma 3.1.7. *The operator D is self-adjoint on the domain given in Definition 3.1.2. D has a discrete point spectrum with eigenvalues of finite multiplicity.*

Proof: By Lemma 3.1.5 we have that D is closed and symmetric.

Let P be the projection on the kernel of D and let R be the operator of restriction of sections in $\tilde{E}$ to sections in E. Then the adjoint R^* of R is the operator of extension by 0. The operator $D+P$ has an inverse, given by

$$(D+P)^{-1}f = Pf + (1-P)R\tilde{D}^{-1}R^*(1-P)f.$$

Since $\ker(\tilde{D})$ consists of extensions of elements of $\ker(D)$ it follows that $R^*(1-P)f$ is orthogonal to $\ker(\tilde{D})$. Thus $\tilde{D}^{-1}R^*(1-P)f$ exists and belongs to $L^2(\tilde{M}, \tilde{E})$. Further, since it satisfies the equation

$$\tilde{D}\tilde{D}^{-1}R^*(1-P)f = R^*(1-P)f,$$

it follows like in the proof of Theorem 3.1.6 that the restriction of $\tilde{D}^{-1}R^*(1-P)f$ to Z belongs to the non-negative eigenspace of A and that the component in $\ker(A)$ is constant on the cylinder. Since it further belongs to $\mathcal{D}(\tilde{D})$ it follows that $R\tilde{D}^{-1}R^*(1-P)f \in \mathcal{D}(D)$. It follows that $(D+P)^{-1}$ is a right inverse of $D+P$. Further, $(D+P)^{-1}$ is by construction everywhere defined. Since its graph is contained in the transpose of the graph of the injective symmetric operator $D+P$, it is closeable. But an everywhere defined closeable operator is closed, so $(D+P)^{-1}$ is bounded by the closed graph theorem. A symmetric operator with a bounded right inverse is always self-adjoint. This proves that $D+P$ is self-adjoint and thus also that D is self-adjoint.

That D has a discrete point spectrum with eigenvalues of finite multiplicity follows since P has finite rank and $(D+P)^{-1}$ is compact by the compactness of $\tilde{D}^{-1}$. □

Remark 3.1.8. All the main results of this section hold with only minor changes if S is replaced with another augmentation. In particular Lemma 3.1.7 holds in the case where some of the wedge singularities are augmented using local augmentations and the scattering matrix is changed to the corresponding relative scattering matrix.

3.2 Extensions of Hilbert Spaces.

The Atiyah-Patodi-Singer boundary conditions are closely related to the extension of a manifold with boundary and product structure in a neighborhood of the boundary to a manifold with cylindrical ends. This was observed and used already in [2] and has since then been an important starting point for generalizations of the Atiyah-Patodi-Singer boundary conditions. See for example [22] and [31]. Also the approach of this paper is based on an extension, though we have chosen consistently to make use of boundary conditions rather than of open ends.

When we imposed ideal slow-growing Atiyah-Patodi-Singer boundary conditions on a wedge singularity we used that the restriction of the Dirac operator on $\mathbb{R}^{2,\sigma}_{\times} \times Y$ to $\ker(B_N)$ is conjugate to a boundary value problem and imposed Atiyah-Patodi-Singer boundary conditions. There is however no natural geometric extension of $\mathbb{R}^{2,\sigma}_{\times} \times Y$ corresponding to those boundary conditions.

What we can do is to extend the Hilbert space $L^2(\mathbb{R}^{2,\sigma}_{\times} \times Y, E)$ together with some spaces of sections and functions. This kind of extensions will turn out to play a crucial role in the generalization of this theory to manifolds with corners of codimension 3 and arbitrary gluings or to manifolds with corners of codimension 4 and the canonical gluing. In this paper we will use it as a technique for studying the signature complex in Section 3.3. In the relevant cases in this paper only geometric cylinders are attached.

Let

$$\mathcal{H} := L^2(\tilde{M}, \tilde{E}). \tag{3.2.1}$$

Definition 3.2.1. Let $\tilde{D}_{\text{slow,max}}$ be the realization of $\tilde{D}$ defined on the domain

$$\mathcal{D}(\tilde{D}_{\text{slow,max}}) := \{f \in H^1(\tilde{M}, \tilde{E}) \mid \forall N : \lim_{r \to 0} \|f(r)\|_{L^2(N \times Y, \tilde{E})} = O(r^{-\frac{1}{2}})\}. \tag{3.2.2}$$

Definition 3.2.2. We say that an element $f \in L^2(\tilde{M}, \tilde{E})$ is *smooth* if for all $k \in \mathbb{N}$, $f \in \mathcal{D}(\tilde{D}^k_{\text{slow,max}})$.

Then we may define

$$\mathcal{H}_R := \mathcal{H} \oplus L^2(Z \times [1, R+1], F) \oplus \bigoplus L^2(Y \times [-R, 0], \ker(B_N)), \tag{3.2.3}$$

$$\mathcal{H}_\infty := \mathcal{H} \oplus L^2(Z \times [1, \infty), F) \oplus \bigoplus L^2(Y \times (-\infty, 0], \ker(B_N)). \tag{3.2.4}$$

The Hilbert space $\mathcal{H}_\infty$ is related to the non-smooth space

$$\tilde{M}_\infty := \tilde{M} \cup_Z (Z \times [1, \infty)) \cup_{Y_1 \sqcup \cdots \sqcup Y_k} ((Y_1 \sqcup \cdots \sqcup Y_k) \times (-\infty, 0]),$$

where $Y_1, \ldots, Y_k$ runs over the spaces Y at the various corners. $\tilde{M}_\infty$ is in a natural way a σ-compact Hausdorff space with a Borel measure. Further the pointwise squared norm,

$$|\cdot|^2 : \mathcal{H}_\infty \mapsto L^1(\tilde{M}_\infty)$$

is well defined. The space $\mathcal{H}_\infty$ is however not in any natural way the space of L^2-sections in a bundle over $\tilde{M}_\infty$.

Definition 3.2.3. Let $f \in \mathcal{H}_\infty$. The *support* of f, $\text{supp}(f)$ is given by

$$\text{supp}(f) := \tilde{M}_\infty \setminus \bigcup \{U \in \tilde{M}_\infty \mid U \text{ is open and } \int_U |f|^2(x)dx = 0\}.$$

We say that f has compact support if $\text{supp}(f)$ is compact.

Definition 3.2.4. The space $C^\infty(\mathcal{H}_\infty)$ of *smooth sections* in $\mathcal{H}_\infty$ is the subspace of $f \in \mathcal{H}_\infty$ such that each of the components of f in (3.2.4) are smooth, such that f extends to a smooth section in the extension of $\tilde{E}$ to $\tilde{M} \cup_Z (Z \times [1,\infty))$ and such that for each wedge singularity, if we let $P_{\ker(B_N)}$ be the projection on $\ker(B_N)$, defined in a neighborhood of the singularity, the sections $r^{\frac{1}{2}} P_{\ker(B_N)} f(r,\cdot)$ and the restriction of f to $L^2(Y \times (-\infty,0], \ker(B_N))$ glue together to a smooth section in $L^2(Y \times (-\infty,\varepsilon), \ker(B_N))$ for some $\varepsilon > 0$.

Definition 3.2.5. Let P_R be the orthogonal projection on $\mathcal{H}_R$ in $\mathcal{H}_\infty$. The space $C^\infty(\mathcal{H}_R)$ of smooth sections in $\mathcal{H}_R$ is given by

$$C^\infty(\mathcal{H}_R) := P_R C^\infty(\mathcal{H}_\infty).$$

Definition 3.2.6. The space $C_0^\infty(\mathcal{H}_\infty)$ is the subspace of $C^\infty(\mathcal{H}_\infty)$ of elements with compact support. It can be supplied with an inductive limit topology using H^k-norms of sections with fixed support. The space $\mathcal{D}'(\mathcal{H}_\infty)$ of *currents with values in* $\mathcal{H}^\infty$ is the dual space of $C_0^\infty(\mathcal{H}_\infty)$.

Definition 3.2.7. The Dirac operator $D_{\infty,0}$ defined on $C_0^\infty(\mathcal{H}_\infty)$ is the Direct sum of the operators $\tilde{D}_{\text{slow,max}}$,

$$\gamma(\frac{\partial}{\partial u} + A) \quad ; \text{ defined on } L^2(Z \times [1,\infty), F),$$

and the operators

$$\nu(\frac{\partial}{\partial r} + B_Y) \quad ; \text{ defined on } L^2(Y \times (-\infty,0], \ker(B_N)).$$

The operator $D_{R,0}$ is the restriction of $D_{\infty,0}$ to $C^\infty(\mathcal{H}_R)$.

Lemma 3.2.8. *The operator $D_{\infty,0}$ is essentially self-adjoint.*

Proof: This follows by constructing the resolvent using the resolvent of $\tilde{D}$ and of Dirac operators on the cylinders, cut-off operators and analytic perturbation theory. □

Let D_∞ be the unique self-adjoint extension of $D_{\infty,0}$. We denote the domain of D_∞ by $\mathcal{D}(D_\infty)$. Further we let $\ker_\infty(D_\infty)$ be the space of smooth sections of $\mathcal{H}_\infty$, such that the restriction to each cylinder (but not to $\tilde{M}$) is bounded and such that $D_\infty f = 0$. Finally we let

$$\mathcal{D}_\infty(D_\infty) := \mathcal{D}(D_\infty) + \ker_\infty(D_\infty), \tag{3.2.5}$$
$$\mathcal{D}_\infty(D_\infty^2) := \mathcal{D}(D_\infty^2) + \ker_\infty(D_\infty). \tag{3.2.6}$$

It now follows exactly like for a manifold with boundary that

$$\mathcal{D}(\tilde{D}) = \{f_{|\tilde{M}} \mid f \in \mathcal{D}_\infty(D_\infty) \text{ and } D_\infty f \in \mathcal{H}\}.$$

A section $f \in \mathcal{D}(\tilde{D})$ can be extended uniquely to a section $\tilde{f} \in \mathcal{D}_\infty(D_\infty)$ satisfying that $D_\infty \tilde{f} \in \mathcal{H}$. This holds because $\tilde{D}$ is augmented with respect to the scattering matrix.

Lemma 3.2.9. *Let* $f \in \mathcal{H}_\infty$ *be a section with compact support, which is orthogonal to* $\ker_\infty(D_\infty)$*. Then there exists* $g \in \mathcal{D}(D_\infty)$ *such that* $D_\infty g = f$*.*

Proof: Let R be such that $f \in \mathcal{H}_R$. We may impose APS boundary conditions augmented with respect to the scattering matrix on $D_{R,0}$. The resulting self-adjoint operator D_R has a discrete point spectrum and $\ker(D_R) \cong \ker_\infty(D_\infty)$, where the isomorphism is by unique extension and restriction. Consequently f is contained in $\ker(D_R)^\perp$. Thus there exists $g_R \in \mathcal{D}(D_R)$ with $D_R g_R = f$. Now, g_R may be extended to $g_\infty \in \mathcal{D}_\infty(D_\infty)$ such that $D_\infty g_\infty \in \mathcal{H}_R$. Thus $D_\infty g_\infty = f$.

Finally, the limit value of g_∞ corresponds to the limit value of a harmonic section ω. Thus $g := g_\infty - \omega$ satisfies the claims of the lemma. □

3.3 The de Rham and Signature Complexes.

In this section E will be the bundle $\Lambda^*(T^*M \otimes \mathbb{C})$ supplied with the Levi-Civita connection and the canonical Hermitian structure induced by the Riemannian metric on M.

The vector-bundle $\tilde{E}$ will be the bundle of differential forms on $\tilde{M}$. Notice that since functions are glued trivially in the corners, this bundle is not glued using the gluing from Lemma 1.1.14. This follows from point 6) of Lemma 1.1.14. In all of this section we will assume:

Assumption 3.3.1. *Each angle* σ_j *at the corners belongs to the interval* $(0, 3\pi)$*.*

Lemma 3.3.2. *The projection on* p*-forms preserves* $\mathcal{D}(\tilde{D}_{\mathrm{slow,max}})$ *for* $p = 0, \ldots, n$*. Further, for each* N *we have* $\ker(B_N) = \{0\}$*.*

Proof: Locally, in a neighborhood of a wedge singularity, E may be decomposed

$$E = \{\Lambda^*(Y) \oplus \nu\delta\Lambda^*(Y)\} \oplus \{(\delta - i\nu)\Lambda^*(Y)\} \oplus \{(\delta + i\nu)\Lambda^*(Y)\}. \tag{3.3.1}$$

Clearly the projection on p-forms preserves this decomposition. The gluing operator U is of the form $\Lambda(V)$, where V is a rotation with angle $\pi - \sigma$ in the plane spanned by ν and δ. Explicitly,

$$U\nu = \cos(\pi - \sigma)\nu + \sin(\pi - \sigma)\delta,$$

$$U\delta = \cos(\pi - \sigma)\delta - \sin(\pi - \sigma)\nu.$$

Explicit computation now shows that (3.3.1) is a decomposition into eigenspaces of U, with eigenvalues $1, e^{-i(\pi-\sigma)}$ and $e^{i(\pi-\sigma)}$, respectively. The operators B_N and B_Y^2 send each of the terms in (3.3.1) into itself. In particular (3.3.1) respects $\mathcal{D}(\tilde{D})$.

For the 1-eigenspace we see that for $k \in \mathbb{Z}$:

$$s_{k,1} = \frac{2\pi k + i\log(1)}{\sigma + \pi} \in \frac{2\pi}{\sigma + \pi}\mathbb{Z}.$$

Since $\sigma < 3\pi$ it follows that $|s_{k,1}| \in (0, \frac{1}{2}]$ does not occur. Thus on this space, $\mathcal{D}(\tilde{D}_{\mathrm{slow,max}}) = W^{2,1}$. It immediately follows that it can be split into p-forms.

For the other eigenspaces we notice that B_N and B_Y^2 preserve both spaces and that $\nu(\frac{\partial}{\partial r}+\frac{1}{r}B_N)$ commutes with the projection on $(p-1)$-forms in $\Lambda^*(Y)$. Further a local complement of $W^{2,1}$ is spanned by solutions of $\nu(\frac{\partial}{\partial r}+\frac{1}{r}B_N)\varphi=0$, so it can be split into p-forms. It thus follows that the projection on p-forms preserves $\mathcal{D}(\tilde{D}_{\text{slow,max}})$.

Finally we notice that for $\sigma\in(0,3\pi)$ also $s_{k,e^{-i(\pi-\sigma)}}\neq\frac{1}{2}$ and $s_{k,e^{i(\pi-\sigma)}}\neq\frac{1}{2}$ for any k. Thus $\ker(B_N)=\{0\}$ and there is no augmentation in the wedge singularities. □

Let in the following M_∞ be like in Section 3.2. Then, since $\ker(B_N)=\{0\}$ for all wedge singularities, M_∞ is a manifold with cylindrical ends and wedge singularities. In addition to the objects defined in Section 3.2 we let d_∞ and d_∞^* denote the exterior differential and its adjoint on M_∞.

Lemma 3.3.3. *Each of the operators d_∞ and d_∞^* are defined on $\mathcal{D}_\infty(D_\infty)$. Further, $d_\infty\mathcal{D}_\infty(D_\infty)$ is orthogonal to $d_\infty^*\mathcal{D}_\infty(D_\infty)$.*

Proof: By Lemma 3.3.2 it is enough to check that if $\omega\in\mathcal{D}_\infty(D_\infty)$ is a form of pure degree, then $d_\infty\omega$ and $d_\infty^*\omega$ both belong to $\mathcal{H}_\infty$. But this is clear since $D_\infty\omega\in\mathcal{H}_\infty$ and d_∞ and d_∞^* map ω to forms of different degrees. In order to check that the images of d_∞ and d_∞^* are orthogonal to each other it is enough to check that the images of $\mathcal{D}_\infty(D_\infty^2)$ are orthogonal. This follows since $\mathcal{D}(D_\infty^2)$ is a core for D_∞. For $\omega\in\mathcal{D}_\infty(D_\infty^2)$, $\omega'\in\mathcal{D}_\infty(D_\infty^2)$

$$\begin{aligned}\langle d_\infty\omega, d_\infty^*\omega'\rangle &= \sum_{p=0}^{n-2}\left\langle d_\infty\omega_p, d_\infty^*\omega'_{p+2}\right\rangle\\ &= \sum_{p=0}^{n-2}\left\langle D_\infty\omega_p, D_\infty\omega'_{p+2}\right\rangle\\ &= \sum_{p=0}^{n-2}\left\langle D_\infty^2\omega_p, \omega'_{p+2}\right\rangle\\ &= 0.\end{aligned}$$

The last equation holds since D_∞^2 preserves the degree of forms. The integration by parts did not course a contribution from ∞ because D_∞ maps $\mathcal{D}_\infty(\tilde{D}_\infty)$ into $\mathcal{H}_\infty$. □

Let in the following $\mathcal{D}(\tilde{M},\tilde{E})=C_0^\infty(\tilde{M}^\circ,\tilde{E})$ and let $\mathcal{D}'(\tilde{M},\tilde{E})$ be the dual space of $\mathcal{D}(\tilde{M},\tilde{E})$.

Lemma 3.3.4. *For every closed form $h\in\mathcal{D}(\tilde{D}_{\text{slow,max}})$ there exists $\xi\in W^{2,1}(\tilde{M},\tilde{E})$ and $\eta\in\mathcal{D}'(\tilde{M},\tilde{E})$ such that η vanishes on $(\frac{1}{2},1]\times Z$ and*

$$h=\xi+\tilde{d}\eta.$$

Further, for every co-closed form $h'\in\mathcal{D}(\tilde{D}_{\text{slow,max}})$ there exists $\xi'\in W^{2,1}(\tilde{M},\tilde{E})$ and $\eta'\in\mathcal{D}'(\tilde{M},\tilde{E})$ such that η' vanishes on $[\frac{1}{2},1]\times Z$ and

$$h'=\xi'+\tilde{d}^*\eta'.$$

Proof: Let $a = \sigma + \pi$. We decompose

$$H_*(\mathbb{R}^{2,a}_\times \times Y, \mathbb{C}) = H_0(\mathbb{R}^{2,a}_\times, \mathbb{C}) \otimes H_*(Y, \mathbb{C}) \oplus H_1(\mathbb{R}^{2,a}_\times, \mathbb{C}) \otimes H_*(Y, \mathbb{C}),$$
$$H^*(\mathbb{R}^{2,a}_\times \times Y, \mathbb{C}) = H^0(\mathbb{R}^{2,a}_\times, \mathbb{C}) \otimes H^*(Y, \mathbb{C}) \oplus H^1(\mathbb{R}^{2,a}_\times, \mathbb{C}) \otimes H^*(Y, \mathbb{C}).$$

By the de Rham isomorphism, the elements of $H^0(\mathbb{R}^{2,a}_\times, \mathbb{C}) \otimes H^*(Y)$ are exactly the cohomology classes $[\omega]$ such that

$$\langle \sigma, \omega \rangle = 0$$

for all $\sigma \in H_1(\mathbb{R}^{2,a}_\times, \mathbb{C}) \otimes H_*(Y)$. By the growth condition $\varphi(r, \theta) = O(r^{-\frac{1}{2}})$, holding for all $\varphi \in \mathcal{D}(\tilde{D}_{\text{slow,max}})$ it follows that all closed elements of $\mathcal{D}(\tilde{D}_{\text{slow,max}})$ are locally cohomologous to elements of the form $1 \wedge h_Y$, where h_Y is a harmonic form on Y.

The first part of the lemma now follows by using suitable cutoff functions.

The second part of the lemma follows by the first and Hodge duality. □

Theorem 3.3.5. *There is a direct sum decomposition*

$$\mathcal{H}_\infty = \ker(D^2_\infty) \oplus \overline{\operatorname{Im}(d_\infty)} \oplus \overline{\operatorname{Im}(d^*_\infty)}. \tag{3.3.2}$$

All harmonic forms in $\mathcal{D}_\infty(D_\infty)$, $\mathcal{D}(D)$ and $\mathcal{D}(\tilde{D})$ are closed and co-closed.

Proof: By self-adjointness of D_∞ it follows that

$$\mathcal{H}_\infty = \ker(D^2_\infty) \oplus \overline{\operatorname{Im}(D_\infty)}. \tag{3.3.3}$$

Combining this with Lemma 3.3.3 gives (3.3.2). Now assume $\omega \in \mathcal{D}_\infty(D_\infty)$ is a harmonic p-form. If $f \in C^\infty_0(\mathcal{H}_\infty)$

$$\langle \omega, d_\infty f \rangle = \langle \omega, d_\infty f_{p-1} \rangle = \langle \omega, D_\infty f_{p-1} \rangle = 0.$$

Since all harmonic forms can be split into harmonic forms of pure degree it follows that for ω an arbitrary harmonic form

$$\langle \omega, d_\infty f \rangle = 0 \quad \text{and} \quad \langle \omega, d^*_\infty f \rangle = 0.$$

This implies that $d\omega = d^*\omega = 0$. By the extension properties, also harmonic forms in $\mathcal{D}(\tilde{D})$ and $\mathcal{D}(D)$ are closed and co-closed. □

Lemma 3.3.6. *Let f be a closed form in $\mathcal{H}_\infty$. Then there is a decomposition*

$$f = f_0 + \tilde{d}\eta,$$

where $f_0 \in \ker(D_\infty)$ and $\eta \in \mathcal{D}'(\mathcal{H}_\infty)$.

Proof: For $R > 0$, let P_R be the orthogonal projection on $\mathcal{H}_R$ and let Q_R be the orthogonal projection on $\ker(D_R)^\perp$ in $\mathcal{H}_R$. By Lemma 3.2.9 there exists $g_\infty(R) \in \mathcal{D}(D_\infty)$ such that $D_\infty g_\infty(R) = Q_R P_R f$.

Let V be the annihilator of $\ker_\infty(D_\infty)$ in $C_0^\infty(\mathcal{H}_\infty)$. For $\varphi \in V$, pick $R' < R$ big enough such that $\varphi \in \mathcal{H}_{R'} \subseteq \mathcal{H}_R$. By Lemma 3.2.9 there exists $\psi \in \mathcal{D}(D_\infty)$ with $D_\infty \psi = \varphi$. Thus

$$\langle \varphi, g_\infty(R) \rangle = \langle D_\infty \psi, g_\infty(R) \rangle .$$

Using symmetry of D_∞ gives

$$= \langle \psi, D_\infty g_\infty(R) \rangle = \langle \psi, Q_R P_R f \rangle .$$

Now, $Q_R P_R f \in P_R f + P_R \ker_\infty(D_\infty)$ and $\|Q_R P_R f\|_{\mathcal{H}_\infty} \leq \|f\|_{\mathcal{H}_\infty}$. In particular the $\mathcal{H}_R$-norm of the error term in $P_R \ker_\infty(D_\infty)$ is bounded for fixed R' and $R \to \infty$. Since $\ker_\infty(D_\infty)$ is finite dimensional this proves that $P_{R'} Q_R P_R f$ has an accumulation point for $R \to \infty$. Thus there exists a sequence R_m such that for all $\psi \in \mathcal{H}_\infty$, $\langle \psi, Q_{R_m} P_{R_m} f \rangle$ is convergent. It follows that also the sequence $\{\langle \varphi, g_\infty(R_m) \rangle\}$ is convergent. Consequently the sequence $\{g_\infty(R_m)\}$ is convergent in the dual space V^* of V. We notice that a complement of V in $C_0^\infty(\mathcal{H}_\infty)$ can be given and is a finite dimensional space of dimension $\dim(\ker_\infty(D_\infty))$.

The split exact sequence

$$0 \mapsto V \mapsto C_0^\infty(\mathcal{H}_\infty) \mapsto C_0^\infty(\mathcal{H}_\infty)/V \mapsto 0$$

gives rise to a split exact sequence

$$0 \mapsto (C_0^\infty(\mathcal{H}_\infty)/V)^* \mapsto \mathcal{D}'(\mathcal{H}_\infty) \mapsto V^* \mapsto 0.$$

Thus the dual of $C_0^\infty(\mathcal{H}_\infty)/V$ is canonically identified with the annihilator of V in $\mathcal{D}'(\mathcal{H}_\infty)$. Since a complement of V is finite dimensional and has $\ker_\infty(D_\infty)$ as dual, the annihilator of V in $\mathcal{D}'(\mathcal{H}_\infty)$ can be canonically identified with $\ker_\infty(D_\infty)$. Consequently the limit

$$\lim_{m \to \infty} g_\infty(R_m)$$

is well defined in $\mathcal{D}'(\mathcal{H}_\infty)$ up to an element of $\ker_\infty(D_\infty)$. Let $\eta \in \mathcal{D}'(\mathcal{H}_\infty)$ be such an element. Then $D_\infty \eta = Q_\infty f$. Further, since f is closed we get

$$\langle f, d^* \eta \rangle = \lim_{m \to \infty} \langle f, d^* g_\infty(R_m) \rangle = 0.$$

Thus in fact $f = f_0 + d\eta$, as claimed. □

As in [2] an important step is to find the relation between harmonic forms in the domain of $\tilde{D}^2$ and the cohomology of M.

Let $\sigma > 0$ be given. The diffeomorphism $\kappa : \mathbb{R}_\times^{2,\sigma+\pi} \mapsto \mathbb{R}^2 \setminus \{0\}$ given by

$$\kappa([r, \theta]) = \begin{pmatrix} r \cos \frac{2\pi\theta}{\sigma+\pi} \\ r \sin \frac{2\pi\theta}{\sigma+\pi} \end{pmatrix}$$

extends by continuity to a homeomorphism $\bar{\kappa}$ from the completion of $\mathbb{R}_\times^{2,\sigma+\pi}$ to $\mathbb{R}^2$. Using $\bar{\kappa}$ we may thus give $\tilde{M}$ a canonical differentiable structure. Let M_1 denote the smoothened version of $\tilde{M}$. Let g_1 be an arbitrary smooth Riemannian metric on M_1 such that the inclusion $\tilde{M} \mapsto M_1$ is an isometry on $[\frac{1}{2}, 1] \times Z$. Then (M_1, g_1) is a smooth Riemannian manifold with boundary and product structure in a neighborhood of the boundary.

Now let $\bar{M}$ be the extension of M_1 to a manifold with cylindrical ends and let $\bar{E} \mapsto \bar{M}$ be the bundle of differential forms on $\bar{M}$. Finally let $\bar{d}$, $\bar{d}^*$ and $\bar{D}$ be the operators of exterior differentiation, its adjoint and the Dirac operator $\bar{D} = \bar{d} + \bar{d}^*$, defined on $\bar{M}$.

Let

$$\ker(\tilde{D})_0 := \{f \in \ker(\tilde{D}) \mid f_{|Z} \in \ker(A)^{\perp}\}.$$

There are maps

$$\imath^* : H^*(\bar{M}) \mapsto H^*(\tilde{M}),$$

$$[\cdot] : \ker(\tilde{D})_0 \mapsto H^*(\tilde{M}^\circ).$$

Here the first map is pullback and the second is the association of a cohomology class to a harmonic form. All cohomology spaces are taken to have complex coefficients.

Lemma 3.3.7. *The image of* $[\cdot]$ *coincides with* $\imath^* H^*_{\mathrm{comp}}(\bar{M})$. *Here* $H^*_{\mathrm{comp}}(\bar{M})$ *is the cohomology with compact support.*

Proof: Since $\bar{M}$ is a manifold with cylindrical ends homotopy equivalent to M, by [2, Proposition 4.9] it suffices to prove, that the image of the space $\ker(\tilde{D})_0$ in $H^*(\tilde{M})$ is isomorphic to the image of the space $\ker(\bar{D})$ in $H^*(\tilde{M})$. Here $\bar{D}$ is the Dirac operator on $\bar{M}$.

First assume that $\tilde{\omega} \in \ker(\tilde{D})_0$. Then $\tilde{d}\tilde{\omega} = 0$ by Lemma 3.3.5 and by Lemma 3.3.4, $\tilde{\omega}$ is cohomologous to a form $\xi \in W^{2,1}(\tilde{M}, \tilde{E})$, which is equal to $\tilde{\omega}$ on $[\frac{1}{2}, 1]$. The inclusion $\imath : \tilde{M} \mapsto \bar{M}$ induces an isomorphism $\imath^* : W^{2,1}(M_1, \bar{E}) \mapsto W^{2,1}(\tilde{M}, \tilde{E})$. Thus $(\imath^*)^{-1}(\xi)$ is a closed L^2-form in $W^{2,1}(M_1, \bar{E})$. Further this form extends harmonically to a form $\jmath_0(\tilde{\omega})$ on $\bar{M}$. Let $\mathbf{h}$ be the harmonic component of $\jmath_0(\tilde{\omega})$. By a theorem of de Rham and Kodeira [11, Theorem 25], there exists a current ζ, such that $\jmath_0(\tilde{\omega}) = \mathbf{h} + \tilde{d}\zeta$. Pulling back and using that the cohomology can be computed from the space of currents, we get that $\imath^*(\mathbf{h})$ induces the same cohomology class as $\tilde{\omega}$.

On the other hand any harmonic L^2-form $\bar{\omega}$ on $\bar{M}$ can be pulled back to a closed $W^{2,1}$-form $\tilde{\omega}$ on $\tilde{M}$. The harmonic component of $\tilde{\omega}$ then induces the same cohomology class as the pullback of $\bar{\omega}$ by Lemma 3.3.6.

This completes the proof of the lemma. □

Theorem 3.3.8. *The space* $\ker(\tilde{D})_0$ *is canonically isomorphic to the image of* $H^*_{\mathrm{comp}}(\overline{M})$ *in* $H^*(\overline{M})$.

Proof: By Lemma 3.3.7 it suffices to prove that the restriction of the pullback $\imath^*$: $H^*(\overline{M}) \mapsto H^*(\tilde{M}^\circ)$ to $H^*_{\mathrm{comp}}(\overline{M})$ is injective. That means that it suffices to prove that if some $f \in C_0^\infty(\Lambda^*(\overline{M}))$ satisfies that $\imath^*(f) = dg$ for some $g \in C^\infty(\Lambda^*(\tilde{M}^\circ))$, then for some $g' \in C^\infty(\Lambda^*(\overline{M}))$, $f = dg'$.

To this end notice that there exists a diffeomorphism ϕ of $\overline{M}$ homotopic to the identity such that the support of $\phi^*(f)$ is contained in the interior of M. Further there exists a diffeomorphism ϕ' between the interior M° of M and $\overline{M}$ such that $\phi = \phi'$ on the support of f'. Thus f is cohomologous to some f' with support in M°. If $\imath^*(f') = \tilde{d}g'$, also $f = ((\phi')^{-1})^*(f') = \bar{d}((\phi')^{-1})^*(g')$, so f is cohomologous to zero. □

Corollary 3.3.9. *For the signature complex we have*

$$\mathrm{Index}_0(\tilde{D}) = \mathrm{sign}(M) \tag{3.3.4}$$

and for the de Rham complex we have

$$\mathrm{Index}_0(\tilde{D}) = \chi(M). \tag{3.3.5}$$

Proof: The identity (3.3.4) holds because the signature is by definition the signature of $\langle \tau \cdot, \cdot \rangle$ on the image of $H^*(M, Z)$ in $H^*(M)$, which is isomorphic to the image of $H^*_{\mathrm{comp}}(\bar{M})$ in $H^*(\bar{M})$. For (3.3.5) we notice that since Z is a closed manifold of odd dimension, $\chi(Z) = 0$. Thus $\chi(M) = \chi((M, Z)) + \chi(Z) = \chi((M, Z))$. Now the long exact sequence

$$H^*(M, Z) \mapsto H^*(M) \mapsto H^*(Z)$$

gives that

$$0 = \chi(Z) = \chi(\mathrm{Im}(H^*(M) \mapsto H^*(Z))) + \chi(\mathrm{Im}(H^*(Z)) \mapsto H^*(M, Z)).$$

By averaging over two ways to compute $\mathrm{Index}(D)_0$ we get

$$\begin{aligned}
\mathrm{Index}(D)_0 &= \frac{1}{2}\left(\chi(H^*(M)) - \chi(\mathrm{Im}(H^*(M) \mapsto H^*(Z)))\right) \\
&+ \frac{1}{2}\left(\chi(H^*(M, Z)) - \chi(\mathrm{Im}(H^*(Z) \mapsto H^*(M, Z))\right) \\
&= \frac{1}{2}\left(\chi(M) + \chi(M, Z) - \chi(Z)\right) \\
&= \chi(M).
\end{aligned}$$

□

The signature and de Rham complexes allow the same analysis as in [2] to be carried out.

Lemma 3.3.10. *For the signature and de Rham complexes we have*

$$\dim(\ker(S_\pm - 1)) = \frac{1}{2}\dim H^*(Y).$$

Proof: In this proof we use $\pm$ for the signature complex and ev/odd for the de Rham complex. Since the involutions corresponding to the two complexes commute, any combitation can be taken.

The Lefschetz duality theorem [15, Section 28] gives that the diagram

$$\begin{array}{ccccccccc} \mapsto & H^{q-1}(M) & \overset{j}{\mapsto} & H^{q-1}(Z) & \overset{\delta}{\mapsto} & H^q(M,Z) & \mapsto & H^q(Z) & \mapsto \\ & (-1)^{q-1}\zeta\cap\downarrow & & (\delta\zeta)\cap\downarrow & & \zeta\cap\downarrow & & \zeta\cap\downarrow & \\ \mapsto & H_{n-q+1}(M,Z) & \overset{\partial}{\mapsto} & H_{n-q}(Z) & \overset{j^*}{\mapsto} & H_{n-q}(M) & \mapsto & H_{n-q}(M,Z) & \mapsto \end{array} \tag{3.3.6}$$

is commutative and that the vertical arrows are isomorphisms. The horizontal maps make up exact sequences. Here ζ is the fundamental class of M, the horizontal maps are induced by inclusions and restrictions, and δ and ∂ are the connecting homomorphisms. The integer q runs from 0 to $\dim(M)$. A diagram chase shows that $(\delta\zeta)\cap$ is an isomorphism between $\mathrm{Im}(j)$ and its orthogonal complement. In particular

$$\dim(\mathrm{Im}(j)) \leq \frac{1}{2}\dim(H^*(Z)). \tag{3.3.7}$$

Every $\varphi \in \ker(A_\pm)$ is of the form $\varphi = \omega \pm \tau\omega$, where $\omega \in \Lambda^*(Z)$. Thus the pullback ω of φ vanishes if and only if φ vanishes. Since restrictions of harmonic sections on $\tilde{M}$ to Z factors through j it follows

$$\dim(\ker(S_+ - 1)) \leq \frac{1}{2}\dim(H^*(Z)), \tag{3.3.8}$$

$$\dim(\ker(S_- - 1)) \leq \frac{1}{2}\dim(H^*(Z)). \tag{3.3.9}$$

On the other hand $\ker(S_+ - 1) \oplus \ker(S_- - 1)$ is a Lagrangian subspace for ν, so

$$\begin{aligned} \dim(\ker(S_+ - 1) \oplus \ker(S_- - 1)) &= \dim\left(\ker(S_+ - 1) \oplus \nu\ker(S_+ + 1)\right) \\ &= \dim(\ker(A_+)) \\ &= \dim(H^*(Z)). \end{aligned}$$

Combining this with (3.3.8) and (3.3.9) immediately gives

$$\dim\ker(S_+ - 1) = \dim\ker(S_+ + 1) = \frac{1}{2}\dim(H^*(Z)). \tag{3.3.10}$$

This proves the lemma for the signature complex. In order to handle the de Rham complex, notice that we may split $\Lambda^*(M)$ into the direct sum of two Clifford bundles

$$\Lambda^*(M) = \left(\Lambda^{+\mathrm{ev}}(M) \oplus \Lambda^{-\mathrm{odd}}(M)\right) \oplus \left(\Lambda^{-\mathrm{ev}}(M) \oplus \Lambda^{+\mathrm{odd}}(M)\right). \tag{3.3.11}$$

Now, j is injective on each of the $\pm$-spaces and maps the odd/ev spaces to complementary subspaces of $H^*(Y)$. It thus follows like above

$$\dim\ker(S_{+\mathrm{ev}} - 1) + \dim\ker(S_{-\mathrm{odd}} - 1) = \frac{1}{2}\dim(H^*(Z)), \tag{3.3.12}$$

$$\dim\ker(S_{+\mathrm{odd}}-1)+\dim\ker(S_{-\mathrm{ev}}-1)=\frac{1}{2}\dim(H^*(Z)). \quad (3.3.13)$$

Now since $\jmath(\mathrm{Im}(S_+-1))=\jmath(\mathrm{Im}(S_--1))$ it follows that $\jmath\mathrm{Im}(S_{+\mathrm{ev}}-1)=\jmath\mathrm{Im}(S_{-\mathrm{ev}}-1)$. Further, since $\jmath$ is injective on each of those spaces it follows that

$$\dim\ker(S_{+\mathrm{ev}}-1)=\dim\ker(S_{-\mathrm{ev}}-1).$$

Thus $\dim(\ker(S_{\mathrm{odd}}-1))=\frac{1}{2}\dim\ker(H^*(Z))$, and the lemma follows for the de Rham complex also. □

From Lemma 3.3.10 we conclude

Corollary 3.3.11. *For the de Rham and signature complexes we have*

$$\mathrm{tr}\,(S_+)=0.$$

□

4 Computation of the Index.

Let in the following $\tilde{E}\mapsto\tilde{M}$ be a Dirac bundle over an even-dimensional Riemannian manifold $\tilde{M}$ with boundary, product structure around the boundary and wedge singularities of codimension 2. Let $\tilde{D}$ be the self-adjoint realization of the associated Dirac operator with slow-growing ideal Atiyah-Patodi-Singer boundary conditions augmented with respect to the scattering matrix.

We will assume that $\tilde{E}$ is a super bundle, i.e. that a parallel self-adjoint involution $\tau\in C^\infty(End(\tilde{E}))$, which anti-commutes with Clifford multiplication and which preserves $\mathcal{D}(\tilde{D})$, is given. With respect to the ± 1 eigenspaces of τ, $\tilde{D}$ has the following decomposition

$$\tilde{D}=\begin{pmatrix}0 & \tilde{D}_-\\ \tilde{D}_+ & 0\end{pmatrix}. \quad (4.0.1)$$

The index, we will compute, is that of $\tilde{D}_+$.

4.1 Heat Kernel Estimates.

Lemma 4.1.1. *The operator $e^{-t\tilde{D}^2}$ is of trace class. For each $t>0$ we have*

$$\mathrm{Index}(\tilde{D}_+)=\mathrm{tr}\,(\tau e^{-t\tilde{D}^2})=\int_{\tilde{M}}\mathrm{tr}\,(\tau e^{-t\tilde{D}^2}(x,x))dx. \quad (4.1.1)$$

In particular

$$\mathrm{Index}(\tilde{D}_+)=\lim_{t\to 0}\int_{\tilde{M}}\mathrm{tr}\,(\tau e^{-t\tilde{D}^2}(x,x))dx. \quad (4.1.2)$$

Proof: By the semi-group property of $e^{-t\tilde{D}^2}$ it suffices to prove that $e^{-t\tilde{D}^2}$ is a Hilbert-Schmidt operator for each $t > 0$. Let for some small $\varepsilon > 0$:

$$U_\varepsilon := \{x \in \tilde{M} \mid \exists \text{ a wedge } Y : \operatorname{dist}(x, Y) < \varepsilon \text{ or } \operatorname{dist}(x, Z) < \varepsilon\}.$$

For $x \in \tilde{M} \setminus U_\varepsilon$ and $v^* \in \tilde{E}^*_{|x}$, let $\delta_x \otimes v^*$ be the distribution with values in $\tilde{E}^*$ given by

$$(\delta_x \otimes v^*)(\varphi) = \langle \varphi(x), v^* \rangle \, .$$

By elliptic regularity and self-adjointness of $e^{-t\tilde{D}^2}$, $e^{-t\tilde{D}^2}$ maps distributions of the form $\delta_x \otimes v$ into $L^2(\tilde{M}, \tilde{E})$, and there is an estimate

$$\|e^{-t\tilde{D}^2}(\delta_x \otimes v^*)\|_{L^2(\tilde{M},\tilde{E})} \leq C(\varepsilon, t) \|v\|_{\tilde{E}^*_{|x}}.$$

This immediately gives that the component of $e^{-t\tilde{D}^2}$ mapping $L^2(\tilde{M} \setminus U_\varepsilon, \tilde{E})$ into $L^2(\tilde{M}, \tilde{E})$ is a Hilbert Schmidt operator. By symmetry of the heat kernel, also the component mapping $L^2(\tilde{M}, \tilde{E})$ into $L^2(\tilde{M} \setminus U_\varepsilon, \tilde{E})$ is a Hilbert Schmidt operator. Like above it also follows that for any differential operator P with smooth coefficients, the component of $Pe^{-t\tilde{D}^2}$ mapping $L^2(\tilde{M} \setminus U_\varepsilon, \tilde{E})$ to $L^2(\tilde{M}, \tilde{E})$ is a Hilbert Schmidt operator.

It remains to prove that the component of $e^{-t\tilde{D}^2}$ mapping U_ε into U_ε is a Hilbert Schmidt operator. Take ε small enough such that $U_{4\varepsilon}$ can be identified with a disjoint union of neighborhoods of 0 in closed model wedges and neighborhoods of the boundary of half-cylinders. Let X be the union of those closed model wedges and half-cylinders and let E_X be the extension by the product structure to X of $\tilde{E}_{|U_{|4\varepsilon}}$. Let D_X be the associated Dirac operator on X.

Now let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi(r) = 1$ for $r \in [0, \varepsilon]$, $\varphi(r) = 0$ for $r > 2\varepsilon$ and let $\psi \in C^\infty(\mathbb{R})$ be a function such that $\psi(r) = 1$ for $r \in \operatorname{supp}(\varphi)$ and such that $\psi(r) = 0$ for $r \geq 3\varepsilon$. Let $\mathcal{M}_\varphi$ and M_ψ denote the operators of multiplication by φ and ψ, respectively, (either in $\tilde{M}$ or in X). By Duhamels principle it follows

$$M_\varphi e^{-t\tilde{D}^2} M_\varphi - M_\varphi e^{-tD_X^2} M_\varphi = \int_0^t M_\varphi \frac{\partial}{\partial s} e^{-s\tilde{D}^2} M_\psi e^{-(t-s)D_X^2} M_\varphi ds \tag{4.1.3}$$

$$= -\int_0^t M_\varphi e^{-s\tilde{D}^2} \left(\tilde{D}^2 M_\psi - M_\psi D_X^2\right) e^{-(t-s)D_X^2} M_\varphi ds \tag{4.1.4}$$

Since $\tilde{D}^2 M_\psi - M_\psi D_X^2$ is a differential operator with compact support away from the boundary it follows that (4.1.3) is a Hilbert Schmidt operator. Thus it suffices to prove that $M_\varphi e^{-tD_X^2} M_\varphi$ is a Hilbert Schmidt operator. This can be done by splitting D_X^2 into a direct sum of operators on the eigenspaces of B_Y^2 and exploiting that the gluing operator has only finitely many eigenvalues. This gives that $e^{-tD_X^2}$ is a direct sum of the type $\bigoplus e^{-t\mu^2} e^{-tD_j^2}$, where j runs over a finite index set and $\sum_\mu e^{-t\mu^2}$ is convergent. Further each D_j is a Dirac operator on a cone, so that $M_\varphi e^{-tD_j^2} M_\varphi$ is a Hilbert Schmidt operator.

Since $\tilde{D}$ has a discrete point spectrum with eigenvalues of finite multiplicity and because $\tilde{D}$ commutes with $\tilde{D}^2$ and anti-commutes with τ, the restriction of $\tilde{D}$ to each eigenspace of $\tilde{D}^2$ anti-commutes with the restriction of τ, and except for in $\ker(\tilde{D})$ this gives that each eigenvalue of $\tilde{D}^2$ contributes with a zero to (4.1.1). Finally $\ker(\tilde{D}^2)$ contributes to (4.1.1) with $\mathrm{Index}(\tilde{D}_+)$. □

4.2 Localized Index Contributions.

Using finite propagation speed or Duhamels principle it follows that the limit (4.1.2) can be split into two contributions:

a) The interior contribution I_{int} given by

$$I_{\mathrm{int}} := \int_{\tilde{M}} a_D(x)dx, \tag{4.2.1}$$

where $a_D(x)$ is the zero order term in the local heat trace expansion

$$e^{-t\tilde{D}^2}(x,x) \sim \sum_{k=-\infty}^{\dim(\tilde{M})} a_k(x)t^{-\frac{k}{2}}.$$

The term $a_D(x)$ is the same as in the local index formula for closed manifolds. See for example [3].

b) A joint contribution coming from the various boundaries and wedge singularities. This contribution is further the same as the boundary contribution coming from a Dirac operator on the disjoint union of the corresponding model spaces. Notice that since the scattering matrix S mixes contributions from the various boundary components, each boundary component or wedge singularity can not be treated separately.

We will consider each boundary component as a wedge with $Y = Z$, $N = \{0\}$ and $B_N = 0$. We notice that on the model space $\sqcup \mathbb{R}_+ \times N \times Y$, the Dirac operator splits into a sum of Dirac operators on $\bigoplus \ker(B_N) \cap \ker(B_Y)$, and $(\bigoplus \ker(B_N) \cap \ker(B_Y))^{\perp}$. Further the mixing of boundary conditions from various boundary components only takes place in $\bigoplus \ker(B_N) \cap \ker(B_Y)$. On $\bigoplus \ker(B_N) \cap \ker(B_Y)$ the index contribution from the boundary is the same as for an operator of the form $\gamma \frac{\partial}{\partial u}$ defined on $L^2((-\infty, 0], \bigoplus \ker(B_N) \cap \ker(B_Y))$ with the restriction of τ as involution and the scattering matrix as boundary condition. This contribution is well known [31] and is given by

$$I_{\mathrm{scat}} := \frac{1}{2}\mathrm{tr}\,(S_+), \tag{4.2.2}$$

where S_+ is the restriction of S to $\ker(\tau - 1)$.

On the space $(\bigoplus \ker(B_N) \cap \ker(B_Y))^\perp$ the boundary conditions do not mix the various components, so we can consider each $N \times Y$ separately. If $Y = Z$ is a boundary component, the boundary contribution to the index is known from [2] and is given by

$$I_{\mathrm{bd}} := -\frac{1}{2}\eta(A_{Z+}, 0), \tag{4.2.3}$$

where A_Z is the induced Dirac operator on Z, A_{Z_+} denotes the restriction of A_{Z+} to $\ker(\tau - 1)$ and $\eta(A_{Z_+}, 0)$ denotes the η-invariant of A_{Z_+}.

On the wedges we again split the contribution to the index into contributions from $\ker(B_N) \cap \ker(B_Y)^\perp$, $\ker(B_N)^\perp \cap \ker(B_Y)^\perp$ and $\ker(B_N)^\perp \cap \ker(B_Y)$.

Lemma 4.2.1 and Lemma 4.2.2 have been stated separately because Lemma 4.2.1 holds in high generality, whereas Lemma 4.2.2 relies on the fact that $\dim(Y)$ is even.

Lemma 4.2.1. *The index contribution from* $\ker(B_N)^\perp \cap \ker(B_Y)^\perp$ *vanishes.*

Proof: We notice that the Dirac operator is locally of the form

$$D = \nu\left(\frac{\partial}{\partial r} + \frac{1}{r}B_N + \frac{1}{2r} + B_Y\right).$$

Further, D can be decomposed into eigenspaces of τ:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

This operator decomposes into a Direct sum of operators on the eigenspaces of B_Y^2. On each of those eigenspaces a small computation shows that the operator νB_Y conjugates D_+D_- into D_-D_+. Further, on an eigenspace of B_Y^2, νB_Y commutes with B_N and preserves the growth rate of sections. Consequently the operator $\nu B_Y|B_Y|^{-1}$ preserves the domain of D and D^2 and interchanges D_+D_- and D_-D_+. Let P be the projection on $\ker(B_N)^\perp \cap \ker(B_Y)^\perp$. It follows that

$$\int_N\int_Y \operatorname{tr}(\tau e^{-tD^2})((r,n,y),(r,n,y))dydn =$$
$$\int_N\int_Y \operatorname{tr}(e^{-tD_-D_+})((r,n,y),(r,n,y)) - \operatorname{tr}(e^{-tD_+D_-})((r,n,y),(r,n,y))dydn = 0.$$

This proves the lemma. □

Lemma 4.2.2. *The contribution to the index from* $\ker(B_N) \cap \ker(B_Y)^\perp$ *vanishes.*

Proof: Let B_{Y_+} and B_{Y-} be the restrictions of B_Y to the ± 1 eigenspaces of τ. Let τ_Y be the canonical involution on Y with respect to some orientation of Y and the structure of Clifford multiplication from $\tilde{E}$. Then τ_Y commutes with τ and anti-commutes with B_Y. Thus the spectra of B_{Y+} and $B_{Y\_}$ are symmetric. On the other hand the operator ν of Clifford multiplication in the radial direction at the singularity conjugates B_{Y_+} into $-B_{Y\_}$. This gives that B_{Y_+} and B_{Y-} are conjugate. Consequently, after having locally conjugated the restriction of $\tilde{D}$ to $\ker(B_N)$ into a Dirac operator on a piece of a cylinder, we get

$$\left(\nu(\frac{\partial}{\partial u}+B_Y)\right)^2=-\frac{\partial^2}{\partial u^2}+B_Y^2.$$

On the non-zero spectrum of B_Y, the restrictions of this operator to the the ± 1 eigenspaces of τ are thus conjugate and further have conjugate boundary conditions. This gives that the difference of the heat kernels vanishes to all orders for $t\to 0$, so the contribution to the index vanishes. □

Finally we consider the contribution I_{wedge} from the space $\ker(B_N)^\perp\cap\ker(B_Y)$. Here we notice that $\ker(B_Y)$ is a Clifford module over $\mathbb{R}_+\times N$ with respect to the Clifford module structure on $\mathbb{R}_+\times N\times Y$. We will fix some orientation on $\mathbb{R}_+\times N$ and let τ_N be the image of the volume form in the Clifford algebra with respect to that orientation.

The action of the gluing operator $U_{\ker(B_Y)}$ on $\ker(B_Y)\mapsto\mathbb{R}_+\times N$ is further the restriction of the gluing operator on $\tilde{E}$, considered as an operator in $L^2(Y,E_{|Y})$, to $\ker(B_Y)$. $U_{\ker(B_Y)}$ commutes with τ and τ_N and ν. Now let φ be a joint eigensection of B_N, τ and τ_N. Then $V:=\text{span}\,\{\varphi,\nu\varphi\}$ is preserved by B_N, τ, τ_N and ν. Thus this is a subspace of $\ker(B_Y)$ invariant under ν, τ_N, and thereby under Clifford multiplication, and $U_{\ker(B_Y)}$. Parallel transport of V in the radial direction gives a Dirac sub-bundle of $\ker(B_Y)$. Thus also $V^\perp$ is a Dirac sub-bundle of $\ker(B_Y)$. In this way the Dirac operator D decomposes to a Direct sum of Dirac operators in 2-dimensional vector bundles $V_1,\dots,V_k$ over $\mathbb{R}_+\times N$. We decompose $V_i=V_{i+}\oplus V_{i-}$ into the ± 1 eigenspaces of τ_N.

Since τ and τ_N are commuting self-adjoint involutions in two dimensional bundles and both anti-commute with ν it follows that on each V_i, either $\tau=\tau_N$ or $\tau=-\tau_N$.

We write this as $\tau=\text{tr}\,(\tau_{|V_{i+}})\tau_N$. Let D_i be the Dirac operator on V_i. Then it follows

$$\text{tr}\,(\tau e^{-tD_i^2})((r,n),(r,n))=\text{tr}\,(\tau_{|V_{i+}})\text{tr}\,(\tau_N e^{-tD_i^2})((r,n),(r,n)). \tag{4.2.4}$$

Now, except from that $\ker(B_N)\cap\ker(B_Y)$ has been removed and B_N is not necessarily a spin operator, the last term is exactly as in [9], and the same computation goes through. See also [36]. It follows

$$I_{\text{wedge}}=-\frac{1}{2}\sum_i\text{tr}\,(\tau_{|V_{i+}})\eta(B_{N,i,+},0). \tag{4.2.5}$$

Here $B_{N,i,+}$ is the restriction of B_N to $V_i\cap\ker(\tau_N-1)$.

Another way to write (4.2.5) is to label $B_{N,i,+}$ according to the eigenvalues of $U_{\ker(B_Y)}$. If we fix V_i, $U_{|\ker(B_Y)}$ has complex eigenvalues α_+ and α_- in the $+1$ and -1 eigenspaces of τ_N, respectively. We notice that $B_{N,i,+}$ depends only on $\alpha+$ and write $B_{N,i,+}=B_{\alpha_+}$. In the following we denote by V_{α_+} the bundle $\ker(B_Y)\cap\ker(U-\alpha_+)\cap\ker(\tau_N-1)$. It follows

$$I_{\text{wedge}}=-\frac{1}{2}\sum_{\alpha_+\in\text{spec}(U_{\ker(B_Y)}|\ker(\tau_N-1))}\text{tr}\,(\tau_{|V_{\alpha_+}})\eta(B_{\alpha_+},0). \tag{4.2.6}$$

We have proved:

Theorem 4.2.3. *Let $\tilde{M}$ be a Riemannian manifold with boundary and isolated wedge singularities of codimension 2 such that $\tilde{M}$ has product structure in a neighborhood of the boundary and of the wedge singularities. Further let $\tilde{E}$ be a Dirac bundle over $\tilde{M}$ respecting the product structure and let $\tilde{D}$ be the realization of the associated Dirac operator given by imposing slow-growing ideal Atiyah-Patodi-Singer boundary conditions. If τ is a parallel self-adjoint involution in $C^\infty(\tilde{M}, \tilde{E})$ anti-commuting with Clifford multiplication and $\tilde{D}_+$ is the restriction of $\tilde{D}$ to* $\ker(\tau - 1)$ *then*

$$\begin{aligned}\mathrm{Index}(\tilde{D}_+) = \int_{\tilde{M}} a_D(x)dx - \frac{1}{2}\eta(A_+, 0) + \frac{1}{2}\mathrm{tr}\,(S_+) \\ - \frac{1}{2} \sum_{\alpha+ \in \mathrm{spec}(U_{\ker(B_Y)}|\ker(\tau_N - 1))} \mathrm{tr}\,(\tau_{|V_{\alpha+}})\eta(B_{\alpha+}, 0). \quad (4.2.7)\end{aligned}$$

The terms are defined around (4.2.1), (4.2.2), (4.2.3) and (4.2.6), respectively.

In the case where $\tilde{M}$ is the extension of a manifold M with corners we use the extension and restriction properties of elements of $\ker(\tilde{D})$ to prove an index theorem for manifolds with corners of codimension 2.

Corollary 4.2.4. *Let M be a manifold with corners of codimension 2 and product structure in a neighborhood of the boundary and corners. Further let $E \mapsto M$ be a Dirac bundle respecting the product structure and let D be the realization of the associated Dirac operator with the generalized Atiyah-Patodi-Singer boundary conditions defined by (3.1.2). If τ is a parallel self-adjoint involution in $C^\infty(M, End(E))$ anti-commuting with Clifford multiplication we have:*

$$\begin{aligned}\mathrm{Index}(D_+) = \int_{M} a_D(x)dx - \frac{1}{2}\eta(A_+, 0) + \frac{1}{2}\mathrm{tr}\,(S_+) \\ - \frac{1}{2} \sum_{\alpha_+ \in \mathrm{spec}(U_{\ker(B_Y)}|\ker(\tau_N - 1))} \mathrm{tr}\,(\tau_{|V_{\alpha_+}})\eta(B_{\alpha_+}, 0). \quad (4.2.8)\end{aligned}$$

Proof: This follows by Theorem 4.2.3 since $\ker(D) \cong \ker(\tilde{D})$. Because $\tilde{M} \setminus M$ has product structure, a_D vanishes identically on $\tilde{M} \setminus M$. □

5 Another Approach to Index Theory.

The way we have proceeded in order to construct boundary conditions on a manifold with corners, which give rise to a good index theorem, is by no means unique. It is just the simplest one to handle. In this section we present another construction, which is just as natural.

5.1 Boundary Conditions on $\tilde{M}'$.

In the proof of the self-adjointness of D we made essential use of the self-adjointness of $\tilde{D}$ and the extension property of harmonic sections. Further the extension property of harmonic sections on M to $\tilde{M}$ was important for proving the index theorem. These two important properties can however be obtained in other ways. Below we will define another extension $\tilde{M}'$ of M by gluing on a piece of a cylinder over each boundary component. This extension also allows an extension $\tilde{E}'$ of E. A self-adjoint Dirac operator $\tilde{D}'$ on $\tilde{M}'$ can be constructed using boundary conditions, which are not local, but more local than the boundary conditions on M.

We will restrict attention to the case, where the boundary has a decomposition into two manifolds Z_1 and Z_2 with boundary, which intersect in their boundaries only, such that none of Z_1 and Z_2 have self-intersections at the boundary. This was an irrelevant assumption in the other case, but here it will simplify things. In the end of this section we explain how to proceed without this assumption.

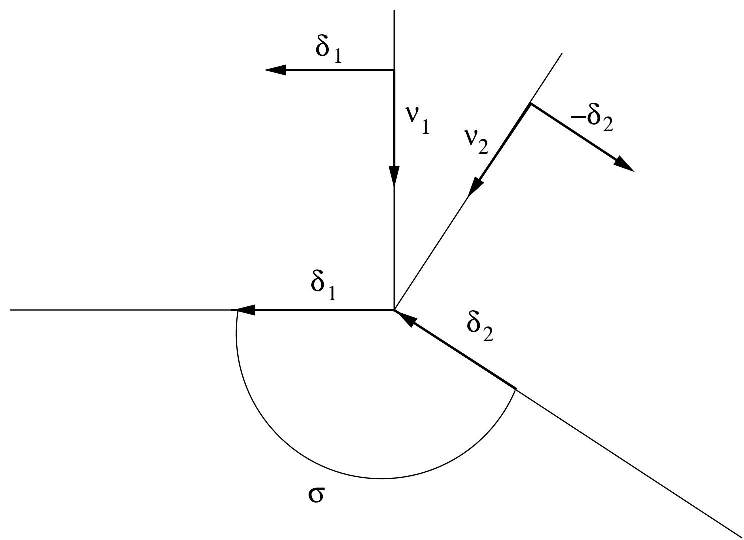

Fig5: A neighborhood of a corner component in $\tilde{M}$.

In the following let

$$\tilde{M}' = M \cup_{Z_1} (Z_1 \times [0,1]) \cup_{Z_2} (Z_2 \times [0,1]) \tag{5.1.1}$$

and let $\tilde{E}'$ be the obvious extension of E to a vector-bundle on $\tilde{M}'$, supplied with the product connection, Hermitian structure and structure of Clifford multiplication. To each Z_j there is an associated Dirac operator $A_j = \nu_j D_j$, which is a self-adjoint operator with Atiyah-Patodi-Singer boundary conditions augmented with respect to the scattering matrix. This augmentation has the crusial property that it commutes both with τ and ν_j. This follows from Lemma 2.1.6. Consequently τ and ν preserve the domain of A_j since they commute with the induced Dirac operators at the boundaries of each Z_j, and therefore also preserve the positive and negative spectral subspaces for the induced Dirac operators.

For most other augmentations the finer details of the analysis of $\tilde{D}$ break down because ν_j does not preserve the domain of A_j. Let in the following Q_{j+} be the projection on the strictly positive spectral subspace for A_j. Further let E_j be the restriction of $\tilde{E}'$ to $Z_j \times [0,1]$. We also write E_j for the extension of E_j by the product structure to $Z_j \times \mathbb{R}$.

We may define a domain $\mathcal{D}_0(\tilde{D}')$ by

$$\mathcal{D}_0(\tilde{D}') := \{f \in W^{2,1}(\tilde{M}', \tilde{E}') \mid \forall j, u > 0 : f_{|Z_j \times \{u\}} \in \mathcal{D}(A_j) \text{ and } (1-Q_{j+})f_{|Z_j\times\{1\}} = 0\}.$$

Lemma 5.1.1. *$\tilde{D}'$ is symmetric on $\mathcal{D}_0$.*

Proof: It suffices to prove that the domain

$$\mathcal{D}_{00}(\tilde{D}') := \{f \in W^{2,1}(\tilde{M}', \tilde{E}') \mid \forall j, u > 0 : f_{|Z_j \times \{u\}} \in \mathcal{D}(A_j),\ (1-Q_{j+})f_{|Z_j\times\{1\}} = 0 \text{ and } \exists \text{ a neighborhood } U \text{ of the corners of } M : f_{|U} = 0\}.$$

is dense in $\mathcal{D}_0$. This can be proved by decomposing into the eigenspaces of B_Y^2 in a neighborhood of the corners and proceeding like in Lemma 1.2.2. □

Lemma 5.1.2. *On $Z_j \times \mathbb{R}$ the norms $\|\cdot\|_{H^{2,1}}$ and $\|\cdot\|_{W^{2,1}}$ are equivalent on the domain:*

$$\mathcal{D} := \{f \in W^{2,1}(Z_j \times \mathbb{R}, E_j) \mid \forall u \in \mathbb{R} : f(u,\cdot) \in \mathcal{D}(A_j)\}.$$

Proof: Let $\mathcal{D}_0$ be the domain

$$\mathcal{D}_0 := \{f \in W^{2,2}(Z_j \times \mathbb{R}, E_j) \mid \forall u \in \mathbb{R} : f(u,\cdot) \in \mathcal{D}(A_j^2)\}.$$

Then $\mathcal{D}_0$ is dense in $\mathcal{D}$ with respect to the H^1-norm. This can be seen by decomposing sections into eigenspaces of A_j^2 and approximating the components with smooth functions. Consequently it suffices to show that the norms dominate each other on $\mathcal{D}_0$. By Lemma 1.2.1 we have that $\|\cdot\|_{H^{2,1}} \le C\|\cdot\|_{W^{2,1}}$, so it suffices to prove the opposite inequality.

Let D_j be the Dirac operator in E_j. Then D_j is of the form $D_j = \gamma(\frac{\partial}{\partial u} + A_j)$. If $f \in \ker(\gamma \pm i)$ this gives for some $c > 0$

$$\|D_j f\|^2 = \|\frac{\partial}{\partial u} f\|^2 + \|A_j f\|^2 \ge c\|\nabla f\|^2 - \|f\|^2. \tag{5.1.2}$$

If now $f \in \mathcal{D}_0$ we may decompose $f = f_+ + f_-$ into the components of $f \in \ker(\gamma \pm i)$. Then

$$\|D_j f\|^2 = \langle D_j^2 f, f\rangle = \langle D_j^2 f_-, f_-\rangle + \langle D_j^2 f_+, f_+\rangle$$

$$= \|D_j f_-\|^2 + \|D_j f_+\|^2.$$

Together with (5.1.2) this proves the lemma. □

Again we can use polar coordinates around the corners. We get the operators $\bigoplus B_N$ and $\bigoplus B_Y$, defined on $\coprod(Y \times [0, \sigma + \pi])$. B_N is given the same Atiyah-Patodi-Singer boundary conditions as $\bigoplus_j A_j$. First we notice that on the orthogonal complement of $\bigoplus \ker(B_Y)$ each corner can be handled separately and the analysis goes through exactly like for the operator $\tilde{D}$. A lemma, which requires some care is though:

Lemma 5.1.3. *Let F be a Dirac bundle respecting the product structure on a model corner $C \times Y$. On the domain*

$$\mathcal{D}_0 = \{f \in W^{2,1}(C \times Y, F) \mid (1 - Q_+)f_{|\partial C} = 0\},$$

the $W^{2,1}$ and $H^{2,1}$-norms are equivalent. Here Q_+ is the projection on the positive spectral subspace for $-\delta D_Y$, where δ is the outward-pointing normal at the boundary.

Proof: It is enough to prove the lemma for sections in $W^{2,2}(C \times Y, F)$, which in addition vanish in the corner and have vanishing normal derivatives at the boundary. If f is such a section we may compute

$$\begin{aligned}
\|\nabla f\|^2 &= \|\nabla^C f\|^2 + \|\nabla^Y f\|^2 \\
&\leq \big\langle (\nabla^C)^* \nabla^C f, f \big\rangle + C(\|B_Y f\|^2 + \|f\|^2) \\
&\leq (1 + C)\left(\big\langle D^2 f, f \big\rangle + \|f\|^2\right) \\
&= (1 + C)\left(\|Df\|^2 - \langle (-\delta D_Y) f, f \rangle_{\partial C \times Y} + \|f\|^2\right).
\end{aligned}$$

Since $\langle (-\delta D_Y) f, f \rangle_{\partial(C \times Y)} \geq 0$ this term can be dropped and the desired Sobolev inequality holds. □

Like for $\tilde{D}$, $\tilde{D}'$ is given slow-growing ideal Atiyah-Patodi-Singer boundary conditions augmented with respect to the scattering matrix. In $\bigoplus \ker(B_Y)$ the scattering matrices S_j mix the boundary conditions at different corners. This does not affect the self-adjointness of the restriction of the operator $\bigoplus B_N$ to $\bigoplus \ker(B_Y)$, nor does it affect that $\bigoplus B_N$ has a discrete point spectrum with eigenvalues of finite multiplicity. It however means that when we impose slow-growing ideal boundary conditions, asymptotics of sections in $\mathcal{D}(\tilde{D}')$ at the various corners are not independent.

Finally the space $\bigoplus \ker(A_j) \oplus \ker(\bigoplus B_N) \cap \ker(\bigoplus B_Y)$ is augmented with respect to the scattering matrix S defined exactly like in Section 2. This gives a self-adjoint realization $\tilde{D}'$.

Definition 5.1.4. We say that $\tilde{D}'$ is given *slow-growing Atiyah-Patodi-Singer boundary conditions of level 2 augmented with respect to the scattering matrices.*

It follows like for $\tilde{D}$ that $\tilde{D}'$ has a discrete point spectrum, that $e^{-t(\tilde{D}')^2}$ is a trace class operator for all $t > 0$, that if $\dim(M)$ is even, $\tilde{D}'$ has a decomposition like (4.0.1) and that $\mathrm{Index}(\tilde{D}'_+)$ can be computed using (4.1.2).

Theorem 5.1.5. *Let M be an even dimensional manifold with corners of codimension 2 and product structure around the boundary and corners and let $E \mapsto M$ be a Dirac bundle over M respecting the product structure. Let $\tilde{E}' \mapsto \tilde{M}'$ be the the extension* *of $E \mapsto M$ given by (5.1.1) and the product structure. Finally let $\tilde{D}'$ be the self-adjoint realization of the Dirac operator associated to $\tilde{E}'$, given slow-growing Atiyah-Patodi-Singer boundary conditions of level 2 augmented with respect to the scattering matrices. Let τ be a parallel self-adjoint involution in $\tilde{E}'$ anti-commuting with Clifford multiplication and let $\tilde{D}'_+$ be the restriction of $\tilde{D}'$ to $\ker(\tau - 1)$. Further let $\tau_N = -i\nu\delta$ and let $\tau_Y = \tau_N\tau$. Then we have the following index theorem:*

$$\mathrm{Index}(\tilde{D}'_+) = \int_M a_D(x)dx - \frac{1}{2}\sum_j \eta(A_{Z_j+}, 0) + \frac{1}{2}\mathrm{tr}\,(S_+) - \frac{1}{2}\eta(\tau_Y B_{N+|\oplus\ker(B_Y)}, 0). \quad (5.1.3)$$

Proof: By (4.1.2) we get that the index can be split into an interior contribution, a contribution from the boundary and a contribution from the corners (except from the augmentation, which mixes corners and boundaries). Because of the product structure, on the cylinders the interior contribution as well as the contribution from boundary components of the form $\partial Z_j \times [0,1]$ vanish. Further the boundary components of the form $Z_j \times \{1\}$ can be treated exactly like an Atiyah-Patodi-Singer boundary in [2]. This gives that the contribution from the non-zero spectrum of A_j is $-\frac{1}{2}\eta(A_{j+}, 0)$ as claimed. Lemma 4.2.1 and Lemma 4.2.2 hold with the same proofs as for $\tilde{D}$. Consequently the contribution from the corners of M comes from $\ker(B_Y)$ alone. The contribution from $\ker(B_Y) \cap \ker(B_N)^\perp$ is the same as from a cone except from that the super-structure is different. Thus the results of [9] gives the claimed contribution. Finally $\ker(\bigoplus B_Y) \cap \ker(\bigoplus B_N) \oplus \bigoplus \ker(A_j)$ gives the contribution $\frac{1}{2}\mathrm{tr}\,(S_+)$ because the restriction to $\tilde{D}'$ and τ to this space is conjugate to the corresponding contribution for an Atiyah-Patodi-Singer boundary. This proves the theorem. □

We proceed by computing the contribution from the corners explicitly. For each corner $C \times Y$ with associated Dirac operator B_Y we define an operator T_Y on $\ker(B_Y)$ by letting T_Y be multiplication by $\sigma + \pi$, where σ is the angle of C. Let T be the direct sum of the T_Y. Then T is self-adjoint and commutes with the operator $\nu\delta$, but in general T need not satisfy commutation relations with S_1 and S_2.

The operator B_N is given by $B_N = -\nu\delta\frac{\partial}{\partial\theta} - \frac{1}{2}$. Consequently solutions of $B_N u = \lambda u$ are locally of the form

$$u = e^{(\lambda+\frac{1}{2})\theta\nu\delta}\varphi$$

for some $\varphi \in \ker(B_Y)$. Further the boundary conditions give that

$$S_1\varphi = \varphi, \quad (5.1.4)$$

$$S_2 e^{(\lambda+\frac{1}{2})T\nu\delta}\varphi = e^{(\lambda+\frac{1}{2})T\nu\delta}\varphi. \quad (5.1.5)$$

In particular

$$S_1 e^{-(\lambda+\frac{1}{2})T\nu\delta} S_2 e^{(\lambda+\frac{1}{2})T\nu\delta}\varphi = \varphi. \quad (5.1.6)$$

On the other hand, if (5.1.6) is satisfied we notice that the solution space for φ for fixed λ has the solution space of (5.1.4) and (5.1.5) as a Lagrangian subspace for $\nu\delta$. This follows by the following general lemma.

Lemma 5.1.6. *Assume that U and V are unitary and self-adjoint operators on a Hilbert space and that , is unitary, anti-self-adjoint and anti-commutes with both U and V. Then* $\ker(U-1)\cap\ker(V-1)$ *is a Lagrangian subspace of* $\ker(UV-1)$.

Proof: The equation $UV\varphi = \varphi$ gives that $U\varphi = V\varphi$ since U is unitary and self-adjoint. Consequently the space $W := \operatorname{span}\{\varphi, V\varphi\}$ is closed under application of U and V. Further $UVV\varphi = U\varphi = V\varphi$, so W is contained in $\ker(UV-1)$. So is $W + , W$ since , commutes with UV. Since , interchanges the ± 1 eigenspaces for U it follows that $\ker(U-1)\cap(W+ , W)$ is a Lagrangian subspace for $W + , W$. But $\ker(U-1)\cap\ker(UV-1) = \ker(U-1)\cap\ker(V-1)$. The lemma follows since φ can be an arbitrary element of $\ker(UV-1)$. □

The above immediately gives the general result

Corollary 5.1.7. *The spectrum of B_N is given by*

$$\operatorname{spec}(B_N) = \{\lambda\in\mathbb{R} \mid \exists\varphi\neq 0 : S_1 e^{-(\lambda+\frac{1}{2})T\nu\delta} S_2 e^{(\lambda+\frac{1}{2})T\nu\delta}\varphi = \varphi\}.$$

Further the multiplicity of $\lambda\in\operatorname{spec}(B_N)$ is given by

$$\operatorname{mult}(\lambda) = \frac{1}{2}\dim\ker\left(S_1 e^{-(\lambda+\frac{1}{2})T\nu\delta} S_2 e^{(\lambda+\frac{1}{2})T\nu\delta} - 1\right).$$

□

If T commutes with S_2, Corollary 5.1.7 can be refined to give

$$\operatorname{spec}(B_N) = \{\lambda\in\mathbb{R} \mid \exists\varphi\neq 0 : S_1 S_2 e^{(2\lambda+1)T\nu\delta}\varphi = \varphi\}. \tag{5.1.7}$$

If T further commutes with S_1, (5.1.7) decomposes into the eigenspaces for S_1S_2. For each eigenspace of S_1S_2 it further decomposes into eigenspaces for T such that we get a union of spectra corresponding to the elementary case that $S_1S_2 = e^{i\beta}I$ and $T = aI$ are multiples of the identity. In this case we get

$$\operatorname{spec}((B_N)_{|\ker(S_1S_2-e^{i\beta})\cap\ker(T-a)\cap\ker(\nu\delta\pm i)}) = \{\lambda \mid (2\lambda+1)a \in \mp\beta + 2\pi\mathbb{Z}\}.$$

Explicitly

$$\operatorname{spec}((B_N)_{|\ker(S_1S_2-e^{i\beta})\cap\ker(T-a)\cap\ker(\nu\delta\pm i)}) = \mp\frac{\beta}{2a} - \frac{1}{2} + \frac{\pi}{a}\mathbb{Z}. \tag{5.1.8}$$

The operator τ_Y commutes with S_1S_2 and T, so (5.1.8) suffices to compute the contribution from the corner. We will however not do so before we have reached a deeper understanding of the corner term, such that we can write it up in a senseful way.

The condition that T commutes with S_1 and S_2 is satisfied in the applications in this paper, but is still completely unreasonable. We here give a lemma that reduces the general case to the case, where T commutes with S_1 and S_2. First we notice that the operator of application of $e^{-\frac{\theta}{2}\nu\delta}$ conjugates B_N into the operator $B_N' := -\nu\delta\frac{\partial}{\partial\theta}$ with boundary Lagrangians $\ker(S_1-1)$ and $\ker(e^{-\frac{1}{2}T\nu\delta}S_2e^{\frac{1}{2}T\nu\delta}-1)$. In a neighborhood of the boundary $\theta = 0$ this is a Dirac type operator. Further this conjugation of S_2 corresponds to that we identify the copies of $\ker(B_Y)$ seen from each boundary component at the corner using the universal gluing operator defined in Lemma 1.1.14.

Lemma 5.1.8. *The η-invariant $\eta(\tau_Y B_{N+}, 0)$ is equal to the η-invariant of the Dirac type operator $-\tau_Y \nu\delta\frac{\partial}{\partial\theta}$ defined in $\ker(B_Y) \mapsto [0, 2\pi]$ and augmented with respect to S_1 and $S_2' := e^{-\frac{1}{2}T\nu\delta}S_2 e^{\frac{1}{2}T\nu\delta}$.*

Proof: Let $B_N' = B_N'(T')$ be the Dirac type operator $-\nu\delta\frac{\partial}{\partial\theta}$ defined on sections of the bundle $\bigoplus \ker(B_Y) \mapsto [0, T'(Y)]$, where T' is defined like T, and is the operator of multiplication by $T'(Y)$ on each $\ker(B_Y)$.

In a neighborhood of $\theta = 0$, B_N' is locally like a Dirac type operator on a piece of a cylinder over a point. By the same proof as in [30, Section 2], it follows that the η-invariant does not depend on the length of the attached cylinder modulo $\mathbb{Z}$ for fixed boundary conditions. Further the dimension of the kernel of B_N' remains constant under variation of the length of the cylinder, since by Corollary 5.1.7, it is simply half the dimension of $\ker(S_1 e^{-\frac{1}{2}T\nu\delta}S_2 e^{\frac{1}{2}T\nu\delta} - 1)$. Thus the η-invariant remains constant under variation of the length of the attached cylinder. Further the η-invariant is invariant under scaling of all angles simultaneously, (with fixed boundary conditions, which do *not* depend on the scaling), since this just changes the spectrum by a factor. Thus the η-invariant is constant under any combination of scalings and prolongnations of the cylindrical piece. Consequently we may compute

$$\eta(\tau_Y B_{N+}, 0) = \eta(\tau_Y B_N'(T), 0) = \lim_{s\to\infty} \eta(\tau_Y B_{N+}'(\frac{T + 2\pi s}{s}), 0) = \eta(\tau_Y B_{N+}'(2\pi I), 0).$$

This proves the lemma. □

Remark 5.1.9. The question, whether one should consider the contribution from the corner as a function of S_2 or of $e^{-\frac{1}{2}T\nu\delta}S_2 e^{\frac{1}{2}T\nu\delta}$ depends on, whether one has glued $E_{|Z}$ at the corners or not. If $E_{|Z}$ is glued using the canonical gluing. The correct identification of the spaces in which S_1 and S_2 live has already been made, and the contribution morally does not depend on the angles. If however we consider the corner from the inside of M, the spaces where S_1 and S_2 live are differently identified, and the corner contribution depends on the angles through the eigenvalues of $S_1 e^{-\frac{1}{2}T\nu\delta}S_2 e^{\frac{1}{2}T\nu\delta}$.

In what follows we will use the notation

$$[\beta] = \begin{cases} \beta - 2\pi k & ; \beta - 2\pi k \in (-\pi, \pi) \\ 0 & ; \beta \in \pi + 2\pi\mathbb{Z} \end{cases}, \tag{5.1.9}$$

The η-invariant of an operator with periodic spectrum is well known [26] and [16, Example 1.13.1]. If an operator Q has spectrum

$$p\,(\beta + \pi\mathbb{Z})$$

then

$$\eta(Q, 0) = \frac{-1}{\pi}\,[2\beta + \pi]\,. \tag{5.1.10}$$

In the case of the operator $\tau_Y B_{N+}$, Lemma 5.1.8 gives that we may replace T by 2π and S_2 by S_2' and still get the same η-invariant. Further in this case (5.1.7) gives

$$\operatorname{spec}(B_N) = \{\lambda \in \mathbb{R} \mid \exists\varphi \neq 0 : S_1 S_2' e^{2\pi(2\lambda+1)\nu\delta}\varphi = \varphi\}. \tag{5.1.11}$$

On $\ker(\tau_N - 1) \cap \ker(\tau - 1)$ we have that $\nu\delta = i$. Thus this simplifies to

$$\{\lambda \in \mathbb{R} \mid \exists \varphi \neq 0 : S_1 S_2' e^{4\pi\lambda i}\varphi = \varphi\}.$$

Thus λ is in the spectrum if and only if $e^{4\pi i\lambda} = e^{-i\gamma}$, where $e^{i\gamma}$ is an eigenvalue of $S_1 S_2'$. Expressed differently, $4\pi\lambda$ is of the form $-(\beta - \pi) + 2\pi\mathbb{Z}$, where $e^{i\beta}$ is an eigenvalue of $-S_1 S_2'$. It follows that the spectrum of B_N on $\ker(\tau_N - 1) \cap \ker(\tau - 1)$ is the union over $e^{i\beta} \in \mathrm{spec}(-S_1 S_2)$ of

$$-\frac{1}{4\pi}(\beta - \pi) + \frac{1}{2}\mathbb{Z} = \frac{1}{2\pi}\left(-\frac{1}{2}(\beta - \pi) + \pi\mathbb{Z}\right). \tag{5.1.12}$$

Now, S_1 anti-commutes with τ_N and τ_Y and ν anti-commutes with τ_N but commutes with τ_Y. Finally ν commutes with $S_1 S_2'$, so it follows that

$$\nu S_1(-S_1 S_2')S_1(-\nu) = -S_2' S_1 = -(S_1 S_2')^*. \tag{5.1.13}$$

Thus the spectrum of $-S_1 S_2'$ on $\ker(\tau_N - 1) \cap \ker(\tau_Y + 1)$ is the adjoint of the spectrum of $-S_1 S_2'$ on $\ker(\tau_N - 1) \cap \ker(\tau_Y - 1)$. This courses $-(\beta - \pi)$ to change sign, so it follows that the spectrum of B_N on $\ker(\tau_N - 1) \cap \ker(\tau_Y + 1)$ is the negative of (5.1.12). Thus the η-invariant has the opposite sign. The factor of τ_Y in $\tau_Y B_N$ makes up for that, so it follows

$$\eta(\tau_Y B_{N+}, 0) = 2\eta(B_{N++}, 0),$$

where B_{N++} is the restriction of B_N to $\ker(\tau - 1) \cap \ker(\tau_N - 1)$. Further, by (5.1.12) and (5.1.10) we get

$$\eta(B_{N++}, 0) = \frac{-1}{2\pi} \sum_{e^{i\beta} \in \mathrm{spec}(-S_{1++} S_{2++}')} [\beta]. \tag{5.1.14}$$

The last factor of $\frac{1}{2}$ is because of the multiplicity part of Corollary 5.1.7. Now we may refine the index theorem for $\tilde{D}_+'$:

Theorem 5.1.10. *Let M be an even dimensional manifold with corners of codimension 2 and product structure around the boundary and corners and let $E \mapsto M$ be a Dirac bundle over M respecting the product structure. Let $\tilde{E}' \mapsto \tilde{M}'$ be the the extension of $E \mapsto M$ given by (5.1.1) and the product structure. Finally let $\tilde{D}'$ be the self-adjoint realization of the Dirac operator associated to $\tilde{E}'$, given slow-growing Atiyah-Patodi-Singer boundary conditions of level 2 augmented with respect to the scattering matrices. Let τ be a parallel self-adjoint involution in $\tilde{E}'$ anti-commuting with Clifford multiplication and let $\tilde{D}_+'$ be the restriction of $\tilde{D}'$ to* $\ker(\tau - 1)$. *Then we have the following index theorem:*

$$\begin{aligned} \mathrm{Index}(\tilde{D}_+') = &\int_M a_D(x)dx - \frac{1}{2}\sum_j \eta(A_{Z_j+}, 0) + \frac{1}{2}\mathrm{tr}\,(S_+) \\ &+ \frac{1}{2\pi} \sum_{\beta \in \mathrm{spec}((-S_{1++}e^{-\frac{1}{2}T\nu\delta}S_{2++}e^{\frac{1}{2}T\nu\delta})|\ker(\tau-1)\cap\ker(\tau_N-1))} [\beta]. \end{aligned} \tag{5.1.15}$$

Remark 5.1.11. If the boundary components of M have self-intersections, we can still prove self-adjointness of $\tilde{D}'$ and prove an index formula. In this case there is (morally) only one scattering matrix S_1. We can however get an additional one by imposing boundary conditions in the middle of each angle interval corresponding to the condition that sections should be continuous. In this way the construction of boundary conditions and computation of the index contribution becomes equivalent to what we have done above, just on a space of twice the dimension.

Remark 5.1.12. The class of operators B_N on $\ker(B_Y)$ is unitarily equivalent to the class of operators $Q\frac{\partial}{\partial t}$ on $L^2([-\pi,\pi],\ker(B_Y))$, where Q is antisymmetric and the boundary conditions are arbitrary self-adjoint boundary conditions. The spectrum of $Q\frac{\partial}{\partial t}$ can be computed directly and is a union of shifted periodic spectra. Consequently this conjugation gives a more direct way to compute the η-invariant of parts of B_N.

6 Special Cases and Applications.

In this section we work out the corner contribution in Theorem 4.2.3 in some special cases. The methods are the same as in Section 5 and we maintain a lot of notation from Section 5. The cases are however simpler and instead of scattering matrices we have gluing operators.

6.1 The twisted Spin Complex.

We recall that the corner contribution from Theorem 4.2.3 is given by

$$-\frac{1}{2}\sum_{\alpha_+\in\mathrm{spec}(U_{\ker(B_Y)}|\ker(\tau_N-1))}\mathrm{tr}\,(\tau_{\alpha+})\eta(B_{\alpha+},0). \tag{6.1.1}$$

Further we have from (1.2.4) that the spectrum of the operator B_{α_+} in question is given by

$$\mathrm{spec}(B_{\alpha_+})=\frac{2}{\sigma+\pi}\left(\frac{i\log(\alpha_+)}{2}-\frac{\sigma+\pi}{4}+\pi\mathbb{Z}\right). \tag{6.1.2}$$

Thus (5.1.10) gives that, with notation from (5.1.9), we have

$$\eta(B_{\alpha_+},0)=\frac{-1}{\pi}\left[i\log(\alpha_+)-\frac{\sigma+\pi}{2}+\pi\right]. \tag{6.1.3}$$

By (6) of Lemma 1.1.14 we see that for the universal gluing operator, the only value of α_+ is $\alpha_+=e^{i\frac{\pi-\sigma}{2}}$. Thus

$$\eta(B_{\alpha_+},0)=\frac{-1}{\pi}\left[\frac{\sigma-\pi}{2}-\frac{\sigma+\pi}{2}+\pi\right]=0. \tag{6.1.4}$$

We have proved:

Lemma 6.1.1. *If* $\tilde{E} \mapsto \tilde{M}$ *is glued using the universal gluing operator, the contribution from* $(\ker(B_N) \cap \ker(B_Y)^{\perp})$ *from the corner vanishes.*

The universal gluing operator corresponds to the twisted spin bundle, where the twisting bundle is glued trivially around the singularities.

6.2 The de Rham and Signature Complex.

Also the deRham and signature complexes are relevant examples to work out. We only consider the case $\sigma \in (0, 3\pi)$, for which the results from Section 3.3 are valid.

Lemma 6.2.1. *The contribution from the corner for the signature complex vanishes. In particular we have for a manifold with corners of codimension 2*

$$\mathrm{sign}(M) = \int_M L - \frac{1}{2}\eta(A_{Z+}, 0),$$

where A_{Z+} *is the induced Dirac operator on the boundary.*

Proof: Let (r, θ) be polar coordinates on C, let δ be Clifford multiplication by $\frac{1}{r}\frac{\partial}{\partial\theta}$ and let ν be Clifford multiplication by $\frac{\partial}{\partial r}$.

By the proof of Lemma 3.3.2 we know that the decomposition

$$\Lambda^*(C \times Y) = (i + \nu\delta)\Lambda^*(Y) \oplus (-i \oplus \nu\delta)\Lambda^*(Y) \oplus (\delta - i\nu)\Lambda^*(Y) \oplus (\delta + i\nu)\Lambda^*(Y) \tag{6.2.1}$$

is a decomposition of $\Lambda^*(C \times Y)$ into joint eigenspaces of the gluing operator U and $\nu\delta$ to the eigenvalues $1, 1, e^{-i(\pi-\sigma)}, e^{i(\pi-\sigma)}$ and $i, -i, -i, i$, respectively. This gives that the restriction U_+ of U to $\ker(\tau_N - 1) = \ker(\nu\delta - i)$ has two eigenvalues, 1 and $e^{i(\pi-\sigma)}$, each occurring with the same multiplicity. We compute using (1.2.4) and (5.1.10)

$$\eta(B_{[1]}, 0) = \frac{-1}{\pi}[-\frac{\sigma + \pi}{2} + \pi], \tag{6.2.2}$$

$$\eta(B_{[e^{i(\pi-\sigma)}]}, 0) = \frac{-1}{\pi}\left[\frac{\sigma - \pi}{2}\right]. \tag{6.2.3}$$

Since $-\frac{\sigma+\pi}{2} + \pi = -\frac{\sigma}{2} + \frac{\pi}{2} = -\left(\frac{\sigma-\pi}{2}\right)$ it follows that the sum of those η-invariants vanishes. Further τ_Y acts identically on each of those spaces since it commutes with ν and δ. Consequently the corner contribution vanishes.

By Corollary 3.3.9 and the fact that the Scattering matrix anti-commutes with ν the index theorem for the signature complex can be rewritten

$$\mathrm{sign}(M) + \mathrm{tr}\,(S_+) = \int_M L - \frac{1}{2}\eta(A_{Z+}, 0) + \frac{1}{2}\mathrm{tr}\,(S_+). \tag{6.2.4}$$

Since by Corollary 3.3.11 the scattering term vanishes, we are done. □

Theorem 6.2.2. *(The Gauß-Bonnet theorem) Assume $\sigma \in (0, 3\pi)$ for all angles at the corners. Then the corner contribution from the de Rham complex is given by*

$$\frac{1}{\pi}\chi(Y)\left[-\frac{\sigma+\pi}{2}+\pi\right],$$

where χ denotes the Euler characteristic and σ runs over the angles. Further $\eta(A_{Z+}, 0) = 0$ in this case and the scattering term vanishes. Thus we have

$$\chi(M) = \int_M e + \left(\sum_{N\times Y} \frac{\pi - \sigma(N)}{2\pi}\right)\chi(Y). \tag{6.2.5}$$

In particular, if $\dim(M) = 2$ and all angles are in the interval $(0, 2\pi)$ we recover the Gauss-Bonnet theorem [12]:

$$\chi(M) = \int_M e + \sum_{\sigma} \frac{\pi - \sigma}{2\pi}. \tag{6.2.6}$$

Proof: The de Rham complex is handled like the signature complex. The gluing operator U is the same, τ is replaced by the parity involution ρ, $\tau_N = -i\nu\delta$ is preserved and τ_Y is given by the convention $\tau_Y := \tau_N\rho$. In this case, however, ρ is the parity operator on $\Lambda^*(Y)$ on $(i+\nu\delta)\Lambda^*(Y)$ and minus the parity operator on $\Lambda^*(Y)$ on $(\delta + i\nu)\Lambda^*(Y)$. Consequently (6.2.2) and (6.2.3) must be subtracted rather than added so

$$-\frac{1}{2}\sum_{\alpha+\in \mathrm{spec}(U_{\ker(B_Y)}|\ker(\tau_N-1))} \mathrm{tr}\,(\tau_{|V_{\alpha+}})\eta(B_{\alpha+}, 0) = -\frac{1}{2}\times 2\chi(Y)\eta(B_1, 0)$$

$$= \frac{1}{\pi}\chi(Y)\left[-\frac{\sigma+\pi}{2}+\pi\right].$$

This is the claimed contribution from the corner.

That the η-invariant of Z vanishes in this case follows since the the parity operator on $\Lambda^*(Z)$ commutes with the parity operator and anti-commutes with A_Z. Thus A_{Z+} has a symmetric spectrum and a vanishing η-invariant.

The scattering term vanishes by Corollary 3.3.11. Thus $\mathrm{Index}(\tilde{D}) = \mathrm{Index}_0(\tilde{D})$ and the scattering term on the right hand side vanishes. □

Remark 6.2.3. The same trick as in [16, Section 2.7.7] can be applied to extend the Gauß-Bonnet theorem to the case where there is no product structure on the boundaries away from the corners. Passing to the limit of such problems gives the Gauß-Bonnet theorem for manifolds with corners with the restriction on the structure close to the corners that the angles along the corners must be constant.

Remark 6.2.4. If ∂M splits into two components without self-intersections, a global proof giving the vanishing of the corner contribution for the signature complex from the fact that the signature of a boundary vanishes, applies. In the general case this proof can though not be applied.

6.3 The Splitting Formula for η-Invariants.

The splitting formula for η-invariants of closed manifolds into η-invariants of manifolds with boundary and product structure around the boundary is well-known. Different proofs have been given by Bunke [5], Brüning & Lesch [4], Wojciechowski [13], [44], [45], Mazzeo & Melrose [27], [28] and Müller [31]. It is also well-known how the η-invariant behaves under variation of the boundary conditions. The case, where only the augmentation is varied, is handled in [26] and [30]. A case of very general pseudo-differential boundary conditions is handled in [46].

Subtracting Theorem 4.2.3 from Theorem 5.1.10 gives a new proof of the splitting formula. Like for all other proofs, an integer valued term, given in terms of indices and scattering matrices, which is not very accessible, remains. We here give another construction, which is a rather direct proof of the splitting formula for the η-invariant, here given in a setup compatible with the index theorem.

Let Z be a closed odd-dimensional manifold and let $E \mapsto Z \times (-\infty, \infty)$ be a Dirac bundle over an infinite cylinder over Z respecting the product structure. Let D be the Dirac operator associated to E. Then D has the usual product decomposition:

$$D = \nu \left(\frac{\partial}{\partial u} + A \right).$$

If further E is a super-bundle we may define A_+ as usual. Now assume that Z has a decomposition $Z = Z_1 \cup_Y Z_2$ into manifolds with product structure in a neighborhood of the boundary. We set

$$\tilde{M}' = Z \times [-2, 0] \cup_{Z_1 \sqcup Z_2} ((Z_1 \sqcup Z_2) \times [0, 1]).$$

The pullback of E to $\tilde{M}'$ is a Dirac bundle on $\tilde{M}'$ respecting the local product structure. Further $\tilde{M}'$ is a special case of the manifold $\tilde{M}'$ from Section 5. Therefore we also denote the Dirac operator on $\tilde{M}'$ by $\tilde{D}'$. The induced Dirac operators on Z_i we denote by A_i and the corresponding scattering matrices we denote by S_i. The scattering matrix of $\tilde{D}'$ we denote by S. With this notation we have:

Theorem 6.3.1. *We have*

$$\eta(A_+, 0) - \eta(A_{1+}, 0) - \eta(A_{2+}, 0) = 2\mathrm{Index}(\tilde{D}'_+) - \mathrm{tr}\,(S_+(0)) - \frac{1}{\pi} \sum_{e^{i\beta} \in \mathrm{spec}(-S_{1++} S_{2++})} [\beta].$$

Proof: This is an immediate consequence of Theorem 5.1.10. The local formulas vanish because of the local product structure everywhere and the η-invariants appear with different signs because of the different orientations at the boundaries. □

This splitting formula corresponds to the known splitting formulae modulo $\mathbb{Z}$. The integer-valued terms we are not able to compare. It would be desirable to find another way to compute the index in order to get more information about the integer-valued term. This however seems to be difficult since the ideal boundary conditions in the corner are not compatible with relevant algebraic operations. The most troublesome part of the problems comes from the space $\ker(B_N) \cap \ker(B_Y)^\perp$, which gives rise to an infinite dimensional space of possible singularities not accounted for in the scattering matrix.

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