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Equivalence of Sobolev Spaces.*

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Contents

Abstract

We repair the proof of equivalence of certain L^2-Sobolev spaces on manifolds with bounded curvature of all orders from [4]. The results are extended to generalized compatible Dirac operators, fractional order Sobolev spaces and weighted Sobolev spaces. A certain way of doing coordinate free computations is presented.
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0 Introduction

Let $E \mapsto M$ be a Hermitian vector bundle supplied with a Hermitian connection ∇. A natural operator to consider is the Bochner Laplacian $\Delta = \nabla^*\nabla$. Thus we can define Sobolev spaces $H^{2k}(E)$ by

$$H^{2k}(E) = \{f \in L^2(E) \mid \forall 0 \leq j \leq k : \Delta^k f \in L^2(E)\}.$$

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Similarly we can define Sobolev spaces in terms of ∇ itself

$$W^l(E) = \{f \in L^2(E) \mid \forall 0 \le j \le l : \nabla^j f \in L^2(T^*M^{\otimes j} \otimes E)\}.$$

Solutions of partial differential equations involving Δ are normally found in the H^{2k}-spaces. Further use of the solutions however often requires that they belong to some W^l-space. It is an important problem to find conditions on M and E, under which those two families of Sobolev spaces are equivalent, or under which the Sobolev norms are equivalent on the space $C_0^\infty(E)$ of smooth sections with compact support. It turns out that the norms are equivalent on $C_0^\infty(E)$ if M has bounded curvature of all orders and that $H^{2k}(E)$ and $W^{2k}(E)$ are equivalent if in addition M is complete.

A clever proof of that was given in [4] and was later extended to a more general situation in [3]. The proof uses the Bochner-Weizenböck formula together with some commutater estimates. Unfortunately a wrong identity [4, (1.7)] was used for the commutater estimates. The mistake was discovered by Ulrich Bunke. After that other methods have been applied in order to prove Sobolev imbedding theorems on open manifolds. See for example [3, Lemma 1.19], where a lower bound on the injectivity radius is assumed and [9], where the method of [3, Lemma 1.19] is extended to manifolds with bounded curvature of arbitrary order. This other method, which involves a special covering of M by geodesic balls, is well suited for generalizing all sorts of elliptic estimates to open manifolds, but when a lower bound on the injectivity radius is not assumed, it does not give the best possible theorems in the L^2-theory. Thus it is worth the effort repairing the proof of [4]. The main strength (and at the same time the main limitation) is that it is completely adapted to L^2-theory and gives the best possible L^2-estimates in very high generality with relatively little effort.

In Section 2 we repair the proof of [4] in the generality of Bochner Laplacians. The decisive lemma is Lemma 2.1, which replaces [4, (1.7)]. We push the method far enough in order to define L^2-Sobolev spaces of any real order and demonstrate how to prove that suitable differential operators map between those spaces as expected. In Section 3 we extend the method to Dirac type operators, for which it is in some sense more natural. In Section 4 we consider the case of weighted Sobolev spaces. Under mild assumptions on the weight the method can be extended to prove equivalence of weighted Sobolev spaces also. Finally in Section 5 we use self-adjointness properties of operators on complete Riemannian manifolds in order to get further equivalences, among others that the space of smooth sections with compact support is dense in distributionally defined Sobolev spaces. The case of weighted Sobolev spaces follows surprisingly easily from the case of non-weighted Sobolev spaces under very reasonable assumptions on the weights.

We have also used this paper as an opportunity for presenting some methods for coordinate free manipulation of tensors on manifolds using higher parallel tensors. These methods have been very harshly received wherever we have presented them, but they simplify computations so much that they certainly deserve to be taken seriously. The complicated commutaters in this paper are just perfectly suited for presenting those methods. There is no particular new result involved. It is just a way to proceed doing coordinate free computation, which limits the computations in local coordinates and the number of proofs by induction. The idea is that higher tensor products of vector bundles contain an abundance of canonical parallel sections. Expressing formulas in terms of those give coordinate free and vector field free expressions, which we can differentiate covariantly without getting the typical mess of terms of secondary importance, expressing the invariance properties.

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1 Invariantly Defined Spaces.

Let M be a Riemannian manifold and let $E \mapsto M$ be a Hermitian vector-bundle supplied with a Hermitian structure h and a Hermitian connection ∇. Together with the Riemannian metric and the Levi-Civita connection on the tangent bundle, h and ∇ induce Hermitian structures and Hermitian connections on all bundles in the algebra of bundles generated by $1_{\mathbb{C}}$, E, E^*, TM and T^*M. Isomorphisms of bundles will be required to preserve the Hermitian structure as well as the connection. For example $E \otimes E^* \cong End(E)$ is an isomorphism in this sense. In this section we define a number of spaces of sections and operators, which we will need later on.

Definition 1.1. The space $C_b^\infty(E)$ is the space of smooth sections s of E such that for all $k \in \mathbb{N}_0$, the section $\nabla^k s$ is bounded.

Lemma 1.2. *If M is a Riemannian manifold with metric $g \in C^\infty(T^*M \otimes T^*M) \cong C^\infty(End(TM, T^*M))$ we have*

$$g \in C_b^\infty(End(TM, T^*M)), \tag{1.1}$$
$$g^{-1} \in C_b^\infty(End(T^*M, TM)). \tag{1.2}$$

Further if $E \mapsto M$ is a Hermitian vector-bundle supplied with a Hermitian connection we have

$$h \in C_b^\infty(End(E, E^*)), \tag{1.3}$$
$$h^{-1} \in C_b^\infty(End(E^*, E)). \tag{1.4}$$

Proof: Because g is parallel and is bounded with respect to itself, (1.1) follows. But since g is parallel, also g^{-1} is parallel and it follows that also (1.2) holds. The statements about h are similar since the fact that the connection is Hermitian is the same as to say $\nabla h = 0$. $\square$

Definition 1.3. If E, F are Hermitian vector bundles supplied with Hermitian connections, both denoted by ∇, and $m \in \mathbb{N}_0$, the space $\mathrm{Diff}^m_{\mathrm{bd}}(E, F)$ is the space of differential operators P of the form

$$P = \sum_{j=0}^{m} \xi_j \nabla^j,$$

where for $j = 0, \dots, m$, $\xi_j \in C_b^\infty(End((T^*M)^{\otimes j} \otimes E, F))$.

Lemma 1.4. *We have*

$$C_b^\infty(End(E,F)) = \mathrm{Diff}^0_{\mathrm{bd}}(E,F). \tag{1.5}$$

Further, if $P \in \mathrm{Diff}^{m_1}_{\mathrm{bd}}(E,F)$*,* $Q \in \mathrm{Diff}^{m_2}_{\mathrm{bd}}(F,F')$*, then* $QP \in \mathrm{Diff}^{m_1+m_2}_{\mathrm{bd}}(E,F')$*.*

Proof: This is easily checked. □

Definition 1.5. Let E be a Hermitian vector-bundle with a compatible connection. For $k \in \mathbb{N}_0$, let $W^{2,k}_0(E)$ be the completion of $C_0^\infty(E)$ with respect to the norm

$$\|f\|_{W^{2,k}} := \sqrt{\sum_{j=0}^k \|\nabla^j f\|_{L^2((T^*M)^{\otimes j}\otimes E)}}. \tag{1.6}$$

Further let $W^{2,k}(E)$ be the space

$$W^{2,k}(E) := \{f \in L^2(E) \mid \forall j = 0,\dots,k : \nabla^j f \in L^2((T^*M)^{\otimes j} \otimes E)\}. \tag{1.7}$$

Here ∇ is applied iteratively in the distributional sense.

Lemma 1.6. *Let E and F be Hermitian vector bundles supplied with Hermitian connections. The spaces $W^{2,k}_0(E)$ and $W^{2,k}(E)$ are Hilbert spaces and $W^{2,k}_0(E)$ is a closed subspace of $W^{2,k}(E)$. Further, if $m \leq k$ and $P \in \mathrm{Diff}^m_{\mathrm{bd}}(E,F)$, P is bounded as an operator*

$$P : W^{2,k}(E) \mapsto W^{2,k-m}(F)$$

and $PW^{2,k}_0(E) \subseteq W^{2,k-m}_0(F)$.

Proof: All claims are easily checked. □

We notice that if the completion of M is a manifold with a non-empty smooth boundary, $W^{2,1}_0(1_{\mathbb{C}} \mapsto M)$ and $W^{2,1}(1_{\mathbb{C}} \mapsto M)$ do not coincide, so it is possible that $W^{2,1}_0(E)$ is a proper subspace of $W^{2,k}(E)$.

If ∇ is a connection in a vector bundle E, we may define its curvature $R^\nabla \in C^\infty(T^*M \otimes T^*M \otimes End(E))$ by

$$R^\nabla(X \otimes Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \tag{1.8}$$

where $X, Y \in C^\infty(TM)$. If a specific connection ∇ is given on E we write R^E for R^∇. Further we write R for the curvature tensor associated to the Levi-Civita connection on M.

The adjoint of R^E with respect to the canonical pairing of E and E^* is the tensor $-R^{E^*}$. On the other hand the curvature on E^* can be obtained by conjugating by the Hermitian structure, which is parallel. This immediately gives that if $R^E \in C_b^\infty(T^*M \otimes T^*M \otimes End(E))$ then also $R^{E^*} \in C_b^\infty(T^*M \otimes T^*M \otimes End(E))$.

Curvatures behave additively under formation of tensor products, so in particular, if $R \in C_b^\infty(T^*M \otimes T^*M \otimes End(TM))$ and $R^E \in C_b^\infty(T^*M \otimes T^*M \otimes End(E))$, it follows

$$R^{(T^*M)^{\otimes k}\otimes E} \in C_b^\infty(T^*M \otimes T^*M \otimes End(T^*M^{\otimes k} \otimes E)).$$

We make aware of some complicated but elementary facts. Let $E = E_1 \otimes E_2 \otimes \cdots \otimes E_k$ be a tensor product of Hermitian vector bundles with Hermitian connections. Assume that $E_j \cong End(E_i, F)$ and that $j \neq i$. Then the section

$$A_{ij} \in C^\infty(End(E, E_1 \otimes \cdots \otimes \hat{E}_j \otimes E_{j+1} \otimes \cdots \otimes E_{i-1} \otimes F \otimes E_{i+1} \otimes \cdots \otimes E_k)), \quad (1.9)$$

given by applying an element of E_j to an element of E_i, is bounded and parallel. In particular $A_{ij} \in C_b^\infty$.

Let E, F be Hermitian vector bundles and let τ be a parallel element of $C^\infty(E)$. Then the element $\tau\otimes$ of $C^\infty(End(F, E \otimes F))$ given by

$$\tau \otimes (f) = \tau \otimes f$$

is parallel. In particular, since parallel sections are bounded, $\tau\otimes \in C_b^\infty(F, E \otimes F)$.

Now consider the trace $\mathrm{tr} \in C^\infty(End(E)^*)$. We may write the trace as

$$\mathrm{tr} = A_{12}, \quad (1.10)$$

where we have used the isomorphism $End(E) \cong E \otimes E^*$. If there is any doubt about, on which factors of a tensor product the trace should be applied, we write $\mathrm{tr}_{\,i,j}$. Similarly, for example g_i denotes the operator of applying g to the i'th factor of a tensor product.

2 The Bochner Laplacian.

In this section we consider the Bochner Laplacian $\Delta = \nabla^*\nabla = -\mathrm{tr}_{\,1,2} g_1^{-1} \nabla^2$ in a Hermitian vector-bundle E supplied with the Hermitian connection ∇ over a Riemannian manifold M. Sobolev spaces related to Δ are defined by

$$H^{2,2k}(M,E) := \{f \in L^2(M,E) \mid \forall j = 0, \dots, k : \Delta^j f \in L^2(M,E)\}, \quad (2.1)$$
$$H_0^{2,2k}(M,E) := \text{ Closure of } C_0^\infty(M,E) \text{ in } H^{2,2k}(M,E). \quad (2.2)$$

The first substitute for the commutater in Dodziuk is:

Lemma 2.1. *Let E be a Hermitian vector bundle over a Riemannian manifold M and assume that*

$$R \in C_b^\infty(T^*M \otimes T^*M \otimes End(TM))$$

and

$$R^E \in C_b^\infty(T^*M \otimes T^*M \otimes End(E)).$$

Then we have

$$\nabla^*\nabla\nabla - \nabla\nabla^*\nabla \in \mathrm{Diff}^1_{\mathrm{bd}}(M, E, T^*M \otimes E).$$

Proof: Let $g \in C^\infty(M, End(TM, T^*M))$ denote the Riemannian structure on M. Then we have

$$\nabla^* = -\mathrm{tr}_{\,1,2} g^{-1} \nabla.$$

We further introduce parallel sections $S_{i,j}$ in $End(T^*M \otimes \cdots \otimes T^*M \otimes E)$. $S_{i,j}$ interchanges the $i'th$ and $j'th$ factor of T^*M. Then we have

$$(1 - S_{1,2})\nabla^2 = R,$$

where R is the curvature on $T^*M \otimes \cdots \otimes T^*M \otimes E$. Let g_i^{-1} be the parallel section of application of g^{-1} to the i'th factor of T^*M. Then we may compute

$$\nabla^*\nabla\nabla - \nabla\nabla^*\nabla = \mathrm{tr}_{\,1,2} g_1^{-1}\nabla^3 - \mathrm{tr}_{\,2,3} g_2^{-1}\nabla^3$$

$$= \mathrm{tr}_{\,1,2} g_1^{-1}\nabla^3 - \mathrm{tr}_{\,1,2} g_1^{-1} S_{2,3} S_{1,2}\nabla^3$$

$$= \mathrm{tr}_{\,1,2} g_1^{-1}\nabla^3 - \mathrm{tr}_{\,1,2} g_1^{-1} S_{2,3}\nabla^3 + \mathrm{tr}_{\,1,2} g_1^{-1} S_{2,3} R\nabla$$

$$= \mathrm{tr}_{\,1,2} g_1^{-1}\nabla^3 - \mathrm{tr}_{\,1,2} g_1^{-1}\nabla^3 + \mathrm{tr}_{\,1,2} g_1^{-1}\nabla R + \mathrm{tr}_{\,1,2} S_{2,3} g_1^{-1} R\nabla$$

$$= \mathrm{tr}_{\,1,2} g_1^{-1}\nabla R + \mathrm{tr}_{\,1,2} g_1^{-1} S_{2,3} R\nabla \in \mathrm{Diff}^1_{\mathrm{bd}}(M, E, T^*M \otimes E).$$

This proves the lemma. □

Theorem 2.2. *Let E be a Hermitian vector bundle over a Riemannian manifold M and assume that*

$$R \in C_b^\infty(T^*M \otimes T^*M \otimes End(TM)),$$

and

$$R^E \in C_b^\infty(T^*M \otimes T^*M \otimes End(E)).$$

Then for every $k \in \mathbb{N}$ we have the equivalence

$$H_0^{2,2k}(M, E) = W_0^{2,2k}(M, E).$$

Proof: Clearly the theorem is true for $k = 0$. Assume the theorem holds for $k-1$. Let $f \in C_0^\infty(M, E)$ then

$$\left\langle \nabla^{2k} f, \nabla^{2k} f \right\rangle = \left\langle (\nabla^*)^{2k}\nabla^{2k} f, f \right\rangle. \tag{2.3}$$

Now, $T_k := \nabla^*\nabla\nabla - \nabla\nabla^*\nabla$ satisfies that $T_k \in \mathrm{Diff}^1_{\mathrm{bd}}(M, T^*M^{\otimes 2k-2} \otimes E, T^*M^{\otimes 2k-1} \otimes E)$ by Lemma 2.1. Consequently (2.3) can be rewritten

$$\left\langle (\nabla^*)^{2k-1}\nabla\nabla^*\nabla^{2k-1} f, f \right\rangle - \left\langle T_k \nabla^{2k-2} f, \nabla^{2k-1} f \right\rangle. \tag{2.4}$$

The last term can be estimated by

$$\left\langle \nabla^{2k-1} f, \nabla^{2k-1} f \right\rangle = \left\langle (\nabla^*)^{2k-1} \nabla^{2k-1} f, f \right\rangle .$$

If $k = 1$ this is just $\langle \Delta f, f \rangle \leq \frac{1}{2} \left(\|f\|^2 + \|\Delta f\|^2 \right)$, otherwise it equals

$$= \left\langle (\nabla^*)^{2k-2} \nabla \nabla^* \nabla^{2k-2} f, f \right\rangle - \left\langle T_{k-1} \nabla^{2k-3} f, \nabla^{2k-2} f \right\rangle .$$

By induction the last term can by estimated by $\|f\|^2_{H^{2,2(k-1)}}$. For the first term we apply Lemma 2.1 iteratively, such that we each time get a term of the same type as the last one, which can be estimated by $\|f\|^2_{H^{2,2(k-1)}}$ by induction. We end up with

$$\left\langle (\nabla^*)^{2k-2} \nabla^{2k-2} \Delta f, f \right\rangle \leq \frac{1}{2} \left(\|\nabla^{2(k-1)} \Delta f\|^2 + \|\nabla^{2(k-1)} f\|^2 \right)$$

$$\leq \|f\|^2_{H^{2,2k}} .$$

Now we return to the first term of (2.4). Again we apply Lemma 2.1 iteratively and each time get a commutater, which can be handled like the second term of (2.4). We end up with

$$\left\langle (\nabla^*)^{2k-1} \nabla^{2k-1} \Delta f, f \right\rangle = \left\langle \Delta f, (\nabla^*)^{2k-1} \nabla^{2k-1} f \right\rangle . \tag{2.5}$$

Using Lemma 2.1 iteratively we again get a number of terms, which can be estimated by $\|f\|_{H^{2,2(k-1)}}$ by induction. We end up with the term

$$\left\langle \nabla^{2(k-1)} \Delta f, \nabla^{2(k-1)} \Delta f \right\rangle \leq \|\Delta f\|^2_{H^{2,2(k-1)}} \leq \|f\|^2_{H^{2,2k}} .$$

This proves the theorem. □

We consider Δ as a self-adjoint operator on the domain given by the Friedrich's extension of Δ. We define for $f \in C_0^\infty(M, E)$

$$\|f\|^2_{H^k_{0,1}} := \sum_{j=0}^{k} \|\Delta^{j/2} f\|^2_{L^2},$$

and let $H^k_{0,1}(M, E)$ be the completion of $C_0^\infty(M, E)$ with respect to $\|\cdot\|_{H^k_{0,1}}$. By the spectral calculus for Δ it is easy to see that $\|\cdot\|_{H^k}$ and $\|\cdot\|_{H^k_{0,1}}$ are equivalent as norms on $C_0^\infty(M, E)$ when k is even. Further,

Theorem 2.3. *$\|\cdot\|_{H^{2,k}_{0,1}}$ is equivalent to $\|\cdot\|_{W^{2,k}}$ on $W_0^{2,k}$, also for odd k.*

Proof: Exactly like the proof of Theorem 2.2. We omit the details. □

Using the spectral calculus for Δ we can also define $H_0^{2,s}(E)$ for all $s \in R_+$ as the closure of $C_0^\infty(E)$ in the domain of Δ^s. We can also as usual define $H^{2,-s}(E)$ as the dual of $H_0^{2,s}(E)$. Standard commutater estimates together with Lemma 2.1 and Theorem 2.2 suffice to prove that $\mathrm{Diff}^k_{\mathrm{bd}}(E, E)$ consists of bounded operators $H_0^{2,s}(E) \mapsto H_0^{2,s-k}(E)$. We prove this in the following proposition:

Proposition 2.4. *For all $s \in \mathbb{R}$, ∇ maps $H^{2,s}(E)$ into $H^{2,s-1}(T^*M \otimes E)$. Further, if $\Psi \in C_b^\infty(End(E,F))$ for some Hermitian vector bundle F with a Hermitian connection, Ψ maps $H^{2,s}(E)$ into $H^{2,s}(F)$.*

Proof: We already have this for $s = 1$. We consider the case $s > 1$. By Lemma 2.1 we have

$$[\Delta + 1, \nabla] = [\Delta, \nabla] \in \mathrm{Diff}^1_{\mathrm{bd}}(E, T^*M \otimes E).$$

Consequently for each λ

$$[(\Delta + \lambda)^{-1}, \nabla] = -(\Delta + \lambda)^{-1}[\Delta, \nabla](\Delta + \lambda)^{-1}$$

is bounded as an operator $H^{2,k}(E) \mapsto H^{2,k+3}(E)$ with

$$\|[(\Delta + \lambda)^{-1}, \nabla]\|_{H^{2,k},H^{2,k+3}} \leq C\|\mathrm{Im}(\lambda)\|^{-1}$$

for some $C < \infty$. Consequently, for a suitable curve Γ passing around $[0, \infty)$ with $|\mathrm{Im}(\Gamma(\xi))| = 1$ for $|\xi| \gg 1$ we may estimate

$$[(\Delta + 1)^{-s}, \Delta] = \left[\frac{-1}{2\pi i}\int_\Gamma (\xi + 1)^{-s}(\Delta - \xi)^{-1} d\xi, \Delta\right]$$

$$= \frac{1}{2\pi i}\int_\Gamma (\xi + 1)^{-s}(\Delta - \xi)^{-1}[\Delta, \nabla](\Delta - \xi)^{-1} d\xi.$$

Thus for $t \geq 0$, $s > 1$, $[(\Delta+1)^{-s}, \nabla]$ is bounded as an operator $H^t \mapsto H^{\lfloor t\rfloor+3}$. Here $\lfloor t\rfloor$ denotes the biggest integer $\leq t$. Thus $[(\Delta+1)^s, \nabla] = -(\Delta+1)^s[(\Delta+1)^{-s}, \nabla](\Delta+1)^s$ is bounded as an operator $H^t(E) \mapsto H^{\lfloor t-2s\rfloor+3-2s}(T^*M \otimes E)$. Thus we may estimate for $f \in C_0^\infty(E)$, considered as an element of $H^{s+1}(E)$:

$$\|(\Delta + 1)^{\frac{s}{2}}\nabla f\| \leq \|\nabla(\Delta + 1)^{\frac{s}{2}} f\| + \|[\nabla, (\Delta + 1)^{\frac{s}{2}}]f\|.$$

The commutater maps $H^{s+1}(E)$ into $H^{\lfloor s-s\rfloor+4-s} \subseteq L^2$ for $0 \leq s \leq 4$. Further the first term can be estimated directly

$$\|\nabla(\Delta + 1)^{\frac{s}{2}} f\| = \sqrt{\left\langle\nabla(\Delta + 1)^{\frac{s}{2}} f, \nabla(\Delta + 1)^{\frac{s}{2}} f\right\rangle}$$

$$\leq \|(\Delta + 1)^{\frac{s+1}{2}} f\|^{\frac{1}{2}} \cdot \|(\Delta + 1)^{\frac{s}{2}} f\|^{\frac{1}{2}}.$$

The integration by parts is justified since $(\Delta+1)^{\frac{s}{2}} f \in \mathcal{D}(\Delta) \subset W_0^{2,1}(E)$. Thus the statement about ∇ holds for $s \leq 4$. In the same way as above, since $[\Delta, \Psi] \in \mathrm{Diff}^1_{\mathrm{bd}}(E, F)$ it follows that Ψ maps $H^s(E)$ into $H^s(F)$ for $1 \leq s \leq 4$.

Now, if the statement about ∇ holds for s we compute

$$[(\Delta + 1)^{\frac{s}{2}}, \nabla] = (\Delta + 1)^{\frac{s-2}{2}}[\Delta, \nabla] + [(\Delta + 1)^{\frac{s-2}{2}}, \nabla](\Delta + 1).$$

It follows by induction that $[(\Delta+1)^{\frac{s}{2}}, \nabla]$ maps $H^{s+1}(E)$ into $L^2(E)$. The induction step for Ψ is similar.

Consequently the proposition holds for all $s \geq 1$. By duality the proposition holds for all $s \leq 0$. Finally, if $0 < s < 1$, we write $f = (\Delta+1)g$ for some $g \in H^{s+2}$. It follows immediately

$$\nabla f = \nabla g + \Delta\nabla g + [\nabla, \Delta]g \in H^{s-1}.$$

This finishes the proof of the proposition. $\square$

3 Twisted Dirac Operators.

Let M be a Riemannian manifold with a Riemannian metric $g \in C^\infty(T^*M \otimes T^*M)$ and let $E \mapsto M$ be a Dirac bundle. That means a Hermitian vector bundle supplied with a connection and a structure $\mathbf{c} \in C^\infty(End(\mathrm{Cliff}(TM) \otimes E, E))$ of Clifford multiplication, such that $\mathbf{c}$, the Hermitian structure $h \in C^\infty(E^* \otimes E^*)$ and the connection ∇ are compatible in the way:

$$\nabla \mathbf{c} = 0, \tag{3.1}$$

$$\nabla h = 0, \tag{3.2}$$

$$h(\mathbf{c}_x(X)v, w) = -h(v, \mathbf{c}_x(X)w), \quad ; x \in M, X \in T_xM \text{ and } v, w \in E_x \tag{3.3}$$

$$h(\mathbf{c}_x(X)v, \mathbf{c}_x(X)v) = |X|^2 h(v, v) \quad ; x \in M, X \in T_xM \text{ and } v, w \in E_x. \tag{3.4}$$

If (3.1), (3.2), (3.3) and (3.4) are satisfied we define the associated generalized compatible Dirac operator D by

$$D := \mathbf{c} g^{-1} \nabla. \tag{3.5}$$

Now assume that F is a Hermitian vector bundle with Hermitian structure h_F, supplied with a connection ∇^F. The Hermitian structure $h_{F\otimes E}$ on $F \otimes E$ is given by C^∞-linearity and the condition

$$h_{F\otimes E}(f_1 \otimes e_1, f_2 \otimes e_2) := h_F(f_1, f_2) h(e_1, e_2) \quad ; e_1, e_2 \in C^\infty(E); f_1, f_2 \in C^\infty(F). \tag{3.6}$$

Similarly the connection and a structure of Clifford multiplication are given by

$$\nabla(f \otimes e) := (\nabla f) \otimes e + f \otimes \nabla e, \tag{3.7}$$

$$\mathbf{c}_{F\otimes E}(X \otimes f \otimes e) := f \otimes \mathbf{c}(X \otimes e). \tag{3.8}$$

With this structure, $F \otimes E$ is a Dirac bundle, and we may define the Dirac operator $D_{F\otimes E}$.

Definition 3.1. Let E be a Dirac bundle. We let $H_0^{2,k}$ be the completion of $C_0^\infty(E)$ with respect to the norm

$$\|f\|_{H^{2,k}} := \sqrt{\sum_{j=0}^{k} \|D^j f\|_{L^2(E)}}.$$

Further we let $H^{2,k}(E)$ be the space

$$H^{2,k}(E) = \{f \in L^2(E) \mid \forall j = 0, \dots, k : D^j f \in L^2(E)\}.$$

Also $H^{2,k}(E)$ is supplied with the norm $\|f\|_{H^{2,k}}$.

Lemma 3.2. *Let E be a Dirac bundle. Then $H^{2,k}_0(E)$ and $H^{2,k}(E)$ are Hilbert spaces, $H^{2,k}_0(E)$ is a closed subspace of $H^{2,k}(E)$ and there are bounded inclusions*

$$W^{2,k}(E) \hookrightarrow H^{2,k}(E), \tag{3.9}$$
$$W^{2,k}_0(E) \hookrightarrow H^{2,k}_0(E). \tag{3.10}$$

Further, if M is complete we have the identity

$$H^{2,k}_0(E) = H^{2,k}(E). \tag{3.11}$$

Proof: That $H^{2,k}_0(E)$ and $H^{2,k}(E)$ are Hilbert spaces is easily checked. The inclusions (3.9) and (3.10) follow since $\mathbf{c} \circ g^{-1} \in C^\infty_b(End(T^*M \otimes E, E))$ so that $D \in \mathrm{Diff}^1_{\mathrm{bd}}(E)$. Thus, by Lemma 1.4 for $j \in \mathbb{N}_0$, $D^j \in \mathrm{Diff}^j_{\mathrm{bd}}(E)$ and by Lemma 1.6, D^j maps $W^{2,k}$ continuously into $L^2(E)$. This immediately gives that the inclusions are bounded.

If M is complete all powers of D are essentially self-adjoint by [2]. Thus the domain of the closure $\overline{D^k}$ of D^k coincides with the domain of $(D^k)^*$. Further it follows that $\bar{D}^k = \overline{D^k}$, such that $D^j f \in L^2(E)$ for all f in the domain of $\overline{D^k}$ and for all $j = 0, \dots, k$. Consequently $H^{2,k}_0(E) = \mathcal{D}(\overline{D^k}) = \mathcal{D}((D^k)^*) = H^{2,k}(E)$. $\square$

The Bochner-Weizenböck curvature tensor $R^{BW} \in C^\infty(End(E))$ is given by

$$D^2 = \nabla^*\nabla + R^{BW}.$$

Explicitly, if $\{e_i\}$ is an orthonormal basis for T_xM, $R^{BW}_{|x}$ is given by

$$R^{BW}_{|x} = \frac{1}{2}\sum_{i \neq j} \mathbf{c}(e_i)\mathbf{c}(e_j) R^E_{|x}(e_i \otimes e_j). \tag{3.12}$$

Lemma 3.3. *If $R^E \in C^\infty_b(T^*M \otimes T^*M \otimes End(E))$, then $R^{BW} \in C^\infty_b(End(E))$.*

Proof: The formula (3.12) and that fact that $R^E_{e_ie_i} = 0$ gives that we may rewrite

$$R^{BW} = (\mathbf{c}_{1,2}g_1^{-1})(\mathbf{c}_{2,3}g_2^{-1})A_{2,4}A_{4,6}\mathrm{Id}_{T^*M} \otimes \mathrm{Id}_{T^*M} \otimes R^E. \tag{3.13}$$

Here $\mathbf{c}_{ij}$ denotes the section $\mathbf{c}$ of Clifford multiplication applied to the i'th and j'th entry, the i'th entry is skipped and the result is inserted in the j'th. The lemma immediately follows. $\square$

Lemma 3.4. *Let E be a Dirac bundle and assume that*

$$R \in C^\infty_b(T^*M \otimes T^*M \otimes End(TM))$$

and

$$R^E \in C^\infty_b(T^*M \otimes T^*M \otimes End(E)).$$

Let D be the Dirac operator defined on $(T^*_{\mathbb{C}}M)^{\otimes k} \otimes E$ for any $k \in \mathbb{N}_0$. For each $p \in \mathbb{N}$ we have

$$[\nabla, D^p] \in \mathrm{Diff}^{p-1}_{\mathrm{bd}}((T^*_{\mathbb{C}})^{\otimes k} \otimes E, (T^*_{\mathbb{C}})^{k+1} \otimes E).$$

Proof: We may assume without loss of generality that $k = 0$. Let $S_{i,j}$ be the permutation of the i'th and j'th entry like in Lemma 2.1. Then since $\mathbf{c}$ and g^{-1} are parallel,

$$\nabla D - D\nabla = \mathbf{c}_{2,3} g_2^{-1} \nabla^2 - \mathbf{c}_{1,3} g_1^{-1} \nabla^2 \tag{3.14}$$

$$= \mathbf{c}_{2,3} g_2^{-1} (1 - S_{1,2}) \nabla^2 \tag{3.15}$$

$$= \mathbf{c}_{2,3} g_2^{-1} R. \tag{3.16}$$

This proves the lemma in the case $p = 1$. For $p > 1$ it follows by the formula

$$[\nabla, D^p] = \sum_{q=0}^{p-1} D^q [\nabla, D] D^{p-q-1}$$

and Lemma 1.4. $\square$

We now have the tools for (re-)proving the main theorem in the case of Dirac operators.

Theorem 3.5. *Let E be a Dirac bundle. If the curvatures R and R^E satisfy*

$$R \in C_b^\infty(T^*M \otimes T^*M \otimes End(TM)),$$

$$R^E \in C_b^\infty(T^*M \otimes T^*M \otimes End(E)).$$

Then for each $k \in N_0$,

$$W_0^{2,k}(E) = H_0^{2,k}(E).$$

Proof: Since $C_0^\infty(E)$ is dense in both spaces, it suffices to prove that the norms are equivalent on $C_0^\infty(E)$. We already have that there exists a constant $C(k)$ such that

$$\|f\|_{H^{2,k}(E)} \leq C(k) \|f\|_{W^{2,k}(E)} \quad ; f \in C_0^\infty(E).$$

Thus it suffices to prove that there exists a constant $C'(k)$ such that

$$\|f\|_{W^{2,k}(E)} \leq C'(k) \|f\|_{H^{2,k}(E)}.$$

For $k = 0$ this is trivial. For $k = 1$ this is standard: Using the Bochner-Weizenböck formula one gets for $f \in C_0^\infty(E)$

$$\langle \nabla f, \nabla f \rangle_{L^2(T^*M \otimes E)} = \langle \nabla^* \nabla f, f \rangle_{L^2(E)}$$

$$= \left\langle (D^2 - R^{BW}) f, f \right\rangle_{L^2(E)} = \langle Df, Df \rangle_{L^2(E)} - \left\langle R^{BW} f, f \right\rangle_{L^2(E)}$$

$$\leq (1 + \|R^{BW}\|_\infty) \|f\|^2_{H^{2,1}(E)}.$$

We will iterate this computation and use induction. Thus we assume that the theorem is true for $k-1$. In order to get through with the induction step we need a further induction in a variable $p \in \mathbb{N}$. The induction hypothesis in p is that there exists a constant $C(k, p-1, R, R^E)$ such that

$$\|\nabla^k f\|^2_{L^2((T^*M)^k \otimes E)} \leq \|D^{p-1}\nabla^{k-p+1} f\|^2_{L^2((T^*M)^{\otimes k-p+1}\otimes E)} + C(k, p-1, R, R^E)\|f\|_{H^{2,k-1}(E)}.$$

This induction hypothesis is trivial for $p = 1$. Now, if $p \leq k$, we may estimate

$$\|D^{p-1}\nabla^{k-p+1} f\|_{L^2((T^*M)^{\otimes k-p+1}\otimes E)} \leq \|\nabla D^{p-1}\nabla^{k-p} f\|_{L^2((T^*M)^{\otimes k-p+1}\otimes E)} + \|[\nabla, D^{p-1}]\nabla^{k-p} f\|_{L^2((T^*M)^{\otimes k-p+1}\otimes E)}.$$

For $p = 1$, $[\nabla, D^{p-1}]$ vanishes. For $p > 1$, since $[\nabla, D^{p-1}] \in \mathrm{Diff}^{p-2}_{\mathrm{bd}}$ it follows by induction in k that the last term can be estimated by a multiple of $\|f\|_{H^{2,k-1}(E)}$. The first term can again be handled by the Bochner-Weizenböck formula:

$$\|\nabla D^{p-1}\nabla^{k-p} f\|_{L^2((T^*M)^{\otimes k-p+1}\otimes E)} = \left\langle D^{p+1}\nabla^{k-p} f, D^{p-1}\nabla^{k-p} f\right\rangle_{L^2((T^*M)^{\otimes k-p}\otimes E)} - \left\langle R^{BW} D^{p-1}\nabla^{k-p} f, D^{p-1}\nabla^{k-p} f\right\rangle_{L^2((T^*M)^{\otimes k-p}\otimes E)}. \tag{3.17}$$

By induction in k the second term can be estimated by a multiple of $\|f\|_{H^{2,k}(E)}$. This establishes the induction hypothesis in p.

For $p = k$ the first term can trivially be estimated by $\|f\|_{H^{2,k}(E)}$. This establishes the induction step in k, and the proof of the theorem is completed. □

The Bochner-Weizenböck formula gives easily that the H^k_0-spaces in this section coincide with the H^k_0-spaces in Section 2. Using the operator $|D|$ we can again construct fractional order Sobolev spaces and prove that fractional order Sobolev spaces are mapped by $\mathrm{Diff}^k_{\mathrm{bd}}(E, E)$ like one would expect it. These fractional order Sobolev spaces coincide with the ones defined in Section 2. This can be proved using the Bochner-Weizenböck formula and commutater estimates similar to those in the proof of Proposition 2.4.

4 Weighted Sobolev Spaces.

Let E be a Hermitian vector bundle and let ξ be a measurable section in $End(E)$, taking values in the pointwise positive endomorphisms with respect to the Hermitian structure. We assume that ξ is locally bounded from above and below and define

$$L^2_\xi(E) = \{f \in L^{2,\mathrm{loc}}(E) \mid \xi^{\frac{1}{2}} f \in L^2(E)\}.$$

The inner product on $L^2_\xi(E)$ is given by

$$\langle f, g\rangle_{L^2_\xi(E)} = \int_M \langle \xi f, g\rangle_{E_{|x}}\, dx.$$

The section $\xi \in End(E)$ naturally extends to $\xi \in (T^*_{\mathbb{C}}M)^{\otimes j} \otimes E$ by tensoring with 1. Using L^2_ξ, for each $k > 0$ we get a Sobolev space norm

$$\|f\|_{W^{2,k}_\xi(E)} = \sqrt{\sum_{j=0}^{k} \|\nabla^j f\|_{L^2_\xi((T^*_{\mathbb{C}}M)^{\otimes j}\otimes E)}} \, . \tag{4.1}$$

Let $W^{2,k}_{0,\xi}(E)$ be the completion of $C^\infty_0(E)$ with respect to $\|\cdot\|_{W^{2,k}_\xi}$. Further we define

$$W^{2,k}_\xi(E) = \{f \in L^2_\xi(E) \mid \forall j = 0, \dots k : \nabla^j f \in L^2_\xi((T^*_{\mathbb{C}}M)^{\otimes j} \otimes E)\}.$$

Again $W^{2,k}_{0,\xi}(E)$ and $W^{2,k}_\xi(E)$ are Hilbert Spaces and $W^{2,k}_{0,\xi}(E) \subseteq W^{2,k}_\xi(E)$.

If E is in addition a Dirac bundle, we may define $H^{2,k}_{0,\xi}(E)$ and $H^{2,k}_\xi(E)$ like above with D in place of ∇. Again there are the obvious inclusions

$$W^{2,k}_\xi(E) \hookrightarrow H^{2,k}_\xi(E), \tag{4.2}$$

$$W^{2,k}_{0,\xi}(E) \hookrightarrow H^{2,k}_{0,\xi}(E). \tag{4.3}$$

Due to commutaters between $\xi^{\frac{1}{2}}$ and differential operators the proof of Theorem 3.5 becomes more complicated in the case of weighted Sobolev spaces, and it only goes through under additional assumptions on ξ. We will assume:

Assumption 4.1. *The distribution $\nabla\xi^{\frac{1}{2}}$ belongs to $L^{\infty,loc}(End(E, T^*M \otimes E))$ and the section*

$$\xi^{-\frac{1}{2}}(\nabla\xi^{\frac{1}{2}}) \tag{4.4}$$

is bounded. Further, $\xi^{\frac{1}{2}}$ commutes with Clifford multiplication.

Lemma 4.2. *If Assumption 4.1 holds, $\xi^{-\frac{1}{2}}[D, \xi^{\frac{1}{2}}] \in L^\infty(End(E))$.*

Proof: We compute

$$D\xi^{\frac{1}{2}} = \mathbf{c}g^{-1}\nabla\xi^{\frac{1}{2}} = \mathbf{c}g^{-1}\xi^{\frac{1}{2}}\nabla + \mathbf{c}g^{-1}(\nabla\xi^{\frac{1}{2}})$$

$$= \xi^{\frac{1}{2}}D + \mathbf{c}g^{-1}(\nabla\xi^{\frac{1}{2}}).$$

Since $\mathbf{c}$ commutes with $\xi^{\frac{1}{2}}$, it also commutes with $\xi^{-\frac{1}{2}}$. Consequently

$$\xi^{-\frac{1}{2}}[D, \xi^{\frac{1}{2}}] = \mathbf{c}g^{-1}\xi^{-\frac{1}{2}}(\nabla\xi^{\frac{1}{2}}).$$

This proves the lemma. $\square$

Remark : If $M = \mathbb{R}$ typical weight functions like $(1+x^2)^{\frac{s}{2}}$ and $e^{a\sqrt{1+x^2}}$ satisfy (4.1). For the interior of a compact manifold with a smooth boundary, (4.1) is however rarely satisfied and the theory for weighted spaces does therefore not extend the results of the previous section in this case. The examples of weight functions on the real line immediately give examples of weight functions on a complete Riemannian manifold M. Let namely β be a weight function on $\mathbb{R}$ and let $\xi(x) = \beta(1 + d(x, q))$, where q is a fixed point in M and d denotes the Riemannian distance. Then, almost everywhere

$$\nabla \xi^{\frac{1}{2}} = (\beta^{\frac{1}{2}})'(1 + d(x, q))\nabla d(x, q).$$

Further the estimate

$$\frac{|d(x,q) - d(x',q)|}{d(x,x')} \leq 1$$

gives that $\nabla d(x, q)$ is bounded uniformly in norm wherever it is defined.

Theorem 4.3. *Let E be a Dirac bundle. If the curvatures R and R^E satisfy*

$$R \in C_b^\infty(T^*M \otimes T^*M \otimes End(TM)),$$

$$R^E \in C_b^\infty(T^*M \otimes T^*M \otimes End(E)),$$

and additionally ξ satisfies Assumption 4.1, then for each $k \in N_0$,

$$W_{0,\xi}^{2,k}(E) = H_{0,\xi}^{2,k}(E).$$

Proof: As in the proof of Theorem 3.5 it suffices to prove that there exist a constant $C'(k)$ such that for $f \in C_0^\infty(E)$,

$$\|f\|_{W_\xi^{2,k}(E)} \leq C'(k)\|f\|_{H_\xi^{2,k}(E)}. \tag{4.5}$$

For $k = 0$ this is trivial and for $k = 1$ we may compute

$$\langle \nabla f, \nabla f \rangle_{L^2_\xi(T^*M\otimes E)} = \left\langle \xi^{\frac{1}{2}}\nabla f, \xi^{\frac{1}{2}}\nabla f \right\rangle_{L^2(T^*M\otimes E)}$$

$$= \left\langle \nabla \xi^{\frac{1}{2}} f - (\nabla \xi^{\frac{1}{2}})f, \nabla \xi^{\frac{1}{2}} f - (\nabla \xi^{\frac{1}{2}})f \right\rangle_{L^2(T^*M\otimes E)}$$

$$\leq 2\left(\left\langle \nabla \xi^{\frac{1}{2}} f, \nabla \xi^{\frac{1}{2}} f \right\rangle_{L^2(T^*M\otimes E)} + \left\langle (\nabla \xi^{\frac{1}{2}})f, (\nabla \xi^{\frac{1}{2}})f \right\rangle_{L^2(T^*M\otimes E)}\right)$$

$$\leq 2\left(\left\langle (D^2 - R^{BW})\xi^{\frac{1}{2}} f, \xi^{\frac{1}{2}} f \right\rangle_{L^2(E)} + \|\xi^{-\frac{1}{2}}(\nabla \xi^{\frac{1}{2}})\|^2_{L^\infty(T^*M\otimes End(E))}\|f\|^2_{L^2_\xi}\right).$$

The term containing the Bochner-Weizenböck curvature courses no problems. After an integration by parts the remaining term is of the form

$$\left\langle D\xi^{\frac{1}{2}}f, D\xi^{\frac{1}{2}}f\right\rangle_{L^2(E)} \leq 2\left(\|\xi^{\frac{1}{2}}Df\|^2_{L^2(E)} + \|[D,\xi^{\frac{1}{2}}]f\|^2_{L^2(E)}\right).$$

Here the first term is just $\|Df\|_{L^2_\xi(E)}$. By Lemma 4.2 the second term can be estimated by a multiple of $\|f\|_{L^2_\xi}$. Thus the theorem holds for $k=1$. Again we will iterate this computation, and we will need an additional induction in a variable $p \in \mathbb{N}$. The induction hypothesis in p is that there exists a constant $C_{k,p-1} = C(k,\xi,p-1,R,R^E)$ such that

$$\|\nabla^k f\|^2_{L^2_\xi((T^*M)^{\otimes k}\otimes E)} \leq \|D^{p-1}\nabla^{k-p+1}f\|^2_{L^2_\xi((T^*M)^{\otimes k-p+1}\otimes E)} + C_{k,p-1}\|f\|^2_{H^{2,k-1}_\xi(E)}.$$

This hypothesis is trivial for $p=1$. Now, by induction in $1 \leq p \leq k$:

$$\|\nabla^k f\|^2_{L^2_\xi((T^*M)^{\otimes k}\otimes E)} \leq \|D^{p-1}\nabla^{k-p+1}f\|^2_{L^2_\xi((T^*M)^{\otimes k-p+1}\otimes E)} + C_{k,p-1}\|f\|^2_{H^{2,k-1}_\xi(E)}.$$

Further, if $p \leq k$ we may estimate

$$\begin{aligned}\|D^{p-1}\nabla^{k-p+1}f\|_{L^2_\xi((T^*M)^{\otimes k-p+1})} \leq{}& \|\nabla D^{p-1}\nabla^{k-p}f\|_{L^2_\xi((T^*M)^{\otimes k-p+1})}\\ &+ \|[\nabla, D^{p-1}]\nabla^{k-p}f\|_{L^2_\xi((T^*M)^{\otimes k-p+1})}.\end{aligned}$$

Since $[\nabla, D^{p-1}] \in \mathrm{Diff}^{p-2}_{\mathrm{bd}}$ it follows by induction in k that the last term is bounded from above by a constant multiple of $\|f\|_{H^{2,k-1}_\xi(E)}$. Again the first term can be handled exactly like in the case $k=1$. This establishes the induction hypothesis in p.

For $p = k$ the first term can trivially be estimated by a constant multiple of $\|f\|_{H^{2,k}(E)}$. This completes the induction step in k and proves the theorem. □

5 Further Equivalences.

In this section we assume that M is *complete* and that the curvatures R and R^E are bounded with all derivatives.

In the non-weighted case we have $H^{2,k}_0(E) = H^{2,k}(E)$. Together with the other inclusions this gives a circle of inclusions such that

$$W^{2,k}_0(E) = H^{2,k}_0(E) = H^{2,k}(E) \supseteq W^{2,k}(E) \supseteq W^{2,k}_0(E).$$

Consequently all Sobolev spaces are equivalent.

With only a slight strengthening of Assumption 4.1, this can be extended to weighted Sobolev spaces:

Assumption 5.1. *For all k the distribution $\nabla^k\xi^{\frac{1}{2}}$ belongs to $L^{\infty,loc}(End(E,T^*M\otimes E))$ and the sections*

$$\xi^{-\frac{1}{2}}(\nabla^k\xi^{\frac{1}{2}}) \tag{5.1}$$

are all bounded. Further, $\xi^{\frac{1}{2}}$ commutes with Clifford multiplication.

Lemma 5.2. *If Assumption 5.1 holds, $C_0^\infty(E)$ is dense in $W_\xi^{2,k}(E)$ and in $H_\xi^{2,k}(E)$ for all k.*

Proof: Take $\varphi_j \in C_0^\infty(E)$ such that $\|\varphi_j - \xi^{\frac{1}{2}} f\|_{W^{2,k}(E)} \to 0$ for $j \to \infty$. This is possible by Theorem 3.5 and since $C_0^\infty(E)$ is dense in $H^k(E)$ for all k. For $k = 0$ the lemma is easily checked. For $k > 0$ we use induction. Consider

$$\|\nabla^k (f - \xi^{-\frac{1}{2}} \varphi_j)\|_{L^2_\xi(T^*M^{\otimes k} \otimes E)} \leq$$

$$\|\xi^{\frac{1}{2}} \nabla^k (f - \xi^{-\frac{1}{2}} \varphi_j)\|_{L^2(T^*M^{\otimes k} \otimes E)} \leq$$

$$\|\nabla^k (\xi^{\frac{1}{2}} f - \varphi_j)\|_{L^2(T^*M^{\otimes k} \otimes E)} + \sum_p \begin{pmatrix} k \\ p \end{pmatrix} \|(\nabla^p \xi^{\frac{1}{2}}) \nabla^{k-p} (f - \xi^{-\frac{1}{2}} \varphi_j)\|_{L^2(T^*M^{\otimes k} \otimes E)}$$

$$\leq \|\xi^{\frac{1}{2}} f - \varphi_j\|_{W^{2,k}(E)} + \sum_{p=1}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \|\xi^{-\frac{1}{2}} \nabla^p \xi^{\frac{1}{2}}\|_\infty \|f - \xi^{-\frac{1}{2}} \varphi_j\|_{W_\xi^{2,k-p}(E)}.$$

The term $(\nabla^p \xi^{\frac{1}{2}}) \nabla^{k-p} (f - \xi^{-\frac{1}{2}})$ means the symmetrization of the tensor product in the factors of T^*M. By induction it follows that the terms in the sum converge towards 0. Thus $\xi^{-\frac{1}{2}} \varphi_j \to f$. The proof that $C_0^\infty(E)$ is dense in $H^{2,k}(E)$ is similar. □

Thus if ξ satisfies Assumption 5.1 we again have the circle of inclusions

$$W_{0,\xi}^{2,k}(E) = W_\xi^{2,k}(E) \subseteq H_\xi^{2,k}(E) = H_{0,\xi}^{2,k}(E) = W_{0,\xi}^{2,k}(E),$$

which implies that all spaces are equivalent.

Remark 1. By results of [9] it follows that Lemma 5.2 holds for a great class of weight functions satisfying Assumption 4.1 but not Assumption 5.1.

We will conclude this paper by demonstrating how the weighted theory can be used for studying properties of functions of D, considered as operators in L^2_ξ. These results overlap with the results of [9], but each method gives information not provided by the other. The following is of course well known. See for example [6]. The operator $\xi^{-\frac{1}{2}}$ is an isometry $L^2(E) \mapsto L^2_\xi(E)$. Consequently the operator

$$\xi^{-\frac{1}{2}} D \xi^{\frac{1}{2}} = D + \xi^{-\frac{1}{2}} [D, \xi^{\frac{1}{2}}]$$

is essentially self-adjoint on $C_0^\infty(E)$ considered as an unbounded operator in $L^2_\xi(E)$. If we let

$$A := \xi^{-\frac{1}{2}} D \xi^{\frac{1}{2}},$$

$$B := \xi^{-\frac{1}{2}} [D, \xi^{\frac{1}{2}}],$$

we thus have that A is a self-adjoint operator in $L^2_\xi(E)$, that B is a bounded operator in $L^2_\xi(E)$ and that $D = A - B$. Thus

$$(D-\lambda)^{-1} = (A-\lambda)^{-1}(I - B(A-\lambda)^{-1})^{-1}$$

is a bounded operator if $\mathrm{dist}(\lambda, \mathrm{spec}(A)) > \|B\|_\infty$. In the case where $B(A-\lambda)^{-1}$ is compact (which tends to be the case if ξ has sub-exponential growth and does not oscillate too much) we can further use analytic perturbation theory [11] in order to establish that $\mathrm{spec}_{L^2_\xi(E)}(D) \subseteq \mathrm{spec}_{L^2(E)}(D) \cup V$, where V is some discrete set of points. From the resolvent other functions of D can be constructed using contour integrals, and in this way a variety of partial differential equations involving D can be solved. The imbedding theorems for weighted Sobolev spaces now provide more precise information on the solutions.

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