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Brown's Spectral Distribution Measure for R-diagonal Elements in Finite von Neumann Algebras

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Abstract

In 1983 L. G. Brown introduced a spectral distribution measure for non-normal elements in a finite von Neumann algebra $\mathscr{M}$ with respect to a fixed normal faithful tracial state τ. In this paper we compute Brown's spectral distribution measure in case T has a polar decomposition $T = UH$ where U is a Haar unitary and U and H are $*$-free. (When $\operatorname{Ker} T = \{0\}$ this is equivalent to that (T, T^*) is an R-diagonal pair in the sense of Nica and Speicher.) The measure μ_T is expressed explicitly in terms of the S-transform of the distribution μ_{T^*T} of the positive operator T^*T. In case T is a circular element, i.e., $T = (X_1 + iX_2)/\sqrt{2}$ where (X_1, X_2) is a free semicircular system, then $\operatorname{sp} T = \bar{D}$, the closed unit disk, and μ_T has constant density $1/\pi$ on $\bar{D}$.

1 Introduction

In 1995 Nica and Speicher introduced the class of R-diagonal pairs in non-commutative probability spaces (see [10]). A pair (a, b) in the non-commutative probability space (A, φ) is called R-diagonal if the (2-dimensional) R-transform $R_{\mu_{(a,b)}}$ of the joint distribution $\mu_{(a,b)}$ of (a, b) is of the form

$$R_{\mu_{(a,b)}}(z_1, z_2) = \sum_{j=1}^{\infty} \alpha_j (z_1 z_2)^j + \sum_{j=1}^{\infty} \alpha_j (z_2 z_1)^j$$

for arbitrary complex numbers α_j. An R-diagonal element is a random variable in a non-commutative $*$-probability space such that (a, a^*) is an R-diagonal pair. In [10] Nica and Speicher prove that if T is an R-diagonal element in some tracial non-commutative C^*-probability space then T has the same $*$-distribution as a product UH where U and H

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are $*$-free random variables in some tracial non-commutative C^*-probability space, U is a Haar unitary and H is positive. When this happens H and $|T|$ have the same distribution, and the $*$-distribution of T is uniquely determined by the distribution of $T^*T = |T|^2$. In this paper we restrict to the case of tracial non-commutative W^*-probability spaces. This is not an essential restriction since a tracial C^*-probability space can always be embedded in a tracial W^*-probability space via the GNS representation.

L. G. Brown introduced in the paper [3] a spectral distribution measure μ_T for not necessarily normal operators T in a von Neumann algebra $\mathscr{M}$ with a faithful normal tracial state τ. The main purpose of this paper is to compute the spectrum $\operatorname{sp} T$ as well as the Brown measure μ_T for every R-diagonal element T in $(\mathscr{M}, \tau)$. We find a general expression for μ_T in terms of the S-transform of the distribution of T^*T and in particular, we find that the support of μ_T for an R-diagonal element T is given by

$$\operatorname{supp} \mu_T = \left\{ \lambda \in \mathbb{C} \mid \|T^{-1}\|_2^{-1} \leqslant |\lambda| \leqslant \|T\|_2 \right\} \tag{1.1}$$

in case $\operatorname{Ker} T = \{0\}$ and $T^{-1} \in L^2(\mathscr{M}, \tau)$. Otherwise $\operatorname{supp} \mu_T$ is the closed disk with radius $\|T\|_2$. The spectrum $\operatorname{sp} T$ coincides with $\operatorname{supp} \mu_T$ unless $T^{-1} \in L^2(\mathscr{M}, \tau) \setminus \mathscr{M}$ in which case $\operatorname{supp} \mu_T$ is the annulus (1.1), while $\operatorname{sp} T$ is the full closed disk with radius $\|T\|_2$. A key step in the proof is to show, that when a and b are $*$-free elements in $\mathscr{M}$ and $\tau(a) = \tau(b) = 0$, then the spectral radius of ab is $\|a\|_2 \, \|b\|_2$.

The paper is organized as follows. In Section 2 we list, for easy reference, the theory we need in this paper. In Section 3 we derive the basic properties of R-diagonal elements in finite von Neumann algebras. In Section 4 we give a complete description of the spectrum and the Brown measure of an R-diagonal element, and in Section 5 we compute concrete examples of Brown measures.

2 Preliminaries and Notation

We use the notation $(\mathscr{M}, \tau)$ to denote a tracial non-commutative W^*-probability space, i.e., a von Neumann algebra $\mathscr{M}$ with a normal faithful tracial state τ. When needed we assume that $\mathscr{M}$ acts as a von Neumann algebra on its associated GNS Hilbert space $L^2(\mathscr{M})$. When clarity demands it we write $\hat{a}$ to denote the element $a \in \mathscr{M}$ as an element of $L^2(\mathscr{M})$. We let $\|\cdot\|_2$ denote the norm arising from the inner product $\langle \hat{a}, \hat{b} \rangle = \tau(b^*a)$ on $L^2(\mathscr{M})$. For h a positive element in $\mathscr{M}$ we let μ_h be the unique compactly supported probability measure on $\mathbb{R}$ such that $\tau(h^n) = \int_{\mathbb{R}} t^n \, d\mu_h(t)$ and we extend $\|\cdot\|_2$ to "inverse" positive elements by the formula $\|h^{-1}\|_2 := (\int_{\mathbb{R}} t^{-2} \, d\mu_h(t))^{1/2} \in [0, \infty]$ for all $h \geqslant 0$. (We use the conventions $1/0 = \infty$ and $1/\infty = 0$ when computing these integrals.) This definition agrees with the previous if h is invertible. By $\operatorname{sp} a$ we denote the spectrum of a and $r(a)$ denotes the spectral radius of a. A symmetry is a self-adjoint unitary.

For a measure μ we let $\operatorname{supp} \mu$ denote the support of μ and if f is a function, μ_f is the image measure of μ induced by f. The name inv stands for the map $z \mapsto z^{-1}$ on $\mathbb{C} \setminus \{0\}$, and sq is the map $z \mapsto z^2$. If μ is supported on $\mathbb{R}$ we let $\widetilde{\mu}$ be the symmetrization of μ, i.e., $\widetilde{\mu}(A) = (\mu(-A) + \mu(A))/2$. The point measure centered at α is δ_α and dm, $d\lambda$ are used to denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{C}$ respectively. By $\times_{\mathrm{p}}$ we denote polar set product: $A \times_{\mathrm{p}} B = \{ae^{i\theta} \mid a \in A, \theta \in B\}$ and we say that f is a radial density function for the measure μ if the absolute continuous part of μ (with respect to Lebesgue measure) is given by $f(|\lambda|)\,d\lambda$. By $B(a,r)$ we denote the open ball with radius r centered at a.

We let Δ denote the Fuglede–Kadison-determinant on $(\mathscr{M},\tau)$, cf. [4] and let L denote $\log \Delta$. For easy reference we state the most important properties of Δ (expressed in terms of the L-function): for an arbitrary element a in $\mathscr{M}$ we have

$$L(a) = \int_{\mathbb{R}} \log t \, d\mu_{|a|}(t) \in [-\infty, \infty[$$

and $L(a) = L(a^*a)/2 = L(a^*)$. If b is an element in $\mathscr{M}$ then $L(ab) = L(a) + L(b)$, if u is a unitary $L(u) = 0$, and if z is a scalar $L(z1) = \log|z|$. If (a_n) is a sequence of positive elements, $a_n \geqslant a \geqslant 0$ and $a_n \to a$ in norm then $L(a_n) \to L(a)$, and if a is invertible then $L(a) = \tau(\log|a|)$. In particular $L(\exp\, a) = \operatorname{Re}\tau(a)$ for $a \geqslant 0$. In fact the formula

$$L(\exp\, a) = \operatorname{Re}\tau(a) \tag{2.1}$$

holds for all a in $\mathscr{M}$ (use [4, Lemma 3 with $H = 1$]). The functions L and Δ are continuous on the invertible elements in $(\mathscr{M}, \|\cdot\|)$ and in general upper semicontinuous on $(\mathscr{M}, \|\cdot\|)$.

For an arbitrary element a in $\mathscr{M}$ the function $\lambda \mapsto L(a - \lambda 1)$ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C}\setminus \operatorname{sp} a$ and the Riesz construction applied to $(2\pi)^{-1}L(a-\lambda 1)$ gives a regular positive probability measure (denoted) μ_a. We call this measure the *Brown measure* for a, cf. [3]. It is defined by

$$\mu_a = \frac{1}{2\pi}\nabla^2 L(a - \lambda 1)\, d\operatorname{Re}\lambda\, d\operatorname{Im}\lambda$$

where ∇^2 is the Laplace operator $\partial^2/\partial(\operatorname{Re}\lambda)^2 + \partial^2/\partial(\operatorname{Im}\lambda)^2$ in the distribution sense. (The notation $d\operatorname{Re}\lambda\, d\operatorname{Im}\lambda$ will often be replaced by $d\lambda$ or $dm_2(\lambda)$.) The Brown measure has the following properties: μ_a is the unique compactly supported measure (on the Borel measurable sets) that fulfils $L(a - \lambda 1) = \int_{\mathbb{C}} \log|z - \lambda|\, d\mu_a(z)$ for (almost) all complex numbers λ. The support of μ_a is contained in $\operatorname{sp} a$, and for p any natural number we have $\tau(a^p) = \int_{\mathbb{C}} z^p\, d\mu_a(z)$. Furthermore $\mu_{ab} = \mu_{ba}$ for arbitrary a and b in $\mathscr{M}$, and if f is analytic in a neighbourhood of $\operatorname{sp} a$, $\mu_{f(a)} = (\mu_a)_f$. As consequences we have $\mu_{a^{-1}} = (\mu_a)_{\mathrm{inv}}$ if a is invertible and $\mu_{bab^{-1}} = \mu_a$ whenever b is invertible. If a is normal, μ_a is the trace composed with the spectral measure for a hence the notation μ_a agrees with the previous introduced notation for positive elements, and the Brown measure for a Haar unitary is the Haar measure on $\mathbb{T}$.

By a non-commutative probability space (A,φ) we mean a unital algebra A (over the complex numbers) equipped with a linear functional φ such that $\varphi(1) = 1$. If A is a von Neumann algebra and φ is a normal state, (A,φ) is called a non-commutative W^*-probability space. We refer to [14] for the basics of free probability theory. For easy reference we restate some of the notation and nomenclature: By a° we denote the centered part $a - \varphi(a)$ of a, if $\varphi(a) = 0$ we say that a is centered. We call the numbers $\varphi(a^p)$ $(p = 1, 2, \dots)$ the moments of a, and the distribution μ of a is the linear functional $\mu\colon \mathbb{C}[X] \to \mathbb{C}$ determined by $\mu(P) = \varphi(P(a))$ for all P in $\mathbb{C}[X]$. If all the odd moments of a vanishes we say that a is symmetric distributed. If (B, ψ) is a non-commutative probability space, $a \in A$, $b \in B$ we write $a \sim_{\mathrm{D}} b$ when a and b have the same distribution. If (A, φ) and (B, ψ) are $*$-probability spaces the notation $a \sim_{*\mathrm{D}} b$ means that a and b have the same $*$-distribution. If (A, φ) is a tracial non-commutative $*$-probability space, $a, u \in A$, u is a unitary we have that uau^* and a have the same $*$-distribution. Note that if a, b, c, d are random variables, $a \sim_{*\mathrm{D}} b$, $c \sim_{*\mathrm{D}} d$, a, c are $*$-free and b, d are $*$-free, then $ac \sim_{*\mathrm{D}} bd$. When forming free products of non-commutative probability spaces we often have a natural choice of functionals, and in such cases we omit specifying the functionals, i.e., $A * B$ is an abbreviation for $(A, \varphi) * (B, \psi)$. By an isomorphism $\Phi\colon (A, \varphi) \to (B, \psi)$ we mean an isomorphism $\Phi\colon A \to B$ such that $\varphi = \psi \circ \Phi$. If $(\mathscr{N}, \omega)$ is a finite non-commutative W^*-probability space with a faithful trace ω, $a \in \mathscr{M}$, $b \in \mathscr{N}$ and $a \sim_{*\mathrm{D}} b$ then there exists a surjective $*$-isomorphism $\Phi\colon (W^*(a), \tau) \to (W^*(b), \omega)$ such that $\Phi(a) = b$. By the notation $i_1 \neq i_2 \neq \cdots \neq i_n$ we mean $i_1 \neq i_2$, $i_2 \neq i_3, \dots, i_{n-1} \neq i_n$. A product $x_1 \cdots x_n$ where $x_j \in A_{i_j}$, $i_1 \neq i_2 \neq \cdots \neq i_n$ is called an alternating product.

If a is a self-adjoint random variable in $\mathscr{M}$ there is a unique measure μ_a supported on $\operatorname{sp} a$ fulfilling $\tau(a^n) = \int_{\mathbb{R}} t^n \, d\mu_a(t)$. Faithfulness of τ implies that $\operatorname{supp} \mu_a = \operatorname{sp} a$. If μ is a compactly supported probability measure on $\mathbb{R}$ the distribution of the identity map id in $(L^\infty(\operatorname{supp} \mu, \mu), \int \cdot \, d\mu)$ has the same moments as μ, hence given a compactly supported probability measure on $\mathbb{R}$ this measure represents the distribution of a self-adjoint element in some finite non-commutative W^*-probability space. If $\operatorname{supp} \mu \subseteq [0, \infty[$ then μ corresponds to the distribution of a positive element. Using measures in place of distributions we get the analytic version of the theory of R- and S-transforms: if μ is a compactly supported measure on $\mathbb{R}$ the Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} \, d\mu(x) \tag{2.2}$$

is defined and analytic on $\mathbb{C} \setminus \operatorname{supp} \mu$ and $G_\mu(z) \sim z^{-1}$ as $|z| \to \infty$. It is seen from (2.2) that $\operatorname{Im} z \cdot \operatorname{Im} G_\mu(z) \leqslant 0$ for all z. These properties are the main tools used to determine the Cauchy transform when solving quadratic equations. The Cauchy transform is invertible in a neighbourhood of ∞ and the R-transform of μ is obtained from the inverse function ${G_\mu}^{-1}$ as $\mathscr{R}_\mu(z) = {G_\mu}^{-1}(z) - z^{-1}$ or equivalently

$$z = G_\mu\bigl(z^{-1}(1 + z\mathscr{R}_\mu(z))\bigr) \tag{2.3}$$

for z in a neighbourhood of 0, $z \neq 0$. If μ is symmetric a simple computation shows that $G_{\mu_{\mathrm{sq}}}(z) = G_\mu(\sqrt{z})/\sqrt{z}$. If $\int_{\mathbb{R}} t \, d\mu(t) \neq 0$ we have the analytic version of the S-transform too, cf. [14, Section 3.6]: The ψ-function is analytic in a neighbourhood of 0 and is given by

$$\psi_\mu(u) = \int_{\mathbb{R}} \frac{tu}{1-tu}\, d\mu(t).$$

Since ${\psi_\mu}'(0) = \int_{\mathbb{R}} t\, d\mu(t) \neq 0$, ψ_μ is invertible (with inverse χ_μ) in a neighbourhood of 0. Then the S-transform of μ is given by $\mathscr{S}_\mu(w) = (w+1)\chi_\mu(w)/w$ and is analytic in a neighbourhood of 0, cf. [6]. The Cauchy transform G_μ and ψ_μ are related by the formula

$$u(1+\psi_\mu(u)) = G_\mu(u^{-1}) \tag{2.4}$$

for u in a neighbourhood of 0, $u \neq 0$. Now define $z = z(u) = u(1+\psi_\mu(u))$ for $u \approx 0$. It follows from (2.3) and (2.4) that

$$\frac{1}{u} = \frac{1}{z}(1 + z\mathscr{R}_\mu(z))$$

and hence that $z/u - 1 = z\mathscr{R}_\mu(z) = \psi_\mu(u)$. Then $u = \chi_\mu(\psi_\mu(u)) = \chi_\mu(z\mathscr{R}_\mu(z))$ and also

$$z\mathscr{R}_\mu(z)\mathscr{S}_\mu\big(z\mathscr{R}_\mu(z)\big) = (1+z\mathscr{R}_\mu(z))\chi_\mu(z\mathscr{R}_\mu(z)) = \frac{z}{u}\cdot u = z.$$

This relation is valid for z in a neighbourhood of 0. This proves the following connection between $\mathscr{R}_\mu$ and $\mathscr{S}_\mu$, first established in [9]:

$$z\mathscr{S}_\mu(z) = \big(z\mathscr{R}_\mu(z)\big)^{\langle -1\rangle} \tag{2.5}$$

where $(\,\cdot\,)^{\langle -1\rangle}$ means inversion with respect to composition. (Also note that if one of the functions $\mathscr{R}_\mu$, ψ_μ and $\mathscr{S}_\mu$ is analytic in a neighbourhood of 0, the other two functions are analytic too. Therefore (2.5) holds for all distributions μ with $\mu(X) \neq 0$ because we can apply the standard trick of truncating power series to polynomials.)

3 Basic Properties of R-diagonal Elements in Finite von Neumann Algebras

Proposition 3.1. ([11], Theorem 4.5.) *Let x and y be free self-adjoint symmetric distributed elements in a tracial non-commutative W^*-probability space $(\mathscr{M},\tau)$. Then xy is R-diagonal.*

We shall use the following immediate corollary of Proposition 3.1.

Corollary 3.2. *Let $(\mathscr{M},\tau)$ be as above and let a and x be free self-adjoint symmetric distributed elements in $\mathscr{M}$, such that $a^2 = 1$. Then ax is R-diagonal. Hence ax has the same $*$-distribution as uh, where u and h are $*$-free elements in a non-commutative W^*-probability space $(\mathscr{N},\omega)$, u is a Haar unitary and h is positive with the same distribution as $|x|$.*

Lemma 3.3. *Let (A,φ) be a non-commutative probability space, and let a and x be free symmetric distributed random variables in A. Suppose that $a^2 = 1$. Define*

$$\begin{aligned} P_{\mathrm{e}} &= \{p \mid p \text{ is an even polynomial, } \varphi(p(x)) = 0\} \\ P_{\mathrm{o}} &= \{p \mid p \text{ is an odd polynomial}\} \\ P &= P_{\mathrm{e}} \cup P_{\mathrm{o}} \end{aligned}$$

and let T be the set of products $a^{m_0} p_1(x) a \cdots p_k(x) a^{m_k}$ where $k \in \mathbb{N}$, $p_1, \ldots, p_k \in P$, $m_0, m_k \in \{0,1\}$ and $m_0 + (k-1) + m_k + \sum_{j=1}^{n} \deg\, p_j \in 2\mathbb{N}$.

Then $\mathrm{alg}(1, ax, xa) = \mathrm{span}(\{1\} \cup T)$ *and* $\mathrm{alg}(1, ax, xa)^\circ = \mathrm{span}\, T$.

Proof: The last statement is an immediate consequence of the first statement. Put $B = \mathrm{alg}(1, ax, xa)$, $S = \mathrm{span}(\{1\} \cup T)$. Then $1, ax, xa \in S$. We show that $axS, xaS, Sax, Sxa \subseteq S$ and it is enough to prove that $axT, xaT, Txa, Tax \subseteq S$.

Let $t = a^{m_0} p_1(x) a \cdots p_k(x) a^{m_k} \in T$. If $m_0 = 1$ then $axt = axap_1(x)a \cdots p_k(x)a^{m_k} \in T$ and if $m_0 = 0$ then

$$axt = axp_1(x)ap_2(x)a \cdots p_k(x)a^{m_k}. \tag{3.1}$$

If $\deg p_1$ is even then $axt \in T$ because $\mathrm{id} \cdot p_1 \in P_{\mathrm{o}}$ and

$$\begin{aligned} &1 + (k-1) + m_k + \deg(\mathrm{id} \cdot p_1) + \sum_{j=2}^{k} \deg p_j \\ &\qquad = m_0 + (k-1) + m_k + \sum_{j=1}^{k} \deg p_j + 2 \in 2\mathbb{N}. \end{aligned}$$

Otherwise $\deg p_1$ is odd and we rewrite (3.1) to

$$\begin{aligned} axt &= axp_1(x)ap_2(x) \cdots p_k(x)a^{m_k} - \varphi(xp_1(x))p_2(x)a \cdots p_k(x)a^{m_k} \\ &\quad + \varphi(xp_1(x))p_2(x)a \cdots p_k(x)a^{m_k} \\ &= a(xp_1(x))^\circ ap_2(x)a \cdots p_k(x)a^{m_k} + \varphi(xp_1(x))p_2(x)a \cdots p_k(x)a^{m_k}. \end{aligned}$$

Here $p_2(x)a \cdots p_k(x)a^{m_k} \in T$ because

$$\begin{aligned} &(k-2) + m_k + \sum_{j=2}^{k} \deg p_j \\ &\qquad = m_0 + (k-1) + m_k + \sum_{j=1}^{k} \deg p_j - (1 + \deg p_1) \in 2\mathbb{N}. \end{aligned}$$

Since $\deg(\mathrm{id}\cdot p_1)^\circ = \deg(\mathrm{id}\cdot p_1) = 1+\deg p_1$, we infer that $a(xp_1(x))^\circ ap_2(x)\cdots p_k(x)a^{m_k} \in T$ because

$$\begin{aligned}1 + (k-1) + m_k + \deg(\mathrm{id}\cdot p_1)^\circ + \sum_{j=2}^{k} \deg p_j \\ = m_0 + (k-1) + m_k + \sum_{j=1}^{k} \deg p_j + 2 \in 2\mathbb{N}.\end{aligned}$$

This shows that $axt \in S$.

Summing up we have proved that $axT \subseteq S$. The same argument applies to show that $xaT, Tax, Txa \subseteq S$. We conclude that S contains $1, ax, xa$ and is stable under multiplication by ax and xa. But B is the smallest subspace of A with this property whence $B \subseteq S$.

We remain to prove that $S \subseteq B$, and it suffices to prove that $T \subseteq B$. If p is an even polynomial then $p(s) = q(s^2)$ for some polynomial q, and

$$\begin{aligned}p(x) &= q(xaax) \in B,\\ ap(x)a &= q(ax^2a) \in B.\end{aligned}$$

If p is an odd polynomial then $p(s) = sq(s^2) = q(s^2)s$ for some polynomial q and

$$\begin{aligned}ap(x) &= axq(xaax) \in B,\\ p(x)a &= q(xaax)xa \in B.\end{aligned}$$

It is then easy to see that an arbitrary element in T can be written as a product of elements of the forms $p(x), ap(x)a$ (p even) and $ap(x), p(x)a$ (p odd). We conclude that $S \subseteq B$. $\square$

Lemma 3.4. *Let (A,φ) be a non-commutative probability space and let $a, (x_i)_{i\in I}$ be symmetric distributed random variables in A such that $(x_i)_{i\in I}$ is a free family and $\{a\}$ and $\{x_i \mid i \in I\}$ are free sets in A. Suppose that $a^2 = 1$.*

Then the sets $\{ax_i, x_ia\}$ $(i \in I)$ are free.

Proof: Put $A_i = \mathrm{alg}(1, ax_i, x_ia)$. Lemma 3.3 shows that

$$A_i^\circ \subseteq \mathrm{span}\big(S_i \cup aS_i \cup S_ia \cup aS_ia \cup S_iaS_i \cup \cdots\big)$$

where $S_i = \mathrm{alg}(1, x_i)^\circ$. To show freeness of $\{ax_i, x_ia\}$ $(i \in I)$ it suffices to show that the product

$$X_1 \cdots X_n \tag{3.2}$$

is centered whenever $n \in \mathbb{N}$, $X_j \in A_{i_j}^\circ$ $(j = 1,\dots,n)$ and $i_1 \neq \cdots \neq i_n$. By linearity of φ it is sufficient to assume that X_j is a word in $S_{i_j} \cup aS_{i_j} \cup S_{i_j}a \cup \cdots$ (for all j). In this case the product (3.2) consists of alternating occurrences of words from the sets $\{a\}$, S_{i_j} $(j = 1,\dots,n)$, and the freeness assumptions implies that the product (3.2) is centered. $\square$

We note that the condition that all odd moments vanishes is necessary: If $\varphi(x^{2m-1}) \neq 0$ and $\varphi(y^{2n-1}) \neq 0$ for some m, n then $\varphi(x^{2m-1}aay^{2n-1}) = \varphi(x^{2m-1}y^{2n-1}) = \varphi(x^{2m-1})\varphi(y^{2n-1}) \neq 0$, but $\varphi(x^{2m-1}a) = 0$ and $\varphi(ay^{2n-1}) = 0$. If $\varphi(x^{2m-1}) = 0$ for all $m \in \mathbb{N}$ but $\varphi(y^{2n-1}) \neq 0$ for some n then $\varphi((xa)(ay^{2n-1})(xa)(ay^{2n-1})) = \varphi(x^2)\varphi(y^{2n-1})^2$ which in general is different from 0.

The condition that $a^2 = 1$ is necessary too: if a, x, y are suitably chosen $*$-free unitaries in $L(\mathbb{Z}_4 * \mathbb{Z}_2 * \mathbb{Z}_2)$ then $(xa)(ayya)(ax) = 1$ but xa, ax and $ayya = a^2$ are centered. This shows that $\{ax, xa\}$ and $\{ay, ya\}$ are not free.

Proposition 3.5. ([10], [11].) *Let r and s be $*$-free R-diagonal elements in a tracial non-commutative W^*-probability space $(\mathscr{M}, \tau)$. Then*

(i) *$r + s$ is R-diagonal.*

(ii) *The distribution of $|r + s|$ can be obtained from the distributions of $|r|$ and $|s|$ by the formula*

$$\widetilde{\mu}_{|r+s|} = \widetilde{\mu}_{|r|} \boxplus \widetilde{\mu}_{|s|}.$$

where $\widetilde{\mu}$ denotes the symmetrization of a measure μ on $\mathbb{R}$.

Proof: (i) follows immediately from the definition of R-diagonal elements in [10] and the fact that

$$R_{r+s,r^*+s^*}(z_1, z_2) = R_{r,r^*}(z_1, z_2) + R_{s,s^*}(z_1, z_2)$$

whenever r and s are $*$-free elements, cf. [8].

(ii) This can be extracted from Proposition 5.2 in [11]. For convenience of the reader, we include a different proof based on Corollary 3.2 and Lemma 3.4. We can choose a tracial non-commutative W^*-probability space $(\mathscr{N}, \omega)$, which contains three self-adjoint elements a, x and y, such that (a, x, y) is a free family, $a^2 = 1$ and a, x and y are symmetric distributed with

$$\mu_x = \widetilde{\mu}_{|r|}, \qquad \mu_y = \widetilde{\mu}_{|s|}.$$

By Corollary 3.2 and Lemma 3.4 $r' = ax$ and $s' = ay$ are $*$-free R-diagonal elements with the same $*$-distributions as r and s respectively. Hence (r', s') has the same (joint) $*$-distribution as (r, s), so without loss of generality, we may assume that $r = ax$, $s = ay$ and $\tau = \omega$. Since x and y are free and symmetric, $x + y$ is symmetric distributed and $\mu_{x+y} = \mu_x \boxplus \mu_y$. We have $r + s = a(x + y)$ and thus $|r + s| = |x + y|$ whence

$$\widetilde{\mu}_{|r+s|} = \widetilde{\mu}_{|x+y|} = \mu_{x+y} = \mu_x \boxplus \mu_y = \widetilde{\mu}_{|x|} \boxplus \widetilde{\mu}_{|y|} = \widetilde{\mu}_{|r|} \boxplus \widetilde{\mu}_{|s|}.$$

This proves (ii). $\square$

Proposition 3.6. ([10].) *Let r and s be $*$-free R-diagonal elements in a finite non-commutative W^*-probability space $(\mathscr{M},\tau)$. Then*

(i) *rs is R-diagonal.*

(ii) *The distribution of $|rs|$ can be obtained from the distributions of $|r|$ and $|s|$ by the formula*

$$\mu_{|rs|^2} = \mu_{|r|^2} \boxtimes \mu_{|s|^2}.$$

Proof: (i) This is a special case of Theorem 1.5 in [10].

(ii) This can be extracted from Corollary 1.8 in [10], but for convenience of the reader we include a direct proof. We can choose 4 $*$-free elements u, v, h and k in a tracial non-commutative W^*-probability space $(\mathscr{N},\omega)$ such that u and v are Haar unitaries, h and k are positive elements with the same distributions as $|r|$ and $|s|$ respecitively. Then $r' = uh$ and $s' = vk$ are $*$-free R-diagonal elements with the same $*$-distributions as r and s respectively, and hence (r',s') has the same (joint) $*$-distribution as (r,s). Thus, without loss of generality, we may assume that $r = uh$, $s = vk$ and $\tau = \omega$. Since ω is a trace, we have for all natural numbers p

$$\omega(|rs|^{2p}) = \omega((v^*h^2vk^2)^p).$$

Thus $|rs|^2$ has the same distribution as $(v^*h^2v)k^2$ and since v^*h^2v and k^2 are free, we get

$$\mu_{|rs|^2} = \mu_{v^*h^2v} \boxtimes \mu_{k^2} = \mu_{h^2} \boxtimes \mu_{k^2},$$

where the last equality follows from the trace property of ω. This proves (ii). $\square$

In the next lemma we collect some well-known facts about freeness obtained by encapsulating sets with Haar unitaries.

Lemma 3.7. *Let (A,φ) be a non-commutative $*$-probability space, let u be a Haar unitary in A. Assume that S is a set in A such that S and $\{u\}$ are $*$-free.*

Then for any natural number n we have that

(i) *the sets S, uSu^*, $u^2S(u^*)^2, \dots$ are $*$-free,*

(ii) *the sets S, $uSu^*, \dots, u^{n-1}S(u^*)^{n-1}$, $\{u^n\}$ are $*$-free,*

(iii) *the sets $uSu^*, \dots, u^nS(u^*)^n$, $\{u^n\}$ are $*$-free.*

Proof: Put $A_0 = \mathrm{alg}(\{1\} \cup S \cup S^*)$ and for any natural number n put

$$A_n = \mathrm{alg}\big(\{1\} \cup u^nSu^{-n} \cup u^nS^*u^{-n}\big) = u^nA_0u^{-n}.$$

Note that $(A_n)^\circ = u^nA_0^\circ u^{-n}$.

Consider an alternating product $x_1 \cdots x_p$ of centered elements from $A_0, A_1, \dots$, i.e., $i_1 \neq \cdots \neq i_p$ and $x_j = u^{i_j} y_j u^{-i_j}$ for some $y_1, \dots, y_p \,\epsilon\, A_0^\circ$. Then

$$x_1 \cdots x_p = u^{i_1} y_1 u^{i_2 - i_1} y_2 \cdots u^{i_p - i_{p-1}} y_p u^{-i_p}$$

where $i_2 - i_1 \neq 0, \dots, i_p - i_{p-1} \neq 0$. The $*$-freeness assumption on $\{u\}$ and S gives that $\varphi(x_1 \cdots x_p) = 0$. This proves (i).

Let $A_{-1} = \mathrm{alg}(u^n, (u^*)^n)$. Suppose that $x_1 \cdots x_p$ is an alternating product of centered elements from $A_{-1}, A_0, \dots, A_{n-1}$. This means that $x_j \,\epsilon\, A_{i_j}^\circ$ for $i_1 \neq i_2 \neq \cdots \neq i_p$, $i_j \,\epsilon\, \{-1, 0, 1, \dots, n-1\}$. Since $A_{-1}^\circ = \mathrm{span}\{u^{nq} \mid q \,\epsilon\, \mathbb{Z} \setminus \{0\}\}$ it is sufficient to consider the case where x_j is of the form u^{nq} whenever $i_j = -1$. If $i_j \neq -1$ we have $x_j = u^{i_j} y_j u^{-i_j}$ for some $y_j \,\epsilon\, A_0^\circ$, and we assume without loss of generality that $y_j \neq 0$.

We show that the occurring y's are separated by elements from $\mathrm{alg}(u, u^*)^\circ$. There are two cases: Either two neighbouring y's come from consecutive x's, otherwise there is precisely one element of the form u^{nq} ($q \,\epsilon\, \mathbb{Z} \setminus \{0\}$) between the corresponding x's. In the first case the y's in question are y_j and y_{j+1}, for some j. But then $y_j u^{i_{j+1} - i_j} y_{j+1}$ is a subword of $x_1 \cdots x_p$ hence the y's are separated. Otherwise we have for some j that $i_j, i_{j+2} \,\epsilon\, \{0, \dots, n-1\}$ and $i_{j+1} = -1$. Then $x_{j+1} = u^{nq}$ for some non-zero integer q and $y_j u^{nq - i_j + i_{j+2}} y_{j+2}$ is a subword of $x_1 \cdots x_p$. But $nq - i_j + i_{j+2} \neq 0$ for any non-zero integer q because $i_j, i_{j+2} \,\epsilon\, \{0, 1, \dots, n-1\}$. We conclude that in this case the y's are separated by an element of the form u^r ($r \,\epsilon\, \mathbb{Z} \setminus \{0\}$).

We have thus shown that $x_1 \cdots x_p$ is an alternating product of centered elements from $\mathrm{alg}(u, u^*)$ and A_0 respectively, and this shows (ii). (iii) follows by the same proof as for (ii), since $i_j, i_{j+2} \,\epsilon\, \{1, \dots, n\}$ and $q \,\epsilon\, \mathbb{Z} \setminus \{0\}$ also implies that $nq - i_j + i_{j+2} \neq 0$. $\square$

The same method can be used to show that for example S, uSu^*, u^3, $u^5 S (u^*)^5$ are $*$-free.

Lemma 3.8. *Let (A, φ) be a non-commutative probability space, let a be a random variable in A satisfying $\varphi(a) = \cdots = \varphi(a^{n-1}) = 0$ and $a^n = 1$. Assume that S is a set in A such that $\{a\}$ and S are free.*

Then $S, aSa^{n-1}, a^2 S a^{n-2}, \dots, a^{n-1} S a$ are free.

Proof: As in the proof of Lemma 3.7 we put $A_0 = \mathrm{alg}(\{1\} \cup S)$, $A_j = \mathrm{alg}(\{1\} \cup a^j S a^{n-j})$ for $j = 1, \dots, n-1$, and note that $A_j = a^j A_0 a^{n-j}$, $A_j^\circ = a^j A_0^\circ a^{n-j}$. Let $x_1 \cdots x_p$ be an alternating product of centered elements from $A_0, \dots, A_{n-1}$, i.e., $x_j \,\epsilon\, A_{i_j}^\circ$, $i_1 \neq \cdots \neq i_p$. Then $x_j = a^{i_j} y_j a^{n-i_j}$ for some $y_j \,\epsilon\, A_0^\circ$ ($j = 1, \dots, p$) and $x_1 \cdots x_p = a^{i_1} y_1 a^{n - i_1 + i_2} \cdots y_p a^{n - i_p}$. Since $i_1 \neq \cdots \neq i_p$ it follows from the assumptions on a that $x_1 \cdots x_p$ is an alternating product of centered elements, and the freeness assumption on S and $\{a\}$ implies then that $x_1 \cdots x_p$ is centered. This shows that $S, aSa^{n-1}, \dots, a^{n-1} S a$ are free. $\square$

Lemma 3.9. ([10].) *Let u and a be $*$-free elements in a finite non-commutative W^*-probability space, such that u is a Haar unitary. Then ua and au are R-diagonal elements.*

Proof: Since u is R-diagonal this follows from Theorem 1.5 in [10]. $\square$

Proposition 3.10.

(i) *Let r be an R-diagonal element and let p be a natural number. Then r^p is R-diagonal and*

$$\mu_{|r^p|^2} = \underbrace{\mu_{|r|^2} \boxtimes \cdots \boxtimes \mu_{|r|^2}}_{p \text{ factors}} \qquad (= \mu_{|r|^2}^{\boxtimes p}.)$$

(ii) *If r is R-diagonal and invertible, then r^{-1} is R-diagonal and $\mu_{|r^{-1}|}$ is the image measure $(\mu_{|r|})_{\mathrm{inv}}$ of $\mu_{|r|}$ by the inversion map $t \mapsto t^{-1}$ on $\mathbb{R} \setminus \{0\}$.*

Proof: (i) Without loss of generality we may assume that $r = uh$, where u and h are $*$-free, u is a Haar unitary and $h \geqslant 0$. Then

$$(uh)^p = u^p\big((u^*)^{p-1}hu^{p-1}\big)\cdots(u^*hu)h.$$

It is clear that u^p is a Haar unitary. By Lemma 3.7 u^p is $*$-free from the remaining p factors in the above product. Hence by Lemma 3.9 $(uh)^p$ is R-diagonal. Moreover

$$|(uh)^p|^2 = h(u^*hu)\cdots\big((u^*)^{p-1}hu^{p-1}\big)^2\cdots(u^*hu)h$$

so by the trace property, $|(uh)^p|^2$ has the same distribution as

$$(u^*hu)\cdots\big((u^*)^{p-1}hu^{p-1}\big)^2\cdots(u^*hu)h^2,$$

so by Lemma 3.7

$$\mu_{|(uh)^p|^2} = \mu_y \boxtimes \mu_{h^2}$$

where $y = (u^*hu)\cdots\big((u^*)^{p-1}hu^{p-1}\big)^2\cdots(u^*hu) = u^*|(uh)^{p-1}|^2u$. Thus by the trace property

$$\mu_{|(uh)^p|^2} = \mu_{|(uh)^{p-1}|^2} \boxtimes \mu_{h^2},$$

and hence, (i) follows by induction in p.

(ii) Again, we may assume that $r = uh$, as in (i). Hence $r^{-1} = h^{-1}u^*$. Since u^* is a Haar unitay and $*$-free from h^{-1}, we get from Lemma 3.9 that r^{-1} is R-diagonal. Moreover $|r^{-1}| = uh^{-1}u^*$ has the same distribution as $h^{-1} = |r|^{-1}$. This proves (ii). $\square$

Example 3.11. Lemma 3.8 enables us to compute the Brown measure of an element ah where a and h are free random variables in some non-commutative W^*-probability space $(\mathscr{M}, \tau)$, $a^2 = 1$, $a = a^*$, $\tau(a) = 0$ and h is positive: We first note that ah is symmetric distributed and that $\mu_{ah} = \mu_{h^{1/2}ah^{1/2}}$ whence μ_{ah} is supported in $\mathbb{R}$, and then we compute its square (which is supported in $[0, \infty[$): $(\mu_{ah})_{\mathrm{sq}} = \mu_{ahah} = \mu_{aha} \boxtimes \mu_h = \mu_h \boxtimes \mu_h$ because aha and h are positive and free, cf. Lemma 3.8. Then μ_{ah} is the symmetrization of $(\mu_h \boxtimes \mu_h)_{z \mapsto \sqrt{z}}$.

Example 3.12. Using Lemma 3.8 we can state the distribution of the real and imaginary parts of an R-diagonal random variable $T = uh = ax$ (where u, h, a and x are random variables as stated in Corollary 3.2). We first note that $T \sim_{*\mathrm{D}} -iT$ whence $\mathrm{Re}\, T$ and $\mathrm{Re}\, -iT = \mathrm{Im}\, T$ have the same distribution. Then $2\mathrm{Re}\, T = T + T^* = ax + xa$ and it is straightforward to verify that the odd moments of $2\mathrm{Re}\, T$ vanishes. Lemma 3.8 yields that x and axa are free hence $x + axa$ is symmetric distributed. In computing the even moments we note that $(ax + xa)^2 = (x + axa)^2$ and we conclude that $ax + xa \sim_{\mathrm{D}} x + axa$. This gives $\mu_{2\mathrm{Re}\, T} = \mu_{ax+xa} = \mu_{x+axa} = \mu_x \boxplus \mu_{axa} = \mu_x \boxplus \mu_x$, cf. Application 1.3 in [11].

4 Brown Measures of R-diagonal Elements

Proposition 4.1. *Let $(\mathscr{M}, \tau)$ be a non-commutative von Neumann probability space with a faithful trace τ. Let a and b be $*$-free centered elements in $\mathscr{M}$.*

Then the spectral radius, $r(ab)$, of ab is $\|a\|_2 \, \|b\|_2$.

Proof: We can without loss of generality assume that $\|a\|_2 = \|b\|_2 = 1$. Put $\mathscr{M}_a = W^*(a)$, $\mathscr{M}_b = W^*(b)$ and let $\mathscr{M}_j^\circ$ denote the set of centered elements of $\mathscr{M}_j$, $j = a, b$. It is no loss of generality to assume that $\mathscr{M} = \mathscr{M}_a * \mathscr{M}_b$ and that $\mathscr{M}_a * \mathscr{M}_b$ acts on its GNS Hilbert space $(\mathscr{H}, \xi) = (L^2(\mathscr{M}), \hat{1})$. Let $(\mathscr{H}_a, \xi_a)$ and $(\mathscr{H}_b, \xi_b)$ be the GNS spaces of $(\mathscr{M}_a, \tau_a)$ and $(\mathscr{M}_b, \tau_b)$ ($\tau_j = \tau|_{\mathscr{M}_j}$, $j = a, b$). Then by [12, Section 1] $(\mathscr{H}, \xi) = (\mathscr{H}_a, \xi_a) * (\mathscr{H}_b, \xi_b)$, i.e.,

$$\mathscr{H} = \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ j_1 \neq \cdots \neq j_n}} \mathscr{H}_{j_1}^\circ \otimes \cdots \otimes \mathscr{H}_{j_n}^\circ,$$

where $\mathscr{H}_j^\circ = \{\xi_j\}^\perp \subseteq \mathscr{H}_j$. Note also, that

$$\mathscr{H}_{j_1}^\circ \otimes \cdots \otimes \mathscr{H}_{j_n}^\circ = [\mathscr{M}_{j_1}^\circ \cdots \mathscr{M}_{j_n}^\circ \xi], \qquad j_1 \neq \cdots \neq j_n,$$

where $[S]$ denotes the closed linear span of a set S. Put

$$\begin{aligned}
&\mathscr{K}_0 = \mathbb{C}\xi, \\
&\mathscr{K}_n = [\underbrace{\mathscr{M}_a^\circ \mathscr{M}_b^\circ \cdots}_{n} \xi], \qquad \mathscr{L}_n = [\underbrace{\mathscr{M}_b^\circ \mathscr{M}_a^\circ \cdots}_{n} \xi], \qquad n \in \mathbb{N}, \\
&\mathscr{K} = \bigoplus_{n=0}^{\infty} \mathscr{K}_n, \qquad \mathscr{L} = \bigoplus_{n=1}^{\infty} \mathscr{L}_n.
\end{aligned}$$

Then clearly $\mathscr{H} = \mathscr{K} \oplus \mathscr{L}$. Since $a \in \mathscr{M}_a^\circ$ and $b \in \mathscr{M}_b^\circ$, we have

$$ab\mathscr{K}_n \subseteq \mathscr{K}_{n+2}, \qquad n = 0, 1, 2, \ldots$$

and hence $ab(\mathscr{K}) \subseteq \mathscr{K}$. Therefore the 2×2-matrix representation of ab corresponding to the decomposition $\mathscr{H} = \mathscr{K} \oplus \mathscr{L}$ is

$$ab = \begin{pmatrix} R & S \\ 0 & T \end{pmatrix}$$

where $R = ab|_{\mathscr{K}}$, $S = P_{\mathscr{K}} ab|_{\mathscr{L}}$, $T = P_{\mathscr{L}} ab|_{\mathscr{L}}$, and $P_{\mathscr{K}}$, $P_{\mathscr{L}}$ denotes orthogonal projections onto $\mathscr{K}$ respectively $\mathscr{L}$. We have $R(\mathscr{K}_n) \subseteq \mathscr{K}_{n+2}$, and the restriction of R to $\mathscr{K}_n$ is given by

$$R(a_1 b_1 a_2 b_2 \cdots) = aba_1 b_1 a_2 b_2 \cdots$$

which corresponds to tensoring from the left by $a\xi_a \otimes b\xi_b \in \mathscr{H}_a^\circ \otimes \mathscr{H}_b^\circ$ on $\mathscr{H}_a^\circ \otimes \mathscr{H}_b^\circ \otimes \cdots$. Since $\|a\|_2 = \|b\|_2 = 1$, R maps $\mathscr{K}_n$ isometrically into $\mathscr{K}_{n+2}$, and hence R is an isometry of $\mathscr{K}$ into $\mathscr{K}$. In particular $\|R^p\| = 1$ for all p in $\mathbb{N}$. Since

$$(ab)^* = \begin{pmatrix} R^* & 0 \\ S^* & T^* \end{pmatrix},$$

$\mathscr{L}$ is invariant under $(ab)^* = b^* a^*$ and $T^* = b^* a^*|_{\mathscr{L}}$. Using $\|a^*\|_2 = \|a\|_2 = 1$, $\|b^*\|_2 = \|b\|_2 = 1$ we get, as above, that T^* maps $\mathscr{L}_n$ isometrically into $\mathscr{L}_{n+2}$ for any $n \geqslant 1$, and hence T^* is an isometry. In particular $\|T^p\| = \|(T^*)^p\| = 1$ for all p in $\mathbb{N}$. Since

$$(ab)^p = \begin{pmatrix} R^p & 0 \\ 0 & T^p \end{pmatrix} + \sum_{r=0}^{p-1} \begin{pmatrix} 0 & R^{p-r-1} S T^r \\ 0 & 0 \end{pmatrix},$$

we have

$$1 \leqslant \|(ab)^p\| \leqslant 1 + p\|S\| \leqslant 1 + p\|ab\|$$

hence $r(ab) = \lim_{p\to\infty} \|(ab)^p\|^{1/p} = 1$ as desired. □

Corollary 4.2. *If T is an R-diagonal element, then*

$$\|T^p\| \leqslant (1+p)\|T\|\,\|T\|_2^{p-1}$$

for every $p = 1, 2, \ldots$.

Proof: By Corollary 3.2 we can assume that $T = ax$, for some free self-adjoint and symmetric distributed random variables a, x, where $a^2 = 1$. In particular a and x are centered, $\|a\|_2 = 1$ and $\|x\|_2 = \|T\|_2$, so by the proof of Proposition 4.1, we get

$$\left\| \left(\frac{T}{\|T\|_2} \right)^p \right\| \leqslant 1 + p \left\| a \cdot \frac{x}{\|x\|_2} \right\| = 1 + p \cdot \frac{\|T\|}{\|T\|_2}$$

hence $\|T\|^p \leqslant \|T\|_2^p + p\|T\|\,\|T\|_2^{p-1} \leqslant (1+p)\|T\|\,\|T\|_2^{p-1}$. □

Lemma 4.3. *Let $(\mathscr{M}, \tau)$ be a non-commutative von Neumann probability space with a faithful trace τ. Suppose that $a \in \mathscr{M}$ has vanishing moments, i.e., $\tau(a^n) = 0$ for $n \in \mathbb{N}$.*

Then $\Delta(1-a) = 1$ if $r(a) \leqslant 1$.

Proof: We first assume that $r(a) < 1$. Then $b = \log(1-a)$ is well defined, and expanding log in a power series we obtain from (2.1)

$$\log \Delta(1-a) = \operatorname{Re} \tau(b) = -\operatorname{Re} \tau\Big(\sum_{n=1}^\infty \frac{a^n}{n} \Big) = -\operatorname{Re} \sum_{n=1}^\infty \frac{\tau(a^n)}{n} = 0.$$

Next suppose that $r(a) = 1$. Then $1 - ta \to 1 - a$ in norm as $t \to 1^-$ and upper semicontinuity of Δ gives

$$\Delta(1-a) \geqslant \limsup_{t \to 1^-} \Delta(1 - ta) = 1.$$

The reverse inequality follows from the maximum principle for subharmonic functions and the fact that the mapping $\lambda \mapsto \Delta(a - \lambda 1)$ is subharmonic on $\mathbb{C}$: For $s > 1$ we have

$$\Delta(1-a) \leqslant \max_{|\lambda| \leqslant s} \Delta(\lambda 1 - a) = \max_{|\lambda|=s} s\Delta(1 - \lambda^{-1}a) = s.$$

We conclude that $\Delta(1-a) = 1$. $\square$

Theorem 4.4. *Let $(\mathscr{M}, \tau)$ be a non-commutative von Neumann probability space with a faithful trace τ. Let u and h be $*$-free random variables in $\mathscr{M}$, u a Haar unitary, $h \geqslant 0$ and assume that the distribution μ_h for h is not a Dirac measure.*

Denote by μ_{uh} the Brown measure for uh. Then

(i) *μ_{uh} is rotation invariant and*

$$\operatorname{supp} \mu_{uh} = [\|h^{-1}\|_2^{-1}, \|h\|_2] \times_{\mathrm{p}} [0, 2\pi[.$$

(ii) *The S-transform $\mathscr{S}_{\mu_{h^2}}$ of the distribution of h^2 has an analytic continuation to (a neighbourhood of) $]\mu_h(\{0\}) - 1, 0]$, $\mathscr{S}_{\mu_{h^2}}(]\mu_h(\{0\}) - 1, 0]) = [\|h\|_2^{-2}, \|h^{-1}\|_2^2[$ and $\mathscr{S}'_{\mu_{h^2}} < 0$ on $]\mu_h(\{0\}) - 1, 0[$.*

(iii) *$\mu_{uh}(\{0\}) = \mu_h(\{0\})$ and*

$$\mu_{uh}\big(B(0, \mathscr{S}_{\mu_{h^2}}(t-1)^{-1/2})\big) = t, \qquad \text{for } t \in\,]\mu_h(\{0\}), 1]. \tag{4.1}$$

(iv) *μ_{uh} is the only rotation symmetric probability measure satisfying* (iii).

In our attempt to compute $L(uh - \lambda 1)$ in order to compute the Brown measure μ_{uh} for uh, it is computationally more convenient to convert uh to a product ax of free self-adjoint symmetric distributed elements and compute $L(x + \lambda a)$. The idea in this computation is to use Lemma 3.3 in [6] to factorize $x + \lambda a$ and separate computations involving the distributions of a and x: We then have to compute Fuglede–Kadison determinants of functions of a and x only. It turns out that it is not in general possible to factorize $x + \lambda a$ but we can state a set of λ's for which this is possible. For these specified values of λ we compute $L(x + \lambda a)$ in terms of the distribution of x and this information enables us to state the absolute continuous part of μ_{uh}.

Proof: Let $T = uh$. If ρ is a complex number of modulus 1 then $\rho T = (\rho u)h \cong uh = T$ because $\rho u \cong u$ and u and h are $*$-free. Therefore the spectrum of T, the map $\lambda \mapsto L(T - \lambda 1)$, the support of the Brown measure μ_T of T and the measure μ_T are rotation symmetric. Applying Corollary 3.2 we infer that T has the same $*$-distribution as ax, where a and x are free self-adjoint symmetric distributed random variables, a is a $L(\mathbb{Z}_2)$-symmetry with distribution $\mu_a = (\delta_{-1}+\delta_1)/2$ and the distribution μ of x is determined by $\mu_{\mathrm{sq}} = \mu_{h^2}$. Define

$$k_x(s) = \tau((1-sx)^{-1}) = \int_{\mathbb{R}} \frac{1}{1-sw}\, d\mu(w),$$

$$f(v) = k_x(iv) = \int_{\mathbb{R}} \frac{1}{1-ivw}\, d\mu(w) \overset{(*)}{=} \int_{\mathbb{R}} \frac{1}{1+ivw}\, d\mu(w) = \int_{\mathbb{R}} \frac{1}{1+v^2w^2}\, d\mu(w),$$

for $s \in \mathbb{C}\setminus(\operatorname{sp} x)^{-1}$, $v>0$. At $(*)$ we use the fact that μ is symmetric. Note that $0<f<1$ and $f(v)\to\mu(\{0\})$ as $v\to\infty$. Thus we can define g on $]0,\infty[$ by

$$g(v) = \frac{1-f(v)}{v^2 f(v)} = \int_{\mathbb{R}} \frac{w^2}{1+v^2w^2}\, d\mu(w) \Big/ \int_{\mathbb{R}} \frac{1}{1+v^2w^2}\, d\mu(w). \tag{4.2}$$

We show that g is strictly decreasing and $g(]0,\infty[) =]\|h^{-1}\|_2^{-2}, \|h\|_2^2[$. Observe first that $g>0$ hence we can do logarithmic differentiation of g:

$$\frac{d}{dv}\log g(v) = \frac{g'(v)}{g(v)} = -\frac{2(1-f(v))f(v)+vf'(v)}{v(1-f(v))f(v)}. \tag{4.3}$$

It follows that we can show monotonicity of g by showing the numerator in (4.3) is positive: for $v>0$ we find

$$\begin{aligned}
&2(1-f(v))f(v) - vf'(v)\\
&\qquad = 2\int_{\mathbb{R}} \frac{v^2t^2}{1+v^2t^2}\, d\mu(t) \int_{\mathbb{R}} \frac{1}{1+v^2s^2}\, d\mu(s) - 2v^2 \int_{\mathbb{R}} \frac{t^2}{(1+v^2t^2)^2}\, d\mu(t)\\
&\qquad = v^2 \int_{\mathbb{R}^2} \frac{t^2+s^2}{(1+v^2t^2)(1+v^2s^2)}\, d\mu\times\mu(s,t)\\
&\qquad\quad - v^2 \int_{\mathbb{R}^2} \Big(\frac{t^2}{(1+v^2t^2)^2} + \frac{s^2}{(1+v^2s^2)^2}\Big)\, d\mu\times\mu(s,t)\\
&\qquad = v^4 \int_{\mathbb{R}^2} \frac{(t^2-s^2)^2}{(1+v^2t^2)^2(1+v^2s^2)^2}\, d\mu\times\mu(s,t).
\end{aligned} \tag{4.4}$$

Since μ_h is not a Dirac measure, $\operatorname{supp}\mu\times\mu \nsubseteq \{(x,y) \mid |x|=|y|\}$ whence the expression in Equation (4.4) is strictly positive. This shows that $g'<0$ on $]0,\infty[$. The image of g can be computed using formula (4.2). We observe that

$$\int_{\mathbb{R}} \frac{v^2}{1+v^2w^2}\, d\mu(w) \to \int_{\mathbb{R}} \frac{1}{w^2}\, d\mu(w) = \|h^{-1}\|_2^2, \qquad \int_{\mathbb{R}} \frac{v^2w^2}{1+v^2w^2}\, d\mu(w) \to 1$$

as $v \to \infty$ and

$$\int_{\mathbb{R}} \frac{w^2}{1+v^2w^2}\,d\mu(w) \to \int_{\mathbb{R}} w^2\,d\mu(w) = \|h\|_2^2, \qquad \int_{\mathbb{R}} \frac{1}{1+v^2w^2}\,d\mu(w) \to 1$$

as $v \to 0$, and collecting these results we obtain $g(]0,\infty[) =]\|h^{-1}\|_2^{-2}, \|h\|_2^2[$. It follows from Morera's Theorem that f (hence g and $1/g$) is analytic in a neighbourhood of $]0,\infty[$.

We define $\lambda = \lambda(v)$ by $\lambda^2 = g(v)$ and $k_{\lambda a}$ by

$$k_{\lambda a}(t) = \tau((1-t\lambda a)^{-1}) = \frac{1}{1-t^2\lambda^2}$$

for $t \neq \pm\lambda^{-1}$. Using $s = iv$, $t = i/(\lambda^2 v)$ we get $k_{\lambda a}(t) = 1 - f(v)$, and we are able to compute the product

$$\begin{aligned}
(1-sx)&\Big(1 - \frac{((1-sx)^{-1})^\circ((1-t\lambda a)^{-1})^\circ}{k_x(s)k_{\lambda a}(t)}\Big)(1-t\lambda a) \\
&= (1-sx)\Big(1 - \frac{(1-sx)^{-1} - k_x(s)}{k_x(s)} \cdot \frac{(1-t\lambda a)^{-1} - k_{\lambda a}(t)}{k_{\lambda a}(t)}\Big)(1-t\lambda a) \\
&= \frac{-iv}{1-f(v)}(x+\lambda a).
\end{aligned}$$

Then we are able to compute

$$\begin{aligned}
L(T-\lambda 1) &= L(x+\lambda a) \\
&= L(1-sx) + L\Big(1 - \frac{((1-sx)^{-1})^\circ}{k_x(s)} \cdot \frac{((1-t\lambda a)^{-1})^\circ}{k_{\lambda a}(t)}\Big) \\
&\quad + L(1-t\lambda a) - \log v + \log(1-f(v)).
\end{aligned}$$

Observe that

$$\begin{aligned}
\|((1-sx)^{-1})^\circ\|_2^2 &= \|(1-sx)^{-1}\|_2^2 - |\tau((1-sx)^{-1})|^2 \\
&= \int_{\mathbb{R}} \frac{1}{1+|s|^2w^2}\,d\mu(w) - f(v)^2 = \int_{\mathbb{R}} \frac{1}{1+v^2w^2}\,d\mu(w) - f(v)^2 \\
&= f(v)(1-f(v))
\end{aligned}$$

and that $\|((1-t\lambda a)^{-1})^\circ\|_2^2 = f(v)(1-f(v))$. Freeness of a and x implies that Proposition 4.1 applies to the product $((1-sx)^{-1})^\circ \cdot ((1-t\lambda a)^{-1})^\circ$ and we infer that

$$r\big(((1-sx)^{-1})^\circ \cdot ((1-t\lambda a)^{-1})^\circ\big) = f(v)(1-f(v)) = k_x(s)k_{\lambda a}(t)$$

Using the freeness assumption on a and x we see that $((1-sx)^{-1})^\circ \cdot ((1-t\lambda a)^{-1})^\circ$ has vanishing moments, and it follows then from Lemma 4.3 that

$$L\Big(1 - \frac{((1-sx)^{-1})^\circ}{k_x(s)} \cdot \frac{((1-t\lambda a)^{-1})^\circ}{k_{\lambda a}(t)}\Big) = 0.$$

Invertibility of $1 - sx$ and $1 - t\lambda a$ implies that we can compute $L(1 - sx)$ and $L(1 - t\lambda a)$:

$$L(1 - t\lambda a) = \tau(\log|1 - t\lambda a|) = \frac{1}{2}\log\frac{1 + \lambda^2 v^2}{\lambda^2 v^2},$$
$$L(1 - sx) = \int_{\mathbb{R}} \log|1 - sw|\, d\mu(w) = \frac{1}{2}\int_{\mathbb{R}} \log(1 + v^2 w^2)\, d\mu(w),$$

and we have

$$L(T - \lambda 1) = \frac{1}{2}\int_{\mathbb{R}} \log(1 + v^2 w^2)\, d\mu(w) + \frac{1}{2}\log\frac{\lambda^2}{1 + v^2\lambda^2} \tag{4.5}$$

for $\lambda = \lambda(v) \in \,]\|h^{-1}\|_2^{-1}, \|h\|_2[$. The mapping $v \mapsto \lambda(v)$ is analytic in a neighbourhood of $]0, \infty[$ hence of class C^2 on $]0, \infty[$. It then follows from Equation (4.5) that the mapping $v \mapsto L(T - \lambda(v)1)$ is real valued and of class C^2 on $]0, \infty[$. In addition $\lambda'(v) > 0$ for $v > 0$ hence v is a C^2-function of λ. It follows that $\lambda \mapsto L(T - \lambda 1)$ is a C^2-function on $\mathbb{C} \setminus \{0\}$.

We are now in a position to compute the Brown measure on circular annuli: Let $0 < v_1 < v_2 < \infty$ and put $\alpha = g(v_2)^{1/2}$, $\beta = g(v_1)^{1/2}$. Then

$$\begin{aligned}
&\mu_T([\alpha, \beta] \times_{\mathrm{p}} [0, 2\pi[) \\
&\qquad = \frac{1}{2\pi} \iint_{[\alpha,\beta] \times_{\mathrm{p}} [0,2\pi[} \nabla^2 H(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2 \\
&\qquad = \frac{\beta}{2\pi}\int_0^{2\pi} \operatorname{grad} H(\beta\cos\theta, \beta\sin\theta) \cdot \tbinom{\cos\theta}{\sin\theta}\, d\theta \\
&\qquad\quad - \frac{\alpha}{2\pi}\int_0^{2\pi} \operatorname{grad} H(\alpha\cos\theta, \alpha\sin\theta) \cdot \tbinom{\cos\theta}{\sin\theta}\, d\theta \\
&\qquad \overset{(\dagger)}{=} \frac{\beta}{2\pi}\int_0^{2\pi} K'(\beta)\tbinom{\cos\theta}{\sin\theta} \cdot \tbinom{\cos\theta}{\sin\theta}\, d\theta - \frac{\alpha}{2\pi}\int_0^{2\pi} K'(\alpha)\tbinom{\cos\theta}{\sin\theta} \cdot \tbinom{\cos\theta}{\sin\theta}\, d\theta \\
&\qquad = \beta K'(\beta) - \alpha K'(\alpha),
\end{aligned}$$

where $H(\lambda_1, \lambda_2) = L(x + (\lambda_1 + i\lambda_2)a)$, $K(\lambda) = H(\lambda, 0)$. At ($\dagger$) we use the fact that H is rotation symmetric. We are able to compute $K'(\lambda)$:

$$\begin{aligned}
K'(\lambda(v))\lambda'(v) &= \frac{d}{dv}L(x + \lambda(v)a) = \frac{d}{dv}\Big(\int_{\mathbb{R}} \log(1 + v^2 w^2)\, d\mu(w) + \frac{1}{2}\log\frac{\lambda^2}{1 + v^2\lambda^2}\Big) \\
&= \frac{1 - f(v)}{v} + \frac{\lambda'(v)}{\lambda(v)} - \frac{v\lambda(v)^2 + v^2\lambda(v)\lambda'(v)}{1 + v^2\lambda(v)^2} = \lambda'(v)f(v)/\lambda(v),
\end{aligned}$$

hence $f(v) = \lambda(v)K'(\lambda(v))$. This means that $\mu_T([\alpha,\beta] \times_{\mathrm{p}} [0,2\pi[) = f(v_1) - f(v_2)$. Letting v_1 tend to 0 we obtain

$$\mu_T\big([\alpha, \|h\|_2[\times_{\mathrm{p}}[0,2\pi[\big) = 1 - f(v_2)$$

for all $v_2 > 0$. Letting v_2 tend to ∞ we obtain

$$\mu_T\big(]\|h^{-1}\|_2^{-1}, \|h\|_2[\times_{\mathrm{p}}[0,2\pi[\big) = 1 - \mu_h(\{0\}). \tag{4.6}$$

If $\mu(\{0\}) = 0$ we see that $\operatorname{supp}\mu_T = [\|h^{-1}\|_2^{-1}, \|h\|_2] \times_{\mathrm{p}} [0,2\pi[$. Otherwise let p be the orthogonal projection onto $\operatorname{Ker} T$. Then $\tau(p) = \mu_h(\{0\})$ and we can think of T as $T = \left(\begin{smallmatrix} 0 & R \\ 0 & S \end{smallmatrix}\right)$ where the decomposition is with respect to p and $p^\perp := 1 - p$, i.e., $T = pTp^\perp + p^\perp T p^\perp$. Let $\lambda \neq 0$. Then $T - \lambda 1 = \left(\begin{smallmatrix} -\lambda 1 & R \\ 0 & S-\lambda 1 \end{smallmatrix}\right)$ and by Proposition 1.8 in [3] we have

$$\Delta(T) = \Delta\begin{pmatrix} -\lambda 1 & R \\ 0 & S - \lambda 1 \end{pmatrix} = \Delta_1(-\lambda 1)^{\tau(p)}\,\Delta_2(S - \lambda 1)^{\tau(p^\perp)}$$

where Δ_1 and Δ_2 are the Fuglede–Kadison determinants on $p\mathscr{M}p$ and $p^\perp\mathscr{M}p^\perp$ computed with respect to the normalized traces on these two algebras. Put $L_i = \log\Delta_i$, $i = 1, 2$. Then

$$L(T) = \tau(p)L_1(-\lambda 1) + \tau(p^\perp)L_2(S - \lambda 1) = \tau(p)\log|\lambda| + \tau(p^\perp)L_2(S - \lambda 1)$$

hence the Brown measure for T is given by

$$\mu_T = \tau(p)\delta_0 + \tau(1-p)\mu_S$$

where $\delta_0 = (2\pi)^{-1}\nabla^2 \log|\lambda|$ is the Dirac measure at 0 and μ_S is the Brown measure of S relative to $p^\perp\mathscr{M}p^\perp$. Hence $\mu_T(\{0\}) \geqslant \tau(p)$. Combined with Equation (4.6) this gives $\mu_T(B(0, \|h\|_2)) \geqslant 1$, but μ_T is a probability measure and we conclude that $\mu_T(\{0\}) = \mu_h(\{0\})$. Furthermore $\|h^{-1}\|_2^{-1} = 0$ and $\operatorname{supp}\mu_T = [\|h^{-1}\|_2^{-1}, \|h\|_2] \times_{\mathrm{p}} [0,2\pi[$.

For $v > 0$ we have

$$\mu_T\big(B(0,\lambda(v))\big) = 1 - \mu_T\big([\lambda(v), \|h\|_2[\times_{\mathrm{p}}[0,2\pi[\big) = 1 - (1 - f(v)) = f(v),$$

and if $v > 0$ is small then

$$f(v) - 1 = \int_{\mathbb{R}} \frac{-v^2w^2}{1 + v^2w^2}\,d\mu(w) = \int_{\mathbb{R}} \frac{-v^2 w}{1 - (-v^2)w}\,d\mu_{\mathrm{sq}}(w) = \psi_{\mu_{h^2}}(-v^2)$$

which means that $-v^2 = \chi_{\mu_{h^2}}(f(v) - 1)$, hence

$$\mathscr{S}_{\mu_{h^2}}(f(v) - 1) = \frac{f(v)}{f(v) - 1}\chi_{\mu_{h^2}}(f(v) - 1) = \frac{v^2 f(v)}{1 - f(v)} = \frac{1}{\lambda(v)^2} \tag{4.7}$$

for v in a neighbourhood of 0. Since $f' < 0$ on $]0, \infty[$, $f - 1$ is univalent in every $v > 0$ hence we can construct an analytic function F on a neighbourhood of $f(]0, \infty[) - 1 =]\mu_h(\{0\}) - 1, 0[$ such that $f(F(z)) - 1 = z$ for all z in $]\mu_h(\{0\}) - 1, 0[$. This implies that $\mathscr{S}_{\mu_{h^2}}$ can be continued analytically to a neighbourhood of $]\mu_h(\{0\}) - 1, 0[$, (4.7) holds for all $v > 0$ and that

$$\mu_T\big(B(0, \mathscr{S}_{\mu_{h^2}}(f(v) - 1)^{-1/2})\big) = \mu_T\big(B(0, \lambda(v))\big) = f(v)$$

for all $v > 0$. This means that $\mu_T(B(0, \mathscr{S}_{\mu_{h^2}}(t - 1)^{-1/2})) = t$ for all t in $]\mu_h(\{0\}), 1[$. It is shown in [6] that $\mathscr{S}_{\mu_{h^2}}$ is analytic in a neighbourhood of 0 and $\mathscr{S}_{\mu_{h^2}}(0) = \tau(h^2)^{-1} = \|h\|_2^{-2}$. Thus $\mathscr{S}_{\mu_{h^2}}$ has an analytic continuation to a neighbourhood of $]\mu(\{0\}) - 1, 0]$ and a continuity argument shows that $\mu_T(B(0, \|h\|_2)) = 1$. It follows from Equation (4.7) that $\mathscr{S}'_{\mu_{h^2}} < 0$ on $]\mu_h(\{0\}) - 1, 0[$.

We remain to prove uniqueness of μ_T. Suppose that ν is a rotation symmetric probability measure satisfying $\nu(\{0\}) = \mu_h(\{0\})$ and $\nu(B(0, \mathscr{S}_{\mu_{h^2}}(t-1)^{-1/2})) = t$ for all t in $]\mu_h(\{0\}), 1]$. Then $\nu(B(0, r)) = \mu_T(B(0, r))$ for all $r > \|h^{-1}\|_2^{-1}$ because ν is a probability measure. If $\|h^{-1}\|_2^{-1} > 0$ then $\mu_T(\{0\}) = 0$ and

$$\begin{aligned} \nu\big(B(0, \|h^{-1}\|_2^{-1})\big) &= \lim_{\varepsilon \to 0} \nu\big(B(0, \|h^{-1}\|_2^{-1} + \varepsilon)\big) = \lim_{\varepsilon \to 0} \mu_T\big(B(0, \|h^{-1}\|_2^{-1} + \varepsilon)\big) \\ &= \mu_T\big(B(0, \|h^{-1}\|_2^{-1})\big) = 0. \end{aligned}$$

We conclude that $\nu(B(0, r)) = \mu_T(B(0, r))$ for all $r > 0$ which implies that ν and μ_T agree on all circular annuli of the form $[\alpha, \beta[\times_{\mathrm{p}} [0, 2\pi[$ $(0 < \alpha < \beta)$. Since ν and μ_T are rotation symmetric they must be equal on sets of the form $[\alpha, \beta[\times_{\mathrm{p}} [0, 2\pi/n[$ $(0 < \alpha < \beta,\ n \in \mathbb{N})$ hence they are equal on all sets of the form $[\alpha, \beta[\times_{\mathrm{p}} [2\pi\gamma, 2\pi\delta[$ $(0 < \alpha < \beta,\ 0 \leqslant \gamma < \delta \leqslant 1,$ $\gamma, \delta \in \mathbb{Q})$. Since these sets generate the Borel σ-algebra on $\mathbb{C}$ we conclude that $\nu = \mu_T$. $\square$

Corollary 4.5. *With the notation as in Theorem 4.4 we have*

(i) *the function* $F(t) = \mathscr{S}_{\mu_{h^2}}(t - 1)^{-1/2} \colon]\mu_h(\{0\}), 1] \to]\|h^{-1}\|_2^{-1}, \|h\|_2]$ *has an analytic continuation to a neighbourhood of its domain and* $F' > 0$ *on* $]\mu_h(\{0\}), 1[$,

(ii) μ_{uh} *has a radial density function* f *on* $]0, \infty[$ *defined by*

$$f(s) = \begin{cases} \dfrac{1}{2\pi s F'(F^{-1}(s))}, & s \in]\|h^{-1}\|_2^{-1}, \|h\|_2], \\ 0, & \textit{otherwise}. \end{cases} \tag{4.8}$$

The radial density function has an analytic continuation to a neighbourhood of $]\|h^{-1}\|_2^{-1}, \|h\|_2]$.

Proof: (i) follows immediately from Theorem 4.4.

Let $\alpha = \mu(\{0\})$, $\beta = \|h^{-1}\|_2^{-1}$, and define $\nu = \alpha\delta_0 + f(|\lambda|)\, dm_2(\lambda)$ on the Borel-measurable sets. Let $t \,\epsilon\,]\alpha, 1[$. Then

$$\begin{aligned}
\nu\big(B(0, F(t))\big) &= \alpha + \iint\limits_{]0,F(t)[\,\times_{\mathrm{p}}\,[0,2\pi[} f(|\lambda|)\, dm_2(\lambda) = \alpha + 2\pi \int\limits_0^{F(t)} f(r) r\, dr \\
&= \alpha + \lim_{n\to\infty} \int\limits_{\beta+\frac{1}{n}}^{F(t)} (F^{-1})'(r)\, dr = \alpha + \lim_{n\to\infty} \int\limits_{F^{-1}(\beta+\frac{1}{n})}^{t} 1\, ds \\
&= t - \lim_{n\to\infty} F^{-1}(\beta + \tfrac{1}{n}) + \alpha = t.
\end{aligned}$$

The uniqueness of μ_{uh} shows that $\nu = \mu_{uh}$.

Note that $F' \circ F^{-1}$ is analytic in a neighbourhood of $]\|h^{-1}\|_2^{-1}, \|h\|_2]$ and that $F'(F^{-1}(s)) > 0$ for all s in $]\|h^{-1}\|_2^{-1}, \|h\|]$. This implies that f can be continued analytically to a neighbourhood of $]\|h^{-1}\|_2^{-1}, \|h\|_2]$. $\square$

The Corollary shows that the radial density function for the Brown measure is determined by the distribution of h and the formula (4.8).

Now suppose $u, h, k \,\epsilon\, \mathscr{M}$, u is a Haar unitary, $h, k \geqslant 0$, u, h are $*$-free, u, k are $*$-free and that $\mu_{uh} = \mu_{uk}$. Then it follows from Theorem 4.4 that $\|h\|_2 = \|k\|_2$, $\|h^{-1}\|_2 = \|k^{-1}\|_2$ and from Corollary 4.5 that $(F_{h^2}^{-1})'(r) = (F_{k^2}^{-1})'(r)$ for $r \,\epsilon\, I =]\|h^{-1}\|_2^{-1}, \|h\|_2[=]\|k^{-1}\|_2^{-1}, \|k\|_2[$ from which we infer that $F_{h^2}^{-1}(r) - F_{k^2}^{-1}(r)$ is constant on I. But $F_{h^2}^{-1}(r) \to 1$ as $r \to \|h\|_2$ and $F_{k^2}^{-1}(r) \to 1$ as $r \to \|k\|_2 = \|h\|_2$ so that $F_{h^2}^{-1} = F_{k^2}^{-1}$ on I. This implies that $\mathscr{S}_{\mu_{h^2}} = \mathscr{S}_{\mu_{k^2}}$ on $]\mu_{uh}(\{0\}) - 1, 0[$ and we conclude that $\mathscr{S}_{\mu_{h^2}} = \mathscr{S}_{\mu_{k^2}}$ in a neighbourhood of 0 using the Principle of Analytic Continuation. Therefore the distribution of h is uniquely determined by the Brown measure for uh.

Proposition 4.6. *With the notation as in Theorem* 4.4 *we have*

(i) *the Brown measure for uh is the uniform probability measure on $\alpha\mathbb{T}$ (for some $\alpha \geqslant 0$) if and only if $\mu_h = \delta_\alpha$, i.e., h is a scalar. Thus* (i) *in Theorem* 4.4 *holds if μ_h is a Dirac measure,*

(ii) *if h is invertible then* $\operatorname{sp} uh = \operatorname{supp} \mu_{uh}$,

(iii) *if h is not invertible then* $\operatorname{sp} uh = \overline{B(0, \|h\|_2)}$,

(iv) $\operatorname{supp} \mu_{uh} \subsetneqq \operatorname{sp} uh$ *if and only if h is not invertible in $\mathscr{M}$ and* $\|h^{-1}\|_2 < \infty$.

Proof: If h is a scalar $\alpha \geqslant 0$, uh has spectrum $\alpha\mathbb{T}$ and $\mu_{uh} = (\mu_u)_{z \mapsto \alpha z}$ is the uniform probability measure on $\alpha\mathbb{T}$. (In the case $\alpha = 0$ this measure is δ_0.) If, conversely, $\operatorname{sp} h$ has more than one point, Theorem 4.4 applies and we infer that $[\|h^{-1}\|_2^{-1}, \|h\|_2] \subseteq \operatorname{supp} \mu_{uh}$.

Either $\mu_h(\{0\}) \neq 0$ or $\mu_h(\{0\}) = 0$. If $\mu_h(\{0\}) \neq 0$ we have $\|h^{-1}\|_2 = 0 < \|h\|_2$. Otherwise we have

$$\|h^{-1}\|_2 \cdot \|h\|_2 = \Big(\int_{\mathbb{R}} t^{-2}\, d\mu_h(t)\Big)^{1/2} \Big(\int_{\mathbb{R}} t^2\, d\mu_h(t)\Big)^{1/2} \geqslant \int_{\mathbb{R}} 1_{\mathbb{R}\setminus\{0\}}\, d\mu_h(t) = 1.$$

We have equality in Hölders inequality only if the integrands are proportional a.e. w.r.t. μ_h. (This means that $\operatorname{supp}\mu_h = \operatorname{sp} h$ consists of a single point.) Therefore $\|h^{-1}\|_2^{-1} < \|h\|_2$ and we conclude that μ_{uh} is not the uniform measure on $\alpha\mathbb{T}$ for any $\alpha \geqslant 0$. This proves (i).

It follows from Corollary 3.2 that we can replace $T = uh$ by ax where a and x are free self-adjoint symmetric distributed random variables, $a^2 = 1$, $x^2 = h^2$, and using Proposition 4.1 we get $r(uh) = r(ax) = \|a\|_2\, \|x\|_2 = \|h\|_2$ whence $\operatorname{sp} uh \subseteq \overline{B(0, \|h\|_2)}$. If h is invertible then it follows from Proposition 3.10 that $(uh)^{-1}$ is R-diagonal hence $\operatorname{sp}(uh)^{-1} \subseteq \overline{B(0, \|h^{-1}\|_2)}$. Summing up we have

$$\operatorname{sp} uh \subseteq [\|h^{-1}\|_2^{-1}, \|h\|_2] \times_{\mathrm{p}} [0, 2\pi[= \operatorname{supp}\mu_{uh} \subseteq \operatorname{sp} uh,$$

which shows (ii).

Now suppose that h is not invertible. To prove (iii) we assume without loss of generality that $\|h^{-1}\|_2 = \int_{\mathbb{R}} t^{-2}\, d\mu_x(t) < \infty$. Then $\mu_x(\{0\}) = \mu_h(\{0\}) = 0$, whence $\operatorname{Ker} x = \{0\}$, so x has an (unbounded) inverse. Let E be the spectral resolution of x. Then x^{-1} (as an unbounded operator) is given by

$$x^{-1} = \int_{\mathbb{R}} \frac{1}{\lambda}\, dE(\lambda).$$

We introduce approximants: for m a natural number let

$$x_m = \int_{\mathbb{R}} \frac{1}{\lambda} \cdot 1_{]1/m,\infty[}(|\lambda|)\, dE(\lambda).$$

We note that the approximants are centered elements in $(\mathscr{M}, \tau)$, that $\{x_m \mid m \in \mathbb{N}\}$ and $\{a\}$ are free, and that $(x_m)_m$ form a Cauchy sequence with respect to $\|\cdot\|_2$ on $\mathscr{M}$: for $m \leqslant m'$ we have

$$\|x_m - x_{m'}\|_2^2 = \int_{\mathbb{R}} \frac{1}{\lambda^2} \cdot 1_{]1/m',1/m]}(|\lambda|)\, d\mu_x(\lambda) \leqslant \int_{\mathbb{R}} \frac{1}{\lambda^2} \cdot 1_{[0,1/m]}(|\lambda|)\, d\mu_x(\lambda) \to 0$$

as $m \to \infty$ due to the assumption on $\|h^{-1}\|_2$. It follows that for any natural number n, the sequence $((x_m a)^n)_{m=1}^{\infty}$ is Cauchy with respect to $\|\cdot\|_2$: Fix n in $\mathbb{N}$. For m and m' natural numbers we get

$$\begin{aligned}
\|(x_m a)^n - (x_{m'} a)^n\|_2 &= \Big\| \sum_{j=0}^{n-1} (x_{m'} a)^j (x_m a)^{n-j} - (x_{m'} a)^{j+1} (x_m a)^{n-1-j} \Big\|_2 \\
&\leqslant \sum_{j=0}^{n-1} \|(x_{m'} a)^j (x_m - x_{m'})(x_m a)^{n-1-j}\|_2 \\
&= \|x_m - x_{m'}\|_2 \sum_{j=1}^{n-1} \|x_{m'}\|_2^j \, \|x_m\|_2^{n-1-j} \\
&\leqslant \|x_m - x_{m'}\|_2 \cdot n \cdot \|h^{-1}\|_2^{n-1}.
\end{aligned}$$

We denote by T^{-n} the limit of the sequence $((x_m a)^n)_m$ in $(L^2(\mathscr{M}), \|\cdot\|_2)$. We note that $xx_m \to \hat{1}$ in $L^2(\mathscr{M})$ as $m \to \infty$ hence $T(T^{-1}) = \hat{1}$ and $T(T^{-n-1}) = T^{-n}$ for any $n \geqslant 1$. We are able to compute norms of the vectors T^{-n} $(n \,\epsilon\, \mathbb{N})$ too:

$$\begin{aligned}
\|T^{-n}\|_2 = \lim_{m\to\infty} \|(x_m a)^n\|_2 &\overset{(\ddagger)}{=} \lim_{m\to\infty} \|x_m\|_2^n = \lim_{m\to\infty} \Big(\int_{\mathbb{R}} \frac{1}{\lambda^2} \cdot 1_{]1/m,\infty[}(|\lambda|)\, d\mu_x(\lambda) \Big)^n \\
&= \Big(\int_{\mathbb{R}} \frac{1}{\lambda^2} \cdot 1_{]0,\infty[}(\lambda)\, d\mu(\lambda) \Big)^n = \|h^{-1}\|_2^n.
\end{aligned}$$

Freeness of x_m and a implies equality at $(\ddagger)$. We define $f\colon B(0, \|h^{-1}\|_2^{-1}) \to L^2(\mathscr{M})$ by $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n T^{-n-1}$. The series is absolutely convergent hence f is analytic on its domain. Now suppose $T - \lambda_0 1$ is invertible for some λ_0 in $]0, \|h^{-1}\|_2^{-1}[$. Then $\lambda_0 \notin \operatorname{sp} uh$ hence $\lambda_0 \mathbb{T} \cap \operatorname{sp} uh = \emptyset$ because $\operatorname{sp} uh$ is rotation symmetric. Thus $T - \lambda 1$ is invertible for all λ in $\lambda_0 \mathbb{T}$. For $\lambda \,\epsilon\, \lambda_0 \mathbb{T}$ we find

$$(T - \lambda 1) f(\lambda) = \sum_{n=0}^{\infty} (T - \lambda 1)(\lambda^n T^{-n-1}) = \sum_{n=0}^{\infty} (\lambda^n T^{-n} - \lambda^{n+1} T^{-(n+1)}) = \hat{1}$$

and we conclude that $f(\lambda) = ((T - \lambda 1)^{-1})\hat{}$ (for $|\lambda| = \lambda_0$). We next note that the mapping $\theta \mapsto (T - \lambda_0 e^{i\theta} 1)^{-1} \colon [0, 2\pi] \to (\mathscr{M}, \|\cdot\|)$ is continuous, $S = (2\pi)^{-1} \int_0^{2\pi} (T - \lambda_0 e^{i\theta} 1)^{-1}\, d\theta$ belongs to $\mathscr{M}$ and $\hat{S} = (2\pi)^{-1} \int_0^{2\pi} f(\lambda_0 e^{i\theta})\, d\theta = T^{-1}$. We thus obtain $TS\hat{1} = T(\hat{S}) = \hat{1}$ and conclude that $TS = 1$. Finiteness of $\mathscr{M}$ ensures us that $ST = 1$ and we see that T is invertible. This contradicts the assumption that $\lambda_0 \notin \operatorname{sp} T$.

Summing up we have proved that $]0, \|h^{-1}\|_2^{-1}[\;\subseteq \operatorname{sp} uh$. Theorem 4.4 shows that $\operatorname{sp} uh$ contains $[\|h^{-1}\|_2^{-1}, \|h\|_2]$ and we conclude that $\overline{B(0, \|h\|_2)} = \operatorname{sp} uh$. This proves (iii). The last assertion is a reformulation of (ii) and (iii). $\square$

Combining the facts that $\mu_{f(uh)} = (\mu_{uh})_f$ for f a polynomial, reciprocal function and involution with the change-of-variables theorem we have the following

Theorem 4.7. *With the notation as in Theorem* 4.4 *and Corollary* 4.5 *we have*

(i) *If* $\alpha \neq 0$ *then* $f_{\alpha uh}(s) = |\alpha|^{-2} f_{uh}(s/|\alpha|)$ *on* $]0,\infty[$.

(ii) *If* $p \in \mathbb{N}$ *then* $f_{(uh)^p}(s) = p^{-1} f_{uh}(s^{1/p}) s^{2(1/p-1)}$ *on* $]0,\infty[$.

(iii) *If* h *is invertible then* $f_{(uh)^{-1}}(s) = s^{-4} f_{uh}(s^{-1})$ *on* $]0,\infty[$.

(iv) $f_{(uh)^*} = f_{uh}$.

In the next examples we apply the main theorem to give a slight extension of Proposition 3.1 in [1].

Example 4.8. Let h be positive and invertible in $(\mathscr{M},\tau)$. Then $\mathscr{S}_{\mu_h}(-t)\mathscr{S}_{\mu_{h^{-1}}}(t-1) = 1$ for $0 \leqslant t \leqslant 1$.

If h is a scalar the conclusion holds trivially. Otherwise μ_h is not a Dirac measure and Theorem 4.4 applies. Let $t \in\,]\mu_h(\{0\}),1[\, = \,]0,1[$, $s = \mathscr{S}_{\mu_{h^2}}(t-1)^{-1/2}$, $a = \|h^{-1}\|_2^{-1}$, $b = \|h\|_2$. Then $s \in\,]a,b[$, $\operatorname{supp}\mu_{(uh)^{-1}} = [b^{-1},a^{-1}] \times_{\mathrm{p}} [0,2\pi[$ and

$$\begin{aligned} t &= \mu_{uh}(B(0,s)) = \mu_{uh}(]a,s[\,\times_{\mathrm{p}}[0,2\pi[) = \mu_{(uh)^{-1}}(]s^{-1},a^{-1}[\,\times_{\mathrm{p}}[0,2\pi[) \\ &= 1 - \mu_{(uh)^{-1}}(]b^{-1},s^{-1}[\,\times_{\mathrm{p}}[0,2\pi[) \end{aligned}$$

whence $1-t = \mu_{(uh)^{-1}}(B(0,s^{-1}))$. But Proposition 3.10 implies that $(uh)^{-1}$ is R-diagonal so $1-t = \mu_{(uh)^{-1}}\big(B(0,\mathscr{S}_{\mu_{|(uh)^{-1}|^2}}(-t)^{-1/2})\big)$. Note that $\mu_{|(uh)^{-1}|^2} = \mu_{h^{-2}}$ whence

$$\mathscr{S}_{\mu_{h^{-2}}}(-t)^{-1/2} = s^{-1} = \mathscr{S}_{\mu_{h^2}}(t-1)^{1/2}$$

for $0 < t < 1$. Then Proposition 3.2 in [6] applies and we conclude that $\mathscr{S}_{\mu_{h^2}}$ and $\mathscr{S}_{\mu_{h^{-2}}}$ are analytic in a neighbourhood of $[-1,0]$.

From this formula it follows that $\chi_{\mu_{h^2}}(-t)\chi_{\mu_{h^{-2}}}(t-1) = 1$ for $0 < t < 1$ and that $\psi_{\mu_{h^2}}(x) + \psi_{\mu_{h^{-2}}}(x^{-1}) = -1$ for $x < 0$ hence we have the idea for a simple proof of the formula: Let μ be a compactly supported probability measure on $]0,\infty[$. Then ψ_μ has an analytic continuation to $]-\infty,0]$ given by

$$\psi_\mu(x) = \int_{\mathbb{R}} \frac{xs}{1-xs}\, d\mu(s)$$

and analogously for $\mu^{-1} := \mu_{\mathrm{inv}}$:

$$\psi_{\mu^{-1}}(x) = \int_{\mathbb{R}} \frac{xs}{1-xs}\, d\mu^{-1}(s) = \int_{\mathbb{R}} \frac{x}{s-x}\, d\mu(s)$$

for $x \leqslant 0$. Then $\psi_\mu(x) + \psi_{\mu^{-1}}(x^{-1}) = -1$ and it follows that $\chi_{\mu^{-1}}(z) = (\chi_\mu(-1-z))^{-1}$ for all z in $\psi_\mu(]-\infty,0[)$. Observe that $\psi_\mu(z) \to -1$ as $z \to -\infty$ and $\psi_\mu(0) = 0$ hence

$\chi_{\mu^{-1}}(z)\chi_\mu(-1-z) = 1$ and $\mathscr{S}_{\mu^{-1}}(z)\mathscr{S}_\mu(-1-z) = 1$ for $z \in]-1,0[$. The formula shows that $\mathscr{S}_\mu$ has an analytic continuation to a neighbourhood of $[-1,0]$: In a suitably chosen finite non-commutative W^*-probability space $(\mathscr{M},\tau)$ we can find a positive element h whose distribution is μ and the conclusion follows as in the paragraph above.

Example 4.9. Let μ be a probability measure on $[0,\infty[$ and assume that μ is not a Dirac measure. Then μ is the distribution (as a measure) of a positive element h in some finite non-commutative W^*-probability space $(\mathscr{M},\tau)$ with a faithful trace. Note that $\mu_{h^{1/2}}(\{0\}) = \mu_h(\{0\}) = \mu(\{0\})$. Then Theorem 4.4 shows that $\mathscr{S}_\mu{}' < 0$ on $]\mu(\{0\})-1,0[$. But $\mathscr{S}_\mu{}'(0) = -\tau((h-\tau(h))^2)/\tau(h)^3$ hence $\mathscr{S}_\mu{}'(0) = 0$ if and only if h is a scalar, i.e., if and only if μ is a Dirac measure. We conclude that $\mathscr{S}_\mu{}' < 0$ on $]\mu(\{0\})-1,0]$. If $\operatorname{supp}\mu \subseteq]0,\infty[$ the formula derived in Example 4.8 yields that $\mathscr{S}_\mu{}' < 0$ on $[-1,0]$: We obtain $\mathscr{S}_\mu{}'(-t)\mathscr{S}_{\mu^{-1}}(t-1) = \mathscr{S}_\mu(-t)\mathscr{S}'_{\mu^{-1}}(t-1)$ for $0 \leqslant t \leqslant 1$, and it follows that $\mathscr{S}_\mu{}'(-1) = 0$ if and only if $\mathscr{S}'_{\mu^{-1}}(0) = 0$.

Proposition 4.10. *Let $(\mathscr{M},\tau)$ be a tracial non-commutative von Neumann probability space and let $a = uh \neq 0$ be an R-diagonal element in $\mathscr{M}$, i.e., u and h are $*$-free random variables in $\mathscr{M}$, u is a Haar unitary and $h \geqslant 0$. Let b be an invertible element in $\mathscr{M}$ such that a and b are $*$-free.*

Then bab^{-1} is R-diagonal if and only if b is a scalar times a unitary.

Proof: Let $b = \lambda v$ where $\lambda \in \mathbb{C}\setminus\{0\}$ and v is a unitary $*$-free from a. Then $bab^{-1} = vav^* \sim_{*\mathrm{D}} a$ which shows that bab^{-1} is R-diagonal.

Next suppose that bab^{-1} is R-diagonal. Let $b = vk$ be the polar decomposition of b. Invertibility of b ensures that v is unitary. Then $bab^{-1} = vkak^{-1}v^*$ has the same Brown measure as a. If $\mu_{|bab^{-1}|}$ is a Dirac measure then $|bab^{-1}| = \alpha 1$ (for some $\alpha \geqslant 0$) whence $|a| = \alpha 1$ and $|bab^{-1}|^2 = |a|^2$. (Here we used the assumptions that bab^{-1} and a are R-diagonal.) If $\mu_{|bab^{-1}|}$ is not a Dirac measure then $\mu_{|a|}$ is not a Dirac measure, and we infer from Theorem 4.4 that $\mathscr{S}_{\mu_{|bab^{-1}|^2}} = \mathscr{S}_{\mu_{|a|^2}}$ which implies that

$$\tau(|a|^2) = \tau(|bab^{-1}|^2) = \tau\big((vkak^{-1}v^*)^*vkak^{-1}v^*\big) = \tau(|a|^2)\tau(k^2)\tau(k^{-2})$$

by the freeness assumption on a and b. Let μ denote the distribution of k. Then $\mu(\{0\}) = 0$ and

$$1 = \tau(k^2)^{1/2}\tau(k^{-2})^{1/2} = \Big(\int_{\mathbb{R}} t^2\,d\mu(t)\Big)^{1/2}\Big(\int_{\mathbb{R}} t^{-2}\,d\mu(t)\Big)^{1/2}.$$

Equality holds if and only if the integrands are proportional, i.e., $t = \beta t^{-1}$ a.e. w.r.t. μ. ($\beta > 0$ is some constant.) It follows that μ is a Dirac measure thus k is a scalar. $\square$

The argument also shows that if u, u^*bu and b are $*$-free then b is a scalar times a unitary: Note first that u, u^*bu, b^{-1} are $*$-free such that u, u^*bub^{-1} are $*$-free. Then $bub^{-1} = uu^*bub^{-1}$ is R-diagonal and thus b is a scalar times a unitary.

Finally we note that Theorem 4.4(ii) implies that $\lim_{z\to\mu(\{0\})-1+}\mathscr{S}_\mu(z) = \mu(X^{-1}) \in]0,\infty]$.

5 Examples

Example 5.1. (Circular element.) Let x_1, x_2 be a free semicircular system (with the normalization $\tau(x_1^2) = \tau(x_2^2) = 1$) in a tracial W^*-probability space $(\mathscr{M}, \tau)$. Put

$$y = \frac{1}{\sqrt{2}}(x_1 + ix_2). \tag{5.1}$$

Then y is a circular element in the sense of Voiculescu [13], and $\tau(y^*y) = 1$. By [13], y has polar decomposition $y = uh$, where u and h are $*$-free, u is a Haar unitary and $h \geqslant 0$ is quarter-circular distributed:

$$d\mu_h = \frac{1}{\pi}\sqrt{4 - x^2}\, 1_{[0,2]}(x)\, dx.$$

In particular y is R-diagonal. A simple computation shows that

$$\mathscr{S}_{\mu_{h^2}}(z) = \frac{1}{z+1}$$

(see the computation in Example 5.2 below for $c = 1$). Moreover $\|h\|_2 = 1$ and $\|h^{-1}\|_2 = \infty$. Hence by Theorem 4.4 and Proposition 4.6 $\operatorname{sp} y = \{z \in \mathbb{C} \mid |z| \leqslant 1\} =: \bar{D}$, and

$$\mu_y(B(0, \sqrt{t})) = t, \qquad 0 < t < 1.$$

Since μ_y is rotation symmetric, this implies that the Brown measure for y is the uniform distribution on the disk $\bar{D}$ given by

$$d\mu_y = \frac{1}{\pi} 1_{\bar{D}}(z)\, d\operatorname{Re} z\, d\operatorname{Im} z. \tag{5.2}$$

From the random matrix model for a semicircular system (Theorem 1.13 in [13]) one can obtain the following random matrix model for a circular element (cf. Remark 5.1.4 in [14]): Let for n any natural number $Y^{(n)}$ denote the random matrix

$$Y^{(n)} = (Y_{ij})_{i,j=1}^n \tag{5.3}$$

where for each n, $(\operatorname{Re} Y_{ij}, \operatorname{Im} Y_{ij})_{i,j=1}^n$ are $2n^2$ stochastically independent normal distributed centered random variables with variance $(2n)^{-1}$. Then $Y^{(n)}$ converges in $*$-distribution to the circular element (5.1) when we use the states τ_n, $n = 1, 2, \ldots$ on $\operatorname{alg}(Y^{(n)}, (Y^{(n)})^*, 1)$ given by $\tau_n = \mathbb{E} \circ \operatorname{tr}_n$. Here $\mathbb{E}$ is the expectation value and tr_n is the normalized trace on $M_n(\mathbb{C})$. In [5] Ginibre computed the eigenvalue distribution of the random matrix $Y^{(n)}$ for each natural number n, and proved that for $n \to \infty$ this eigenvalue distribution converges weakly to the measure μ_y given by (5.2). However, due to the discontinuity of the Fuglede–Kadison determinant, it appears to be difficult to deduce Equation (5.2) for the Brown measure of a circular element directly from Ginibres result.

Example 5.2. (Free Poisson Distribution.) We consider a one-parameter family of Free Poisson distributions, cf. Section 3.7 in [14].

Let $c > 0$ and consider the measure

$$\nu_c = \max\{1-c, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{]a,b[}(x)\, dx,$$

where $a = (\sqrt{c}-1)^2$, $b = (\sqrt{c}+1)^2$. Then

$$\begin{aligned} \mathscr{R}_{\nu_c}(z) &= \frac{c}{1-z}, \\ G_{\nu_c}(z) &= \frac{z + (1-c) \pm \sqrt{(c-1)^2 - 2(c+1)z + z^2}}{2z}, \\ \psi_{\nu_c}(z) = \frac{G_{\nu_c}(z^{-1})}{z} - 1 &= \frac{1 - z(c+1) - \sqrt{1 - 2z(c+1) + z^2(c-1)^2}}{2z}. \end{aligned}$$

We are able to compute the S-transform too:

$$\mathscr{S}_{\nu_c}(z) = \frac{1}{z+c}.$$

Now consider a $*$-free pair (u, h) of elements in $(\mathscr{M}, \tau)$ where u is a Haar unitary and $h \geqslant 0$ with distribution given by $\mu_{h^2} = \nu_c$. The Brown measure μ_{uh} for uh is completely described by Theorem 4.4. We find then $\|h\|_2 = \sqrt{c}$ and using Corollary 4.5 we obtain an expression for the radial density f_{uh} of μ_{uh}:

$$f_{uh}(s) = \frac{1}{\pi} \cdot 1_{]\|h^{-1}\|_2^{-1}, \sqrt{c}[}(s).$$

If $c \geqslant 1$ then $\mu_{uh}(\{0\}) = 0$ hence μ_{uh} has no point masses. Since $\mu_{uh}(B(0, \sqrt{c})) = 1$ we conclude that $\|h^{-1}\|_2^{-1} = \sqrt{c-1}$.

If $0 < c < 1$ then $\mu_{uh}(\{0\}) = 1 - c$ and $\|h^{-1}\|_2^{-1} = 0$.

Finally we note that if h is quarter circular distributed with $\tau(h^2) = 1$ then $\mu_{h^2} = \nu_1$. Hence $c = 1$ gives the circular element treated in Example 5.1.

Example 5.3. (Bernoulli Distribution.) Let u and p be $*$-free random variables in $(\mathscr{M}, \tau)$, and assume that u is a Haar unitary, p is a projection with trace $\alpha \in\,]0,1[$. Then μ_p is the Bernoulli distribution with parameter α, i.e., $\mu_p = (1-\alpha)\delta_0 + \alpha\delta_1$ and the S-transform $\mathscr{S}_{\mu_{p^2}}$ of $p^2 = p$ is $\mathscr{S}_{\mu_p}(z) = (z+1)/(z+\alpha)$ for $z \neq -\alpha$, cf. Example 3.6.7 in [14]. Using Corollary 4.5 we obtain an expression for the radial density f_{up} of the Brown measure μ_{up} for up:

$$f_{up}(s) = \frac{1-\alpha}{\pi(1-s^2)^2} \cdot 1_{]0,\sqrt{\alpha}[}(s).$$

We can check that this gives the complete information:

$$\mu_{up}\big(B(0,\sqrt{\alpha})\setminus\{0\}\big) = \iint\limits_{B(0,\sqrt{\alpha})\setminus\{0\}} f_{up}(|\lambda|)\,d\lambda = 2\pi\int_0^{\sqrt{\alpha}} s f_{up}(s)\,ds$$
$$= 2(1-\alpha)\int_0^{\sqrt{\alpha}} \frac{s}{(1-s^2)^2}\,ds = \alpha,$$

which together with $\mu_{up}(\{0\}) = \mu_p(\{0\}) = 1-\alpha$ gives $\mu_{up}(B(0,\sqrt{\alpha})) = 1$.

In the two examples below, we compute the Brown measures for certain linear combinations of Haar unitaries. Let $u_1,\dots,u_n$ $(n \geqslant 2)$ be $*$-free generating Haar unitaries in $L(\mathbb{F}_n)$, let $\alpha_1,\dots,\alpha_n \in \mathbb{C}\setminus\{0\}$ and put

$$T = \alpha_1 u_1 + \cdots + \alpha_n u_n.$$

By the addition property of $*$-free R-diagonal elements we infer that T is R-diagonal. Next we note that the $*$-distribution (hence the spectrum and the Brown measure) for T only depends on $|\alpha_1|,\dots,|\alpha_n|$: Let $\rho_j = \alpha_j/|\alpha_j|$ $(j = 1,\dots,n)$ and observe that $\rho_j u_j$ has the same $*$-distribution as u_j. This implies that $\rho_j u_j \cong u_j$ whence

$$T = |\alpha_1|\rho_1 u_1 + \cdots + |\alpha_n|\rho_n u_n \cong |\alpha_1| u_1 + \cdots + |\alpha_n| u_n.$$

We therefore assume that $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n$.

We are able to compute the Brown measure explicitly in two cases: the case $\alpha_1 = \cdots = \alpha_n = 1$ and the case $n = 2$.

Lemma 5.4. *For any a in $\mathbb{C}\setminus\,]-1,1[$ we have*

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{a^2-x^2}\,dx = \pi(1-\sqrt{1-a^{-2}}), \tag{5.4}$$
$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{a-x}\,dx = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{a+x}\,dx = \pi(a-\sqrt{a^2-1}).$$

Proof: We first note that $\int_{-1}^{1}\frac{\sqrt{1-x^2}}{a-x}\,dx = \int_{-1}^{1}\frac{\sqrt{1-x^2}}{a+x}\,dx$ and that $\frac{1}{a^2-x^2} = \frac{1}{2a}\big(\frac{1}{a-x}+\frac{1}{a+x}\big)$ (for $|x| < a$) so that $\int_{-1}^{1}\frac{\sqrt{1-x^2}}{a^2-x^2}\,dx = \frac{1}{a}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{a-x}\,dx$. Therefore we only need to prove (5.4).

The case $a^2 = 1$ is straightforward so we next assume that $a > 1$. Then for $|x| \leqslant 1$ we have $(1-(x/a)^2)^{-1} = \sum_{n=0}^{\infty}(x/a)^{2n}$ and

$$\begin{aligned}
\int_{-1}^{1} \frac{\sqrt{1-x^2}}{a^2-x^2}\,dx &= \sum_{n=0}^{\infty} \frac{1}{a^{2n+2}} \int_{-1}^{1} x^{2n}\sqrt{1-x^2}\,dx = \sum_{n=0}^{\infty} \frac{2}{a^{2n+2}} \int_{0}^{1} x^{2n}\sqrt{1-x^2}\,dx \\
&= \sum_{n=0}^{\infty} \frac{1}{a^{2n+2}} \int_{0}^{1} t^{n-\frac{1}{2}}\sqrt{1-t}\,dt = \sum_{n=0}^{\infty} \frac{1}{a^{2n+2}} B\big(n+\tfrac{1}{2},\tfrac{3}{2}\big) \\
&= \pi \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{2n+2}} \binom{\frac{1}{2}}{n+1} = \pi\Big(1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{2n}} \binom{\frac{1}{2}}{n}\Big) \\
&= \pi(1-\sqrt{1-a^{-2}}),
\end{aligned}$$

where B denotes the Beta function. Since $1-a^{-2} \in \mathbb{C}\setminus]-\infty,0]$ for $a \in \mathbb{C}\setminus[-1,1]$ the right hand side of (5.4) is analytic on $\mathbb{C}\setminus[-1,1]$. It follows from Morera's Theorem that the left hand side of (5.4) is analytic on $\mathbb{C}\setminus[-1,1]$, and we conclude that (5.4) is valid for all a in $\mathbb{C}\setminus[-1,1]$. $\square$

Example 5.5. (Sum of Haar unitaries.) We show below that the Brown measure for $T = u_1 + \cdots + u_n$ is rotation invariant, has support equal to $\overline{B(0,\sqrt{n})}$ ($= \operatorname{sp} T$) and has radial density

$$f_T(r) = \begin{cases} \dfrac{n^2(n-1)}{\pi(n^2-r^2)^2}, & 0 < r < \sqrt{n}, \\ 0, & \text{otherwise.} \end{cases}$$

To prove this we first compute some R- and S-transforms of a family of distributions. Then we use Corollary 4.5 to compute the Brown measure of T.

For $c > 1$ we let $s_c = 2\sqrt{c-1}$ and define the measure ν_c by

$$\nu_c = \max\{0, \frac{2-c}{2}\}(\delta_{-c}+\delta_c) + \frac{c\sqrt{s_c^2-x^2}}{2\pi(c^2-x^2)} \cdot 1_{]-s_c,s_c[}(x)\,dx.$$

It is easily seen that ν_c is a symmetric, compactly supported measure on $\mathbb{R}$ and it follows from Lemma 5.4 that ν_c has total mass 1 for every $c > 1$. Fix $z > 0$ large and apply Lemma 5.4 to compute:

$$G_{\nu_c}(z) = \frac{z(c-2) - c\sqrt{z^2-4(c-1)}}{2(c^2-z^2)},$$

which we can invert to obtain the R-transform of ν_c:

$$\mathscr{R}_{\nu_c}(z) = c\,\frac{\sqrt{1+4z^2}-1}{2z} \tag{5.5}$$

for z in a neighbourhood of 0. By ν_c^2 we denote the measure $(\nu_c)_{\mathrm{sq}}$. Then

$$G_{\nu_c^2}(z) = \frac{G_{\nu_c}(\sqrt{z})}{z} = \frac{(c-2)z \pm c\sqrt{z^2 - 4(c-1)z}}{2z(c^2 - z)}$$

from which we obtain:

$$\begin{aligned}
\psi_{\nu_c^2}(z) &= \frac{G_{\nu_c^2}(z^{-1})}{z} - 1 = \frac{c - 2c^2 z - c\sqrt{1 - 4z(c-1)}}{2(c^2 z - 1)}, \\
\mathscr{R}_{\nu_c^2}(z) &= \frac{zc^2 - c + c\sqrt{z^2c^2 - 2z(c-2) + 1}}{2z}, \\
\mathscr{S}_{\nu_c^2}(z) &= \frac{z + c}{c^2(z+1)},
\end{aligned}$$

for z in a neighbourhood of 0. (The formulas hold in the case $c = 1$ too. In this case the transforms are the transforms of the distribution of a generating symmetry in $L(\mathbb{Z}_2)$.) The formula (5.5) shows that $(\nu_c)_{c \geqslant 1}$ is a semigroup with respect to free additive convolution.

Let h be a positive element in $(\mathscr{M}, \tau)$ with distribution

$$\mu_h = \max\{0, 2 - c\}\delta_c + \frac{c\sqrt{4(c-1) - x^2}}{\pi(c^2 - x^2)} \cdot 1_{[0, 2\sqrt{c-1}[}(x)\, dx \tag{5.6}$$

and suppose that u is a Haar unitary in $\mathscr{M}$ $*$-free from h. Then $\mu_{h^2} = \nu_c^2$ and we can state the Brown measure for uh using Theorem 4.4 and Corollary 4.5: $\mu_{uh}(\{0\}) = \mu_h(\{0\}) = 0$ and the radial density is

$$f_{uh}(s) = \frac{c^2(c-1)}{\pi(c^2 - s^2)^2} \cdot 1_{]0, \sqrt{c}[}(s).$$

In the case $T = u_1 + \cdots + u_n$ we use Proposition 3.5 to compute the distribution of $|T|$: $\widetilde{\mu}_{|T|} = (\widetilde{\mu}_1)^{\boxplus n}$ and hence

$$\mathscr{R}_{\widetilde{\mu}_{|T|}}(z) = n\mathscr{R}_{\widetilde{\mu}_1}(z) = n\,\frac{\sqrt{1 + 4z^2} - 1}{2z}.$$

Then it follows that the distribution of $|T|$ is

$$\mu_{|T|} = \frac{n\sqrt{4(n-1) - x^2}}{\pi(n^2 - x^2)} \cdot 1_{[0, 2\sqrt{n-1}[}(x)\, dx. \tag{5.7}$$

We remark, that the symmetrizations of the measures (5.7) for $n = 1, 2, \ldots$ were first studied by Kesten [7] in connection with random walks on free groups. Moreover the continuous family of "Kesten measures" $\widetilde{\mu}_h$ (where μ_h is given by 5.6) as well as the measures ν_c in Example 5.6 below were first studied in [2, Theorem 4.3] with different parametrizations.

Example 5.6. In relation to Example 5.5 we consider the family $(\nu_c)_{c>0}$ of symmetric measures on $\mathbb{R}$ defined by

$$\nu_c = \frac{c\sqrt{t_c^2 - x^2}}{2\pi(c^2 + x^2)} \cdot 1_{]-t_c,t_c[}(x)\,dx$$

where $t_c = 2\sqrt{c+1}$. A straightforward application of Lemma 5.4 shows that ν_c is a probability measure and that

$$G_{\nu_c}(z) = \frac{(c+2)z \pm c\sqrt{z^2 - 4(c+1)}}{2(c^2 + z^2)}.$$

Then we are able to compute related transformations:

$$\begin{aligned}
\mathscr{R}_{\nu_c}(z) &= c\,\frac{1 - \sqrt{1 - 4z^2}}{2z}, \qquad (5.8)\\
G_{\nu_c^2}(z) &= \frac{(c+2)z \pm c\sqrt{z^2 - 4(c+1)z}}{2z(c^2 + z)},\\
\psi_{\nu_c^2}(z) &= \frac{c - 2c^2 z - c\sqrt{1 - 4(c+1)z}}{2(1 + c^2 z)},\\
\mathscr{R}_{\nu_c^2}(z) &= \frac{c - c^2 z - c\sqrt{c^2 z^2 - 2(c+2)z + 1}}{2z},\\
\mathscr{S}_{\nu_c^2}(z) &= \frac{c - z}{c^2(1 + z)}.
\end{aligned}$$

The formula (5.8) shows that $(\nu_c)_{c\geqslant 0}$ is a one-parameter convolution semigroup with respect to free additive convolution. (We let ν_0 denote the point measure δ_0.)

If u and h are $*$-free random variables in a non-commutative W^*-probability space $(\mathscr{M}, \tau)$, u is a Haar unitary, and h is positive and has distribution

$$\mu_h = \frac{c\sqrt{4(c+1) - x^2}}{\pi(c^2 + x^2)} \cdot 1_{[0,2\sqrt{c+1}[}(x)\,dx$$

then $\mu_{h^2} = \nu_c^2$, $\|h\|_2^2 = \tau(h^2) = (\mathscr{S}_{\nu_c^2}(0))^{-1} = c$, $\|h^{-1}\|_2^{-1} = 0$, $\mu_{uh}(\{0\}) = 0$ and the Brown measure for uh has radial density

$$f_{uh}(x) = \frac{c^2(c+1)}{\pi(c^2 + x^2)^2} \cdot 1_{]0,\sqrt{c}[}(x).$$

It is straightforward to check that $\int_{B(0,\sqrt{c})} f_{uh}(|\lambda|)\,d\lambda = 1$.

Example 5.7. (Linear combination of two Haar unitaries.) Suppose that α and β are real numbers such that $0 < \alpha < \beta$. The Brown measure μ_T for $T = \alpha^{1/2}u_1 + \beta^{1/2}u_2$ is supported on $[\sqrt{\beta-\alpha}, \sqrt{\beta+\alpha}] \times_{\mathrm{p}} [0, 2\pi[$, rotation invariant and has radial density function

$$f(r) = \frac{2}{\pi} \cdot \frac{(\alpha+\beta)r^4 - 2(\beta-\alpha)^2 r^2 + (\beta-\alpha)^2(\alpha+\beta)}{\big(r^4 - 2(\alpha+\beta)r^2 + (\beta-\alpha)^2\big)^2} \cdot 1_{]\sqrt{\beta-\alpha}, \sqrt{\alpha+\beta}[}(r).$$

We apply Proposition 3.5 to compute the distribution of T^*T: $\widetilde{\mu}_{|T|} = \widetilde{\mu}_{\alpha^{1/2}1} \boxplus \widetilde{\mu}_{\beta^{1/2}1}$ and hence

$$\begin{aligned}
\mathscr{R}_{\widetilde{\mu}_{|T|}}(z) &= \frac{\sqrt{1+4\alpha z^2} + \sqrt{1+4\beta z^2} - 2}{2z}, \\
G_{\widetilde{\mu}_{|T|}}(z) &= \frac{\pm z}{\sqrt{(z^2 - (\alpha+\beta))^2 - 4\alpha\beta}}, \\
G_{\mu_{|T|^2}}(z) &= \frac{G_{\widetilde{\mu}_{|T|}}(\sqrt{z})}{\sqrt{z}} = \frac{\pm 1}{\sqrt{(z-(\alpha+\beta))^2 - 4\alpha\beta}}, \\
\psi_{\mu_{|T|^2}}(z) &= \frac{1}{\sqrt{(1-(\alpha+\beta)z)^2 - 4\alpha\beta z^2}} - 1, \\
\mathscr{R}_{\mu_{|T|^2}}(z) &= \frac{z(\alpha+\beta) - 1 + \sqrt{4\alpha\beta z^2 + 1}}{z}, \\
\mathscr{S}_{\mu_{|T|^2}}(z) &= \frac{(\alpha+\beta)(z+1) - \sqrt{4\alpha\beta(z+1)^2 + (\alpha-\beta)^2}}{(\alpha-\beta)^2 z},
\end{aligned}$$

and using Corollary 4.5 we obtain an expression for the radial density on $]\|h^{-1}\|_2^{-1}, \|h\|_2[=]\|h^{-1}\|_2^{-1}, \sqrt{\alpha+\beta}[$:

$$f_T(r) = \frac{2}{\pi} \cdot \frac{(\alpha+\beta)r^4 - 2(\beta-\alpha)^2 r^2 + (\beta-\alpha)^2(\alpha+\beta)}{\big(r^4 - 2(\alpha+\beta)r^2 + (\beta-\alpha)^2\big)^2}. \tag{5.9}$$

Note that the expression in (5.9) is positive for all r in $]\sqrt{\beta}-\sqrt{\alpha}, \sqrt{\beta}+\sqrt{\alpha}[$ and for ρ in this interval we find

$$\begin{aligned}
\iint\limits_{]\rho,\sqrt{\alpha+\beta}[\times_{\mathrm{p}}[0,2\pi[} f_T(|\lambda|)\, d\lambda &= 2\pi \int_{\rho}^{\sqrt{\alpha+\beta}} r f_T(r)\, dr = 2\Big[\frac{(\beta-\alpha)^2 - (\alpha+\beta)r^2}{r^4 - 2(\alpha+\beta)r^2 + (\beta-\alpha)^2}\Big]_{\rho}^{\sqrt{\alpha+\beta}} \\
&= \frac{2\rho^2(\rho^2 - (\alpha+\beta))}{\rho^4 - 2(\alpha+\beta)\rho^2 + (\beta-\alpha)^2}.
\end{aligned}$$

This expression is 1 only when $\rho = \sqrt{\beta-\alpha}$, which means that $\|h^{-1}\|_2^{-1} = \sqrt{\beta-\alpha}$ and that the radial density function f_T for μ_T is

$$f_T(r) = \frac{2}{\pi} \cdot \frac{(\alpha+\beta)r^4 - 2(\beta-\alpha)^2 r^2 + (\beta-\alpha)^2(\alpha+\beta)}{\big(r^4 - 2(\alpha+\beta)r^2 + (\beta-\alpha)^2\big)^2} \cdot 1_{]\sqrt{\beta-\alpha}, \sqrt{\alpha+\beta}[}(r).$$

Using the Stieltjes inversion formula we obtain an expression for the density of the distribution of $|T|^2$:

$$f_{\mu_{|T|^2}}(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}} \cdot 1_{]a,b[}(x)$$

where $a = (\sqrt{\beta} - \sqrt{\alpha})^2$, $b = (\sqrt{\alpha} + \sqrt{\beta})^2$. We have $\int_{\mathbb{R}} f_{\mu_{|T|^2}}\, dm = 1$ and we conclude that $d\mu_{|T|^2} = f_{\mu_{|T|^2}}\, dm$. (This holds in the case $\alpha = \beta$ too, which is easily verified by inspecting formula (5.7).) We note that $|T|^2$ has an arcus sinus distribution.

We see that $\operatorname{sp}|T|^2 = [a, b]$ which contains 0 if and only if $\alpha = \beta$, thus T is invertible if and only if $\alpha \neq \beta$. We conclude using Proposition 4.6 that $\operatorname{sp} T = [\sqrt{\beta - \alpha}, \sqrt{\beta + \alpha}] \times_{\mathrm{p}} [0, 2\pi[$.

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