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Mixed Moments of Voiculescu's Gaussian Random Matrices

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Abstract

It is showed that the convergence of mixed moments of independent, selfadjoint, Gaussian random matrices, proved by Voiculescu, holds almost surely, and not only in the mean. This is used to obtain asymptotic lower (respectively upper) bounds on the maximum (respectively minimum) of the spectrum of S^*S, for Gaussian random matrices S with operator entries; in particular those studied in [HT2]. Finally a new proof is presented of the result by S. Wassermann, that free group factors can be embedded into ultra products of matrix algebras.

Introduction

One of the main features of Voiculescu's free probability theory, is the fact (proved in [Vo1]), that the joint distribution of a semi-circular system (cf. Definition 1.2 below), can be approximated by the joint distributions of corresponding systems of independent large selfadjoint Gaussian random matrices. More precisely, let $\mathrm{SGRM}(n, \frac{1}{n})$ denote the class of selfadjoint Gaussian random $n \times n$ matrices studied by Voiculescu in [Vo1] (cf. Definition 2.1 below), let $\mathbb{E}$ denote expectation, and let tr_n denote the normalized trace on $M_n(\mathbb{C})$. Then for independent random matrices $X_1, X_2, \dots, X_r$ from $\mathrm{SGRM}(n, \frac{1}{n})$, Voiculescu proved, that for any p in $\mathbb{N}$ and any $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$,

$$\mathbb{E} \circ \mathrm{tr}_n\big[X_{i_1} X_{i_2} \cdots X_{i_p}\big] \to \tau(x_{i_1} x_{i_2} \cdots x_{i_p}), \quad \text{as } n \to \infty, \tag{0.1}$$

where $\{x_1, x_2, \dots, x_r\}$ is a semi-circular family in a C^*-probability space $(\mathcal{B}, \tau)$ (cf. Section 1).

The result corresponding to (0.1) for a single selfadjoint Gaussian random matrix and a single semi-circular element, was proved already in the 1950's by Wigner (cf. [Wi1]-[Wi3]). Since then, several people have strengthened Wigner's result in various ways;

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cf. [OU] for a survey of this development. Most notably, it follows from work of Arnold (cf. [Ar]), that the trace of the powers of a selfadjoint, Gaussian random matrix, converge almost surely (and not only in the mean!), to the corresponding moments of a semi-circular element (see also [HT1, Proposition 3.6]). In this paper we show (in Section 3) that this result generalizes to families of independent selfadjoint Gaussian random matrices and corresponding semi-circular systems. In other words, it is shown that the convergence (0.1) proved by Voiculescu holds almost surely, and not only in the mean. The method leading to this result is based on the derivation of the following explicit combinatorial formula for the mixed moments of independent random matrices from SGRM$(n, \frac{1}{n})$:

$$\mathbb{E} \circ \mathrm{tr}_n\big[X_{i_1}X_{i_2}\cdots X_{i_p}\big] = \sum_{\gamma\in\Gamma(i_1,i_2,\dots,i_p)} n^{-2\sigma(\gamma)}. \tag{0.2}$$

Here $\Gamma(i_1, i_2, \dots, i_p)$ is the set of permutations γ in the symmetric group S_p, which satisfy the conditions: $\gamma(j) \neq j$, $\gamma(\gamma(j)) = j$ and $i_j = i_{\gamma(j)}$, for all j in $\{1, 2, \dots, p\}$. Moreover, $\sigma(\gamma) = \frac{1}{2}(p+1-d(\gamma))$, where $d(\gamma)$ is the quantity introduced by Voiculescu in [Vo1] (cf. Definition 2.12 below). Combining (0.1) and (0.2) with the fact that $\sigma(\gamma) \in \mathbb{N}_0$, for all γ (cf. Proposition 2.15 below), it follows that

$$\mathbb{E} \circ \mathrm{tr}_n\big[X_{i_1}X_{i_2}\cdots X_{i_p}\big] = \tau(x_{i_1}x_{i_2}\cdots x_{i_p}) + O\big(\tfrac{1}{n^2}\big), \tag{0.3}$$

which means that the convergence in (0.1) is as fast as $O(\frac{1}{n^2})$. This fact (together with a few further considerations) implies the almost sure convergence version of (0.1).

We remark that recently the almost sure convergence version of (0.1) has been proved independently by F. Hiai and D. Petz, using different methods (cf. [HiP]). Their argument is based on the study of random unitary matrices. We mention also, that Voiculescu proved in [Vo1, Theorem 3.9], that the mixed moments of independent random unitary matrices converge, in probability, to the corresponding mixed moments of a family of free Haar unitaries.

The formula (0.2) is derived by virtue of the combinatorial techniques and results introduced in [HT1] and [HT2]. We spend Section 2 adapting these techniques to the situation considered in this paper. In particular we prove the following version of (0.2), for only one Gaussian random $n \times n$ matrix:

$$\mathbb{E} \circ \mathrm{tr}_n\big[X^{2p}\big] = \sum_{\gamma\in\Gamma_p} n^{-2\sigma(\gamma)}. \tag{0.4}$$

Here Γ_p is the set of permutations in the symmetric group S_{2p}, which are of order two, and do not have any fixed points (cf. Definition 2.6). Formula (0.4) should (after renormalization) be considered as the "selfadjoint version" of the formula obtained in [HT2, Corollary 1.12].

In Section 4, we generalize the almost sure convergence result obtained in Section 3 to operator valued random matrices of the form:

$$S_n = \sum_{i=1}^{r} a_i \otimes Q_i(X_1, \dots, X_t), \tag{0.5}$$

where $Q_1, \dots, Q_r$ are polynomials in t non-commuting variables, $X_1, \dots, X_t$ are independent elements of $\mathrm{SGRM}(n, \frac{1}{n})$ and $a_1, \dots, a_r$ are bounded operators between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. This is subsequently used to obtain lower (respectively upper) asymptotic bounds on the maximum (respectively minimum) of the spectrum $\mathrm{sp}(S_n^*S_n)$ of $S_n^*S_n$. Specificly we get the following bounds:

$$\liminf_{n\to\infty} \big(\max \big\{ \mathrm{sp}(S_n^*S_n) \big\} \big) \geq \max \big\{ \mathrm{sp}(s^*s) \big\}, \quad \text{almost surely,}$$

and

$$\limsup_{n\to\infty} \big(\min \big\{ \mathrm{sp}(S_n^*S_n) \big\} \big) \leq \min \big\{ \mathrm{sp}(s^*s) \big\}, \quad \text{almost surely,}$$

where s is the "limit object" obtained by replacing in (0.5) the random matrices $X_1, \dots, X_t$ by a semi-circular system $x_1, \dots, x_t$.

In [HT2], we studied operator valued random matrices S_n of the form:

$$S_n = \sum_{j=1}^{r} a_j \otimes \big(\tfrac{1}{\sqrt{2}} (X_j' + iX_j'') \big),$$

where $X_1', X_1'', \dots, X_r', X_r''$ are $2r$ independent elements of $\mathrm{SGRM}(n, \frac{1}{n})$, and the operators $a_1, \dots, a_r$ satisfy the conditions:

$$\sum_{j=1}^{r} a_j^* a_j = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \quad \text{and} \quad \sum_{j=1}^{r} a_j a_j^* \leq \mathbf{1}_{\mathcal{B}(\mathcal{K})}, \tag{0.6}$$

for some constant c in $]1, \infty[$. Under the assumption that the C^*-algebra generated by $\{a_j^* a_k \mid 1 \leq j, k \leq r\}$ is exact, we found in [HT2, Theorem 0.1] upper (respectively lower) bounds on the maximum (respectively minimum) of $\mathrm{sp}(S_n^*S_n)$. In the case where the inequality in (0.6) is replaced by an equality, we apply, in Section 5, the results of Section 4 to show that the bounds found in [HT2] are also lower (respectively upper) bounds for the maximum (respectively minimum) of $\mathrm{sp}(S_n^*S_n)$, proving altogether, that in this situation we actually have convergence of the maximum and minimum of $\mathrm{sp}(S_n^*S_n)$. Specificly we find that

$$\lim_{n\to\infty} \max\{\mathrm{sp}(S_n^*S_n)\} = (\sqrt{c}+1)^2, \quad \text{almost surely,}$$

and

$$\lim_{n\to\infty} \min\{\mathrm{sp}(S_n^*S_n)\} = (\sqrt{c}-1)^2, \quad \text{almost surely.}$$

Finally, in Section 6, we use the almost sure convergence result obtained in Section 3 to give a new proof of the fact, proved by S. Wassermann, that for any m in $\{2, 3, 4, \dots\} \cup \{\infty\}$, the free group factor $\mathcal{L}(F_m)$ can be embedded into the ultra product of matrix algebras.

In Section 1 we recapture a few fundamental definitions from free probability theory, that are used throughout the paper.

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1 Preliminaries on Free Probability

During the 1980's, D.V. Voiculescu founded and developed the so called *free probability theory*. Voiculescu's basic idea was to translate fundamental concepts from "classical" probability theory, (such as random variable, distribution, independence etc.) into corresponding notions in an operator algebraic framework.

The basic object in free probability theory is a non-commutative probability space, which is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital complex algebra, and ϕ is a linear, complex valued functional on $\mathcal{A}$, taking the value 1 at the unit of $\mathcal{A}$. In this paper, we shall consider exclusively the so called C^*-probability spaces, which are non-commutative probability spaces $(\mathcal{A}, \phi)$, such that $\mathcal{A}$ is a unital C^*-algebra and ϕ is a state on $\mathcal{A}$. If $\mathcal{A}$ is a W^*-algebra and ϕ is a normal state on $\mathcal{A}$, we say that $(\mathcal{A}, \phi)$ is a W^*-probability space. Given a non-commutative probability space $(\mathcal{A}, \phi)$, the elements of $\mathcal{A}$ are called *random variables*. The *distribution* of a random variable a in $(\mathcal{A}, \phi)$ is the sequence of moments $\phi(a^p)$, $p \in \mathbb{N}$, of a w.r.t. ϕ. The *joint distribution* of a family $(a_j)_{j\in J}$ of random variables, is the set $\{\phi(a_{j_1}a_{j_2}\cdots a_{j_p}) \mid p \in \mathbb{N},\ j_1, j_2, \dots, j_p \in J\}$ of all mixed moments w.r.t. ϕ of the elements a_j, $j \in J$. In free probability theory, the concept of independence from "classical" probability theory is replaced by the notion of *freeness*:

1.1 Definition. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, and let $(\mathcal{A}_j)_{j\in J}$ be a family of unital subalgebras of $\mathcal{A}$. Then the algebras $\mathcal{A}_j$, $j \in J$, are called free (w.r.t. ϕ), if $\phi(a_1a_2\cdots a_n) = 0$, whenever $n \in \mathbb{N}$, and we have random variables $a_1, a_2, \dots, a_n$, satisfying that $\phi(a_1) = \phi(a_2) = \cdots = \phi(a_n) = 0$, and that $a_1 \in \mathcal{A}_{j_1}, a_2 \in \mathcal{A}_{j_2}, \dots, a_n \in \mathcal{A}_{j_n}$, where $j_1, j_2, \dots, j_r \in J$, such that $j_1 \neq j_2$, $j_2 \neq j_3$, $\dots$, $j_{n-1} \neq j_n$. $\square$

The elements of a family $(a_j)_{j\in J}$ of random variables are called free, if the unital subalgebras they generate (one by one) are free. Random variables of particular interest are the so called *semi-circular* and *circular elements*:

1.2 Definition. Let $(\mathcal{A}, \phi)$ be a C^*-probability space.

(a) A random variable x in $(\mathcal{A}, \phi)$ is called a semi-circular element, if x is selfadjoint, and the moments of x w.r.t. ϕ, equal those of the (standard) semi-circle distribution, i.e., the probability measure on $\mathbb{R}$ with density $x \mapsto \frac{1}{2\pi}\sqrt{4-x^2}\cdot 1_{[-2,2]}(x)$ w.r.t. Lebesgue measure.

(b) A *semi-circular system* in $(\mathcal{A}, \phi)$, is a family $(x_j)_{j\in J}$ of free random variables, such that each x_j is a semi-circular element.

(c) A *circular system* in $(\mathcal{A}, \phi)$ is a family $(y_j)_{j\in J}$ of random variables, satisfying that the family $\{2^{-1/2}(y_j + y_j^*) \mid j \in J\} \cup \{-i2^{-1/2}(y_j - y_j^*) \mid j \in J\}$ is a semi-circular system in $(\mathcal{A}, \phi)$. $\square$

The distribution of the semi-circular elements introduced in Definition 1.2(a) has mean 0 and variance 1. More generally, one may define semi-circular elements to be selfadjoint random variables, whose distributions are semi-circle distributions with arbitrary mean and variance. However, throughout this paper, the term semi-circular element refers to the mean 0 – variance 1 case.

For a thorough introduction to free probability theory, we refer to [VDN].

2 Combinatorics for the Moments of Selfadjoint, Gaussian Random Matrices

In [Vo1], Voiculescu studied selfadjoint Gaussian random matrices. In the following definition, we recall the precise description of these matrices. Recall first, that for ξ in $\mathbb{R}$ and σ^2 in $]0,\infty[$, $N(\xi,\sigma^2)$ denotes the Gaussian distribution with mean ξ and variance σ^2.

2.1 Definition. Let $(\Omega,\mathcal{F},P)$ be a (classical) probability space, let n be a positive integer and let $A\colon \Omega \to M_n(\mathbb{C})$ be a complex random $n\times n$ matrix defined on Ω. For k,l in $\{1,2,\dots,n\}$, let $a(k,l)$ denote the entry at position (k,l) of A. We say that A is a (standard) selfadjoint Gaussian random $n\times n$ matrix with entries of variance σ^2, if the following conditions are satisfied:

(i) The entries $a(k,l), 1 \le k \le l \le n$, form a set of $\frac{1}{2}n(n+1)$ independent, complex valued random variables.

(ii) For each k in $\{1,2,\dots,n\}$, $a(k,k)$ is a real valued random variable with distribution $N(0,\sigma^2)$.

(iii) When $k<l$, the real and imaginary parts $\mathrm{Re}(a(k,l))$ and $\mathrm{Im}(a(k,l))$ of $a(k,l)$ are independent identically distributed random variables with distribution $N(0,\frac{1}{2}\sigma^2)$.

(iv) When $k>l$, $a(k,l)=\overline{a(l,k)}$.

We denote by $\mathrm{SGRM}(n,\sigma^2)$ the set of all such random matrices (defined on Ω). □

2.2 Remark. As was noted in [VDN, Lemma 4.1.3], if f and g are independent, identically distributed, real valued random variables with distribution $N(0,\sigma^2)$, then for any m,n in $\mathbb{N}_0$, we have that

$$\mathbb{E}\big((f+ig)^m(f-ig)^n\big)=0, \quad \text{unless} \quad m=n, \tag{2.1}$$

where $\mathbb{E}$ denotes expectation. This can be seen by noting, that the distribution on $\mathbb{C}$ of $f+ig$ is invariant under multiplication by any complex number of norm 1. From (2.1) it follows, in particular, that we have the following fundamental relation between the entries $a(j,k)$, $1\le j,k\le n$, of an element A of $\mathrm{SGRM}(n,\sigma^2)$:

$$\mathbb{E}\big(a(j_1,k_1)a(j_2,k_2)\big)=\sigma^2\cdot\delta_{j_1,k_2}\cdot\delta_{k_1,j_2}, \qquad (1\le j_1,k_1,j_2,k_2\le n), \tag{2.2}$$

with the standard Kronecker delta notation. Note, in particular, that σ^2 denotes the second absolute moment of the entries of an element from $\mathrm{SGRM}(n,\sigma^2)$. □

2.3 Remark. In [HT2], we studied another kind of Gaussian, random matrices defined on a probability space $(\Omega, \mathcal{F}, P)$: A (standard) Gaussian, random $n \times n$ matrix with entries of variance σ^2, is a random matrix $B = (b(j,k))_{1 \le j,k \le n}$, defined on Ω, such that the real valued random variables $\mathrm{Re}(b(j,k)), \mathrm{Im}(b(j,k))$, $1 \le j,k \le n$, form a family of $2n^2$ independent identically distributed random variables, with distribution $N(0, \frac{\sigma^2}{2})$. We let $\mathrm{GRM}(n, \sigma^2)$ denote the set of such random matrices.

There is a close connection between the two sets $\mathrm{SGRM}(n, \sigma^2)$ and $\mathrm{GRM}(n, \sigma^2)$. Indeed, if A_1 and A_2 are two independent elements of $\mathrm{SGRM}(n, \sigma^2)$ (i.e., the entries of A_1 are jointly independent of those of A_2), then $2^{-1/2}(A_1 + iA_2)$ is an element of $\mathrm{GRM}(n, \sigma^2)$. Conversely, if B is an element of $\mathrm{GRM}(n, \sigma^2)$, then $2^{-1/2}(B + B^*)$ and $-i2^{-1/2}(B - B^*)$ are two independent elements of $\mathrm{SGRM}(n, \sigma^2)$. These relations follow by standard arguments, using the convolution properties of the Gaussian distribution. $\square$

In the following, we omit mentioning the underlying probability space $(\Omega, \mathcal{F}, P)$, and it will be understood, that all considered random matrices/variables are defined on this probability space. For any positive integer n, we denote by $\mathbf{1}_n$ the unit of $M_n(\mathbb{C})$. By tr_n we denote the trace on $M_n(\mathbb{C})$ satisfying that $\mathrm{tr}_n(\mathbf{1}_n) = 1$, and we put $\mathrm{Tr}_n = n \cdot \mathrm{tr}_n$. The main task of this section is to find a combinatorial expression for the moments $\mathbb{E} \circ \mathrm{tr}_n[X^{2p}]$, $p \in \mathbb{N}$, of an element X of $\mathrm{SGRM}(n, \frac{1}{n})$. The argument leading to this result is similar to the argument in [HT2] resulting in [HT2, Corollary 1.12].

Throughout the paper, we will be considering families of independent random matrices. By independence between random matrices, we mean classical independence between the entries of the random matrices. More precisely, let $(F_i)_{i \in I}$ be a family of random matrices, and for each i, let $f^{(i)}(j,k)$, $1 \le j,k \le n^{(i)}$, denote the entries of F_i. Then we say that the random matrices F_i, $i \in I$, are independent, if the sets of random variables $\{f^{(i)}(j,k) \mid 1 \le j,k \le n^{(i)}\}$, $i \in I$, are independent in the classical sense. So for example, two random matrices F_1 and F_2 are independent, if the entries of F_1 are jointly independent of the entries of F_2.

2.4 Remark. Let r, n be positive integers, and let $A_1, \dots, A_r$ be arbitrary elements of $\mathrm{SGRM}(n, \sigma^2)$. It shall be useful for us to note that the quantity $\mathbb{E} \circ \mathrm{Tr}_n(A_1 A_2 \cdots A_r)$ is bounded numerically by some constant $K(n, r, \sigma^2)$ depending only on n, r and σ^2, and not on the distributional relations between (the entries of) $A_1, A_2, \dots, A_r$. This result is analogous to the result stated in [HT2, Remark 1.4], and it can be proved similarly. The only difference is, that when working with elements of $\mathrm{SGRM}(n, \sigma^2)$, rather than elements of $\mathrm{GRM}(n, \sigma^2)$, one has to take into account the fact, that diagonal and off-diagonal elements of the matrices do not have the same distribution. We refer the reader to [HT2, Remark 1.4] for further details. $\square$

2.5 Lemma. *Let r, n, q be positive integers, let $A_1, \dots, A_r$ be independent elements of $\mathrm{SGRM}(n, \sigma^2)$, and let $i_1, \dots, i_q$ be elements of $\{1, 2, \dots, r\}$. Then $\mathbb{E}(A_{i_1} A_{i_2} \cdots A_{i_q}) = 0$, unless there exists a permutation γ in S_q, such that for all j in $\{1, 2, \dots, q\}$, $\gamma(j) \neq j$, $\gamma^2(j) = \gamma(\gamma(j)) = j$ and $i_j = i_{\gamma(j)}$. In particular $\mathbb{E}(A_{i_1} A_{i_2} \cdots A_{i_q}) = 0$ for any choice of $i_1, \dots, i_q$, if q is odd.*

Proof. The existence of a permutation γ in S_q with the properties described in the lemma is equivalent to the condition, that the random matrices $A_{i_1}, A_{i_2}, \dots, A_{i_q}$ can be divided into $\frac{q}{2}$ pairs of equal matrices. This, in turn, is equivalent to the condition:

$$\forall i \in \{1,2,\dots,r\}\colon \operatorname{card}\big(\{j \in \{1,2,\dots,q\} \mid A_{i_j} = A_i\}\big) \text{ is even.} \tag{2.3}$$

Thus, we have to show that $\mathbb{E}(A_{i_1}A_{i_2}\cdots A_{i_q}) = 0$, unless the condition (2.3) is satisfied. So assume that $\operatorname{card}(\{j \in \{1,2,\dots,q\} \mid A_{i_j} = A_i\})$ is odd for some i in $\{1,2,\dots,r\}$. Since $-A_i$ is again an element $\mathrm{SGRM}(n,\sigma^2)$, it follows then that

$$\mathbb{E}(A_{i_1}A_{i_2}\cdots A_{i_q}) = -\mathbb{E}(A_{i_1}A_{i_2}\cdots A_{i_q}),$$

and therefore $\mathbb{E}(A_{i_1}A_{i_2}\cdots A_{i_q}) = 0$ as desired. ■

2.6 Definition. Let p be a positive integer. Then by Γ_p, we denote the set of permutations γ in the symmetric group S_{2p}, satisfying that

$$\gamma(j) \neq j \quad\text{and}\quad \gamma^2(j) = \gamma(\gamma(j)) = j \quad\text{for all } j \text{ in } \{1,2,\dots,2p\}. \qquad \square$$

2.7 Definition. Let p be a positive integer, and let γ be an element of Γ_p. We associate to γ a sequence $\Delta(\gamma,n)$, $n \in \mathbb{N}$, of complex numbers as follows:

Let $A_1, A_2, \dots, A_{2p}$ be elements of $\mathrm{SGRM}(n,1)$, with the property that for all j, j' in $\{1,2,\dots,2p\}$, A_j and $A_{j'}$ are independent unless $j = j'$ or $\gamma(j) = j'$ in which cases $A_j = A_{j'}$. We then define:

$$\Delta(\gamma,n) = \mathbb{E}\circ \mathrm{Tr}_n[A_1A_2\cdots A_{2p}]. \qquad \square$$

2.8 Proposition. *Let A be an element of* $\mathrm{SGRM}(n,1)$*. Then for any positive integer p, we have that*

(i) $\mathbb{E}\circ\mathrm{Tr}_n\big[A^{2p-1}\big] = 0$.

(ii) $\mathbb{E}\circ\mathrm{Tr}_n\big[A^{2p}\big] = \sum_{\gamma\in\Gamma_p}\Delta(\gamma,n)$.

Proof. (i) This follows from the last statement of Lemma 2.5.

(ii) Let $A_1, A_2, A_3, \dots,$ be a sequence of independent random matrices from $\mathrm{SGRM}(n,1)$, and note then, that for any s in $\mathbb{N}$, the random matrix $\frac{1}{\sqrt{s}}(A_1 + A_2 + \cdots + A_s)$ is again an element of $\mathrm{SGRM}(n,1)$. Hence, for any s in $\mathbb{N}$, we have that

$$\mathbb{E}\circ\mathrm{Tr}_n[A^{2p}] = s^{-p}\sum_{1\le i_1,i_2,\dots,i_{2p}\le s}\mathbb{E}\circ\mathrm{Tr}_n[A_{i_1}A_{i_2}A_{i_3}\cdots A_{i_{2p}}].$$

It follows here from Lemma 2.5, that for any $2p$-tuple $(i_1,i_2,\dots,i_{2p})$ in $\{1,2,\dots,s\}^{2p}$, $\mathbb{E}\circ\mathrm{Tr}_n[A_{i_1}A_{i_2}\cdots A_{i_{2p}}] = 0$, unless there exists a permutation γ in Γ_p, such that $i_j = i_{\gamma(j)}$ for all j in $\{1,2,\dots,2p\}$. Define then for each γ in Γ_p,

$$M(\gamma,s) = \big\{(i_1,i_2,\dots,i_{2p}) \in \{1,2,\dots,s\}^{2p} \;\big|\; i_j = i_{\gamma(j)},\ j = 1,2,\dots,2p\big\}.$$

It follows then that

$$\mathbb{E}\circ\mathrm{Tr}_n[A^{2p}] = s^{-p} \sum_{(i_1,i_2,\dots,i_{2p})\in\ \bigcup_{\gamma\in\Gamma_p} M(\gamma,s)} \mathbb{E}\circ\mathrm{Tr}_n[A_{i_1}A_{i_2}\cdots A_{i_{2p}}].$$

Note though, that the sets $M(\gamma,s)$, $\gamma\in\Gamma_p$, are not disjoint. However, if we consider, for each γ, the subset

$$\begin{aligned}&\mathcal{D}(\gamma,s)\\ &\quad=\big\{(i_1,\dots,i_{2p})\in\{1,\dots,s\}^{2p}\ \big|\ \forall j,j'\in\{1,\dots,2p\}\colon i_j=i_{j'}\Leftrightarrow j=j' \text{ or } j=\gamma(j')\big\},\end{aligned}$$

then the sets $\mathcal{D}(\gamma,s)$, $\gamma\in\Gamma_p$, are disjoint, and even more so: For any distinct γ_1,γ_2 in Γ_p, $\mathcal{D}(\gamma_1,s)\cap M(\gamma_2,s)=\emptyset$. It follows thus, that

$$\begin{aligned}\mathbb{E}\circ\mathrm{Tr}_n[A^{2p}] = {}& s^{-p}\sum_{\gamma\in\Gamma_p}\ \sum_{(i_1,\dots,i_{2p})\in\mathcal{D}(\gamma,s)} \mathbb{E}\circ\mathrm{Tr}_n[A_{i_1}A_{i_2}\cdots A_{i_{2p}}]\\ &+ s^{-p}\sum_{(i_1,\dots,i_{2p})\in\bigcup_{\gamma\in\Gamma_p} M(\gamma,s)\setminus\mathcal{D}(\gamma,p)} \mathbb{E}\circ\mathrm{Tr}_n[A_{i_1}A_{i_2}\cdots A_{i_{2p}}].\end{aligned}\tag{2.4}$$

Regarding the first term on the right hand side of (2.4), note that if γ is a permutation in Γ_p and $(i_1,i_2,\dots,i_{2p})\in\mathcal{D}(\gamma,s)$, then by the definition of $\Delta(\gamma,n)$, we have that

$$\mathbb{E}\circ\mathrm{Tr}_n[A_{i_1}A_{i_2}\cdots A_{i_{2p}}] = \Delta(\gamma,n).$$

Therefore, the first term on the right hand side of (2.4) equals

$$\sum_{\gamma\in\Gamma_p} s^{-p}\cdot\mathrm{card}(\mathcal{D}(\gamma,s))\cdot\Delta(\gamma,n).$$

Note here, that for any γ in Γ_p, $\mathrm{card}(\mathcal{D}(\gamma,s)) = s(s-1)(s-2)\cdots(s-p+1)$, so that $s^{-p}\cdot\mathrm{card}(\mathcal{D}(\gamma,p))\to 1$ as $s\to\infty$. Hence, the first term on the right hand side of (2.4) tends to

$$\sum_{\gamma\in\Gamma_p}\Delta(\gamma,n),$$

as $s\to\infty$. Since the left hand side of (2.4) does not depend on s, it remains now to show, that the second term on the right hand side of (2.4) tends to 0 as $s\to\infty$. For this, recall from Remark 2.4, that there exists a constant $K(n,2p,1)$, depending only on n,p (not on s), such that

$$\big|\mathbb{E}\circ\mathrm{Tr}_n[A_{i_1}A_{i_2}\cdots A_{i_{2p}}]\big| \le K(n,2p,1),$$

for any $i_1,i_2,\dots,i_{2p}$ in $\mathbb{N}$. Thus, the second term on the right hand side of (2.4) is bounded numerically by the quantity

$$s^{-p}\cdot\mathrm{card}\big(\textstyle\bigcup_{\gamma\in\Gamma_p} M(\gamma,s)\setminus\mathcal{D}(\gamma,s)\big)\cdot K(n,2p,1).$$

Note here, that for any γ in Γ_p, $\mathrm{card}(M(\gamma,s)) = s^p$, and hence

$$\mathrm{card}(M(\gamma,s)\setminus\mathcal{D}(\gamma,s)) = s^p - s(s-1)(s-2)\cdots(s-p+1).$$

Therefore,

$$\begin{aligned}
s^{-p} \cdot \operatorname{card}\big(\textstyle\bigcup_{\gamma\in\Gamma_p} M(\gamma,s) \setminus \mathcal{D}(\gamma,s)\big) &\le s^{-p} \sum_{\gamma\in\Gamma_p} \operatorname{card}(M(\gamma,s)\setminus\mathcal{D}(\gamma,s)) \\
&= s^{-p} \sum_{\gamma\in\Gamma_p} \big(s^p - s(s-1)\cdots(s-p+1)\big) \\
&\to 0, \quad \text{as } s\to\infty.
\end{aligned}$$

This implies that the second term on the right hand side of (2.4) tends to 0 as $s \to \infty$, as desired. ■

It follows from Proposition 2.8, that in order to obtain a combinatorial expression for $\mathbb{E} \circ \mathrm{Tr}_n[A^{2p}]$, we should concentrate on deriving a combinatorial expression for $\Delta(\gamma, n)$, for all γ in Γ_p and n in $\mathbb{N}$. As in [HT2, Theorem 1.11], we shall express $\Delta(\gamma, n)$ in terms of the number of equivalence classes for a certain equivalence relation $\sim_\gamma$ on $\{1, 2, \dots, 2p\}$.

2.9 Definition. Let p be a positive integer, and let γ be an element of Γ_p. Then we denote by $\sim_\gamma$ the equivalence relation on $\{1, 2, \dots, 2p\}$, generated by the expression

$$j \sim_\gamma \gamma(j) + 1, \qquad (j \in \{1, 2, \dots, 2p\}),$$

where addition is formed mod. $2p$. Moreover, for any j in $\{1, 2, \dots, 2p\}$, we denote by $[j]_\gamma$ the $\sim_\gamma$-equivalence class containing j. □

2.10 Remark. The equivalence relation $\sim_\gamma$ was introduced by Voiculescu in [Vo1, Proof of Theorem 2.2]. In [HT2], we studied this equivalence relation in the special case where γ possesses the additional property

$$\gamma(j) = j + 1 \text{ mod. } 2, \qquad (j \in \{1, 2, \dots, 2p\}). \tag{2.5}$$

For γ in Γ_p, (2.5) is equivalent to the existence of a permutation π in S_p, such that $\gamma(2i-1) = 2\pi^{-1}(i)$ for all i in $\{1, 2, \dots, p\}$, and π is uniquely determined by the equation: $\pi(i) = \frac{1}{2}(\gamma(2i) + 1)$, $i \in \{1, 2, \dots, p\}$. In this case, we denote γ by $\hat{\pi}$ (as in [HT2]). It follows then, that the mapping $\pi \mapsto \hat{\pi}$ is a bijection of S_p onto the set of permutations γ in Γ_p for which (2.5) holds (cf. [HT2, Remark 1.7(a)]). □

2.11 Remark. Under the additional assumption (2.5), we determined in [HT2, Remark 1.9], the appearance of $[j]_\gamma$ for an arbitrary j in $\{1, 2, \dots, 2p\}$. However, the argument given in [HT2, Remark 1.9] did not make use of the extra assumption (2.5), and hence we have also in the presently considered more general situation, that

$$[j]_\gamma = \{j_0, j_1, \dots, j_{m-1}\},$$

where $m = \operatorname{card}([j]_\gamma)$, and

$$j_0 = j,\ j_1 = \gamma(j_0) + 1,\ j_2 = \gamma(j_1) + 1,\ \dots,\ j_{m-1} = \gamma(j_{m-2}) + 1,\ j_0 = \gamma(j_{m-1}) + 1,$$

(addition formed mod. $2p$). □

2.12 Definition. Let p be a positive integer, and let γ be a permutation in Γ_p. We then define:

$$\begin{aligned} d(\gamma) &= \mathrm{card}\big(\{[j]_\gamma \mid j \in \{1,2,\dots,2p\}\}\big) \\ \sigma(\gamma) &= \tfrac{1}{2}(p+1-d(\gamma)). \qquad \square \end{aligned}$$

The quantity $d(\cdot)$ was introduced by Voiculescu in [Vo1, Proof of Theorem 2.2], where it was proved, furthermore, that $d(\gamma)$ is always less than or equal to $p+1$. Thus, the quantity $\sigma(\cdot)$ is always non-negative. In [HT2, Theorem 1.13], it was shown, under the additional assumption (2.5), that $p+1-d(\gamma)$ is always an even number, so that $\sigma(\gamma)$ is a non-negative integer. As we shall see in Proposition 2.15, the same conclusion holds in the more general situation considered presently.

2.13 Proposition. *For any positive integer p and any permutation γ in Γ_p, we have that*

$$\Delta(\gamma, n) = n^{d(\gamma)}, \qquad (n \in \mathbb{N}). \tag{2.6}$$

Proof. Let $A_1, A_2, \dots, A_{2p}$ be elements of $\mathrm{SGRM}(n,1)$, satisfying that for any k, k' in $\{1,2,\dots,2p\}$, A_k and $A_{k'}$ are independent, unless $k = k'$ or $k = \gamma(k')$, in which cases $A_k = A_{k'}$. Let $e(i,j)$, $1 \le i,j \le n$, denote the usual $n \times n$ matrix units, and for each k in $\{1,2,\dots,2p\}$, let $a(i,j,k)$, $1 \le i,j \le n$, denote the entries of A_k. It follows then by a standard matrix calculation, that

$$\begin{aligned} &\Delta(\gamma, n) \\ &= \mathbb{E} \circ \mathrm{Tr}_n[A_1 A_2 \cdots A_{2p}] \\ &= \sum_{\substack{1 \le i_1, i_2, \dots, i_{2p} \le n \\ 1 \le j_1, j_2, \dots, j_{2p} \le n}} \mathbb{E}\big[a(i_1,j_1,1)a(i_2,j_2,2)\cdots a(i_{2p},j_{2p},2p)\big]\,\mathrm{Tr}_n\big[e(i_1,j_1)e(i_2,j_2)\cdots e(i_{2p},j_{2p})\big] \\ &= \sum_{1 \le i_1, i_2, \dots, i_{2p} \le n} \mathbb{E}\big[a(i_1,i_2,1)a(i_2,i_3,2)\cdots a(i_{2p},i_1,2p)\big]. \end{aligned} \tag{2.7}$$

It follows here from (2.2) and the assumptions on $A_1, A_2, \dots, A_{2p}$, that for any $i_1, i_2, \dots, i_{2p}$ in $\{1,2,\dots,n\}$, $\mathbb{E}\big[a(i_1,i_2,1)a(i_2,i_3,2)\cdots a(i_{2p},i_1,2p)\big] = 0$, unless we have that

$$\overline{a(i_k, i_{k+1}, k)} = a(i_{\gamma(k)}, i_{\gamma(k)+1}, \gamma(k)), \qquad (1 \le k \le 2p),$$

i.e., unless

$$i_{\gamma(k)} = i_{k+1} \qquad \text{and} \qquad i_{\gamma(k)+1} = i_k, \qquad (1 \le k \le 2p), \tag{2.8}$$

(where $k+1, \gamma(k)+1$ are calculated mod. $2p$). Note also that if (2.8) is satisfied, then $\mathbb{E}\big[a(i_1,i_2,1)a(i_2,i_3,2)\cdots a(i_{2p},i_1,2p)\big] = 1$. By replacing k by $\gamma(k)$, we see that the two conditions in (2.8) are actually equivalent, and thus it follows that

$$\mathbb{E}\big[a(i_1,i_2,1)a(i_2,i_3,2)\cdots a(i_{2p},i_1,2p)\big] = \prod_{k=1}^{2p} \delta_{i_k, i_{\gamma(k)+1}}.$$

Inserting this in (2.7), we conclude that

$$\Delta(\gamma,n) = \mathrm{card}\big(\{(i_1,i_2,\dots,i_{2p}) \in \{1,2,\dots,n\}^{2p} \mid i_k = i_{\gamma(k)+1}, k=1,2,\dots,2p\}\big),$$

where $\gamma(k)+1$ is calculated mod. $2p$. Finally, by application of Remark 2.11, it is straightforward that this cardinality equals $n^{d(\gamma)}$. ■

2.14 Corollary. *Let n be a positive integer, and let A be a random matrix in* $\mathrm{SGRM}(n,1)$. *Then for any p in $\mathbb{N}$, we have that*

$$\mathbb{E}\circ \mathrm{Tr}_n[A^{2p}] = \sum_{\gamma\in\Gamma_p} n^{d(\gamma)}.$$

Proof. This follows immediately by combining Propositions 2.8 and 2.13. ■

By combining the combinatorial formula obtained in Corollary 2.14 with [HT1, Theorem 4.1], we can now give an easy proof of the fact that $\sigma(\gamma)$ is always a non-negative integer.

2.15 Proposition. *Let p be a positive integer, and let γ be a permutation in Γ_p. Then*

$$\sigma(\gamma) = \tfrac{1}{2}\big(p+1-d(\gamma)\big) \in \mathbb{N}_0.$$

Proof. For all p,n in $\mathbb{N}$, put $C(p,n) = \mathbb{E}\circ\mathrm{Tr}_n[A_n^{2p}]$, where A_n is an element of $\mathrm{SGRM}(n,1)$. Recall then from [HT1, Theorem 4.1], that $C(0,n)=n$, $C(1,n)=n^2$, and that, for fixed n, the numbers $C(p,n)$, $p\in\mathbb{N}$, satisfy the recursion formula

$$C(p+1,n) = n\cdot \tfrac{4p+2}{p+2}\cdot C(p,n) + \tfrac{p(4p^2-1)}{p+2}\cdot C(p-1,n), \qquad (p\geq 1).$$

It follows from these facts, and induction on p, that for any p in $\mathbb{N}$, $C(p,n)$ is of the form:

$$C(p,n) = \sum_{q=0}^{[\frac{p}{2}]} \alpha(p,q) n^{p+1-2q},$$

for suitable positive real numbers $\alpha(p,q)$, $q=0,1,\dots,\big[\frac{p}{2}\big]$, (not depending on n). On the other hand, we know from Corollary 2.14, that

$$C(p,n) = \mathbb{E}\circ\mathrm{Tr}_n[A_n^{2p}] = \sum_{\gamma\in\Gamma_p} n^{d(\gamma)}, \qquad (n,p\in\mathbb{N}).$$

Since two polynomials in $\mathbb{C}[x]$, that coincide on $\mathbb{N}$, are equal, it follows, that for each p in $\mathbb{N}$ and each γ in Γ_p, we must have

$$d(\gamma) = p+1-2q, \quad \text{for some } q \text{ in } \big\{0,1,\dots,\big[\tfrac{p}{2}\big]\big\},$$

and hence that

$$\sigma(\gamma) = \tfrac{1}{2}\big(p+1-d(\gamma)\big) = q, \quad \text{for some } q \text{ in } \big\{0,1,\dots,\big[\tfrac{p}{2}\big]\big\}.$$

This completes the proof. ■

Finally we shall need a generalization of [HT2, Corollary 1.24]. Before stating this result, we recall, that if $\gamma \in \Gamma_p$ and $a, b, c, d \in \{1, 2, \dots, 2p\}$, then we say that (a, b, c, d) is a *crossing* for γ, if $a < b < c < d$, and $\gamma(a) = c$, $\gamma(b) = d$. If γ does not have any crossings, we say that γ is a *non-crossing* permutation.

2.16 Proposition. *Let p be a positive integer, and let γ be a permutation in Γ_p. Then γ is non-crossing if and only if $\sigma(\gamma) = 0$.*

Proof. The proposition was first proved by D. Shlyakhtenko (cf. [Sh, Lemma 2.3]). His argument is based on diagrammatics (graphs). In [HT2, Corollary 1.24], the proposition was proved in the case where γ from Γ_p satisfies the additional property (2.5). The proof given in [HT2] was based on the process of cancelling pairs of neighbors in a permutation. This argument can with no further difficulty be generalized to all permutations in Γ_p, thus supplying an alternative proof of Proposition 2.16. For further details we refer to [HT2, Section 1]. ■

3 Almost Sure Convergence of Mixed Moments

Let n, r, p be positive integers, let $A_1, A_2, \dots, A_r$ be independent random matrices in $\mathrm{SGRM}(n, 1)$, and let $i_1, i_2, \dots, i_p$ be indices from $\{1, 2, \dots, r\}$. For each i in $\{1, 2, \dots, r\}$, put $X_i = \frac{1}{\sqrt{n}} A_i$. The main aim of this section is to show that the mixed moments $\mathrm{tr}_n[X_{i_1} X_{i_2} \cdots X_{i_p}]$ converge, almost surely, as $n \to \infty$, to the corresponding mixed moment $\tau\big(x_{i_1} x_{i_2} \cdots x_{i_p}\big)$, of a semi-circular family $\{x_1, x_2, \dots, x_r\}$ in a C^*-probability space $(\mathcal{B}, \tau)$. The key step in the argument leading to this result is Theorem 3.1 below, in which we determine a combinatorial expression for the mixed moment $\mathbb{E} \circ \mathrm{Tr}_n[A_{i_1} A_{i_2} \cdots A_{i_k}]$.

3.1 Theorem. *Let n, r, p be positive integers, let $A_1, A_2, \dots, A_r$ be independent elements of $\mathrm{SGRM}(n, 1)$, and let $i_1, i_2, \dots, i_p$ be indices from $\{1, 2, \dots, r\}$. For each i in $\{1, 2, \dots, r\}$, put*

$$K(i) = \big\{ j \in \{1, 2, \dots, p\} \bigm| i_j = i \big\}, \quad \text{and} \quad p_i = \mathrm{card}(K(i)), \tag{3.1}$$

and then define

$$\Gamma(i_1, i_2, \dots, i_p) = \begin{cases} \emptyset, & \text{if } p_i \text{ is odd for some } i, \\ \{\gamma \in \Gamma_{p/2} \mid \gamma(K(i)) = K(i), \text{ for all } i\}, & \text{if } p_i \text{ is even for all } i. \end{cases}$$

We then have

$$\mathbb{E} \circ \mathrm{Tr}_n\big[A_{i_1} A_{i_2} \cdots A_{i_p}\big] = \sum_{\gamma \in \Gamma(i_1, i_2, \dots, i_p)} n^{d(\gamma)}, \tag{3.2}$$

where the right hand side is to be thought of as 0, if p_i is odd for some i.

Note before the proof, that in the situation of Theorem 3.1, $p = p_1 + p_2 + \cdots + p_r$. In particular p is even if all p_i are, and hence, in this case, $\Gamma_{p/2}$ is well-defined.

Proof of Theorem 3.1. The proof proceeds along the same lines as the proof of Proposition 2.8.

Note first, that it follows from Lemma 2.5, that $\mathbb{E} \circ \mathrm{Tr}_n\big[A_{i_1}A_{i_2}\cdots A_{i_p}\big] = 0$, unless the random matrices $A_{i_1}, A_{i_2}, \dots, A_{i_p}$ can be divided into $\frac{p}{2}$ pairs of equal matrices. By definition of p_i, this is only possible if p_i is even for all i, and hence the validity of (3.2) follows immediately in the case where p_i is odd for some i.

Thus, we may assume in the following, that p_i is even for all i. For each i in $\{1, 2, \dots, r\}$, let $(A(i, s))_{s\in\mathbb{N}}$ be a sequence of random matrices from $\mathrm{SGRM}(n, 1)$, such that the random matrices in the set $\{A(i, s) \mid i \in \{1, 2, \dots, r\},\ s \in \mathbb{N}\}$ are jointly independent. It follows then from the convolution properties of the Gaussian distribution, that for any i in $\{1, 2, \dots, r\}$ and any s in $\mathbb{N}$, the random matrix

$$s^{-\frac{1}{2}}(A(i, 1) + A(i, 2) + \cdots + A(i, s)),$$

is again an element of $\mathrm{SGRM}(n, 1)$. Due to the independence assumptions, we have even that the joint distribution of the entries in $A_{i_1}, A_{i_2}, \dots, A_{i_k}$ equals that of the entries in

$$s^{-\frac{1}{2}}\big(A(i_1, 1)+\cdots+A(i_1, s)\big), s^{-\frac{1}{2}}\big(A(i_2, 1)+\cdots+A(i_2, s)\big), \dots, s^{-\frac{1}{2}}\big(A(i_p, 1)+\cdots+A(i_p, s)\big),$$

for any s in $\mathbb{N}$. This observation implies, in particular, that for any s in $\mathbb{N}$,

$$\begin{aligned}
&\mathbb{E} \circ \mathrm{Tr}_n[A_{i_1}A_{i_2}\cdots A_{i_p}] \\
&\qquad = s^{-\frac{p}{2}} \cdot \mathbb{E} \circ \mathrm{Tr}_n\big[\big(A(i_1, 1) + \cdots + A(i_1, s)\big)\ \cdots\ \big(A(i_p, 1) + \cdots + A(i_p, s)\big)\big] \\
&\qquad = s^{-\frac{p}{2}} \sum_{1\le l_1, l_2, \dots, l_p\le s} \mathbb{E} \circ \mathrm{Tr}_n\big[A(i_1, l_1)A(i_2, l_2)\cdots A(i_p, l_p)\big].
\end{aligned}$$

Note here, that it follows from Lemma 2.5, that for any $l_1, l_2, \dots, l_p$ in $\{1, 2, \dots, s\}$, we have $\mathbb{E} \circ \mathrm{Tr}_n\big[A(i_1, l_1)A(i_2, l_2)\cdots A(i_p, l_p)\big] = 0$, unless there exists a permutation γ in $\Gamma(i_1, i_2, \dots, i_p)$, such that $l_j = l_{\gamma(j)}$ for all j in $\{1, 2, \dots, p\}$. Hence, if we define for γ in $\Gamma(i_1, i_2, \dots, i_p)$,

$$M(\gamma, s) = \big\{(l_1, \dots, l_p) \in \{1, \dots, s\}^p \ \big|\ l_j = l_{\gamma(j)},\ j = 1, \dots, p\big\},$$

then we have that

$$\begin{aligned}
&\mathbb{E} \circ \mathrm{Tr}_n\big[A_{i_1}A_{i_2}\cdots A_{i_p}\big] \\
&\qquad = s^{-\frac{p}{2}} \sum_{(l_1,\dots,l_p)\in\bigcup_{\gamma\in\Gamma(i_1,\dots,i_p)} M(\gamma,s)} \mathbb{E} \circ \mathrm{Tr}_n\big[A(i_1, l_1)A(i_2, l_2)\cdots A(i_p, l_p)\big].
\end{aligned}$$

Note though, that the sets $M(\gamma, s)$, $\gamma \in \Gamma(i_1, i_2, \dots, i_p)$, are not disjoint. However, if we consider, for each γ in $\Gamma(i_1, i_2, \dots, i_p)$, the subset

$$\mathcal{D}(\gamma, s) = \big\{(l_1, \dots, l_p) \in \{1, \dots, s\}^p \ \big|\ \forall j, j' \in \{1, \dots, p\} \colon l_j = l_{j'} \Leftrightarrow j' = j \text{ or } j' = \gamma(j)\big\},$$

then the sets $\mathcal{D}(\gamma, s)$, $\gamma \in \Gamma(i_1, i_2, \dots, i_p)$, are disjoint, and even more so: For any distinct γ_1, γ_2 in $\Gamma(i_1, i_2, \dots, i_p)$, $M(\gamma_1, s) \cap \mathcal{D}(\gamma_2, s) = \emptyset$. It follows thus, that

$$\mathbb{E} \circ \mathrm{Tr}_n\big[A_{i_1} A_{i_2} \cdots A_{i_p}\big] = T_1(s) + T_2(s), \tag{3.3}$$

where

$$T_1(s) = s^{-\frac{p}{2}} \sum_{\gamma \in \Gamma(i_1, \dots, i_p)} \sum_{(l_1, \dots, l_p) \in \mathcal{D}(\gamma, s)} \mathbb{E} \circ \mathrm{Tr}_n\big[A(i_1, l_1) A(i_2, l_2) \cdots A(i_p, l_p)\big],$$

and

$$T_2(s) = s^{-\frac{p}{2}} \sum_{(l_1, \dots, l_p) \in \bigcup_{\gamma \in \Gamma(i_1, \dots, i_p)} M(\gamma, s) \setminus \mathcal{D}(\gamma, s)} \mathbb{E} \circ \mathrm{Tr}_n\big[A(i_1, l_1) A(i_2, l_2) \cdots A(i_p, l_p)\big].$$

Regarding $T_1(s)$, note that for any γ in $\Gamma(i_1, i_2, \dots, i_p)$ and any $(l_1, l_2, \dots, l_p)$ in $\mathcal{D}(\gamma, s)$, we have by definition of $\Delta(\gamma, n)$ (cf. Definition 2.7), that

$$\mathbb{E} \circ \mathrm{Tr}_n\big[A(i_1, l_1) A(i_2, l_2) \cdots A(i_p, l_p)\big] = \Delta(\gamma, n).$$

Therefore,

$$T_1(s) = s^{-\frac{p}{2}} \sum_{\gamma \in \Gamma(i_1, \dots, i_p)} \mathrm{card}(\mathcal{D}(\gamma, s)) \cdot \Delta(\gamma, n).$$

Here, $\mathrm{card}(\mathcal{D}(\gamma, s)) = s(s-1)(s-2) \cdots (s - \frac{p}{2} + 1)$, so that $s^{-\frac{p}{2}} \mathrm{card}(\mathcal{D}(\gamma, s)) \to 1$, as $s \to \infty$. It follows thus, that

$$T_1(s) \to \sum_{\gamma \in \Gamma(i_1, \dots, i_p)} \Delta(\gamma, n), \quad \text{as } s \to \infty. \tag{3.4}$$

Regarding $T_2(s)$, consider the constant $K(n, p, 1)$ introduced in Remark 2.4, and note then that

$$\begin{aligned} |T_2(s)| &\le s^{-\frac{p}{2}} \cdot \mathrm{card}\big(\textstyle\bigcup_{\gamma \in \Gamma(i_1, \dots, i_p)} M(\gamma, s) \setminus \mathcal{D}(\gamma, s)\big) \cdot K(n, p, 1) \\ &\le s^{-\frac{p}{2}} \cdot K(n, p, 1) \cdot \sum_{\gamma \in \Gamma(i_1, \dots, i_p)} \mathrm{card}(M(\gamma, s) \setminus \mathcal{D}(\gamma, s)). \end{aligned}$$

For any γ in $\Gamma(i_1, i_2, \dots, i_p)$,

$$\begin{aligned} s^{-\frac{p}{2}} \cdot \mathrm{card}(M(\gamma, s) \setminus \mathcal{D}(\gamma, s)) &= s^{-\frac{p}{2}} \big(\mathrm{card}(M(\gamma, s)) - \mathrm{card}(\mathcal{D}(\gamma, s))\big) \\ &= 1 - s^{-\frac{p}{2}} \mathrm{card}(\mathcal{D}(\gamma, s)) \\ &\to 1 - 1 = 0, \end{aligned}$$

as $s \to \infty$. It follows thus, that

$$T_2(s) \to 0, \quad \text{as } s \to \infty. \tag{3.5}$$

Combining (3.3) with (3.4) and (3.5), we obtain that

$$\mathbb{E}\circ \mathrm{Tr}_n\big[A_{i_1}A_{i_2}\cdots A_{i_p}\big] = T_1(s)+T_2(s) \to \sum_{\gamma\in\Gamma(i_1,\dots,i_p)} \Delta(\gamma,n), \quad \text{as } s\to\infty. \tag{3.6}$$

Since the right hand side of (3.6) does not depend on s, we conclude that

$$\mathbb{E}\circ \mathrm{Tr}_n\big[A_{i_1}A_{i_2}\cdots A_{i_p}\big] = \sum_{\gamma\in\Gamma(i_1,\dots,i_p)} \Delta(\gamma,n) = \sum_{\gamma\in\Gamma(i_1,\dots,i_p)} n^{d(\gamma)},$$

where the last equality follows from Theorem 2.13. ■

3.2 Corollary. *Let $n,r,p,\ i_1,i_2,\dots,i_p$, and $\Gamma(i_1,i_2,\dots,i_p)$ be as in Theorem 3.1, but consider now a family $\{X_1,X_2,\dots,X_r\}$ of independent random matrices from* $\mathrm{SGRM}(n,\frac{1}{n})$. *We then have:*

$$\begin{aligned}\mathbb{E}\circ \mathrm{tr}_n\big[X_{i_1}X_{i_2}\cdots X_{i_p}\big] &= \sum_{\gamma\in\Gamma(i_1,\dots,i_p)} n^{-2\sigma(\gamma)} \\ &= \mathrm{card}(\{\gamma\in\Gamma(i_1,\dots,i_p)\mid \gamma \textit{ is non-crossing }\}) + O\left(\tfrac{1}{n^2}\right).\end{aligned} \tag{3.7}$$

Proof. For each i in $\{1,2,\dots,r\}$, put $A_i=\sqrt{n}X_i$, and note then that $A_1,A_1,\dots,A_r$ are independent elements of $\mathrm{SGRM}(n,1)$. By Theorem 3.1, it follows thus that

$$\begin{aligned}\mathbb{E}\circ \mathrm{tr}_n\big[X_{i_1}X_{i_2}\cdots X_{i_p}\big] &= n^{-\frac{p}{2}-1}\cdot \mathbb{E}\circ \mathrm{Tr}_n\big[A_{i_1}A_{i_2}\cdots A_{i_p}\big] = n^{-\frac{p}{2}-1}\sum_{\gamma\in\Gamma(i_1,\dots,i_p)} n^{d(\gamma)} \\ &= \sum_{\gamma\in\Gamma(i_1,\dots,i_p)} n^{-(\frac{p}{2}+1-d(\gamma))} = \sum_{\gamma\in\Gamma(i_1,\dots,i_p)} n^{-2\sigma(\gamma)},\end{aligned}$$

which proves the first equality in (3.7). Regarding the second equality in (3.7), let p_i be as defined in (3.1). Then if p_i is odd for some i in $\{1,2,\dots,r\}$, $\Gamma(i_1,i_2,\dots,i_p)=\emptyset$, and the equality follows trivially. Otherwise, we get the second equality by application of Propositions 2.15 and 2.16. ■

3.3 Corollary. *Let $n,r,p,\ i_1,i_2,\dots,i_p$, $\Gamma(i_1,i_2,\dots,i_p)$ be as in Theorem 3.1, and let $\{X_1,X_2,\dots,X_r\}$ be a family of independent random matrices from* $\mathrm{SGRM}(n,\frac{1}{n})$. *Consider further a semi-circular family $\{x_1,x_2,\dots,x_r\}$ in some C^*-probability space $(\mathcal{B},\tau)$. We then have*

(i) $$\mathrm{card}(\{\gamma\in\Gamma(i_1,i_2,\dots,i_p)\mid \gamma \textit{ is non-crossing }\}) = \tau(x_{i_1}x_{i_2}\cdots x_{i_p}).$$

(ii) $$\mathbb{E}\circ \mathrm{tr}_n\big[X_{i_1}X_{i_2}\cdots X_{i_p}\big] = \tau(x_{i_1}x_{i_2}\cdots x_{i_p}) + O\left(\tfrac{1}{n^2}\right).$$

Proof. It follows from [Vo1, Theorem 2.2], that

$$\mathbb{E}\circ \mathrm{tr}_n\big[X_{i_1}X_{i_2}\cdots X_{i_p}\big] \to \tau(x_{i_1}x_{i_2}\cdots x_{i_p}), \quad \text{as } n\to\infty, \tag{3.8}$$

and combining this with Corollary 3.2, it follows immediately that (i) holds. Plugging then (i) back into (3.7), we get (ii). ■

3.4 Remark. The equation (i) in Corollary 3.3 can also be derived from the "Moment-cumulant formula" of A. Nica and R. Speicher (cf. [NS, 3.5]), recalling that the multi-dimensional R-transform of (the joint distribution of) a semi-circular system $\{x_1, \dots, x_r\}$ is given by:

$$R(z_1, \dots, z_r) = z_1^2 + \cdots + z_r^2,$$

(cf. [NS, Theorem 3.6]). The proof given above of the formula proceeds by combining Voiculescu's celebrated result (3.8) with the combinatorial expression obtained in Corollary 3.2. On the other hand, since neither the Moment-cumulant formula nor the above proof of (3.7), depend upon (3.8), one can obtain a new proof of (3.8), by combining (3.7) and (i). $\square$

Before we can conclude this section by proving almost sure convergence of mixed moments of independent elements of $\mathrm{SGRM}(n, \frac{1}{n})$, we need the following observation

3.5 Lemma. *Let A be an element of $\mathrm{SGRM}(n,1)$, and let F, G be random $n \times n$ matrices, whose entries have moments of all orders and are independent of those of A. Then*

$$\mathbb{E} \circ \mathrm{Tr}_n[AFAG] = \mathbb{E}\big[\mathrm{Tr}_n(F) \cdot \mathrm{Tr}_n(G)\big].$$

Proof. Let $a(i,j), f(i,j), g(i,j)$, $1 \le i, j \le n$, denote the entries of A, F, G respectively, and let $e(i,j)$, $1 \le i, j \le n$, denote the usual $n \times n$ matrix units. By a standard matrix calculation, the independence assumptions and (2.2), it follows then that

$$\begin{aligned}
&\mathbb{E} \circ \mathrm{Tr}_n[AFAG] \\
&= \sum_{\substack{1 \le i_1, \dots, i_4 \le n \\ 1 \le j_1, \dots, j_4 \le n}} \mathbb{E}\big[a(i_1,j_1) f(i_2,j_2) a(i_3,j_3) g(i_4,j_4)\big] \cdot \mathrm{Tr}_n\big[e(i_1,j_1)e(i_2,j_2)e(i_3,j_3)e(i_4,j_4)\big] \\
&= \sum_{1 \le i_1, \dots, i_4 \le n} \mathbb{E}\big[a(i_1,i_2)a(i_3,i_4)\big] \cdot \mathbb{E}\big[f(i_2,i_3)g(i_4,i_1)\big] \\
&= \mathbb{E}\Big[\sum_{1 \le i_1, i_2 \le n} f(i_2,i_2) g(i_1,i_1)\Big] \\
&= \mathbb{E}\big[\mathrm{Tr}_n(F) \cdot \mathrm{Tr}_n(G)\big],
\end{aligned}$$

as desired. $\blacksquare$

3.6 Theorem. *For each n in $\mathbb{N}$, let $X(1,n), X(2,n), \dots, X(r,n)$ be independent elements of $\mathrm{SGRM}(n, \frac{1}{n})$. Moreover, let Q be a non-commutative polynomial in r variables and let $x_1, x_2, \dots, x_r$ be a semi-circular system in a non-commutative probability space $(\mathcal{B}, \tau)$. Then, as $n \to \infty$,*

$$\mathrm{tr}_n\big[Q(X(1,n), X(2,n), \dots, X(r,n))\big] \to \tau\big[Q(x_1, x_2, \dots, x_r)\big], \quad \textit{almost surely.}$$

Proof. By linearity we may assume that Q is a non-commutative monomial, so that

$$\begin{aligned}
Q\big(X(1,n), X(2,n), \dots, X(r,n)\big) &= X(i_1,n)X(i_2,n)\cdots X(i_p,n), \\
Q(x_1, x_2, \dots, x_r) &= x_{i_1} x_{i_2} \cdots x_{i_p},
\end{aligned}$$

for suitable p in $\mathbb{N}$, and $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$. For the sake of short notation, in the following we shall just write Q_n for $Q\big(X(1,n), X(2,n), \dots, X(r,n)\big)$.

From [Vo1, Theorem 2.2] or (ii) in Corollary 3.3, it follows that

$$\mathbb{E} \circ \mathrm{tr}_n(Q_n) \to \tau\big(Q(x_1, x_2, \dots, x_r)\big), \quad \text{as } n \to \infty.$$

Therefore it suffices to show that

$$\mathrm{tr}_n(Q_n) - \mathbb{E} \circ \mathrm{tr}_n(Q_n) \overset{\text{a.s.}}{\to} 0, \quad \text{as } n \to \infty.$$

By the Borel-Cantelli Lemma (cf. for example [Br, Lemma 3.14]), this will follow if we show that for any ϵ in $]0, \infty[$, we have that

$$\sum_{n=1}^{\infty} P\big(|\mathrm{tr}_n(Q_n) - \mathbb{E} \circ \mathrm{tr}_n(Q_n)| > \epsilon\big) < \infty.$$

This, in turn, follows from the Chebychev Inequality, if we show that

$$\sum_{n=1}^{\infty} \mathbb{E}\big(\big|\mathrm{tr}_n(Q_n) - \mathbb{E} \circ \mathrm{tr}_n(Q_n)\big|^2\big) < \infty. \tag{3.9}$$

Note here that

$$\mathbb{E}\big(\big|\mathrm{tr}_n(Q_n) - \mathbb{E} \circ \mathrm{tr}_n(Q_n)\big|^2\big) = \mathbb{E}\big(\big|\mathrm{tr}_n(Q_n)\big|^2\big) - \big|\mathbb{E} \circ \mathrm{tr}_n(Q_n)\big|^2.$$

We consider first the quantity $\big|\mathbb{E} \circ \mathrm{tr}_n(Q_n)\big|^2$. Recalling the form of Q_n, it follows immediately by application of (ii) in Corollary 3.3, that

$$\mathbb{E} \circ \mathrm{tr}_n(Q_n) = \tau\big(Q(x_1, x_2, \dots, x_r)\big) + O\left(\tfrac{1}{n^2}\right),$$

so that, in particular,

$$\big|\mathbb{E} \circ \mathrm{tr}_n(Q_n)\big|^2 = \big|\tau\big(Q(x_1, x_2, \dots, x_r)\big)\big|^2 + O\left(\tfrac{1}{n^2}\right).$$

Turning next to the quantity $\mathbb{E}\big(\big|\mathrm{tr}_n(Q_n)\big|^2\big)$, we introduce for each n in $\mathbb{N}$ an element Z_n of $\mathrm{SGRM}(n, \frac{1}{n})$, satisfying that Z_n is independent of $X(1,n), X(2,n), \dots, X(r,n)$. By Lemma 3.5 (and a normalization consideration), it follows then that

$$\mathbb{E}\big(\big|\mathrm{tr}_n(Q_n)\big|^2\big) = \mathbb{E}\big(\mathrm{tr}_n(Q_n) \cdot \mathrm{tr}_n(Q_n^*)\big) = \mathbb{E} \circ \mathrm{tr}_n(Z_n Q_n Z_n Q_n^*).$$

By another application of Corollary 3.3(ii), it follows thus that

$$\mathbb{E}\big(\big|\mathrm{tr}_n(Q_n)\big|^2\big) = \tau\big(zQ(x_1, x_2, \dots, x_r) zQ(x_1, x_2, \dots, x_r)^*\big) + O\left(\tfrac{1}{n^2}\right),$$

where z is an element of $(\mathcal{B}, \tau)$, satisfying that $\{x_1, x_2, \dots, x_r, z\}$ is a semi-circular family in $(\mathcal{B}, \tau)$.

Taken together, we have now realized that

$$\begin{aligned}
&\mathbb{E}\big(\big|\mathrm{tr}_n(Q_n) - \mathbb{E}\circ\mathrm{tr}_n(Q_n)\big|^2\big)\\
&\quad = \mathbb{E}\big(\big|\mathrm{tr}_n(Q_n)\big|^2\big) - \big|\mathbb{E}\circ\mathrm{tr}_n(Q_n)\big|^2\\
&\quad = \tau\big(zQ(x_1,x_2,\dots,x_r)zQ(x_1,x_2,\dots,x_r)^*\big) - \big|\tau\big(Q(x_1,x_2,\dots,x_r)\big)\big|^2 + O\left(\tfrac{1}{n^2}\right),
\end{aligned}$$

and thus the validity of (3.9) depends on that of the equality

$$\tau\big(zQ(x_1,x_2,\dots,x_r)zQ(x_1,x_2,\dots,x_r)^*\big) = \big|\tau\big(Q(x_1,x_2,\dots,x_r)\big)\big|^2.$$

But this equation follows by a standard calculation, using that $Q(x_1,x_2,\dots,x_r)$ and z are free, and that $\tau(z)=0$, $\tau(z^2)=1$. ■

4 Almost Sure Convergence for Gaussian Random Matrices with Operator Entries

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let r be a positive integer, let $a_1,a_2,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$ and let $\mathcal{A}$ denote the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the set $\{a_i^*a_j \mid 1\le i,j\le r\}\cup\{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$. Furthermore, let t be a positive integer, let $x_1,x_2,\dots,x_t$ be a semi-circular family in some C^*-probability space $(\mathcal{B},\tau)$, and for each n in $\mathbb{N}$, let $X(1,n),X(2,n),\dots,X(t,n)$ be independent elements of $\mathrm{SGRM}(n,\frac{1}{n})$. Finally, let $Q_1,Q_2,\dots,Q_r$ be non-commutative polynomials in t variables, and then define

$$\begin{aligned}
Q(i,n) &= Q_i\big(X(1,n),X(2,n),\dots,X(t,n)\big), \quad (i\in\{1,2,\dots,r\},\ n\in\mathbb{N})\\
q_i &= Q_i\big(x_1,x_2,\dots,x_t\big), \quad (i\in\{1,2,\dots,r\})\\
S_n &= \sum_{i=1}^r a_i\otimes Q(i,n), \quad (n\in\mathbb{N})\\
s &= \sum_{i=1}^r a_i\otimes q_i.
\end{aligned}$$

Note that for each n, $S_n^*S_n$ is a random element of $\mathcal{A}\otimes M_n(\mathbb{C})$, and similarly s^*s is a (non-random) element of the algebraic tensor product $\mathcal{A}\odot\mathcal{B}$. In the first part of this section, we use the results of Section 3 to study, for any state ϕ on $\mathcal{A}$, the asymptotic behavior of $S_n^*S_n$ w.r.t. $\phi\otimes\mathrm{tr}_n$. This is used, in the second part of the section, to obtain asymptotic lower (respectively upper) bounds on the largest (respectively smallest) element of the spectrum of $S_n^*S_n$.

4.1 Lemma. *Let the situation be as described above, and let ϕ be a state on $\mathcal{A}$. Then for any p in $\mathbb{N}$, we have*

(i) $\phi\otimes\mathrm{tr}_n\big((S_n^*S_n)^p\big)\to\phi\otimes\tau\big((s^*s)^p\big)$, *almost surely, as* $n\to\infty$.

(ii) $\mathbb{E}\big[\phi\otimes\mathrm{tr}_n\big((S_n^*S_n)^p\big)\big]\to\phi\otimes\tau\big((s^*s)^p\big)$, *as* $n\to\infty$.

Proof. (i) For any positive integer p, we have that

$$\begin{aligned}
&\phi \otimes \operatorname{tr}_n\big((S_n^* S_n)^p\big) \\
&\quad = \phi \otimes \operatorname{tr}_n\Bigg(\sum_{1\le i_1, j_1, \dots, i_p, j_p \le r} a_{i_1}^* a_{j_1} \cdots a_{i_p}^* a_{j_p} \otimes Q(i_1,n)^* Q(j_1,n) \cdots Q(i_p,n)^* Q(j_p,n)\Bigg) \\
&\quad = \sum_{1\le i_1, j_1, \dots, i_p, j_p \le r} \phi\big(a_{i_1}^* a_{j_1} \cdots a_{i_p}^* a_{j_p}\big) \cdot \operatorname{tr}_n\big(Q(i_1,n)^* Q(j_1,n) \cdots Q(i_p,n)^* Q(j_p,n)\big).
\end{aligned}$$

Note here that for each $2p$-tuple $(i_1, j_1, \dots, i_p, j_p)$ in the set $\{1, 2, \dots, r\}^{2p}$, the product $Q(i_1,n)^* Q(j_1,n) \cdots Q(i_p,n)^* Q(j_p,n)$ is some polynomial in $X(1,n), X(2,n), \dots, X(t,n)$, and hence by Theorem 3.6, we have that

$$\operatorname{tr}_n\big(Q(i_1,n)^* Q(j_1,n) \cdots Q(i_p,n)^* Q(j_p,n)\big) \xrightarrow[n\to\infty]{\text{a.s.}} \tau\big(q_{i_1}^* q_{j_1} \cdots q_{i_p}^* q_{j_p}\big).$$

It follows thus, that

$$\begin{aligned}
\phi \otimes \operatorname{tr}_n\big((S_n^* S_n)^p\big) &\xrightarrow[n\to\infty]{\text{a.s.}} \sum_{1\le i_1, j_1, \dots, i_p, j_p \le r} \phi\big(a_{i_1}^* a_{j_1} \cdots a_{i_p}^* a_{j_p}\big) \cdot \tau\big(q_{i_1}^* q_{j_1} \cdots q_{i_p}^* q_{j_p}\big) \\
&= \phi \otimes \tau\big((s^* s)^p\big),
\end{aligned}$$

and this completes the proof of (i).

(ii) As in the proof of (i), we find, for any p in $\mathbb{N}$, that

$$\begin{aligned}
&\mathbb{E}\big[\phi \otimes \operatorname{tr}_n\big((S_n^* S_n)^p\big)\big] \\
&\quad = \sum_{1\le i_1, j_1, \dots, i_p, j_p \le r} \phi\big(a_{i_1}^* a_{j_1} \cdots a_{i_p}^* a_{j_p}\big) \cdot \mathbb{E} \circ \operatorname{tr}_n\big(Q(i_1,n)^* Q(j_1,n) \cdots Q(i_p,n)^* Q(j_p,n)\big).
\end{aligned}$$

It follows here from [Vo1, Theorem 2.2] or (ii) in Corollary 3.3, that

$$\mathbb{E} \circ \operatorname{tr}_n\big(Q(i_1,n)^* Q(j_1,n) \cdots Q(i_p,n)^* Q(j_p,n)\big) \to \tau\big(q_{i_1}^* q_{j_1} \cdots q_{i_p}^* q_{j_p}\big), \qquad \text{as } n \to \infty,$$

and hence, as in (i), we obtain the desired conclusion. ■

We shall need next, a result from (classical) probability theory. Before stating this result, we recall that a measure μ on $\mathbb{R}$ (with Borel σ-algebra) is called *determinated*, if no other measure on $\mathbb{R}$ has the same moments as μ. All compactly supported measures on $\mathbb{R}$ are determinated. For a more general criteria, cf. [Br, Proposition 8.49]. We recall also, that a sequence (μ_n) of measures on $\mathbb{R}$ is said to *converge weakly* to another measure μ on $\mathbb{R}$, if $\int_{\mathbb{R}} f \, d\mu_n \to \int_{\mathbb{R}} f \, d\mu$, for any function f in $C_b(\mathbb{R}, \mathbb{C})$, the set of continuous, bounded, complex valued functions on $\mathbb{R}$.

4.2 Proposition. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ (with Borel σ-algebra), and let μ be yet another probability on $\mathbb{R}$. Assume that μ and all the μ_n's have moments of all orders, and that μ is determinated. Then if the moments of μ_n converge, as $n \to \infty$, to those of μ, μ_n converges weakly to μ, as $n \to \infty$.*

Proof. Cf. [Br, Theorem 8.48]. ■

Recall that for a selfadjoint element a of a C^*-probability space $(\mathcal{C},\psi)$, there exists a unique probability measure μ_a on $\mathbb{R}$, such that $\mathrm{supp}(\mu_a)\subseteq \mathrm{sp}(a)$, and

$$\int_{\mathrm{sp}(a)} t^p \; d\mu_a(t)=\psi(a^p), \qquad (p\in\mathbb{N}). \tag{4.1}$$

The probability measure μ_a is called the *distribution* of a w.r.t. ψ. Note that there is no ambiguity between this definition and the definition of distribution in Section 1, since μ_a is the unique probability measure on $\mathrm{sp}(a)$, carrying as its moments, the moments of a w.r.t. ψ. The existence and uniqueness of μ_a follow by application of Weierstrass' approximation theorem and Riesz' representation theorem. Note in particular, that (4.1) extends to all continuous functions on $\mathrm{sp}(a)$.

4.3 Theorem. *Let $a_1,\dots,a_r$, $\mathcal{A}$, $X(1,n),\dots,X(t,n)$, $x_1,\dots,x_t$, $(\mathcal{B},\tau)$, S_n and s be as set out in the paragraph preceding Lemma 4.1, and let ϕ be a state on $\mathcal{A}$. Then the set*

$$\mathcal{S}=\Big\{\omega\in\Omega \;\Big|\; \forall f\in C_b(\mathbb{R},\mathbb{C}) : \phi\otimes\mathrm{tr}_n\big[f(S_n^*(\omega)S_n(\omega))\big] \underset{n\to\infty}{\longrightarrow} \phi\otimes\tau\big(f(s^*s)\big)\Big\},$$

is a (generalized) sure event.

Proof. From Lemma 4.1, it follows that the set

$$\mathcal{R}=\bigcap_{p\in\mathbb{N}_0}\Big\{\omega\in\Omega \;\Big|\; \phi\otimes\mathrm{tr}_n\big[\big(S_n^*(\omega)S_n(\omega)\big)^p\big] \underset{n\to\infty}{\longrightarrow} \phi\otimes\tau\big((s^*s)^p\big)\Big\},$$

has probability 1. Therefore it suffices to show that $\mathcal{R}\subseteq\mathcal{S}$.

For this, let ω be an element of $\mathcal{R}$, and for each n in $\mathbb{N}$, let $\mu_{\omega,n}$ denote the distribution of $S_n^*(\omega)S_n(\omega)$ w.r.t. the state $\phi\otimes\mathrm{tr}_n$ on the C^*-algebra $\mathcal{A}\otimes M_n(\mathbb{C})$. Similarly, consider the distribution μ of s^*s w.r.t. the state $\phi\otimes\tau$ on the minimal C^*-algebra tensor product $\mathcal{A}\otimes\mathcal{B}$. Then the assumption that $\omega\in\mathcal{R}$ means exactly that the moments of $\mu_{\omega,n}$ converge, as $n\to\infty$, to those of μ. Since μ is compactly supported, this implies, by Proposition 4.2, that $\mu_{\omega,n}\to\mu$, weakly, as $n\to\infty$, i.e., that

$$\phi\otimes\mathrm{tr}_n\big(f(S_n^*(\omega)S_n(\omega))\big)=\int_{\mathbb{R}} f(u)\; d\mu_{\omega,n}(u) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(u)\; d\mu(u)=\phi\otimes\tau\big(f(s^*s)\big),$$

for any f in $C_b(\mathbb{R},\mathbb{C})$. This shows that $\omega\in\mathcal{S}$. ■

In (the proof of) Proposition 4.5 below, we apply Theorem 4.3 to obtain lower (respectively upper) asymptotic bounds on the largest (respectively smallest) element of the spectrum $\mathrm{sp}(S_n^*S_n)$ of $S_n^*S_n$. For the proof of Proposition 4.5, we need, in addition, the following lemma. This result is presumably well-known, but for completeness, we include a proof in the appendix at the end of the paper.

4.4 Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be C^*-algebras, and let ϕ,ψ be states on $\mathcal{C}$ and $\mathcal{D}$ respectively. If ϕ and ψ are both faithful, then so is the product state $\phi\otimes\psi$ on the minimal (= spatial) tensor product $\mathcal{C}\otimes\mathcal{D}$.* □

4.5 Proposition. *Let $a_1, \dots, a_r$, $\mathcal{A}$, $X(1,n), \dots, X(t,n)$, $x_1, \dots, x_t$, $(\mathcal{B}, \tau)$, S_n and s be as set out in the paragraph preceding Lemma 4.1, and assume in addition that τ is faithful on $\mathcal{B}$. Then on a set with probability 1, we have that*

$$\liminf_{n\to\infty} \big(\max \big\{ \mathrm{sp}(S_n^* S_n) \big\} \big) \geq \max \big\{ \mathrm{sp}(s^* s) \big\},$$

and that

$$\limsup_{n\to\infty} \big(\min \big\{ \mathrm{sp}(S_n^* S_n) \big\} \big) \leq \min \big\{ \mathrm{sp}(s^* s) \big\}.$$

Proof. Consider the sets

$$\mathcal{T} = \Big\{ \omega \in \Omega \;\Big|\; \liminf_{n\to\infty} \big(\max \big\{ \mathrm{sp}\big(S_n^*(\omega) S_n(\omega)\big) \big\} \big) \geq \max \big\{ \mathrm{sp}\big(s^* s\big) \big\} \Big\}$$

and

$$\mathcal{U} = \Big\{ \omega \in \Omega \;\Big|\; \limsup_{n\to\infty} \big(\min \big\{ \mathrm{sp}\big(S_n^*(\omega) S_n(\omega)\big) \big\} \big) \leq \min \big\{ \mathrm{sp}\big(s^* s\big) \big\} \Big\}.$$

We have to show that these sets are (generalized) sure events. For this, note first that since $\mathcal{A}$ is clearly separable, we may choose a faithful state ϕ on $\mathcal{A}$. With this ϕ, it follows then from Theorem 4.3, that the set

$$\mathcal{S} = \Big\{ \omega \in \Omega \;\Big|\; \forall f \in C_b(\mathbb{R}, \mathbb{C}) : \phi \otimes \mathrm{tr}_n \big[f(S_n^*(\omega) S_n(\omega)) \big] \underset{n\to\infty}{\longrightarrow} \phi \otimes \tau \big(f(s^* s) \big) \Big\},$$

is a (generalized) sure event, and it suffices thus to show that both of the sets $\mathcal{T}$ and $\mathcal{U}$ contain $\mathcal{S}$. We shall only prove that $\mathcal{S} \subseteq \mathcal{T}$. The inclusion $\mathcal{S} \subseteq \mathcal{U}$, is proved similarly.

So consider an element ω of $\mathcal{S}$, and assume also (seeking a contradiction) that $\omega \notin \mathcal{T}$, i.e., that

$$\liminf_{n\to\infty} \big(\max \big\{ \mathrm{sp}(S_n^*(\omega) S_n(\omega)) \big\} \big) = \max \big\{ \mathrm{sp}(s^* s) \big\} - \epsilon,$$

for some ϵ in $]0, \infty[$. Put $m = \max \big\{ \mathrm{sp}(s^* s) \big\}$, and then for the considered ϵ, choose a function h in $C_b(\mathbb{R}, \mathbb{C})$, such that $h(u) \geq 0$ for all u in $\mathbb{R}$, such that $\{u \in \mathbb{R} \mid h(u) > 0\} \subseteq [m - \frac{\epsilon}{2}, \infty[$ and such that $\{u \in \mathbb{R} \mid h(u) > 0\} \cap \mathrm{sp}(s^* s) \neq \emptyset$.

Now $h(s^* s)$ is a non-zero, positive element of the minimal C^*-tensor product $\mathcal{A} \otimes \mathcal{B}$ and since $\phi \otimes \tau$ is faithful on $\mathcal{A} \otimes \mathcal{B}$ (since ϕ and τ are both faithful; cf. Lemma 4.4), the number $\delta := \phi \otimes \tau[h(s^* s)]$ is strictly positive.

Since $\omega \in \mathcal{S}$ and $h \in C_b(\mathbb{R}, \mathbb{C})$, we have here that

$$\phi \otimes \mathrm{tr}_n \big[h(S_n^*(\omega) S_n(\omega)) \big] \to \delta, \quad \text{as } n \to \infty,$$

and thus we may choose N in $\mathbb{N}$ such that

$$\phi \otimes \mathrm{tr}_n \big[h(S_n^*(\omega) S_n(\omega)) \big] \geq \tfrac{\delta}{2}, \quad \text{whenever } n \geq N. \tag{4.2}$$

On the other hand we have that

$$\inf_{n\geq N} \big(\max \big\{ \mathrm{sp}(S_n^*(\omega) S_n(\omega)) \big\} \big) \leq \liminf_{n\to\infty} \big(\max \big\{ \mathrm{sp}(S_n^*(\omega) S_n(\omega)) \big\} \big) = m - \epsilon,$$

and thus we may choose n_0 in $\mathbb{N}$, such that $n_0 \geq N$ and

$$\max\left\{\mathrm{sp}(S_{n_0}^*(\omega)S_{n_0}(\omega))\right\} < m - \tfrac{\epsilon}{2}.$$

But since $\mathrm{supp}(h) \subseteq [m - \frac{\epsilon}{2}, \infty[$, this implies that $h = 0$ on $\mathrm{sp}(S_{n_0}^*(\omega)S_{n_0}(\omega))$, and hence $h(S_{n_0}^*(\omega)S_{n_0}(\omega)) = 0$. Since $n_0 \geq N$, this contradicts (4.2), and hence we have obtained the desired contradiction. ■

By minor modifications of the proof just given, we can obtain a slightly stronger result:

4.6 Proposition. *Let $a_1, \dots, a_r$, $\mathcal{A}$, $X(1,n), \dots, X(t,n)$, $x_1, \dots, x_t$, $(\mathcal{B}, \tau)$, S_n and s be as set out in the paragraph preceding Lemma 4.1, and assume in addition that τ is faithful on $\mathcal{B}$. Then with*

$$\mathcal{T} = \Big\{\omega \in \Omega \;\Big|\; \forall g \in C(\mathbb{R},\mathbb{R}) : \liminf_{n\to\infty} \big(\max\big\{\mathrm{sp}\big(g[S_n^*(\omega)S_n(\omega)]\big)\big\}\big) \geq \max\big\{\mathrm{sp}\big(g(s^*s)\big)\big\}\Big\}$$

and

$$\mathcal{U} = \Big\{\omega \in \Omega \;\Big|\; \forall g \in C(\mathbb{R},\mathbb{R}) : \limsup_{n\to\infty} \big(\min\big\{\mathrm{sp}\big(g[S_n^*(\omega)S_n(\omega)]\big)\big\}\big) \leq \min\big\{\mathrm{sp}\big(g(s^*s)\big)\big\}\Big\},$$

we have that $\mathcal{T} = \mathcal{U}$, and moreover, this set is a (generalized) sure event.

Proof. The equation $\mathcal{T} = \mathcal{U}$ follows by noting the following 3 facts:

$$C(\mathbb{R},\mathbb{R}) = \big\{ -g \mid g \in C(\mathbb{R},\mathbb{R})\big\}$$
$$\forall g \in C(\mathbb{R},\mathbb{R}) : \liminf_{n\to\infty} \big(\max\big\{\mathrm{sp}(-g(S_n^*S_n))\big\}\big) = -\limsup_{n\to\infty} \big(\min\big\{\mathrm{sp}(g(S_n^*S_n))\big\}\big)$$
$$\forall g \in C(\mathbb{R},\mathbb{R}) : \max\big\{\mathrm{sp}(-g(s^*s))\big\} = -\min\big\{\mathrm{sp}(g(s^*s))\big\}.$$

It remains thus to show that $\mathcal{T}$ is a (generalized) sure event. This can be proved by a slight modification of the argument given in the proof of Proposition 4.5. ■

4.7 Corollary. *Let $a_1, \dots, a_r$, $\mathcal{A}$, $X(1,n), \dots, X(t,n)$, $x_1, \dots, x_t$, $(\mathcal{B}, \tau)$, S_n and s be as set out in the paragraph preceding Lemma 4.1, and assume in addition that τ is faithful on $\mathcal{B}$. Then on a set with probability 1, we have that*

$$\forall g \in C(\mathbb{R},\mathbb{C}) : \liminf_{n\to\infty} \|g(S_n^*S_n)\| \geq \|g(s^*s)\|. \tag{4.3}$$

Proof. For any g in $C(\mathbb{R},\mathbb{C})$, $|g| \in C(\mathbb{R},\mathbb{R})$, and therefore the statement (4.3) holds for all ω in the set $\mathcal{T}$ from Proposition 4.6. ■

4.8 Remark. Consider the situation described in the paragraph preceding Lemma 4.1. Instead of the family $X(1,n), X(2,n), \dots, X(t,n)$ in $\mathrm{SGRM}(n, \frac{1}{n})$ we might consider a family $Y(1,n), Y(2,n), \dots, Y(t,n)$ in $\mathrm{GRM}(n, \frac{1}{n})$, and instead of the semi-circular family $x_1, x_2, \dots, x_t$, we might consider a circular family $y_1, y_2, \dots, y_t$. Defining then S_n and s as before, but with $X(j,n)$ and x_j replaced by $Y(j,n)$ respectively y_j, we claim that the results 4.1–4.6 are still valid. This follows by recalling from Remark 2.3, that with $X'(j,n) = 2^{-1/2}(Y(j,n) + Y^*(j,n))$ and $X''(j,n) = -i2^{-1/2}(Y(j,n) - Y^*(j,n))$, we have that $X'(1,n), X''(1,n), \dots, X'(t,n), X''(t,n)$ form a family of $2t$ independent elements of $\mathrm{SGRM}(n, \frac{1}{n})$. And similarly, with $x'_j = 2^{-1/2}(y_j + y_j^*)$ and $x''_j = -i2^{-1/2}(y_j - y_j^*)$, the family $x'_1, x''_1, \dots, x'_t, x''_t$ is a semi-circular family. The claim now follows by rewriting the polynomials $Q(Y(1,n), Y(2,n), \dots, Y(r,n))$ and $Q(y_1, y_2, \dots, y_r)$ as some other polynomials of the variables $X'(1,n), X''(1,n), \dots, X'(t,n), X''(t,n)$ respectively $x'_1, x''_1, \dots, x'_t, x''_t$, via the equations: $Y(j,n) = 2^{-1/2}(X'(j,n) + iX''(j,n))$ and $y_j = 2^{-1/2}(x'_j + ix''_j)$. □

5 A Strengthening of the Main Result of [HT2]

Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces and let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$. Assume that

$$\Big\| \sum_{i=1}^r a_i^* a_i \Big\| \le c \qquad \text{and} \qquad \Big\| \sum_{i=1}^r a_i a_i^* \Big\| \le 1, \tag{5.1}$$

for some constant c in $]0, \infty[$, and that the C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ generated by the set $\{a_i^* a_j \mid 1 \le i, j \le r\} \cup \{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$, is exact. Consider furthermore, for each n in $\mathbb{N}$, independent elements $Y(1,n), \dots, Y(r,n)$ of $\mathrm{GRM}(n, \frac{1}{n})$, and define

$$S_n = \sum_{i=1}^r a_i \otimes Y(i,n), \qquad (n \in \mathbb{N}). \tag{5.2}$$

In [HT2, Theorem 4.5], it was proved, that in this setting

$$\limsup_{n\to\infty} \max\big[\mathrm{sp}(S_n^* S_n)\big] \le \big(\sqrt{c} + 1\big)^2, \qquad \text{almost surely.} \tag{5.3}$$

It was proved, furthermore, that if, instead of (5.1), $a_1, \dots, a_r$, satisfy the condition

$$\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \qquad \text{and} \qquad \sum_{i=1}^r a_i a_i^* \le \mathbf{1}_{\mathcal{B}(\mathcal{K})}, \tag{5.4}$$

for some constant c in $[1, \infty[$, then

$$\liminf_{n\to\infty} \min\big[\mathrm{sp}(S_n^* S_n)\big] \ge \big(\sqrt{c} - 1\big)^2, \qquad \text{almost surely,} \tag{5.5}$$

(cf. [HT2, Theorem 8.7]). In this section, we use the results of Section 4 to show, in particular, that if $a_1, \dots, a_r$ satisfy the condition:

$$\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \qquad \text{and} \qquad \sum_{i=1}^r a_i a_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})}, \tag{5.6}$$

for some constant c in $]0, \infty[$, then we can replace lim sup in (5.3) by lim. Moreover, if $c \ge 1$, then we can also replace lim inf in (5.5) by lim.

In order to make use of the results of Section 4 for the particular S_n's introduced in (5.2), we need to determine (under the assumption of (5.6)), the spectrum of s^*s, where $s = \sum_{i=1}^r a_i \otimes y_i$, and $y_1, \dots, y_r$ is a circular system in some C^*-probability space $(\mathcal{B}, \tau)$. We obtain this by determining the distribution of s^*s w.r.t. $\phi \otimes \tau$, for an arbitrary state ϕ on $\mathcal{A}$.

5.1 Lemma. *Let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, let $\mathcal{A}$ denote the (not necessarily exact) C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{a_i^* a_j \mid 1 \le i, j \le r\} \cup \{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$, and let ϕ be a state on $\mathcal{A}$. Consider furthermore a circular system $y_1, \dots, y_r$ in some C^*-probability space $(\mathcal{B}, \tau)$, and put $s = \sum_{i=1}^r a_i \otimes y_i$. Then for any p in $\mathbb{N}$,*

$$\phi \otimes \tau((s^*s)^p) = \sum_{\pi \in S_p^{\mathrm{nc}}} \Big(\sum_{1 \le i_1, \dots, i_p \le r} \phi(a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}) \Big),$$

where S_p^{nc} (as in [HT2, Definition 1.14]) denotes the set of permutations π in S_p, for which the permutation $\hat{\pi}$, described in Remark 2.10, is non-crossing.

Proof. For each n in $\mathbb{N}$, let $Y(1,n), \dots, Y(r,n)$ be independent elements $\mathrm{GRM}(n, \frac{1}{n})$, and define

$$S_n = \sum_{i=1}^r a_i \otimes Y(i,n), \qquad (n \in \mathbb{N}).$$

It follows then from Lemma 4.1(ii) and Remark 4.8, that for each p in $\mathbb{N}$,

$$\mathbb{E}\big[\phi \otimes \mathrm{tr}_n\big((S_n^* S_n)^p\big)\big] \to \phi \otimes \tau\big((s^*s)^p\big), \quad \text{as } n \to \infty. \tag{5.7}$$

On the other hand, it follows from [HT2, Corollary 2.2], that for each p in $\mathbb{N}$,

$$\mathbb{E}\big[\phi \otimes \mathrm{tr}_n\big((S_n^* S_n)^p\big)\big] = \sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})} \Big(\sum_{1 \le i_1, \dots, i_p \le r} \phi(a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}) \Big). \tag{5.8}$$

Recall here from Proposition 2.16, that for any π in S_p, $\sigma(\hat{\pi}) \ge 0$, with equality if and only if $\hat{\pi}$ is non-crossing. Therefore (5.8) implies that

$$\mathbb{E}\big[\phi \otimes \mathrm{tr}_n\big((S_n^* S_n)^p\big)\big] \to \sum_{\pi \in S_p^{\mathrm{nc}}} \Big(\sum_{1 \le i_1, \dots, i_p \le r} \phi(a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}) \Big), \quad \text{as } n \to \infty. \tag{5.9}$$

Combining (5.7) and (5.9), we obtain the desired formula. ■

5.2 Proposition. *Let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, such that (5.6) holds for some constant c in $]0, \infty[$. Let $\mathcal{A}$ denote the (unital) C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the set $\{a_i^* a_j \mid 1 \le i, j \le r\}$, and let ϕ be a state on $\mathcal{A}$. Consider furthermore a circular system $y_1, \dots, y_r$ in some C^*-probability space $(\mathcal{B}, \tau)$, and put $s = \sum_{i=1}^r a_i \otimes y_i$.*

Then, the distribution of s^*s w.r.t. the state $\phi \otimes \tau$ on $\mathcal{A} \otimes \mathcal{B}$, is the Mazchenko-Pastur distribution (also known as the free Poisson distribution) μ_c with parameter c, i.e., the probability measure on $\mathbb{R}$, given by

$$\mu_c = \max\{1 - c, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x) \cdot dx,$$

where $a = (\sqrt{c} - 1)^2$, $b = (\sqrt{c} + 1)^2$, and δ_0 is the Dirac measure at 0.

Proof. Let the situation be as set out in the proposition. In [HT2, Corollary 5.4(i)], it was proved, that in this setting, we have the equation:

$$\sum_{\pi\in S_p^{\mathrm{nc}}}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}}\Big)=\Big[\tfrac{1}{p}\sum_{j=1}^{p}\tbinom{p}{j}\tbinom{p}{j-1}c^j\Big]\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H})},$$

and combining this with Lemma 5.1, it follows that

$$\phi\otimes\tau\big((s^*s)^p\big)=\tfrac{1}{p}\sum_{j=1}^{p}\tbinom{p}{j}\tbinom{p}{j-1}c^j,\qquad(p\in\mathbb{N}).\tag{5.10}$$

Recall finally from [OP] or [HT1, Remark 6.8], that the right hand side of (5.10) is exactly the p'th moment of μ_c, and hence we obtain the desired conclusion. ∎

5.3 Corollary. *Let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, such that* (5.6) *holds for some constant c in $]0,\infty[$. Consider furthermore a circular system $y_1,\dots,y_r$ in some C^*-probability space $(\mathcal{B},\tau)$, and put $s=\sum_{i=1}^r a_i\otimes y_i$. Then, if τ is faithful, the spectrum* $\mathrm{sp}(s^*s)$ *of s^*s is equal to the support of the Mazchenko-Pastur distribution μ_c, i.e.,*

$$\mathrm{sp}(s^*s)=\begin{cases}[(\sqrt{c}-1)^2,(\sqrt{c}+1)^2], & \text{if } c\ge 1,\\ \{0\}\cup[(\sqrt{c}-1)^2,(\sqrt{c}+1)^2], & \text{if } c\in[0,1].\end{cases}$$

Proof. Let $\mathcal{A}$ denote the (unital) C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the set

$$\{a_i^*a_j \mid 1\le i,j\le r\},$$

and note that since $\mathcal{A}$ is clearly separable, we may choose a faithful state ϕ on $\mathcal{A}$. Since τ is assumed faithful on $\mathcal{B}$, it follows then from Lemma 4.4, that $\phi\otimes\tau$ is faithful on $\mathcal{A}\otimes\mathcal{B}$. It is straight-forward to show, that this implies, that for any selfadjoint element b of $\mathcal{A}\otimes\mathcal{B}$, $\mathrm{sp}(b)$ is equal to the support of the distribution of b w.r.t. $\phi\otimes\tau$. In particular, this holds for $b=s^*s$. ∎

We are now able to strengthen the main result in [HT2], in the case where (5.6) holds (cf. [HT2, Theorem 0.1].

5.4 Theorem. *Let $a_1,a_2,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, such that $\sum_{i=1}^r a_i^*a_i=c\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\sum_{i=1}^r a_ia_i^*=\mathbf{1}_{\mathcal{B}(\mathcal{K})}$, for some positive, real number c. Let $\mathcal{A}$ denote the (unital) C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the set $\{a_i^*a_j\mid 1\le i,j\le r\}$, and assume that $\mathcal{A}$ is exact. Consider furthermore, for each n in $\mathbb{N}$, independent elements $Y(1,n),Y(2,n),\dots,Y(r,n)$ of* $\mathrm{GRM}(n,\frac{1}{n})$*, and define $S_n=\sum_{i=1}^r a_i\otimes Y(i,n)$. We then have*

(i) $$\max\big\{\mathrm{sp}(S_n^*S_n)\big\}\to\big(\sqrt{c}+1\big)^2,\quad\textit{almost surely, as } n\to\infty.$$

(ii) *If $c\ge 1$, then*

$$\min\big\{\mathrm{sp}(S_n^*S_n)\big\}\to\big(\sqrt{c}-1\big)^2,\quad\textit{almost surely, as } n\to\infty.$$

(iii) *If $c<1$, then*

$$\min\big\{\mathrm{sp}(S_n^*S_n)\big\}\to 0,\quad\textit{almost surely, as } n\to\infty.$$

Proof. It follows from [HT2, Theorem 0.1], that under the given assumptions, we have

$$\limsup_{n\to\infty}\big(\max\{\mathrm{sp}(S_n^*S_n)\}\big) \le \big(\sqrt{c}+1\big)^2, \quad \text{almost surely} \tag{5.11}$$

and

$$\liminf_{n\to\infty}\big(\min\{\mathrm{sp}(S_n^*S_n)\}\big) \ge \big(\sqrt{c}-1\big)^2, \quad \text{almost surely,} \tag{5.12}$$

if $c \ge 1$.

On the other hand, let $y_1, y_2, \dots, y_r$ be a circular system in a C^*-probability space $(\mathcal{B}, \tau)$, and assume that τ is faithful. Then with $s = \sum_{i=1}^r a_i \otimes y_i$, we have by Remark 4.8 and Corollary 4.5,

$$\liminf_{n\to\infty}\big(\max\{\mathrm{sp}(S_n^*S_n)\}\big) \ge \max\{\mathrm{sp}(s^*s)\}, \quad \text{almost surely,} \tag{5.13}$$

and

$$\limsup_{n\to\infty}\big(\min\{\mathrm{sp}(S_n^*S_n)\}\big) \le \min\{\mathrm{sp}(s^*s)\}, \quad \text{almost surely.} \tag{5.14}$$

By Corollary 5.3, we have here that $\max\{\mathrm{sp}(s^*s)\} = (\sqrt{c}+1)^2$, and that $\min\{\mathrm{sp}(s^*s)\} = (\sqrt{c}-1)^2$, if $c \ge 1$. Combining this with (5.11)-(5.14), we obtain (i) and (ii).

If $c < 1$, then Corollary 5.3 yields that $\min\{\mathrm{sp}(s^*s)\} = 0$, and hence (iii) follows directly from (5.14). ■

Having determined the asymptotic behavior of the largest and smallest elements of the spectrum of $S_n^*S_n$, one might consider the corresponding problem for $S_nS_n^*$. This is settled, however, by applying Theorem 5.4 with a_i replaced by $c^{-\frac{1}{2}}a_i^*$ and c by c^{-1}, recalling that $\mathrm{GRM}(n, \frac{1}{n})$ is invariant under the $*$-operation. If $c > 1$, the case of main interest, it follows in particular by Theorem 5.4(iii), that $\min\{\mathrm{sp}(S_nS_n^*)\} \to 0$, almost surely, as $n \to \infty$. In Proposition 5.6 below, we sharpen this conclusion, even for $a_1, \dots, a_r$ satisfying the condition (5.4), instead of the stronger condition (5.6), which is assumed in Theorem 5.4. In order to handle this more general setting, we need the following

5.5 Lemma. *Let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, such that*

$$\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \qquad \text{and} \qquad \Big\|\sum_{i=1}^r a_ia_i^*\Big\| \le 1, \tag{5.15}$$

for some constant c in $]1, \infty[$. Let further $y_1, \dots, y_r$ be a circular system in some C^*-probability space $(\mathcal{B}, \tau)$, with τ faithful. Then with $s = \sum_{i=1}^r a_i \otimes y_i$, we have that

$$0 \notin \mathrm{sp}(s^*s) \qquad \text{and} \qquad 0 \in \mathrm{sp}(ss^*).$$

Proof. The proof is similar to the proof of [Haa, Lemma 2.3]; the main difference being that presently we consider circular systems, instead of the semi-circular systems considered in [Haa, Lemma 2.3].

Note first that by replacement of a_i by $c^{-\frac{1}{2}}a_i$, we may assume that $a_1, \dots, a_r$ satisfy the condition

$$\sum_{i=1}^r a_i^* a_i = \mathbf{1}_{\mathcal{B}(\mathcal{H})} \qquad \text{and} \qquad \Big\| \sum_{i=1}^r a_i a_i^* \Big\| < 1, \tag{5.16}$$

instead of (5.15).

We note next, that since τ (and hence the corresponding GNS.-representation) is faithful, it follows from [Vo2, Remark 1.8], that we can replace the given circular system $\{y_1, \dots, y_r\}$ in $(\mathcal{B}, \tau)$ by our favorite circular system in our preferred C^*-probability space with faithful state. As our favorite circular system, we choose here the circular system that is naturally obtained from left creation operators on full Fock space. We repeat briefly the construction:

Let $\mathcal{L}$ be a Hilbert space of dimension $2r$ and choose an orthonormal basis

$$\{e_1, \dots, e_r, f_1, \dots, f_r\}$$

for $\mathcal{L}$. Consider then the full Fock space associated to $\mathcal{L}$,

$$\mathcal{T}(\mathcal{L}) = \mathbb{C}\zeta_0 \oplus \Big[\bigoplus_{n=1}^{\infty} \mathcal{L}^{\otimes n} \Big],$$

where ζ_0 is a unit-vector. To each vector h in $\mathcal{L}$, we associate the left creation operator $\ell(h)$ in $\mathcal{B}(\mathcal{T}(\mathcal{L}))$, defined by the equations

$$\begin{aligned} \ell(h)\zeta_0 &= h, \\ \ell(h)(h_1 \otimes \cdots \otimes h_n) &= h \otimes h_1 \otimes \cdots \otimes h_n, \qquad (n \in \mathbb{N},\ h_1, \dots, h_n \in \mathcal{L}). \end{aligned}$$

It is routine to verify the following properties (cf. [VDN, Example 1.5.8]),

$$\ell(h)^*\zeta_0 = 0, \qquad (h \in \mathcal{L}), \tag{5.17}$$

$$\ell(h_1)^*\ell(h_2) = \langle h_2, h_1 \rangle \mathbf{1}_{\mathcal{B}(\mathcal{T}(\mathcal{L}))}, \qquad (h_1, h_2 \in \mathcal{L}). \tag{5.18}$$

Define now,

$$y_i = \tfrac{1}{2}\big(\ell(e_i) + \ell(f_i)^*\big), \qquad (i \in \{1, 2, \dots, r\}).$$

Then it is well-known that the family $\{y_i \mid 1 \le i \le r\}$ is a circular system w.r.t. the vector state $\tau = \langle \cdot\, \zeta_0, \zeta_0 \rangle$, and that τ is faithful on the C^*-algebra $\mathcal{B} = C^*(y_1, \dots, y_r)$ (cf. [Vo2, Remark 1.11 and (proof of) Proposition 2.2]). Thus, as mentioned above, it suffices to prove the lemma with this choice $y_1, \dots, y_r$, $\mathcal{B}$ and τ.

Note then, that in this situation, we have

$$2s = 2\sum_{i=1}^r a_i \otimes y_i = \sum_{i=1}^r a_i \otimes \ell(e_i) + \sum_{i=1}^r a_i \otimes \ell(f_i)^* = v + w,$$

where $v = \sum_{i=1}^r a_i \otimes \ell(e_i)$ and $w = \sum_{i=1}^r a_i \otimes \ell(f_i)^*$.

By (5.18) and (5.16), we have here that

$$v^*v = \sum_{i,j=1}^{r} a_i^*a_j \otimes \ell(e_i)^*\ell(e_j) = \sum_{i=1}^{r} a_i^*a_i \otimes \mathbf{1}_{\mathcal{B}(\mathcal{T}(\mathcal{L}))} = \mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T}(\mathcal{L}))},$$

so that v is an isometry. On the other hand, (5.17) shows that $\mathcal{K} \otimes \zeta_0 \subseteq \ker(v^*)$, so that v^* is not an isometry. Hence v is a non-unitary isometry.

By (5.18) it follows furthermore, that

$$ww^* = \sum_{i,j=1}^{r} a_ia_j^* \otimes \ell(f_i)^*\ell(f_j) = \sum_{i=1}^{r} a_ia_i^* \otimes \mathbf{1}_{\mathcal{B}(\mathcal{T}(\mathcal{L}))},$$

and combining this with (5.16), we see that

$$\|w\|^2 = \|ww^*\| = \Big\| \sum_{i=1}^{r} a_ia_i^* \Big\| < 1.$$

Now, for any vector ξ in $\mathcal{H} \otimes \mathcal{T}(\mathcal{L})$,

$$\|2s\xi\| = \|(v+w)\xi\| \geq \|v\xi\| - \|w\xi\| \geq \|\xi\| - \|w\|\|\xi\| = (1-\|w\|)\|\xi\|,$$

which shows that $4s^*s \geq (1-\|w\|)^2\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T}(\mathcal{L}))}$. Since $\|w\| < 1$, it follows thus that s^*s is invertible in $\mathcal{B}(\mathcal{H} \otimes \mathcal{T}(\mathcal{L}))$, i.e., that $0 \notin \mathrm{sp}(s^*s)$.

Note next that

$$2s = v + w = (\mathbf{1}_{\mathcal{B}(\mathcal{K}\otimes\mathcal{T}(\mathcal{L}))} + wv^*)v. \tag{5.19}$$

Here $\mathbf{1}_{\mathcal{B}(\mathcal{K}\otimes\mathcal{T}(\mathcal{L}))} + wv^*$ is invertible in $\mathcal{B}(\mathcal{K} \otimes \mathcal{T}(\mathcal{L}))$, since $\|wv^*\| \leq \|w\| < 1$. Hence, since v is not invertible in $\mathcal{B}(\mathcal{H} \otimes \mathcal{T}(\mathcal{L}), \mathcal{K} \otimes \mathcal{T}(\mathcal{L}))$, (5.19) shows that s cannot be invertible in $\mathcal{B}(\mathcal{H} \otimes \mathcal{T}(\mathcal{L}), \mathcal{K} \otimes \mathcal{T}(\mathcal{L}))$ either. Since s^*s is invertible, this implies that ss^* cannot be invertible, i.e., that $0 \in \mathrm{sp}(ss^*)$. Indeed, $ss^*(\mathcal{K} \otimes \mathcal{T}(\mathcal{L})) \subseteq s(\mathcal{H} \otimes \mathcal{T}(\mathcal{L}))$, so if ss^* was invertible, s would be both injective and surjective. ■

5.6 Proposition. *Let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, such that $\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\|\sum_{i=1}^r a_ia_i^*\| \leq 1$, for some constant c in $]1, \infty[$. Let $\mathcal{A}$ denote the (unital) C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the set $\{a_i^*a_j \mid 1 \leq i, j \leq r\}$, and assume that $\mathcal{A}$ is exact. Consider furthermore, for each n in $\mathbb{N}$, independent elements $Y(1, n), \dots, Y(r, n)$ of $\mathrm{GRM}(n, \frac{1}{n})$, and define $S_n = \sum_{i=1}^r a_i \otimes Y(i, n)$. Then on a set with probability one,*

$$0 \in \mathrm{sp}(S_nS_n^*), \quad \textit{for all but finitely many } n.$$

Proof. Let $y_1, \dots, y_r$ be a circular system in some C^*-probability space $(\mathcal{B}, \tau)$, and assume that τ is faithful. Then with $s = \sum_{i=1}^r a_i \otimes y_i$, we have according to Lemma 5.5, that $0 \in \mathrm{sp}(ss^*)$. By application of Proposition 4.5 and Remark 4.8, with a_i replaced by a_i^*, and recalling that $\mathrm{GRM}(n, \frac{1}{n})$ as well as the circular system $\{y_1, \dots, y_r\}$ is invariant under the $*$-operation, it follows that

$$\limsup_{n\to\infty} \big(\min \big\{ \mathrm{sp}(S_n S_n^*) \big\} \big) \le 0, \qquad \text{almost surely.} \tag{5.20}$$

On the other hand, it follows from [HT2, Theorem 0.1], that

$$\liminf_{n\to\infty} \big(\min \big\{ \mathrm{sp}(S_n^* S_n) \big\} \big) \ge (\sqrt{c}-1)^2, \qquad \text{almost surely.} \tag{5.21}$$

Let $\mathcal{S}$ denote the set of ω's in Ω, for which (5.20) and (5.21) hold. Then $\mathcal{S}$ is a sure event. Assume that $\omega \in \mathcal{S}$. Then, since $c > 1$, we may choose an N_ω in $\mathbb{N}$, such that $\min\{\mathrm{sp}(S_n(\omega)S_n(\omega)^*)\} < \frac{1}{2}(\sqrt{c}-1)^2$, and $\min\{\mathrm{sp}(S_n(\omega)^*S_n(\omega))\} > \frac{1}{2}(\sqrt{c}-1)^2$, whenever $n \ge N_\omega$. Recalling then that

$$\mathrm{sp}(S_n^*(\omega)S_n(\omega)) \cup \{0\} = \mathrm{sp}(S_n(\omega)S_n^*(\omega)) \cup \{0\},$$

(cf. [KR, Proposition 3.2.8]), it follows that we must have $0 \in \mathrm{sp}(S_n(\omega)S_n(\omega)^*)$, whenever $n \ge N_\omega$. This completes the proof. ∎

6 Semi-circular Families in Ultra Products of Matrix Algebras, and a Result of S. Wassermann

We start this section by recalling the ultra product construction associated to a sequence of finite von Neumann algebras, and a free ultra filter $\mathcal{U}$ on $\mathbb{N}$. For the results stated below about this construction, we refer to [Mc].

For each n in $\mathbb{N}$, let $\mathcal{M}_n$ be a finite von Neumann algebra acting on the Hilbert space $\mathcal{H}_n$, and let τ_n be a normalized trace on $\mathcal{M}_n$. Consider then the direct sum

$$\ell_\infty(\{\mathcal{M}_n\}) = \bigoplus_{n\in\mathbb{N}} \mathcal{M}_n = \big\{ (x_n) \;\big|\; x_n \in \mathcal{M}_n \text{ for all n, and } \sup_{n\in\mathbb{N}} \|x_n\| < \infty \big\}.$$

Then $\ell_\infty(\{\mathcal{M}_n\})$ may be considered as an algebra of "diagonal matrices" acting on the Hilbert space $\oplus_{n\in\mathbb{N}}\mathcal{H}_n$, and using this description, it is easily seen that $\ell_\infty(\{\mathcal{M}_n\})$ is a von Neumann algebra acting on $\oplus_{n\in\mathbb{N}}\mathcal{H}_n$.

Now consider, in addition, a free ultra filter $\mathcal{U}$ on $\mathbb{N}$. Recalling, that any bounded sequence of complex numbers converges along an ultra filter, we then put

$$I_{\mathcal{U}} = \big\{ (x_n) \in \ell_\infty(\{\mathcal{M}_n\}) \;\big|\; \lim_{n\to\mathcal{U}} \|x_n\|_2 = 0 \big\},$$

where $\|x_n\|_2$ denotes the 2-norm of x_n w.r.t. the trace τ_n. Then $I_{\mathcal{U}}$ is a two-sided, $*$-invariant, norm-closed ideal in $\ell_\infty(\{\mathcal{M}_n\})$, and hence we may consider the quotient C^*-algebra

$$\ell_\infty(\{\mathcal{M}_n\}, \mathcal{U}) := \ell_\infty(\{\mathcal{M}_n\})/I_{\mathcal{U}}.$$

Let Ψ denote the quotient $*$-homomorphism of $\ell_\infty(\{\mathcal{M}_n\})$ onto $\ell_\infty(\{\mathcal{M}_n\},\mathcal{U})$. We then have a natural faithful, normalized trace $\tau_{\mathcal{U}}$ on $\ell_\infty(\{\mathcal{M}_n\},\mathcal{U})$ given by

$$\tau_{\mathcal{U}}\big(\Psi[(x_n)]\big) = \lim_{n\to\mathcal{U}} \tau_n(x_n), \qquad \big((x_n) \in \ell_\infty(\{\mathcal{M}_n\})\big).$$

It can be shown, that in the faithful GNS.-representation of $\ell_\infty(\{\mathcal{M}_n\},\mathcal{U})$, associated to $\tau_{\mathcal{U}}$, $\ell_\infty(\{\mathcal{M}_n\},\mathcal{U})$ is a finite von Neumann algebra (in standard form). We call $\ell_\infty(\{\mathcal{M}_n\},\mathcal{U})$ the ultra product of the finite von Neumann algebras $\mathcal{M}_1, \mathcal{M}_2, \dots,$ associated to the free ultra filter $\mathcal{U}$.

Next, let G be a countable, discrete, ICC group, and consider the II_1 factor $\mathcal{L}(G)$ associated with the left regular representation of G on $l_2(G)$. In [Wa1], S. Wassermann proved that if G is, in addition, residually finite (i.e., there exists a decreasing sequence of normal subgroups of G, each of which has finite index in G, and whose intersection is the unit of G), then for any free ultra filter $\mathcal{U}$ on $\mathbb{N}$, $\mathcal{L}(G)$ can be embedded (as a C^*-algebra) into the ultra product of $M_{n_1}(\mathbb{C}), M_{n_2}(\mathbb{C}), M_{n_3}(\mathbb{C}), \dots,$ associated to $\mathcal{U}$, for a suitable sequence $n_1, n_2, n_3, \dots,$ of positive integers. Since the free group F_m on m generators is residually finite for any m in $\{2,3,4,\dots\} \cup \{\infty\}$ (cf. [Wa2, Lemma 3.6]), it follows, in particular, that the free group factors $\mathcal{L}(F_m)$, $2 \le m \le \infty$, can be embedded into ultra products of certain matrix algebras. By application of the results obtained in Section 3, we are able to give a new proof of this result (cf. Corollary 6.3). In fact, we prove a slightly stronger result, namely that the positive integers $n_1, n_2, n_3, \dots,$ may be chosen as $1, 2, 3, \dots$. We achieve this by proving the existence of a semi-circular sequence $\{x_j\}_{j\in\mathbb{N}}$ in the W^*-probability space $(\ell_\infty(\{M_n(\mathbb{C})\},\mathcal{U}), \tau_{\mathcal{U}})$. Once this has been established, we may appeal to the fact (already used in the proof of Lemma 5.5), that under certain faithfulness assumptions, two families of non-commutative random variables with equal joint $*$-distributions, generate $*$-isomorphic W^*-algebras (cf. [Vo2, Remark 1.8]).

We start by stating the following result from [HT1]:

6.1 Lemma. *For each n in $\mathbb{N}$, let X_n be an element of* $\mathrm{SGRM}(n, \frac{1}{n})$*. Then*

$$\lim_{n\to\infty} \|X_n\| = 2, \quad \textit{almost surely.}$$

In particular, $(X_n(\omega))_{n\in\mathbb{N}} \in \ell_\infty(\{M_n(\mathbb{C})\})$, for almost all ω in Ω.

Proof. This was proved in [HT1, Theorem 3.1]. ∎

Now, for each n in $\mathbb{N}$, consider a sequence $(X_{j,n})_{j\in\mathbb{N}}$ of independent elements of $\mathrm{SGRM}(n, \frac{1}{n})$, and for each j in $\mathbb{N}$ define

$$\mathcal{S}_j = \big\{\omega \in \Omega \bigm| (X_{j,n}(\omega))_{n\in\mathbb{N}} \in \ell_\infty(\{M_n(\mathbb{C})\})\big\}.$$

According to Lemma 6.1, $\mathcal{S}_j$ is a sure event for each j, and hence so is the set

$$\mathcal{S} := \bigcap_{j\in\mathbb{N}} \mathcal{S}_j.$$

Consider then also a free ultra filter $\mathcal{U}$ on $\mathbb{N}$, and let $\Psi\colon \ell_\infty(\{M_n(\mathbb{C})\}) \to \ell_\infty(\{M_n(\mathbb{C})\},\mathcal{U})$ denote the quotient $*$-homomorphism. We then define

$$x_j(\omega) = \Psi\big((X_{j,n}(\omega))_{n\in\mathbb{N}}\big), \qquad (\omega \in \mathcal{S},\ j \in \mathbb{N}). \tag{6.1}$$

6.2 Proposition. *Let the situation be as described above. Then for almost all ω in the sure event $\mathcal{S}$, $\{x_j(\omega)\}_{j\in\mathbb{N}}$ is a semi-circular family in $(\ell_\infty(\{M_n(\mathbb{C})\},\mathcal{U}),\tau_{\mathcal{U}})$.*

Proof. Let $\{x'_j\}_{j\in\mathbb{N}}$ be a semi-circular family in some non-commutative probability space $(\mathcal{B},\tau)$. Then for any $p, i_1, i_2, \dots, i_p$ in $\mathbb{N}$, we define:

$$\mathcal{T}(i_1,i_2,\dots,i_p) = \big\{\omega\in\Omega \bigm| \lim_{n\to\infty} \mathrm{tr}_n\big[X_{i_1,n}(\omega)X_{i_2,n}(\omega)\cdots X_{i_p,n}(\omega)\big] = \tau[x'_{i_1}x'_{i_2}\cdots x'_{i_p}]\big\}.$$

By Theorem 3.6, $\mathcal{T}(i_1,i_2,\dots,i_p)$ is a sure event for all $p, i_1, i_2, \dots, i_p$, and hence so is the set

$$\mathcal{T} = \bigcap_{p\in\mathbb{N}}\ \bigcap_{(i_1,\dots,i_p)\in\mathbb{N}^p} \mathcal{T}(i_1,i_2,\dots,i_p).$$

Now, for any ω in $\mathcal{T}\cap\mathcal{S}$ and any $p, i_1, i_2, \dots, i_p$ in $\mathbb{N}$,

$$\begin{aligned}
\tau[x'_{i_1}x'_{i_2}\cdots x'_{i_p}] &= \lim_{n\to\infty} \mathrm{tr}_n\big[X_{i_1,n}(\omega)X_{i_2,n}(\omega)\cdots X_{i_p,n}(\omega)\big]\\
&= \lim_{n\to\mathcal{U}} \mathrm{tr}_n\big[X_{i_1,n}(\omega)X_{i_2,n}(\omega)\cdots X_{i_p,n}(\omega)\big]\\
&= \tau_{\mathcal{U}}\big[\Psi\big((X_{i_1,n}(\omega))_{n\in\mathbb{N}}\cdot(X_{i_2,n}(\omega))_{n\in\mathbb{N}}\ \cdots\ (X_{i_p,n}(\omega))_{n\in\mathbb{N}}\big)\big]\\
&= \tau_{\mathcal{U}}\big[x_{i_1}(\omega)x_{i_2}(\omega)\cdots x_{i_p}(\omega)\big].
\end{aligned}$$

This shows that for any ω in $\mathcal{S}\cap\mathcal{T}$, $\{x_j(\omega)\}_{j\in\mathbb{N}}$ is a semi-circular family in the W^*-probability space $(\ell_\infty(\{M_n(\mathbb{C})\},\mathcal{U}),\tau_{\mathcal{U}})$. ∎

6.3 Corollary. *For any m in $\{2,3,4,\dots\}\cup\{\infty\}$ and any free ultra filter $\mathcal{U}$ on $\mathbb{N}$, there exists an injective $*$-homomorphism $\Phi\colon \mathcal{L}(F_m)\to\ell_\infty(\{M_n(\mathbb{C})\},\mathcal{U})$.*

Proof. Let x_j, $j\in\mathbb{N}$, be as defined in (6.1). According to Proposition 6.2, there exists an ω in Ω, such that $\{x_j(\omega)\}_{j\in\mathbb{N}}$ is a semi-circular family in $(\ell_\infty(\{M_n(\mathbb{C})\},\mathcal{U}),\tau_{\mathcal{U}})$. Let $\mathcal{M}$ denote the W^*-subalgebra of $\ell_\infty(\{M_n(\mathbb{C})\},\mathcal{U})$ generated by $\{x_j(\omega)\}_{j=1}^m$. Then, since $\tau_{\mathcal{U}}$ (and hence the corresponding GNS.-representation) is faithful, it follows from [Vo2, Remark 1.10], that $\mathcal{M}$ is isomorphic, as a W^*-algebra, to $\mathcal{L}(F_m)$. ∎

Appendix. Faithfulness of Product States

Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be unital C^*-algebras, and let ϕ,ψ be states on $\mathcal{C}$ and $\mathcal{D}$ respectively. If ϕ and ψ are both faithful, then so is the product state $\phi\otimes\psi$ on the minimal (= spatial) tensor product $\mathcal{C}\otimes\mathcal{D}$.*

Proof. Recall that the right slice map $R_\phi\colon \mathcal{C}\otimes\mathcal{D}\to\mathcal{D}$, associated to ϕ, is given by:

$$R_\phi(A\otimes B) = \phi(A)\cdot B, \qquad (A\in\mathcal{C},\ B\in\mathcal{D}).$$

Since R_ϕ is positive, and $\phi\otimes\psi = \psi\circ R_\phi$, it suffices thus to show, that R_ϕ is faithful when ϕ is.

In the first part of the proof, we derive a matrix representation for elements of the minimal tensor product $\mathcal{C} \otimes \mathcal{D}$, and determine the corresponding appearance of R_ϕ. We give the argument leading to these matrix representations, by listing a number of claims/facts, that add up to the desired result. Each of the claims/facts can either be found in standard text-books or realized by minor considerations.

(a) Note first that we may assume that $\mathcal{C}$ and $\mathcal{D}$ are unital C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ respectively $\mathcal{B}(\mathcal{K})$, for suitable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

(b) Let $(\xi_i)_{i\in I}$ be an orthonormal basis for $\mathcal{K}$. With respect to this basis, any operator B in $\mathcal{B}(\mathcal{K})$ has a representation as a matrix $B = (b_{ij})_{i,j\in I}$, where $b_{ij} \in \mathbb{C}$ for all i, j in I. This representation is such that

$$B\Big(\sum_{i\in I} t_i\xi_i\Big) = \sum_{i\in I}\Big(\sum_{j\in I} b_{ij}t_j\Big)\xi_i, \qquad \big((t_i)_{i\in I} \in \ell^2(I)\big).$$

If (B_n) is a sequence in $\mathcal{B}(\mathcal{K})$, such that $B_n \to B$ in norm, then $b_{ij}^{(n)} \to b_{ij}$, for all i, j in I, where $B_n = (b_{ij}^{(n)})$ is the matrix representation of B_n (cf. [KR, pp. 147-148]).

(c) Put $\mathcal{H}_i = \mathcal{H}$ for all i in I. Then the Hilbert space $\mathcal{H}\otimes\mathcal{K}$ is isomorphic (as a Hilbert space) to the Hilbert space direct sum $\bigoplus_{i\in I}\mathcal{H}_i$, via the (unitary) operator U, given by

$$U\big((x_i)_{i\in I}\big) = \sum_{i\in I} x_i \otimes \xi_i, \qquad \big((x_i)_{i\in I} \in \textstyle\bigoplus_{i\in I}\mathcal{H}_i\big)$$

(cf. [KR, pp. 148-149]).

(d) Each operator T in $\mathcal{B}(\bigoplus_{i\in I}\mathcal{H}_i)$ has a representation as a matrix $T = (T_{ij})_{i,j\in I}$, where $T_{ij} \in \mathcal{B}(\mathcal{H})$ for all i, j in I. This representation is such that

$$T\big((x_i)_{i\in I}\big) = \Big(\sum_{j\in I} T_{ij}x_j\Big)_{i\in I}, \qquad \big((x_i)_{i\in I} \in \textstyle\bigoplus_{i\in I}\mathcal{H}_i\big).$$

If (T_n) is a sequence in $\mathcal{B}(\bigoplus_{i\in I}\mathcal{H}_i)$, such that $T_n \to T$ in norm, then $T_{ij}^{(n)} \to T_{ij}$ in norm, for all i, j in I, where $T_n = (T_{ij}^{(n)})$ is the matrix representation of T_n. (cf. [KR, pp. 148-149]).

(e) Combining (c) and (d), it follows that any operator T in $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ has a matrix representation $T = (T_{ij})_{i,j\in I}$, where $T_{ij} \in \mathcal{B}(\mathcal{H})$ for all $i, j \in I$. This representation is such that

$$T\Big(\sum_{i\in I} x_i \otimes \xi_i\Big) = \sum_{i\in I}\Big(\sum_{j\in I} T_{ij}x_j\Big) \otimes \xi_i, \qquad \big((x_i)_{i\in I} \in \textstyle\bigoplus_{i\in I}\mathcal{H}_i\big).$$

If (T_n) is a sequence in $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$, such that $T_n \to T$ in norm, then $T_{ij}^{(n)} \to T_{ij}$ in norm, for all i, j in I, where $T_n = (T_{ij}^{(n)})$ is the matrix representation of T_n.

(f) If T from $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ is of the form $T = A\otimes B$, where $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{K})$, then the matrix representation of T from (e) is given by $T = (Ab_{ij})_{i,j\in I}$, where $B = (b_{ij})_{i,j\in I}$ is the matrix representation of B from (b).

(g) Consider now the given C^*-algebras $\mathcal{C}\subseteq\mathcal{B}(\mathcal{H})$ and $\mathcal{D}\subseteq\mathcal{B}(\mathcal{K})$. From (f) and linearity, it follows then, that for any T in the algebraic tensor product $\mathcal{C}\odot\mathcal{D}$, the matrix representation of T from (e) is given by $T = (T_{ij})_{i,j\in I}$, where $T_{ij}\in\mathcal{C}$, for all i,j in I. By the continuity assertion in (e), the same is true for any element T in the minimal tensor product $\mathcal{C}\otimes\mathcal{D}$.

(h) Consider now the right slice map $R_\phi\colon \mathcal{C}\otimes\mathcal{D}\to\mathcal{D}$ associated to ϕ, and let A and B be elements of $\mathcal{C}$ respectively $\mathcal{D}$. It follows then, that with $B = (b_{ij})_{i,j\in I}$ the matrix representation of B from (b),

$$R_\phi(A\otimes B) = R_\phi\big((Ab_{ij})_{i,j\in I}\big) = (\phi(A)b_{ij})_{i,j\in I},$$

where the first equality follows from (f), and the last equality means that $(\phi(A)b_{ij})_{i,j\in I}$ is the matrix representation of $R_\phi(A\otimes B) = \phi(A)B$ from (b).

(i) Let T be an element of the algebraic tensor product $\mathcal{C}\odot\mathcal{D}$, and consider the matrix representation $T = (T_{ij})$ from (e). By (f), (g), (h) and linearity, it follows then that

$$R_\phi(T) = (\phi(T_{ij}))_{i,j\in I},$$

in the sense that $(\phi(T_{ij}))_{i,j\in I}$ is the matrix representation of $R_\phi(T)$ from (b). By the last assertion in (g), continuity of R_ϕ, the continuity assertions in (b) and (e) and continuity of ϕ, it follows then that the same formula holds for any T from $\mathcal{C}\otimes\mathcal{D}$.

Having obtained the desired matrix representations for elements of $\mathcal{C}\otimes\mathcal{D}$ and for R_ϕ, assume now that $T\in\mathcal{C}\otimes\mathcal{D}$, that $T\geq 0$ and that $R_\phi(T) = 0$. Consider then the matrix representation $T = (T_{ij})_{i,j\in I}$ of T from (e). By (i), the assumption that $R_\phi(T) = 0$ means that

$$\phi(T_{ij}) = 0, \qquad (i,j\in I). \tag{6.2}$$

Since $T\geq 0$, it follows that $T_{ii}\geq 0$ for all i in I, and hence, since ϕ is faithful, (6.2) implies that $T_{ii} = 0$ for all i in I. Next, for any distinct i,j in I, consider the 2×2 matrix

$$\begin{pmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{pmatrix}.$$

Since T is positive, so is this matrix, considered as an operator in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$. Knowing that $T_{ii} = T_{jj} = 0$, this implies that also $T_{ij} = T_{ji} = 0$, as desired. ∎

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