ISSN 1398-599X MPS-RR 1999-21 www.maphysto.dk

Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach

Fred Espen Benth[1], Kenneth Hvistendahl Karlsen[2], and Kristin Reikvam[3]

June 25, 1999

Abstract

We study a problem of optimal consumption and portfolio selection in a market where the logreturns of the uncertain assets are not necessarily normally distributed. The natural models then involve pure-jump Lévy processes as driving noise instead of Brownian motion like in the Black and Scholes model. The state constrained optimization problem involves the notion of local substitution and is of singular type. The associated Hamilton-Jacobi-Bellman equation is a nonlinear first order integro-differential equation subject to gradient and state constraints. We prove that the value function of the singular stochastic control problem is the unique constrained viscosity solution of the Hamilton-Jacobi-Bellman equation. To this end, we prove a new comparison (uniqueness) result for the state constraint problem for a class of integro-differential variational inequalities. We generalize our results to the second order case, where we in addition allow for a Brownian motion in the noise term. Here too we are able to prove existence and comparison results for the corresponding second order integro-differential variational inequality. Finally, we discuss related models and present two specific examples. In the first we show that our control problem has an explicit solution when the utility function is of HARA type. In the second example, we consider Merton's problem, which is a special case of our stochastic control problem. We also here provide explicit results for HARA utility.

[1]MaPhySto - Centre for Mathematical Physics and Stochastics, University of Aarhus, Ny Munkegade, DK-8000 Århus C, Denmark and Norwegian Computing Center, PO Box 114 Blindern, N-0314 Oslo, Norway. E-mail: fred.espen.benth@nr.no. MaPhySto is funded by a grant from the Danish National Research Foundation.

[2]Department of Mathematics, University of Bergen, Johs. Brunsgt. 12, N-5008 Bergen, Norway.
E-mail: kennethk@mi.uib.no.

[3]Department of Mathematics, University of Oslo, P. O. Box 1053 Blindern, N-0316 Oslo, Norway.
E-mail: kre@math.uio.no.

1 Introduction

We consider a model of optimal consumption and portfolio selection which captures the notion of local substitution. This optimization problem was first suggested and studied in detail by Hindy and Huang [19] for diffusion processes using verification theorems. Later, Alvarez [1] studied the problem in a viscosity solution framework. A viscosity solution approach has also been pursued by Hindy, Huang, and Zhu [20] for a certain generalization of this problem. The main motivation for the present paper is to generalize the results by Hindy and Huang [19] and Alvarez [1] to *statistically sound* models for the asset price process.

An agent wants to divide her wealth between an uncertain asset with price S_t and a bond B_t with interest rate r. She wants to allocate her wealth and at the same time consume in order to optimize the functional

$$\mathrm{E}\Big[\int_0^\infty e^{-\delta t} U(Y_t^{\pi,C})\,dt\Big].$$

where $\pi = \pi_t$ denotes the fraction of wealth allocated in the uncertain investment and $C = C_t$ is the cumulative consumption at time t. This functional describes the agent's preferences over consumption patterns. The agent's utility is described by U, discounted by the rate δ. The special feature of this problem introduced by Hindy and Huang [19] is the process Y_t modelling the average past consumption. This process will be derived from the total consumption up till time t and a weighting factor (see equation (2.7)). This model says that the agent derives satisfaction from past consumption. In addition, the control problem incorporates the idea of local substitution which says that consumption at nearby dates are almost perfect substitutes. Advancing or delaying consumption has little effect on the consumer's satisfaction. With this model of satisfaction, optimal consumption was shown by [19] to be periodic in the sense of a local time on a boundary. Every time the wealth process hits a boundary, consumption takes place. We have chosen to consider the case of an agent with infinite investment horizon.

The standard model for stock prices in the Black-Scholes world is the geometric Brownian motion

$$S_t = S_0 e^{\mu t + \sigma W_t},$$

where μ is the expected log-return and σ the volatility. This model imposes a normal distribution on the logreturns of an observed stock price. Empirical work by Eberlein and Keller [13] and Rydberg [33] shows that the normal distribution poorly fits the logreturn data. Among other things, the data have heavy tails. They suggest modelling logreturns by generalized hyperbolic distributions, which are shown to fit data extremely well. Barndorff-Nielsen [6] introduces the normal inverse Gaussian distribution which is throughly studied on financial time series by Rydberg [33]. Eberlein and Keller [13] use the hyperbolic distribution. The model for stock prices becomes

$$S_t = S_0 e^{\mu t + L_t},$$

where L_t is a *Lévy process* and L_1 is distributed according to a normal inverse Gaussian law in [6, 33] and a hyperbolic law in [13]. It is worth noticing that in both cases L_t will be a pure jump Lévy process, i.e., it does not have any Brownian motion part in its Lévy-Khintchine representation. The generator of S_t will thus have no second order term, and our control problem – as will be explained later – will be a first order integro-differential variational inequality. We shall assume here that the stock price is driven by a general pure jump Lévy process L_t. However, we will also treat the more general case with a Brownian motion and a pure-jump Lévy process as driving noise in the stock price model.

By the Bellman principle we can associate a Hamilton-Jacobi-Bellman equation (variational inequality) to our optimization problem. This equation is set in an unbounded domain and consists of a nonlinear first order integro-differential equation subject to a gradient constraint, a so-called *integro-differential variational inequality* (see Section 2). Since we allow for consumption processes which are not necessarily absolute continuous with respect to the Lebesgue measure, we have a so-called singular control problem. These problems give rise to a gradient constraint in the variational inequality, see, e.g., Fleming and Soner [14]. In our general set-up, it is natural to consider the variational problem in the framework of *viscosity solutions*, as done by Alvarez [1] for the geometric Brownian motion case. We recall that the notion of viscosity solutions was introduced by Crandal and Lions [9] for first order equations and by Lions [29, 30] for second order equations. The notion of viscosity solutions for integro-differential equations was later pursued by Soner [37, 38] and Sayah [34, 35] for certain problems involving a first order local operator, and by Alvarez and Tourin [2] and Pham [32] for problems involving a second order local operator. For control problems and their associated Hamilton-Jacobi-Bellman equations, this weak solution concept has proven to be extremely useful due to the fact that it allows merely continuous functions to be solutions of fully nonlinear second order partial differential equations. We refer to the user's guide [10], the lecture notes in [4], and the books [3, 5, 14] for an overview of the theory of viscosity solutions and its applications.

For our problem, we need to consider *constrained* viscosity solutions since we are not allowed to consume more than the present wealth, e.g., the control cannot push the wealth process into the negative real line. The notion of constrained viscosity solutions was first introduced by Soner [36, 37] and later Capuzzo-Dolcetta and Lions [12] for first order equations, see also Lasry and Lions [27], Lions and Ishii [23], and Katsoulakis [26] for second order equations. In the present paper, we first prove that the value function of our control problem is a constrained viscosity solution of the associated integro-differential variational inequality (see Section 4). As observed by Lions (see, e.g., [30]), the general fact that value functions of control problems can be characterized as viscosity solutions of certain partial differential equations is a direct consequence of the dynamic programming principle. For singular control problems, however, the classical approach of Lions fails because the state process may jump due to the singular control and it needs thus not stay in a small ball for small t. This problem has usually been circumvented by either relying on the existence of an optimal control (see, e.g., [11, 20]) or by establishing appropriate estimates for the state process (see, e.g., [14]). In [1], Alvarez presented a more direct argument showing that the value function of the singular control problem in [19] is a viscosity solution of the associated variational inequality. We adopt his argument to our singular control problem (where the state process itself can also jump) and its associated integro-differential variational inequality.

Our second result is a comparison principle for the *state constraint* problem for integro-differential variational inequalities, which ensures that the value function is the only solution of our problem, see Section 4. The first comparison principles (uniqueness results) for viscosity solutions were given by Crandall and Lions [9] (see also Crandall, Evans, and Lions [8]) for first order equations. Concerning the uniqueness theory for second order equations (as in Section 5), important contributions are due to Jensen [24], Jensen, Lions, and Souganidis [28], Lions and Souganidis [31], Ishii [22], Jensen [25], and Ishii and Lions [23]. We refer to the user's guide of Crandall, Ishii and Lions [10], the lecture notes of Crandall [7], and the books [3, 5, 14] for an up-to-date overview of the uniqueness machinery for viscosity solutions.

Following the ideas set forth by the general uniqueness theory for viscosity solutions, comparison principles for integro-differential equations were obtained by Soner [37, 38], Sayah [34, 35], Alvarez and Tourin [2], and Pham [32]. Under some assumptions, uniqueness results in the class of bounded uniformly continuous (semiconcave) functions were obtained in [38], see also [37]. The main result of [34] is a comparison theorem between bounded uniformly continuous subsolutions and supersolutions. In [35], this result is extended first to semicontinuous and then to unbounded sub- and supersolutions. In [2], the authors consider nonlinear integro-differential equations of parabolic type and obtain a comparison principle for semicontinuous, bounded and unbounded sub- and supersolutions. In [32], a comparison principle is proved for unbounded sub- and supersolutions of a integro-differential variational inequality associated with the optimal stopping time problem in a finite horizon of a controlled jump-diffusion process.

We consider here a class of integro-differential variational inequalities for which the comparison results in the literature do not (directly) apply. We prove for this class of variational inequalities a comparison theorem between unbounded continuous subsolutions and supersolutions. Inspired by Ishii and Lions [23] in their treatment of general boundary value problems, we handle the gradient constraint by producing strict supersolutions that are close to the supersolution in question. A similar approach has also been used in, e.g., [11] for a singular stochastic control problem (without an integral operator), see also [1]. To handle the state constraint we adapt the proof of Soner [36, 37], which here consists in building a test function so that the minimum associated with the supersolution cannot be on the boundary. When dealing with unbounded domains, it is well known that one has to specify the asymptotic behaviour of the functions being compared. However, due to the choice of a strict supersolution, it is sufficient to restrict our attention to a bounded domain when proving the comparison principle. This fact was also exploited in [1]. In Section 5, we extend our existence and uniqueness results to a class of second order degenerate elliptic integro-differential variational inequalities and point out some possible applications.

If we specialize to a utility function of HARA type, we are able to construct an explicit solution to the control problem. The derivation of our solution is motivated from Hindy and Huang [19]. In the jump process case, however, we are not able to find explicit expressions for all constants, but are only able to state integral equations which must be satisfied. This is the topic of Section 6. In Section 7 we consider a slight simplification of our control problem, namely Merton's problem with consumption. We carry through the calculations for the pure-jump process case, and state the necessary integral equations which must be solved to have a solution. We note that this is also treated by Framstad *et al.* [15], however, with a different model for the stock price than ours. They consider a stock price process which solves a geometric stochastic differential equation with jumps. By a verification theorem they provide an explicit solution of Merton's problem.

In the final section we discuss related models where the price is the solution to a stochastic differential equation with jumps. We show how to relate these models to our results.

For similar and other applications of viscosity solutions in mathematical finance, we refer to the lecture notes by Soner [39] and the references therein.

2 Formulation of the problem and the main result

Let $(\Omega, \mathcal{P}, \mathcal{F})$ be a probability space and $(\mathcal{F}_t)$ a given filtration satisfying the usual assumptions. We consider a financial market consisting of a stock and a bond. Assume that the value of the stock follows the stochastic process

$$S_t = S_0 e^{\mu t + L_t}, \tag{2.1}$$

where μ is a constant and L_t is a pure-jump Lévy process with Lévy-Khintchine decomposition

$$L_t = \xi t + \int_0^t \int_{|z|<1} z\, \tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} z\, N(dt, dz).$$

Here ξ is a constant, $N(dt, dz)$ is Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $dt \times \nu(dz)$, $\nu(dz)$ is a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ with the property

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, z^2)\, \nu(dz) < \infty, \tag{2.2}$$

and $\tilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz)$ is the compensator of N. The measure $\nu(dz)$ is called the Lévy measure. We choose to work with the unique càdlàg version of L_t and denote this also by L_t. By Itô's formula (see, e.g., [21]) we obtain the differential form of S_t:

$$\begin{aligned} dS_t = &\Big(\mu + \xi + \int_{\mathbb{R}\backslash\{0\}} \Big(e^{z\mathbf{1}_{|z|<1}} - 1 - z\mathbf{1}_{|z|<1}\Big)\, \nu(dz)\Big)\, S_t\, dt \\ &+ S_{t-} \int_{\mathbb{R}\backslash\{0\}} \Big(e^{z\mathbf{1}_{|z|<1}} - 1\Big)\, \tilde{N}(dt, dz) + S_{t-} \int_{\mathbb{R}\backslash\{0\}} \Big(e^{z\mathbf{1}_{|z|\geq 1}} - 1\Big)\, N(dt, dz), \end{aligned} \tag{2.3}$$

where $\mathbf{1}_A$ is the indicator function of a measurable set A. For this differential form to be well defined, we need to impose the following additional integrability condition on the Lévy measure:

$$\int_{\mathbb{R}\backslash\{0\}} \left|e^{z\mathbf{1}_{|z|\geq 1}} - 1\right| \nu(dz) < \infty. \tag{2.4}$$

Note that condition (2.4) is effective only when $z \geq 1$ due to (2.2), and says essentially that e^z is $\nu(dz)$ - integrable on $\{|z| \geq 1\}$. Under condition (2.4) we can rewrite the differential form of S_t as

$$(2.5) \qquad dS_t = \Big(\mu + \xi + \int_{I\!R\setminus\{0\}} \left(e^z - 1 - z\mathbf{1}_{|z|<1}\right)\nu(dz)\Big)\, S_t\, dt + S_{t-}\int_{I\!R\setminus\{0\}} \left(e^z - 1\right) \tilde{N}(dt, dz).$$

Note that under condition (2.4), $\int_0^t \mathrm{E}[S_s]\, ds < \infty$.

We let the bond have dynamics

$$dB_t = rB_t\, dt,$$

where $r > 0$ is the interest rate. Assume furthermore that $r < \hat{\mu}$, where we have introduced the short-hand notation

$$(2.6) \qquad \hat{\mu} = \mu + \xi + \int_{I\!R\setminus\{0\}} \left(e^z - 1 - z\mathbf{1}_{|z|<1}\right)\nu(dz).$$

Here, $r < \hat{\mu}$ means that the expected return of the stock is higher then the return of the bond. In (2.6), note that $e^z - 1 - z \geq 0$ for all $z \in I\!R$. Consider an investor who wants to put her money in the stock and the bond so as to maximize her utility. Let $\pi_t \in [0, 1]$ be the fraction of her wealth invested in the stock at time t, and assume that there are no transaction costs in the market.

If we denote her cumulative consumption up to time t by C_t, we have the wealth process $X_t^{\pi,C}$ given as

$$X_t^{\pi,C} = x - C_t + \int_0^t \left(r + (\hat{\mu} - r)\pi_s\right) X_s^{\pi,C}\, ds + \int_0^t \pi_{s-} X_{s-}^{\pi,C} \int_{I\!R\setminus\{0\}} \left(e^z - 1\right) \tilde{N}(ds, dz)$$

where x is the initial wealth. To incorporate the idea of local substitution, Hindy and Huang [19] introduce the process $Y_t^{\pi,C}$ modelling the average past consumption. The process has dynamics

$$(2.7) \qquad Y_t^{\pi,C} = ye^{-\beta t} + \beta e^{-\beta t}\int_{[0,t]} e^{\beta s}\, dC_s,$$

where $y > 0$ and β is a positive weighting factor. We shall frequently use the notation Y_t for $Y_t^{\pi,C}$ and X_t for $X_t^{\pi,C}$. The integral is interpreted pathwise in a Lebesgue-Stieltjes sense. The differential form of Y_t is

$$dY_t = -\beta Y_t\, dt + \beta\, dC_t.$$

The objective of the investor is to find an allocation process π_t^* and a consumption pattern C_t^* which optimizes the expected discounted utility over an investment horizon. We shall here focus on an investor with an infinite investment horizon. We define the value function as

$$(2.8) \qquad V(x, y) = \sup_{\pi, C \in \mathcal{A}_{x,y}} \mathrm{E}\Big[\int_0^\infty e^{-\delta t} U(Y_t^{\pi,C})\, dt\Big],$$

where $\delta > 0$ is the discount factor and $\mathcal{A}_{x,y}$ is a set of admissible controls. Let

$$\mathcal{D} = \Big\{(x, y) \in I\!R^2 : x > 0, y > 0\Big\}.$$

We say that a pair of controls is admissible for $x, y \in \overline{\mathcal{D}}$ and write $\pi, C \in \mathcal{A}_{x,y}$ if:

(c_i) C_t is an adapted process that is right continuous with left-hand limits (càdlàg), nondecreasing, with initial value $C_{0-} = 0$ and satisfies $\mathrm{E}[C_t] < \infty$. Note that C_t refers to the whole process so the inequality is to be understood for all $t \geq 0$.

(c_{ii}) π_t is progressively measurable with values in $[0, 1]$.

(c_{iii}) $X_t^{\pi,C} \geq 0$, $Y_t^{\pi,C} \geq 0$ almost everywhere for all $t \geq 0$.

Note that condition (c_{iii}) introduces a state space constraint into our control problem. The utility function $U : [0,\infty) \to [0,\infty)$ is assumed to have the following properties:

(u_i) $U \in C([0,\infty))$ is nondecreasing and concave.

(u_{ii}) There exists a constant $K > 0$ and $\gamma \in (0,1)$ such that $\delta > k(\gamma)$ and $U(x) \leq K(1+x)^\gamma$ for all nonnegative x, where

$$(2.9)\quad k(\gamma) = \max_{\pi\in[0,1]} \Big[\gamma(r + (\hat{\mu} - r)\pi) + \int_{I\!R\setminus\{0\}} \Big(\big(1+\pi(e^z-1)\big)^\gamma - 1 - \gamma\pi(e^z-1)\Big)\,\nu(dz)\Big].$$

By a Taylor expansion we see that the integral term of $k(\gamma)$ is well-defined in a neighbourhood of zero. The condition (2.4) ensures that the integral is finite outside this neighbourhood, which shows that (2.9) is finite for $\gamma \in (0,1]$. Recall that in the case of no integral operator in (2.9) (see [1]), $k(\gamma)$ maps $[0,\infty)$ onto $[0,\infty)$ with $k(0) = 0$ and is increasing. This is not the case when the integral operator is present. Then $k(\gamma) : (0,\infty) \to I\!R$ can be negative as well as non-monotone. Let us also mention that condition (u_{ii}) guarantees that the value function of the related Merton problem is well-defined, see Section 7.

In this paper we will assume that the dynamic programming principle holds; i.e., for any stopping time τ and $t \geq 0$,

$$(2.10)\qquad V(x,y) = \sup_{\pi,C\in\mathcal{A}_{x,y}} \mathrm{E}\Big[\int_0^{t\wedge\tau} e^{-\delta s} U(Y_s^{\pi,C})\,ds + e^{-\delta(t\wedge\tau)} V(X_{t\wedge\tau}, Y_{t\wedge\tau})\Big],$$

where $a \wedge b = \min(a,b)$. This intuitive but important principle can be proved by using methods from, e.g., [40]. The Hamilton-Jacobi-Bellman equation of our optimization problem is a nonlinear first order integro-differential equation subject to a gradient constraint:

$$(2.11)\qquad \begin{aligned} &\max\Big\{\beta v_y - v_x; U(y) - \delta v - \beta y v_y + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu}-r)\pi)x v_x + \\ &\int_{I\!R\setminus\{0\}} \Big(v(x+\pi x(e^z-1),y) - v(x,y) - \pi x v_x(x,y)(e^z-1)\Big)\,\nu(dz)\Big]\Big\} = 0 \text{ in } \mathcal{D}. \end{aligned}$$

Note that $x + \pi x(e^z - 1) \geq 0$ for all $x \geq 0$ and $z \in I\!R$. In Section 4 we prove that if v is C^2 and sublinearly growing, then (2.11) is well-defined. Moreover, if the value function V satisfies these conditions, then by Itô's formula one can easily prove that V solves (2.11). Although (2.11) only contains first order derivatives, the requirement $V \in C^2$ comes from the fact that the Lévy measure $\nu(dz)$ is possibly singular in zero.

In many applications the value function is not necessarily smooth, or it can be very difficult to prove sufficient regularity. Therefore we introduce an appropriate concept of weak solutions, namely viscosity solutions. With this concept at hand, we are able to prove that the value function V is the (only) solution of (2.11), even when it is not necessarily differentiable. However, if a viscosity solution is sufficiently regular, then, as is well known, it is a solution in the classical sense. The viscosity solution approach is by now a well established approach to control theory problems, see, e.g., the books [14, 3].

Our main result is the following theorem, which follows immediately from the results stated and proved in the Sections 3 and 4:

Theorem 2.1. *The value function V is the unique constrained viscosity solution of the integro-differential variational inequality* (2.11)*, i.e., V is a subsolution of* (2.11) *in $\overline{\mathcal{D}}$ and a supersolution of* (2.11) *in $\mathcal{D}$. The value function V satisfies the growth condition*

$$0 \leq V(x,y) \leq K(1+x+y)^\gamma, \qquad \forall x,y \in \overline{\mathcal{D}}.$$

Moreover, V is uniformly continuous in $\overline{\mathcal{D}}$. If for some $\alpha \in (0,1]$, $\delta > k(\alpha)$, and $U \in C^{0,\alpha}([0,\infty))$, then $V \in C^{0,\alpha}(\overline{\mathcal{D}})$. If $\delta > k(1+\alpha)$ and $U \in C^{1,\alpha}([0,\infty))$, then $V \in C^{1,\alpha}(\overline{\mathcal{D}})$.

Before ending this section, we show that the normal inverse Gaussian Lévy process introduced by Barndorff-Nielsen [6] satisfies the condition in (2.4). First, recall from [6] and [33] that the normal inverse Gaussian distribution is a mean-variance mixture of a normal distribution and an inverse Gaussian with density

$$\mathrm{nig}(x;\alpha,\beta,\mu,\delta)=\frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right)\frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}},$$

where K_1 is the modified Bessel function of the third kind and index 1 given as (for $y>0$)

$$K_1(y)=\frac{1}{2}\int_0^\infty \exp\left(-\tfrac{1}{2}y(x+x^{-1})\right)dx,$$

and $x\in I\!R,\mu\in I\!R,\delta>0,0\le|\beta|\le\alpha$. The parameters have the following meaning; α is the steepness of the distribution, β the asymmetry, μ the location and δ the scale[1]. If $\beta=0$ then the distribution is symmetric. In empirical studies one usually center the data and let $\mu=0$. In this case the Lévy measure is

$$\nu(dz)=\frac{\alpha\delta}{\pi|z|}e^{\beta z}K_1(\alpha|z|)\,dz.$$

For $z\ge 1$, we have

$$(e^z-1)\exp\left(-\tfrac{1}{2}\alpha z(x+x^{-1}\right)\le\exp\left(-\tfrac{1}{2}(\alpha-1)z(x+x^{-1})\right)$$

since $x+x^{-1}\ge 2$ for positive x. By adjusting the α parameter to $\alpha-1$ we have that $(e^z-1)\,\nu(dz)$ for $z\ge 1$ is dominated by another Lévy measure coming from a normal inverse Gaussian Lévy process. On the other hand, when $z\le -1$ we know that $|e^z-1|\le 1$. Since all Lévy measures integrate 1 for $|z|\ge 1$, we have that (2.4) holds whenever $\alpha>1$. In conclusion, when $\alpha>1$ the normal inverse Gaussian Lévy process satisfies the conditions in (2.4).

We recall from empirical studies by Rydberg [33] that the estimated α for two German and two Danish stocks were far greater than 1. For instance, the estimated parameters of Deutsche Bank for day-to-day ticks in the period October 1st, 1989 to December 29th, 1995 (1562 data points) were (see [33]) $(\alpha,\beta,\delta)=(75.49,-4.089,0.012)$. We conclude that a stock price model S_t for Deutsche Bank, where the logreturns are modelled by a normal inverse Gaussian distribution with the parameters above, will fit the framework presented in this paper.

3 Properties of the value function

In this section we prove that the value function V defined in (2.8) possesses certain growth, monotonicity, and regularity properties. The proofs of these results are inspired by the proofs of the corresponding results in [1].

Lemma 3.1. *The value function V is well defined in $\overline{\mathcal{D}}$ and satisfies $0\le V(x,y)\le K(1+x+y)^\gamma$ in $\overline{\mathcal{D}}$. Furthermore, $V(x,y)$ is nondecreasing and concave in $\overline{\mathcal{D}}$.*

Proof. The arguments used to prove that V is nondecreasing and concave on its convex domain are classical and thus omitted. We concentrate here on the growth condition. First, observe that for every $x,y\in\overline{\mathcal{D}}$, $\mathcal{A}_{x,y}$ is nonempty. This is so because for every π_t, $X^{\pi,0}$ is obviously nonnegative. Moreover, since the associated gain $\int_0^\infty e^{-\delta t}U(ye^{-\beta t})\,dt$ is nonnegative, V is also nonnegative. The upper bound is established in the following manner. Let $y>0$ and $\pi,C\in\mathcal{A}_{x,y}$. For $n>0$, consider the stopping time $\tau_n=\inf\{t\ge 0: X_t^{\pi,C}>n\}$. The process $Z_t=X_t+Y_t/\beta$ is bounded away from zero since $Y_t\ge ye^{-\beta t}$. Moreover, Z_t is a solution of

$$dZ_t=\left[(r+(\hat\mu-r)\pi_t)X_t-Y_t\right]dt+\pi_{t-}X_{t-}\int_{I\!R\setminus\{0\}}\left(e^z-1\right)\tilde N(dt,dz)$$

[1]The parameters δ and β are unrelated to the discounting factors in the control problem. The notation of the parameters used here are simply chosen to be consistent with the notation in [6] and [33].

with initial value $z = x + y/\beta$. Applying Itô's formula, the nonnegativity of X_t, Y_t, and the observation that $\frac{X_t}{Z_t}, \pi_t \frac{X_t}{Z_t} \in [0,1]$, we obtain

$$
\begin{aligned}
\mathrm{E}\big[Z^\gamma_{t\wedge\tau_n}\big] &= z^\gamma + \gamma \mathrm{E}\Big[\int_0^{t\wedge\tau_n} Z_s^{\gamma-1}\Big((r + (\hat{\mu} - r)\pi_s)X_s - Y_s\Big)\,ds\Big] \\
&\quad + \mathrm{E}\Big[\int_0^{t\wedge\tau_n}\Big(\int_{I\!R\setminus\{0\}}\Big(\big(Z_s + \pi_s X_s(e^z - 1)\big)^\gamma - Z_s^\gamma - \gamma\pi_s Z_s^{\gamma-1}X_s(e^z-1)\Big)\,\nu(dz)\Big)\,ds\Big] \\
&= z^\gamma + \gamma \mathrm{E}\Big[\int_0^{t\wedge\tau_n} Z_s^{\gamma}\Big((r + (\hat{\mu} - r)\pi_s)\tfrac{X_s}{Z_s} - \tfrac{Y_s}{Z_s}\Big)\,ds\Big] \\
&\quad + \mathrm{E}\Big[\int_0^{t\wedge\tau_n} Z_s^\gamma\Big(\int_{I\!R\setminus\{0\}}\Big(\big(1 + (\pi_s \tfrac{X_s}{Z_s})(e^z - 1)\big)^\gamma - 1 - \gamma(\pi_s\tfrac{X_s}{Z_s})(e^z-1)\Big)\,\nu(dz)\Big)ds\Big] \\
&\le z^\gamma + \mathrm{E}\Big[\int_0^{t\wedge\tau_n} Z_s^\gamma\Big(\gamma\big(r + (\hat{\mu} - r)(\pi_s\tfrac{X_s}{Z_s})\big) \\
&\quad + \int_{I\!R\setminus\{0\}}\Big(\big(1 + (\pi_s \tfrac{X_s}{Z_s})(e^z - 1)\big)^\gamma - 1 - \gamma(\pi_s\tfrac{X_s}{Z_s})(e^z-1)\Big)\,\nu(dz)\Big)\,ds\Big] \\
&\le z^\gamma + \mathrm{E}\Big[\int_0^{t\wedge\tau_n} Z_s^\gamma\,ds\Big]\,k(\gamma),
\end{aligned}
$$

where $k(\gamma)$ is defined in (2.9). Gronwall's lemma now yields $\mathrm{E}\big[Z^\gamma_{t\wedge\tau_n}\big] \le z^\gamma e^{k(\gamma)t}$. Letting $n \to \infty$, we have by Fatou's lemma that

$$
\mathrm{E}\big[Y_t^\gamma\big] \le K(x+y)^\gamma e^{k(\gamma)t}. \tag{3.1}
$$

Note that this bound also holds when $y = 0$ by continuity. The growth condition on the utility function U then implies that (recall $\delta > k(\gamma)$)

$$
\mathrm{E}\Big[\int_0^\infty e^{-\delta t}U(Y_t)\,dt\Big] \le K\int_0^\infty e^{-\delta t}\Big[1 + (x+y)^\gamma e^{k(\gamma)t}\Big]\,dt \le K(1 + x + y)^\gamma.
$$

Maximizing over $\mathcal{A}_{x,y}$ yields the desired upper bound. $\square$

Theorem 3.2. *The value function V is uniformly continuous in $\overline{\mathcal{D}}$. If for some $\alpha \in (0,1]$, $\delta > k(\alpha)$, and $U \in C^{0,\alpha}([0,\infty))$, then $V \in C^{0,\alpha}(\overline{\mathcal{D}})$. Furthermore, if $\delta > k(1+\alpha)$ and $U \in C^{1,\alpha}([0,\infty))$, then $V \in C^{1,\alpha}(\overline{\mathcal{D}})$.*

Proof. We first show how to compare admissible trajectories starting from different points. For $x, y, x', y' \in \overline{\mathcal{D}}$, let $\pi, C \in \mathcal{A}_{x,y}$ and define the stopping time $\tau = \inf\{t \ge 0 : X_t^{x'\pi,C} < 0\}$. When $x' \ge x$ we observe that $\tau = \infty$. Set

$$
\begin{aligned}
C'_t &= C_t\mathbf{1}_{t<\tau} + \big(\Delta X'^{\pi,C}_\tau + X'^{\pi,C}_{\tau-} + C_\tau\big)\mathbf{1}_{t\ge\tau}, \\
,_t &= C_t - C'_t = \big(C_t - \Delta X'^{\pi,C}_\tau - X'^{\pi,C}_{\tau-} - C_\tau\big)\mathbf{1}_{t\ge\tau}.
\end{aligned}
$$

We see that $C'_t = C_t$ and $,_t = 0$ when $x' \ge x$. Since $\Delta X'^{\pi,C}_\tau = -\Delta C_\tau + \pi_{\tau-}X'^{\pi,C}_{\tau-}\big(e^{\Delta L_\tau} - 1\big)$, we can show that $\Delta X'^{\pi,C}_\tau + X'^{\pi,C}_{\tau-} + C_\tau \ge C_{\tau-} + (1 - \pi_{\tau-})X'^{\pi,C}_{\tau-}$. We immediately see that C'_t is nondecreasing since $X'^{\pi,C}_{\tau-} \ge 0$. Similarly we have that $,$ is nondecreasing. We now calculate

$$
\begin{aligned}
X'^{\pi,C}_t\mathbf{1}_{t<\tau} &= X'^{\pi,C}_{t\wedge\tau} + \big(-\Delta X'^{\pi,C}_\tau - X'^{\pi,C}_{\tau-}\big)\mathbf{1}_{t\ge\tau} \\
&= x' - C'_t + \int_0^{t\wedge\tau}(r + (\hat{\mu} - r)\pi_s)X'^{\pi,C}_s\,ds + \int_0^{t\wedge\tau}\pi_{s-}X'^{\pi,C}_{s-}\int_{I\!R\setminus\{0\}}\big(e^z - 1\big)\,\tilde{N}(ds,dz) \\
&= x' - C'_t + \int_0^t (r + (\hat{\mu} - r)\pi_s)X'^{\pi,C}_s\mathbf{1}_{s<\tau}\,ds
\end{aligned}
$$

$$+\int_0^t \pi_{s-} X'^{\pi,C}_{s-} \mathbf{1}_{s<\tau} \int_{I\!R\setminus\{0\}} (e^z - 1)\, \tilde{N}(ds, dz)$$

By uniqueness we have $X'^{\pi,C'}_t = X'^{\pi,C}_t \mathbf{1}_{t<\tau}$. This implies that $X'^{\pi,C'}_t \geq 0$ and, when $x > x'$, $(X - X')^{\pi,\Gamma}_t = X^{\pi,C}_t - X'^{\pi,C'}_t = X^{\pi,C}_t - X'^{\pi,C}_t \mathbf{1}_{t<\tau} \geq 0$. This in particular leads to the conclusion that $\pi, C' \in \mathcal{A}_{x',y'}$ and $\pi, , \in \mathcal{A}_{|x-x'|,|y-y'|}$. Note that this is is trivial when $x' \geq x$. From the explicit form of Y_t, we get $|Y^{\pi,C}_t - Y'^{\pi,C'}_t| \leq |Y - Y'|^{\pi,\Gamma}_t$ and thus

$$\begin{aligned}
\mathrm{E}\Big[\int_0^\infty e^{-\delta t} U(Y^{\pi,C}_t)\, dt\Big] &\leq \mathrm{E}\Big[\int_0^\infty e^{-\delta t} U(Y'^{\pi,C'}_t)\, dt\Big] + \mathrm{E}\Big[\int_0^\infty e^{-\delta t} \omega_U(|Y - Y'|^{\pi,\Gamma}_t)\, dt\Big] \\
&\leq V(x', y') + \omega_V(|x - x'|, |y - y'|),
\end{aligned}$$

where ω_U denotes a modulus of continuity for U. We have used the notation ω_V for the value function when we replace U by ω_U. Maximizing over $\mathcal{A}_{x,y}$ and exchanging x, y and x', y', we obtain $|V(x, y) - V(x', y')| \leq \omega_V(|x - x'|, |y - y'|)$. In the case $U \in C^{0,\alpha}([0, \infty))$, we choose $\omega_U(z) = K z^\alpha$ and since $\delta > k(\alpha)$, we conclude from (3.1) that $\omega_V \leq K(x + y)^\alpha$ in $\overline{\mathcal{D}}$. Hence $V \in C^{0,\alpha}(\overline{\mathcal{D}})$. In general we choose $\omega_U(z) = \inf_{\varepsilon>0}(\varepsilon + K_\varepsilon z^\gamma)$ and obtain as before $\omega_V(z) \leq \inf_{\varepsilon>0}(\varepsilon + K_\varepsilon z^\gamma)$ since $\delta > k(\gamma)$. This implies that V is uniformly continuous. The proof of the $C^{1,\alpha}$ regularity is similar and we therefore omit the details, see instead [1].

□

4 Viscosity solutions

In this section we characterise the value function (2.8) as the unique constrained viscosity solution of the integro-differential variational inequality (2.11). To simplify the presentation, we will on several occasions employ the following notations: $X = (x_1, x_2) \in \overline{\mathcal{D}}$, $D_X = (\partial_{x_1}, \partial_{x_2})$, $G(D_X v) = \beta v_{x_2} - v_{x_1}$, and

$$\begin{aligned}
\mathcal{B}^\pi(X, v) &= \int_{I\!R\setminus\{0\}} \Big(v(x_1 + x_1\pi(e^z - 1), x_2) - v(x_1, x_2) - \pi x_1 v_{x_1}(X)(e^z - 1)\Big)\, \nu(dz), \\
F(X, v, D_X v, \mathcal{B}^\pi(X, v)) &= U(x_2) - \delta v - \beta x_2 v_{x_2} + \max_{\pi\in[0,1]} \Big[(r + (\hat{\mu} - r)\pi) x_1 v_{x_1} + \mathcal{B}^\pi(X, v)\Big].
\end{aligned}$$

Then (2.11) takes the following form

$$\max\Big(G(D_X v); F(X, v, D_X v, \mathcal{B}^\pi(X, v))\Big) = 0 \text{ in } \mathcal{D}. \tag{4.1}$$

Recall that the Lévy measure $\nu(dz)$ is a positive σ-finite measure on $I\!R\setminus\{0\}$ with a possible singularity in zero so that (2.2) holds. We thus need to be more specific about the meaning of the integro-differential operator $\mathcal{B}^\pi$. To this end, define the set

$$C_\ell(\overline{\mathcal{D}}) = \Big\{\phi \in C(\overline{\mathcal{D}}) : \sup_{\overline{\mathcal{D}}} \frac{|\phi(X)|}{(1 + x_1 + x_2)^\ell} < \infty\Big\}, \qquad \ell \geq 0.$$

For any $\kappa \in (0, 1)$, $X \in \overline{\mathcal{D}}$, $\phi \in C_1(\overline{\mathcal{D}})$, $P = (p_1, p_2) \in I\!R^2$, we define

$$\mathcal{B}^{\pi,\kappa}(X, \phi, P) = \int_{|z|>\kappa} \Big(\phi(x_1 + x_1\pi(e^z - 1), x_2) - \phi(X) - \pi x_1 p_1(e^z - 1)\Big)\, \nu(dz).$$

The integrand of $\mathcal{B}^{\pi,\kappa}(X, \phi, P)$ is bounded by $\mathrm{Const}(X, P, \kappa) \cdot \big(1 + |e^z - 1|\big)$ and, thanks to (2.4), the integral is convergent and bounded uniformly in π for every positive κ. For $\kappa \in (0, 1)$, $X \in \overline{\mathcal{D}}$, $\phi \in C^2(\overline{\mathcal{D}})$, we define

$$\mathcal{B}^\pi_\kappa(X, \phi) = \int_{|z|\leq\kappa} \Big(\phi(x_1 + x_1\pi(e^z - 1), x_2) - \phi(X) - \pi x_1 \phi_{x_1}(X)(e^z - 1)\Big)\, \nu(dz).$$

Note that $\phi(x_1+x_1\pi(e^z-1),x_2) = \phi(X)+\phi_{x_1}(X)\big(x_1\pi(e^z-1)\big)+\phi_{x_1x_1}(a,x_2)\big(x_1\pi(e^z-1)\big)^2$, where a is some point on the line between X and $(x_1+x_1\pi(e^z-1),x_2)$. Hence the integrand of $\mathcal{B}^\pi_\kappa(X,\phi)$ is bounded by $\mathrm{Const}(X,\kappa)\cdot|e^z-1|^2$, and the integral is convergent and bounded uniformly in π since every Lévy measure integrates $\frac{1}{z^2}$ in a neighbourhood of zero, see (2.2). Furthermore,

$$\lim_{\kappa\to 0+} \mathcal{B}^\pi_\kappa(X,\phi) = 0. \tag{4.2}$$

We now define for all $\phi \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$ the integro-differential operator $\mathcal{B}^\pi(X,\phi)$ by

$$\mathcal{B}^\pi(X,\phi) := \mathcal{B}^{\pi,\kappa}(X,\phi,D_X\phi) + \mathcal{B}^\pi_\kappa(X,\phi). \tag{4.3}$$

Consequently, the Hamilton-Jacobi-Bellman (4.1) is well defined for all $v \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$. However, in many applications the value function defined in (2.8) is not C^2 or even C^1 (see Sections 3, 6, and 7). and the equation (4.1) should be interpreted in a weaker sense. As discussed in Section 1, we here suitably adopt the notion of constrained viscosity solutions [36, 37, 12]. Constrained viscosity solutions are functions that are supersolutions of (2.11) in $\mathcal{D}$ and subsolutions of (2.11) in $\overline{\mathcal{D}}$. The latter requirement plays the role of a boundary condition, see [36, 37, 12].

The precise definition goes as follows:

Definition 4.1. (i) Let $\mathcal{O} \subset \overline{\mathcal{D}}$. Any $v \in C(\overline{\mathcal{D}})$ is a *viscosity subsolution* (*supersolution*) of (4.1) in $\mathcal{O}$ if and only if we have, for every $X \in \mathcal{O}$ and $\phi \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$ such that X is a global maximum (minimum) relative to $\mathcal{O}$ of $v-\phi$,

$$\max\Big(G(D_X\phi); F(X,v,D_X\phi,\mathcal{B}^\pi(X,\phi))\Big) \geq 0\,(\leq 0). \tag{4.4}$$

(ii) Any $v \in C(\overline{\mathcal{D}})$ is a *constrained viscosity solution* of (4.1) if and only if v is a supersolution of (4.1) in $\mathcal{D}$ and v is a subsolution of (4.1) in $\overline{\mathcal{D}}$.

Hereafter we use the terms *subsolution* and *supersolution* instead of viscosity subsolution and viscosity supersolution, respectively. For $\kappa > 0$, $\phi \in C^2(\overline{\mathcal{D}})$, $v \in C_1(\overline{\mathcal{D}})$ let us introduce the function

$$\begin{aligned} &F(X,v,D_X\phi,\mathcal{B}^{\pi,\kappa}(X,v,D_X\phi),\mathcal{B}^\pi_\kappa(X,\phi)) \\ &\quad = U(x_2) - \delta v - \beta x_2\phi_{x_2} + \max_{\pi\in[0,1]}\Big[(r+(\hat\mu-r)\pi)x_1\phi_{x_1} + \mathcal{B}^{\pi,\kappa}(X,v,D_X\phi) + \mathcal{B}^\pi_\kappa(X,\phi)\Big]. \end{aligned}$$

Note that $\mathcal{B}^{\pi,\kappa}(X,v,D_X\phi)$ and $\mathcal{B}^\pi_\kappa(X,\phi)$ are well defined and bounded independently of π.

We now have an equivalent formulation of viscosity solutions in $C_1(\overline{\mathcal{D}})$.

Lemma 4.1. *Let $v \in C_1(\overline{\mathcal{D}})$ and $\mathcal{O} \subset \overline{\mathcal{D}}$. Then v is a viscosity subsolution (supersolution) of (4.1) in $\mathcal{O}$ if and only if we have, for every $\phi \in C^2(\overline{\mathcal{D}})$ and $\kappa > 0$,*

$$\max\Big(G(D_X\phi); F(X,v,D_X\phi,\mathcal{B}^{\pi,\kappa}(X,v,D_X\phi),\mathcal{B}^\pi_\kappa(X,\phi))\Big) \geq 0\,(\leq 0) \tag{4.5}$$

whenever $X \in \mathcal{O}$ is a global maximum (minimum) relative to $\mathcal{O}$ of $v-\phi$.

Proof. We prove the statement only for the subsolutions, the supersolution case can be proved similarly. Suppose $v \in C_1(\overline{\mathcal{D}})$ satisfies

$$F(X,v,D_X\phi,\mathcal{B}^{\pi,\kappa}(X,v,D_X\phi),\mathcal{B}^\pi_\kappa(X,\phi)) \geq 0, \tag{4.6}$$

where $X \in \mathcal{O}$ is a global maximum relative to $\mathcal{O}$ of $v-\phi$, $\phi \in C^2(\overline{\mathcal{D}})$. Then, since $X \in \mathcal{O}$ is a global maximum, $v(Y)-v(X) \leq \phi(Y)-\phi(X)$ for all $Y \in \mathcal{O}$. Consequently, since $\mathcal{B}^{\pi,\kappa}(X,\phi,D_X\phi) \geq \mathcal{B}^{\pi,\kappa}(X,v,D_X\phi)$, we can use (4.3) and (4.6) to conclude that

$$F(X,v,D_X\phi,\mathcal{B}^\pi(X,\phi)) = F(x,v,D_X\phi,\mathcal{B}^{\pi,\kappa}(X,\phi,D_X\phi),\mathcal{B}^\pi_\kappa(X,\phi)) \geq 0.$$

This implies that v is a subsolution of (4.1) in $\mathcal{O}$ if (4.5) holds.

Conversely, let $v \in C_1(\overline{\mathcal{D}})$ be a subsolution of (4.1) in $\mathcal{O}$ and assume that

$$F(X, v, D_X \phi, \mathcal{B}^\pi(X, \phi)) \geq 0,$$

where $X \in \mathcal{O}$ is a global maximum relative to $\mathcal{O}$ of $v - \phi$, $\phi \in C^2(\overline{\mathcal{D}})$. Let χ_n be a smooth function satisfying $0 \leq \chi_n \leq 1$, $\chi_n(Y) = 1$ for $Y \in \mathcal{N}(X, x_1(e^\kappa - 1 - \frac{1}{n})) \cap \mathcal{O}$, and $\chi_n(Y) = 0$ for $Y \in \mathcal{O} \backslash (\mathcal{N}(X, x_1(e^\kappa - 1)) \cap \mathcal{O})$. Here $\mathcal{N}(X, R)$ denotes the open ball centred in X with radius R. Then define the test function $\psi_n(Y) = \chi_n(Y)\phi(Y) + (1 - \chi_n(Y))v_n(Y)$, where $v_n \in C^2(\overline{\mathcal{D}})$ is such that $v_n \to v$ a.e. in $\mathcal{O} \backslash (\mathcal{N}(X, x_1(e^\kappa - 1)) \cap \mathcal{O})$. Observe that $\psi_n = \phi$ in $\mathcal{N}(X, x_1(e^\kappa - 1 - \frac{1}{n})) \cap \mathcal{O}$, $\psi_n \to \phi$ in $\mathcal{N}(X, x_1(e^\kappa - 1)) \cap \mathcal{O}$, $\psi_n = v_n$ in $\mathcal{O} \backslash (\mathcal{N}(X, x_1(e^\kappa - 1)) \cap \mathcal{O})$, and X is a global maximum relative to $\mathcal{O}$ of $v - \psi_n$. Therefore,

$$\begin{aligned} 0 \leq F(X, v, D_X \psi_n, \mathcal{B}^\pi(X, \psi_n)) &= F(X, v, D_X \phi, \mathcal{B}^{\pi,\kappa}(X, \psi_n, D_X \psi_n), \mathcal{B}^\pi_\kappa(X, \psi_n)) \\ &\to F(X, v, D_X \phi, \mathcal{B}^{\pi,\kappa}(X, v, D_X \phi), \mathcal{B}^\pi_\kappa(X, \phi)), \end{aligned}$$

where we have used Lebesgue's dominated convergence theorem to conclude that

$$\mathcal{B}^{\pi,\kappa}(X, \psi_n, D_X \psi_n) = \mathcal{B}^{\pi,\kappa}(X, v_n, D_X \phi) \to \mathcal{B}^{\pi,\kappa}(X, v, D_X \phi), \quad \mathcal{B}^\pi_\kappa(X, \psi_n) \to \mathcal{B}^\pi_\kappa(X, \phi).$$

This implies that (4.5) holds if $v \in C_1(\overline{\mathcal{D}})$ is a subsolution of (4.1) in $\mathcal{O}$. □

It is convenient to use Definition 4.1 when proving existence of a constrained viscosity solution, whereas the formulation based on Lemma 4.1 is more convenient when proving uniqueness. We also note that Lemma 4.1 is an adaption of a similar lemma in Soner [36], see also Sayah [34].

The following easy result will be useful when proving Theorem 4.3 below.

Lemma 4.2. *If $(x', y') \in \overline{\mathcal{D}}$ and $(x, y) \in \overline{\mathcal{D}}$ satisfy $x = x' - c$ and $y = y' + \beta c$ for some $c > 0$, then $V(x, y) \leq V(x', y')$.*

We next characterize V as a viscosity solution of the Hamilton-Jacobi-Bellman equation (2.11).

Theorem 4.3. *The value function $V(x, y)$ is a constrained viscosity solution of* (2.11).

Proof. We first prove that V is a supersolution in $\mathcal{D}$. Let $\phi \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$ and $(x, y) \in \mathcal{D}$ be a global minimizer of $V - \phi$. Without any loss of generality we may assume that $(V - \phi)(x, y) = 0$. For every $c \in (0, x]$, we choose $C_0 = c$ and $t = 0$ in the dynamic programming principle (2.10), which then yields

$$\phi(x, y) = V(x, y) \geq V(x - c, y + \beta c) \geq \phi(x - c, y + \beta c).$$

Dividing by c and sending $c \to 0$, we conclude

$$\phi_x(x, y) - \beta \phi_y(x, y) \geq 0. \tag{4.7}$$

Let τ_ρ be the exit time from the closed ball $\mathcal{N}_\rho$ with radius ρ and centre at (x, y). By choosing ρ small enough, $\mathcal{N}_\rho \subset \mathcal{D}$. Applying the dynamic programming principle (2.10) with $h \wedge \tau_\rho$, $\pi_t = \pi$, $C_t = 0$, Itô's formula, and the inequality $V \geq \phi$, we obtain

$$\begin{aligned} 0 \geq &\mathrm{E}\Big[\int_0^{h\wedge\tau_\rho} e^{-\delta t} U(Y_t^{\pi,C})\, dt + e^{-\delta(h\wedge\tau_\rho)} V(X_{h\wedge\tau_\rho}, Y_{h\wedge\tau_\rho})\Big] - \phi(x, y) \\ &\geq \mathrm{E}\Big[\int_0^{h\wedge\tau_\rho} e^{-\delta t} \Big\{U(Y_t^{\pi,C}) - \delta\phi - \beta Y_t \phi_y + (r + (\hat{\mu} - r)\pi) X_t \phi_x + \mathcal{B}^\pi((X_t, Y_t), \phi)\Big\}\, dt\Big] \\ &\geq \mathrm{E}\Big[\frac{1 - e^{-\delta(h\wedge\tau_\rho)}}{\delta}\Big] \inf_{(x,y)\in\mathcal{N}_\rho} \Big[U(y) - \delta V - \beta y \phi_y + (r + (\hat{\mu} - r)\pi) x \phi_x + \mathcal{B}^\pi((x, y), \phi)\Big]. \end{aligned}$$

By the right continuity of the paths, $\tau_\rho > 0$ a.s. Hence, by Lebesgue's dominated convergence theorem, $\lim_{h\to 0} \mathrm{E}\left[\frac{1-e^{-\delta(h\wedge\tau_\rho)}}{h}\right] = \delta$. Dividing the inequality by h, sending $h \to 0$, and then sending $\rho \to 0$, we obtain

$$U(y) - \delta V - \beta y\phi_y + (r + (\hat{\mu} - r)\pi)x\phi_x + \mathcal{B}^\pi((x,y),\phi) \leq 0,$$

for every $\pi \in [0,1]$. Hence, from this and (4.7), we have proven that V is a viscosity supersolution.

We now prove that V is a subsolution in $\overline{\mathcal{D}}$. Let $\phi \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$ and $(x,y) \in \overline{\mathcal{D}}$ be a global maximizer of $V - \phi$. Without any loss of generality we may assume $(V-\phi)(x,y) = 0$ and that the maximum is strict. Arguing by contradiction, we suppose that the subsolution inequality (4.4) is violated. Then, by continuity, there is a nonempty open ball $\mathcal{N}$ centred at (x,y) and $\varepsilon > 0$ such that $\beta\phi_y - \phi_x \leq 0$ and

$$U(y) - \delta V - \beta y\phi_y + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu} - r)\pi)x\phi_x + +\mathcal{B}^\pi((x,y),\phi)\Big] \leq -\varepsilon\delta \text{ in } \overline{\mathcal{N}\cap\mathcal{D}},$$

as well as $V \leq \phi - \varepsilon$ on $\partial\mathcal{N}\cap\mathcal{D}$. For $\pi, C \in \mathcal{A}_{x,y}$, let τ^* be the exit time from $\overline{\mathcal{N}\cap\mathcal{D}}$. Since C_t is a singular control with a possible jump at $t = 0$, the state process (X_t, Y_t) might jump out of $\overline{\mathcal{N}\cap\mathcal{D}}$ at once. If the control C_t makes the state process jump out of $\overline{\mathcal{N}\cap\mathcal{D}}$, we know the direction of the jump and from Lemma 4.2 that V is nonincreasing in this direction. However, in our case the Lévy processes itself can cause the state process to jump out of $\overline{\mathcal{N}\cap\mathcal{D}}$. In this case, V is not necessarily nonincreasing in the direction of the jump. To overcome this problem we introduce τ_L, the first time the state process jumps because of the Lévy process, and note that $\tau_L > 0$ a.s. We have now two cases to consider.

If $\tau^* < \tau_L$ we know that the control C_t has made the state process jump out of $\overline{\mathcal{N}\cap\mathcal{D}}$. For $\tau^* \leq 1$, let (x', y') be the intersection between $\partial\mathcal{N}$ and the line between $(X_{\tau-}, Y_{\tau-})$ and (X_τ, Y_τ). Note that the slope vector of this line is $(-1, \beta)$ and that ϕ is nonincreasing along this line in $\overline{\mathcal{N}\cap\mathcal{D}}$. Thanks to Lemma 4.2, we also know that V is nonincreasing along this line in $\overline{\mathcal{D}}$. Hence,

$$V(X_{\tau^*}, Y_{\tau^*}) \leq V(x', y') \leq \phi(x', y') - \varepsilon \leq \phi(X_{\tau^*-}, Y_{\tau^*-}) - \varepsilon.$$

Using the inequalities above and Itô's formula for semimartingales, we obtain (with C_t^c denoting the continuous part of C_t)

$$\begin{aligned}
&\mathrm{E}\Big[\int_0^{1\wedge\tau^*} e^{-\delta t}U(Y_t^{\pi,C})\,dt + e^{-\delta(1\wedge\tau^*)}V(X_{1\wedge\tau^*}, Y_{1\wedge\tau^*})\Big] \\
&\quad\leq \mathrm{E}\Big[\int_0^{1\wedge\tau^*} e^{-\delta t}U(Y_t^{\pi,C})\,dt + e^{-\delta(1\wedge\tau^*)}\phi(X_{1\wedge\tau^*-}, Y_{1\wedge\tau^*-}) - \varepsilon e^{-\delta(1\wedge\tau^*)}\Big] \\
&\quad\leq \phi(x,y) - \varepsilon\mathrm{E}\Big[e^{-\delta\tau^*}\mathbf{1}_{\tau^*\leq 1}\Big] \\
&\qquad + \mathrm{E}\Big[\int_0^{1\wedge\tau^*} e^{-\delta t}\Big\{U(Y_t^{\pi,C}) - \delta\phi - \beta Y_t\phi_y + (r + (\hat{\mu} - r)\pi)X_t\phi_x + \mathcal{B}^\pi((X_t,Y_t),\phi)\Big\}\,dt\Big] \\
&\qquad + \mathrm{E}\Big[\int_0^{1\wedge\tau^*} e^{-\delta t}\big(-\phi_x + \beta\phi_y\big)\,dC_t^c\Big] + \mathrm{E}\Big[\sum_{[0,1]\cap[0,\tau^*)} e^{-\delta t}\big(\phi(X_{t-} + \Delta C_t, Y_{t-}) - \phi(X_{t-}, Y_{t-})\big)\Big] \\
&\quad\leq \phi(x,y) - \varepsilon\mathrm{E}\Big[e^{-\delta\tau^*}\mathbf{1}_{\tau^*\leq 1} + (1 - e^{-\delta(1\wedge\tau^*)})\Big] \leq \phi(x,y) - \varepsilon(1 - e^{-\delta}).
\end{aligned}$$

The dynamic programming principle (2.10) with $t = 1$ gives a contradiction since $(V-\phi)(x,y) = 0$.

If $\tau^* \geq \tau_L$, let τ be a stopping time such that $0 < \tau < \tau^*$. Using that $V \leq \phi$ and Itô's formula for semimartingales, we obtain

$$\mathrm{E}\Big[\int_0^{1\wedge\tau} e^{-\delta t}U(Y_t^{\pi,C})\,dt + e^{-\delta(1\wedge\tau)}V(X_{1\wedge\tau}, Y_{1\wedge\tau})\Big]$$

$$
\begin{aligned}
&\le \mathrm{E}\Big[\int_0^{1\wedge\tau} e^{-\delta t}U(Y_t^{\pi,C})\,dt + e^{-\delta(1\wedge\tau)}\phi(X_{1\wedge\tau},Y_{1\wedge\tau})\Big]\\
&\le \phi(x,y) + \mathrm{E}\Big[\int_0^{1\wedge\tau} e^{-\delta t}\Big\{U(Y_t^{\pi,C}) - \delta\phi - \beta Y_t\phi_y + (r+(\hat\mu - r)\pi)X_t\phi_x + \mathcal{B}^\pi((X_t,Y_t),\phi)\Big\}\,dt\Big]\\
&\quad + \mathrm{E}\Big[\int_0^{1\wedge\tau} e^{-\delta t}\big(-\phi_x+\beta\phi_y\big)\,dC_t^c\Big] + \mathrm{E}\Big[\sum_{[0,1]\cap[0,\tau]} e^{-\delta t}\big(\phi(X_{t-}+\Delta C_t, Y_{t-}) - \phi(X_{t-},Y_{t-})\big)\Big]\\
&\le \phi(x,y) - \varepsilon\mathrm{E}\Big[(1-e^{-\delta(1\wedge\tau)})\Big].
\end{aligned}
$$

The proof is now finished after observing that the dynamic programming principle (2.10) with $t=1$ also in this case gives a contradiction since $(V-\phi)(x,y)=0$. $\square$

We next demonstrate that it is possible to construct strict supersolutions of (4.1) in $\mathcal{D}$. To simplify the presentation, we employ the notations provided by (4.1).

Lemma 4.4. *For $\gamma' > 0$ such that $\delta > k(\gamma')$, let $v \in C_{\gamma'}(\overline{\mathcal{D}})$ be a supersolution of (4.1) in $\mathcal{D}$. Choose $\overline{\gamma} > \max(\gamma,\gamma')$ such that $\delta > k(\overline{\gamma})$, and let*

$$
w = K + \chi^{\overline{\gamma}}, \qquad \chi(X) = \big(1 + x_1 + \tfrac{x_2}{2\beta}\big).
$$

Then for K large enough, $w \in C^\infty(\mathcal{D}) \cap C_{\overline{\gamma}}(\overline{\mathcal{D}})$ is a strict supersolution of (4.1) in $\mathcal{D}$. Moreover, for $\theta \in (0,1]$, the function

$$
v^\theta = (1-\theta)v + \theta w \in C_{\overline{\gamma}}(\overline{\mathcal{D}})
$$

is a strict supersolution of (4.1) in $\mathcal{D}$.

Proof. We first claim that

$$
\max\Big(G(D_X w); F(X, w, D_X w, \mathcal{B}^\pi(X,w))\Big) \le -f, \tag{4.8}
$$

for some strictly positive $f \in C(\overline{\mathcal{D}})$. Observe that $G(D_X w) = \beta w_{x_2} - w_{x_1} = -\frac{\overline{\gamma}}{2}\chi^{\overline{\gamma}-1}$. Next, exploiting that $\frac{x_1}{\chi}, \pi\frac{x_1}{\chi} \in [0,1]$, we have

$$
\begin{aligned}
F(X,&w, D_X w, B(X,w)) = U(x_2) - \delta(K+\chi^{\overline{\gamma}}) - \tfrac12 x_2\overline{\gamma}\chi^{\overline{\gamma}-1} + \max_{\pi\in[0,1]}\Big[\overline{\gamma}(r+(\hat\mu-r)\pi)x_1\chi^{\overline{\gamma}-1}\\
&\quad + \int_{I\!R\setminus\{0\}}\Big(\big(\chi + \pi x_1(e^z-1)\big)^{\overline{\gamma}} - \chi^{\overline{\gamma}} - \overline{\gamma}\pi x_1\chi^{\overline{\gamma}-1}(e^z-1)\Big)\,\nu(dz)\Big]\\
&= U(x_2) - \delta K - \tfrac12 x_2\overline{\gamma}\chi^{\overline{\gamma}-1} + \Big(-\delta + \max_{\pi\in[0,1]}\Big[\overline{\gamma}(r+(\hat\mu-r)\pi)\tfrac{x_1}{\chi}\\
&\quad + \int_{I\!R\setminus\{0\}}\Big(\big(1+\pi\tfrac{x_1}{\chi}(e^z-1)\big)^{\overline{\gamma}} - 1 - \overline{\gamma}\pi\tfrac{x_1}{\chi}(e^z-1)\Big)\,\nu(dz)\Big]\Big)\chi^{\overline{\gamma}}\\
&\le U(x_2) - \delta K + \Big(-\delta + \max_{\pi\in[0,1]}\Big[\overline{\gamma}(r+(\hat\mu-r)\pi)\\
&\quad + \int_{I\!R\setminus\{0\}}\big((1+\pi(e^z-1))^{\overline{\gamma}} - 1 - \overline{\gamma}\pi(e^z-1)\big)\,\nu(dz)\Big]\Big)\chi^{\overline{\gamma}}\\
&= U(x_2) - \delta K + \big(k(\overline{\gamma}) - \delta\big)\chi^{\overline{\gamma}} \le -1
\end{aligned}
$$

by choosing, e.g., $\delta K = 1 + \sup_{\overline{\mathcal{D}}}\big[U(x_2) - \big(\delta - k(\overline{\gamma})\big)\chi^{\overline{\gamma}}\big]$. Note that $\delta K < \infty$ since $\delta > k(\overline{\gamma})$ and $\overline{\gamma} > \gamma$. Consequently, our claim (4.8) holds provided we set $f = \min(1, \frac{\overline{\gamma}}{2}\chi^{\overline{\gamma}-1})$.

Next, we claim that v^θ is a strict supersolution of (4.1) in $\mathcal{D}$. Note that for any $\phi \in C^2$, $X \in \mathcal{D}$ is a global minimum of $v - \phi$ if and only if X is a global minimum of $v^\theta - \phi^\theta$, where $\phi^\theta = (1-\theta)\phi + \theta w$. First, since v is a supersolution of (4.1) in $\mathcal{D}$, we have $G(D_X\phi) = \beta\phi_{x_2} - \phi_{x_1} \le 0$

and hence $G(D_X\phi^\theta) = (1-\theta)(\beta\phi_{x_2} - \phi_{x_1}) + \theta(\beta w_{x_2} - w_{x_1}) \leq -\theta\frac{\overline{\gamma}}{2}\chi^{\overline{\gamma}-1}$. Letting $\pi^* \in [0,1]$ be a maximizer of $(r + (\hat{\mu} - r)\pi)x_1\phi^\theta_{x_1} + \mathcal{B}^\pi(X, \phi^\theta)$, we can calculate as follows

$$
\begin{aligned}
F(X, v^\theta, D_X\phi^\theta, \mathcal{B}^\pi(X, v^\theta)) &= (1-\theta)U(x_2) - \delta(1-\theta)v - \beta x_2(1-\theta)\phi_{x_2} \\
&\quad + (r + (\hat{\mu} - r)\pi^*)x_1(1-\theta)\phi_{x_1} + (1-\theta)\mathcal{B}^{\pi^*}(X, \phi) \\
&\quad + \theta U(x_2) - \delta\theta w - \beta x_2\theta w_{x_2} + (r + (\hat{\mu} - r)\pi^*)x_1\theta w_{x_1} + \theta\mathcal{B}^{\pi^*}(X, w) \\
&\leq (1-\theta)F(X, v, D_X\phi, \mathcal{B}^\pi(X, \phi)) + \theta F(X, w, D_X w, \mathcal{B}^\pi(X, w)) \leq -\theta f.
\end{aligned}
$$

Summing up, we have just shown that

$$
\max\Big(G(D_X\phi^\theta); F(X, v^\theta, D_X\phi^\theta, \mathcal{B}^\pi(X, \phi^\theta)\Big) \leq -\theta f.
$$

□

Following the general viscosity solution technique, we next present a comparison principle for constrained viscosity solutions of (2.11). This comparison principle immediately implies that the value function (2.8) is the *only* solution of (2.11). For orientation, we mention once more that the comparison results in [37, 38, 34, 35, 2, 32] do not apply in our context. Having said this, we do not hesitate to point out that our comparison principle is nevertheless inspired by these results. To simplify the presentation, we use again the notations provided by (4.1).

Theorem 4.5. *Let $\gamma' > 0$ be such that $\delta > k(\gamma')$. Assume $\underline{v} \in C_{\gamma'}(\overline{\mathcal{D}})$ is a subsolution of* (4.1) *in $\overline{\mathcal{D}}$ and $\overline{v} \in C_{\gamma'}(\overline{\mathcal{D}})$ is a supersolution of* (4.1) *in $\mathcal{D}$. Then $\underline{v} \leq \overline{v}$ in $\overline{\mathcal{D}}$.*

Proof. Choose $\overline{\gamma} > \gamma'$ such that $\delta > k(\overline{\gamma})$ and then introduce the function

$$
w = \tilde{K} + \big(1 + x_1 + \tfrac{x_2}{2\beta}\big)^{\overline{\gamma}}.
$$

Now choose $\tilde{K}$ so large that, by Proposition 4.4, $\overline{v}^\theta = (1-\theta)\overline{v} + \theta w$, $\theta \in (0,1]$, is a strict supersolution of (4.1) in $\mathcal{D}$. Instead of comparing $\underline{v}$ and $\overline{v}$, we will compare $\underline{v}$ and $\overline{v}^\theta$. Then by simply sending $\theta \to 0+$, we obtain the desired comparison result $\underline{v} \leq \overline{v}$ in $\overline{\mathcal{D}}$. Observe that

$$
\underline{v}(X) - \overline{v}^\theta(X) \leq \text{Const} \cdot (1 + x_1 + x_2)^{\gamma'} - \theta\big(1 + x_1 + \tfrac{x_2}{2\beta}\big)^{\overline{\gamma}} \to -\infty \text{ as } X \to \infty. \tag{4.9}
$$

In view of (4.9), we can choose $R > 0$ so large that $\underline{v} \leq \overline{v}^\theta$ in $\{x_1, x_2 \geq R\}$. Although $\mathcal{D}$ is unbounded, we can then nevertheless restrict our attention to the bounded domain

$$
\mathcal{K} = \Big\{(x_1, x_2) : 0 < x_1 < R + Re^1,\, 0 < x_2 < R\Big\} \tag{4.10}
$$

and prove that $\underline{v} \leq \overline{v}^\theta$ in $\overline{\mathcal{K}}$. To this end, assume to the contrary that

$$
M := \sup_{\overline{\mathcal{K}}}(\underline{v} - \overline{v}^\theta) = (\underline{v} - \overline{v}^\theta)(Z) > 0 \tag{4.11}
$$

for some $Z \in \overline{\mathcal{K}}$. Observe that we have only the two cases $Z \in (0, R) \times (0, R)$ and $Z \in , _{\mathrm{SC}}$ to consider, where

$$
, _{\mathrm{SC}} = \Big\{(x_1, x_2) : x_1 = 0,\, 0 \leq x_2 < R \text{ or } 0 \leq x_1 < R,\, x_2 = 0\Big\}. \tag{4.12}
$$

is the state constraint boundary restricted by R.

Case I: Let us first consider the case $Z \in , _{\mathrm{SC}}$. The construction presented below is a suitable adaption of the construction of Soner [36, 37]. Since $\partial\mathcal{K}$ is piecewise linear there exist positive constants h, R and a uniformly continuous map $\eta : \overline{\mathcal{K}} \to \mathbb{R}^2$ satisfying

$$
B(X + t\eta(X), Rt) \subset \mathcal{K} \text{ for all } X \in \overline{\mathcal{K}} \text{ and } t \in (0, h]. \tag{4.13}
$$

For any $\alpha > 1$ and $0 < \varepsilon < 1$, define the function $\Phi(X,Y)$ on $\overline{\mathcal{K}} \times \overline{\mathcal{K}}$ by

$$\Phi(X,Y) = \underline{v}(X) - \overline{v}^\theta(Y) - |\alpha(X-Y) + \varepsilon\eta(Z)|^2 - \varepsilon|X-Z|^2. \tag{4.14}$$

Let $M_\alpha = \sup_{\overline{\mathcal{K}}\times\overline{\mathcal{K}}} \Phi(X,Y)$. We then have $M_\alpha \geq M > 0$ for any $\alpha > 1$ and $\varepsilon \leq \varepsilon_0$, where ε_0 is some fixed small number. Let $(X_\alpha, Y_\alpha) \in \overline{\mathcal{K}} \times \overline{\mathcal{K}}$ be a maximizer of Φ, i.e., $M_\alpha = \Phi(X_\alpha, Y_\alpha)$. By (4.13), we assume that α is so large that $Z + \frac{\varepsilon}{\alpha}\eta(Z) \in \mathcal{K}$. The inequality $\Phi(X_\alpha, Y_\alpha) \geq \Phi(Z, Z + \frac{\varepsilon}{\alpha}\eta(Z))$ reads

$$\begin{aligned}
&|\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z)|^2 + \varepsilon|X_\alpha - Z|^2 \\
&\quad \leq \underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha) - \big(\underline{v} - \overline{v}^\theta\big)(Z) + \overline{v}^\theta(Z + \tfrac{\varepsilon}{\alpha}\eta(Z)) - \overline{v}^\theta(Z).
\end{aligned}$$

Since $\underline{v}, -\overline{v}^\theta$ are bounded on $\overline{\mathcal{K}}$, it follows that $|\alpha(X_\alpha - Y_\alpha)|^2$ is bounded uniformly in α and thus $X_\alpha - Y_\alpha \to 0$ as $\alpha \to \infty$. Consequently, for some modulus of continuity $\omega(\cdot)$, we get

$$\begin{aligned}
&|\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z)|^2 + \varepsilon|X_\alpha - Z|^2 \\
&\quad \leq \limsup_{\alpha\to\infty}\big(\underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha)\big) - \big(\underline{v} - \overline{v}^\theta\big)(Z) + \omega(\tfrac{1}{\alpha}) = \omega(\tfrac{1}{\alpha}) \to 0 \text{ as } \alpha \to \infty,
\end{aligned}$$

which implies $\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z) \to 0$ and $X_\alpha, Y_\alpha \to Z$ as $\alpha \to \infty$. Moreover, we have $\limsup_{\alpha\to\infty}\big(\underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha)\big) = M$. Therefore, using also the uniform continuity of η, $Y_\alpha = X_\alpha + \frac{\varepsilon}{\alpha}\eta(Z) + o(\frac{1}{\alpha}) = X_\alpha + \frac{\varepsilon}{\alpha}\eta(X_\alpha) + o(\frac{1}{\alpha})$ and we thus use (4.13) to get $Y_\alpha \in \mathcal{K}$ for α large enough. In fact, we must have $Y_\alpha \in (0,R)\times(0,R)$ for α large enough. Now set

$$\begin{aligned}
\psi(Y) &= \underline{v}(X_\alpha) - |\alpha(X_\alpha - Y) + \varepsilon\eta(Z)|^2 - \varepsilon|X_\alpha - Z|^2, \\
\phi(X) &= \overline{v}^\theta(Y_\alpha) + |\alpha(X - Y_\alpha) + \varepsilon\eta(Z)|^2 + \varepsilon|X_\alpha - Z|^2.
\end{aligned}$$

Finally, set

$$\begin{aligned}
P &= D_X\phi(X_\alpha) = 2\alpha[\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z)] + 2\varepsilon(X_\alpha - Z), \\
Q &= D_Y\psi(Y_\alpha) = 2\alpha[\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z)].
\end{aligned}$$

Since $\overline{v}^\theta - \psi$ takes its minimum at $Y_\alpha \in \mathcal{K}$ and $\overline{v}^\theta$ is a strict supersolution in $\mathcal{K}$, $G(Q) < -\theta f$ and $F(Y_\alpha, \overline{v}^\theta, Q, \mathcal{B}^\pi(Y_\alpha, \psi)) < -\theta f$. Repeating the proof of Lemma 4.1, we see that the latter strict inequality implies

$$F(Y_\alpha, \overline{v}^\theta, Q, \mathcal{B}^{\pi,\kappa}(Y_\alpha, \overline{v}^\theta, Q), \mathcal{B}^\pi_\kappa(Y_\alpha, \psi)) < -\theta f. \tag{4.15}$$

We next claim that $G(P) \leq 0$. Assume to the contrary that $G(P) > 0$. Then it follows that

$$-\theta f > G(Q) - G(P) = \beta(q_2 - p_2) - (q_1 - p_1) = -2\beta\varepsilon\big(x_{\alpha 2} - z_2\big) - 2\varepsilon\big(x_{\alpha 1} - z_1\big),$$

which tends to zero as $\alpha \to \infty$, a contradiction to the fact that f is strictly positive. Thus our claim holds. Then since $\underline{v} - \phi$ takes its maximum at $X_\alpha \in \overline{\mathcal{K}}$ and $\underline{v}$ is a subsolution in $\overline{\mathcal{K}}$, $F(X_\alpha, \underline{v}, P, \mathcal{B}^\pi(X_\alpha, \phi)) \geq 0$. This in turn implies that

$$F(X_\alpha, \underline{v}, P, \mathcal{B}^{\pi,\kappa}(X_\alpha, \underline{v}, P), \mathcal{B}^\pi_\kappa(X_\alpha, \phi)) \geq 0. \tag{4.16}$$

Using (4.15) and (4.16), we can calculate as follows

$$\begin{aligned}
0 &< F(X_\alpha, \underline{v}, P, \mathcal{B}^{\pi,\kappa}(X_\alpha, \underline{v}, P), \mathcal{B}^\pi_\kappa(X_\alpha, \phi)) - F(Y_\alpha, \overline{v}^\theta, Q, \mathcal{B}^{\pi,\kappa}(Y_\alpha, \overline{v}^\theta, Q), \mathcal{B}^\pi_\kappa(Y_\alpha, \psi)) \\
&\leq \big[U(x_{\alpha 2}) - U(y_{\alpha 2})\big] - \delta\big[\underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha)\big] - \beta\big[x_{\alpha 2}\phi_{x_2}(X_\alpha) - y_{\alpha 2}\psi_{y_2}(Y_\alpha)\big] \\
&\quad + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu} - r)\pi)\big[x_{\alpha 1}\phi_{x_1}(X_\alpha) - y_{\alpha 1}\psi_{y_1}(Y_\alpha)\big] \\
&\quad + \big[\mathcal{B}^{\pi,\kappa}(X_\alpha, \underline{v}, P) - \mathcal{B}^{\pi,\kappa}(Y_\alpha, \overline{v}^\theta, Q)\big] + \big[\mathcal{B}^\pi_\kappa(X_\alpha, \phi) - \mathcal{B}^\pi_\kappa(Y_\alpha, \psi)\big]\Big].
\end{aligned} \tag{4.17}$$

Let us start by estimating the integral terms. To this end, observe first that, thanks to (4.2), $\mathcal{B}^\pi_\kappa(X_\alpha,\phi)$ and $\mathcal{B}^\pi_\kappa(Y_\alpha,\psi)$ both tend to zero as $\kappa\to 0$ (for any finite α). Next, for simplicity of presentation, introduce the short-hand notation

$$T^\pi(z;X) = (x_1 + x_1\pi(e^z-1), x_2)$$

and note that $|T^\pi(z;X) - T^\pi(z;Y)| \le |x_1-y_1|\,|e^z-1|$. Then

$$\mathcal{B}^{\pi,\kappa}(X_\alpha,\underline{v},P) - \mathcal{B}^{\pi,\kappa}(Y_\alpha,\overline{v}^\theta,Q) = I_1 + I_2,$$

where, for $A_1=\{\kappa<|z|<1\}$ and $A_2=\{|z|\ge 1\}$,

$$\begin{aligned} I_\ell = \int_{A_\ell} \Big(\big[\underline{v}(T^\pi(z;X_\alpha)) - \overline{v}^\theta(T^\pi(z;Y_\alpha))\big] - \big[\underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha)\big] \\ - \pi\big[x_{\alpha 1}\phi_{x_1}(X_\alpha) - y_{\alpha 1}\psi_{y_1}(Y_\alpha)\big](e^z-1)\Big)\,\nu(dz), \qquad \ell=1,2. \end{aligned} \tag{4.18}$$

We consider first the term I_2. Observe that, for $i=1,2$,

$$\begin{aligned} &\big[x_{\alpha i}\phi_{x_i}(X_\alpha) - y_{\alpha i}\psi_{y_i}(Y_\alpha)\big] \\ &\quad = (x_{\alpha i}-y_{\alpha i})2\alpha[\alpha(x_{\alpha i}-y_{\alpha i}) + \varepsilon\eta_i(Z)] + 2\varepsilon x_{\alpha i}(x_{\alpha i}-z_i)] = \omega_1(\tfrac{1}{\alpha}), \end{aligned} \tag{4.19}$$

for some continuity modulus ω_1. Since obviously $\sup_{\overline{\mathcal{D}}}(\underline{v}-\overline{v}^\theta) \le \sup_{[0,R)\times[0,R)}(\underline{v}-\overline{v}^\theta) \le M$, we get, for some continuity modulus ω_2,

$$\begin{aligned} I_2 &\le \int_{|z|\ge 1} \Big(M + \overline{v}^\theta(T^\pi(z;X_\alpha)) - \overline{v}^\theta(T^\pi(z;Y_\alpha)) - M_\alpha \\ &\qquad\qquad - \big[x_{\alpha 1}\phi_{x_1}(X_\alpha) - y_{\alpha 1}\psi_{y_1}(Y_\alpha)\big](e^z-1)\Big)\,\nu(dz) \\ &\le \Big(M - M_\alpha + \omega_1(\tfrac{1}{\alpha}) + \omega_2(|x_{\alpha 1}-y_{\alpha 1}|)\Big) \int_{|z|\ge 1} |e^z-1|\,\nu(dz) \to 0 \text{ as } \alpha\to\infty, \end{aligned}$$

where we have exploited condition (2.4), estimate (4.19), and that $M_\alpha \to M$ as $\alpha\to\infty$.

We next estimate I_1. To this end, observe that $X_\alpha, Y_\alpha \in [0,R)\times[0,R)$ for α large enough. Consequently, $T^\pi(z;X_\alpha), T^\pi(z;Y_\alpha) \in \overline{\mathcal{K}}$ and thus

$$\Phi\big(T^\pi(z;X_\alpha), T^\pi(z;Y_\alpha)\big) - \Phi(X_\alpha,Y_\alpha) \le 0. \tag{4.20}$$

A calculation reveals that the integrand of I_1 equals

$$\Phi\big(T^\pi(z;X_\alpha), T^\pi(z;Y_\alpha)\big) - \Phi(X_\alpha,Y_\alpha) + \Big(\pi^2[\alpha(x_{\alpha 1}-y_{\alpha 1})]^2 + \varepsilon\pi^2 x_{\alpha 1}^2\Big)(e^z-1)^2,$$

which, thanks to (4.20), is less than or equal to $\Big([\alpha(x_{\alpha 1}-y_{\alpha 1})]^2 + \varepsilon x_{\alpha 1}^2\Big)(e^z-1)^2$. Hence

$$I_1 \le \Big([\alpha(x_{\alpha 1}-y_{\alpha 1})]^2 + \varepsilon x_{\alpha 1}^2\Big) \int_{\kappa<|z|<1} (e^z-1)^2\,\nu(dz).$$

Note that the integral is convergent since every Lévy measure integrates $\frac{1}{z^2}$ in a neighbourhood of zero, see (2.2). Since $\alpha(x_{\alpha 1}-y_{\alpha 1}) \to \varepsilon\eta_1(Z)$ as $\alpha\to\infty$, we conclude that $\lim_{\varepsilon\to 0}\lim_{\alpha\to\infty} I_1 \le 0$.

Finally, in view of the estimates derived above, we send (in that order) $\kappa\to 0$, $\alpha\to\infty$, and $\varepsilon\to 0$ in (4.17) to obtain the desired contradiction

$$\delta\big[\underline{v}(Z) - \overline{v}^\theta(Z)\big] < 0. \tag{4.21}$$

Case II: Let us now consider the case $Z\in(0,R)\times(0,R)$. For any $\alpha>1$ and $0<\varepsilon<1$, define the function $\Phi(X,Y)$ on $\overline{\mathcal{K}}\times\overline{\mathcal{K}}$ by

$$\Phi(X,Y) = \underline{v}(X) - \overline{v}^\theta(Y) - \tfrac{\alpha}{2}|X-Y|^2.$$

Let $M_\alpha = \sup_{\overline{\mathcal{K}}\times\overline{\mathcal{K}}} \Phi(X,Y)$. We then have $M_\alpha \geq M > 0$ for all $\alpha > 1$. Let (X_α, Y_α) be a maximizer so that $M_\alpha = \Phi(X_\alpha, Y_\alpha)$. Next, we note that the inequality $\Phi(X_\alpha, X_\alpha) + \Phi(Y_\alpha, Y_\alpha) \leq 2\Phi(X_\alpha, Y_\alpha)$ implies

$$\frac{\alpha}{2}|X_\alpha - Y_\alpha|^2 \leq \underline{v}(X_\alpha) - \overline{v}(Y_\alpha) + \overline{v}^\theta(X_\alpha) - \overline{v}^\theta(Y_\alpha). \tag{4.22}$$

Consequently, $|X_\alpha - Y_\alpha| \leq K\sqrt{\frac{1}{\alpha}}$, where $K > 0$ is a constant that depends on $\sup_{\overline{\mathcal{K}}} \underline{v}$ and $\sup_{\overline{\mathcal{K}}}(-\overline{v}^\theta)$. Inserting this estimate into (4.22) and using uniform continuity of $\underline{v}, \overline{v}^\theta$ in $\overline{\mathcal{K}}$, we see that $\frac{\alpha}{2}|X_\alpha - Y_\alpha|^2 \to 0$ as $\alpha \to \infty$. Moreover, for a subsequence of (X_α, Y_α) converging to $(\hat{X}, \hat{Y})$, we have $\hat{X} = \hat{Y}$. Using $M \leq M_\alpha$, it then follows that

$$\begin{aligned} 0 &= \limsup_{\alpha\to\infty}\Big\{|\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z)|^2 + \varepsilon|X_\alpha - Z|^2\Big\} \\ &\leq \limsup_{\alpha\to\infty}\Big\{\underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha) - M\Big\} = \underline{v}(\hat{X}) - \overline{v}^\theta(\hat{X}) - M \leq 0. \end{aligned}$$

We thus conclude, passing if necessary to a subsequence, $M_\alpha \to M$ as $\alpha \to \infty$.

Since (4.11) holds and, thanks to Case I, $\underline{v} \leq \overline{v}^\theta$ on $\partial\big\{(0,R)\times(0,R)\big\}$, we conclude that any limit point of (X_α, Y_α) belongs to $(0,R)\times(0,R)$. Hence for large enough α, $X_\alpha, Y_\alpha \in (0,R)\times(0,R)$. Following the classical viscosity theory, let

$$\psi(Y) = \underline{v}(X_\alpha) - \frac{\alpha}{2}|X_\alpha - Y|^2, \qquad \phi(X) = \overline{v}^\theta(Y_\alpha) + \frac{\alpha}{2}|X - Y_\alpha|^2. \tag{4.23}$$

Finally, set

$$P = D_X\phi(X_\alpha) = \alpha(X_\alpha - Y_\alpha), \qquad Q = D_Y\psi(Y_\alpha) = \alpha(X_\alpha - Y_\alpha).$$

Since $\overline{v}^\theta - \psi$ takes its minimum at Y_α and $\overline{v}^\theta$ is a strict supersolution, we have $G(Q) < -\theta f$ and $F(Y_\alpha, \overline{v}^\theta, Q, \mathcal{B}^\pi(Y_\alpha, \psi) < -\theta f$, which also implies

$$F(Y_\alpha, \overline{v}^\theta, Q, \mathcal{B}^{\pi,\kappa}(Y_\alpha, \overline{v}^\theta, Q), \mathcal{B}^\pi_\kappa(Y_\alpha, \psi)) < -\theta f. \tag{4.24}$$

Assume that $G(P) > 0$. Then it follows that $-\theta f > G(Q) - G(P) \equiv 0$, which is a contradiction. Thus, $G(P) \leq 0$. Now since $\underline{v} - \phi$ takes its maximum at X_α and $\underline{v}$ is a subsolution, $F(X_\alpha, \underline{v}, P, \mathcal{B}\phi) \geq 0$, which also implies

$$F(X_\alpha, \underline{v}, P, \mathcal{B}^{\pi,\kappa}(X_\alpha, \underline{v}, P), \mathcal{B}^\pi_\kappa(X_\alpha, \phi)) \geq 0. \tag{4.25}$$

Using (4.24) and (4.25), we get (consult Case I)

(4.26)

$$\begin{aligned} 0 &< F(X_\alpha, \underline{v}, P, \mathcal{B}^{\pi,\kappa}(X_\alpha, \underline{v}, P), \mathcal{B}^\pi_\kappa(X_\alpha, \phi)) - F(Y_\alpha, \overline{v}^\theta, Q, \mathcal{B}^{\pi,\kappa}(Y_\alpha, \overline{v}^\theta, Q), \mathcal{B}^\pi_\kappa(Y_\alpha, \psi)) \\ &\leq \big[U(x_{\alpha 2}) - U(y_{\alpha 2})\big] - \delta\big[\underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha)\big] - \beta\big[x_{\alpha 2}\phi_{x_2}(X_\alpha) - y_{\alpha 2}\psi_{y_2}(Y_\alpha)\big] \\ &\quad + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu} - r)\pi)\big[x_{\alpha 1}\phi_{x_1}(X_\alpha) - y_{\alpha 1}\psi_{y_1}(Y_\alpha)\big] + I_2 + I_2 + \big[\mathcal{B}^\pi_\kappa(X_\alpha, \phi) - \mathcal{B}^\pi_\kappa(Y_\alpha, \psi)\big]\Big], \end{aligned}$$

where I_1, I_2 are defined in (4.18) with ϕ, ψ defined in (4.23). Appealing once more to (4.2), we know that $\mathcal{B}^\pi_\kappa(X_\alpha, \phi)$ and $\mathcal{B}^\pi_\kappa(Y_\alpha, \psi)$ tend to zero as $\kappa \to 0$. Moreover, $\lim_{\alpha\to\infty} I_2 \leq 0$ (consult Case I). To estimate the integral I_1, we note that the integrand equals

$$\Phi\big(T^\pi(z; X_\alpha), T^\pi(z; Y_\alpha)\big) - \Phi(X_\alpha, Y_\alpha) + \pi^2\frac{\alpha}{2}(x_{\alpha 1} - y_{\alpha 1})^2(e^z - 1)^2.$$

Obviously, $T^\pi(z; X_\alpha), T^\pi(z; Y_\alpha) \in \overline{\mathcal{K}}$ and thus $\Phi\big(T^\pi(z; X_\alpha), T^\pi(z; Y_\alpha)\big) - \Phi(X_\alpha, Y_\alpha) \leq 0$. Since $\frac{\alpha}{2}|X_\alpha - Y_\alpha|^2 \to 0$ as $\alpha \to \infty$, we obtain (consult Case I)

$$I_1 \leq \frac{\alpha}{2}(x_{\alpha 1} - y_{\alpha 1})^2 \int_{\kappa<|z|<1} (e^z - 1)^2\,\nu(dz) \to 0 \text{ as } \alpha \to \infty.$$

Sending (in that order) $\kappa \to 0$, $\alpha \to \infty$, and $\varepsilon \to 0$ in (4.26), we obtain as in Case I the contradiction (4.21). This concludes the proof of the theorem. $\square$

5 An extension to the second order case

In this section we generalize our stock price model to also include a Brownian motion term B_t. More precisely, we consider S_t given by

$$S_t = xe^{\mu t + \sigma B_t + L_t}, \tag{5.1}$$

for some constant σ. Here B_t is assumed to be independent of L_t. There are several reasons for studying such a model. First of all, from the Lévy-Khintchine representation we know that every Lévy process can be decomposed into a pure-jump Lévy process and a Wiener process, where the Wiener process is the continuous martingale part. Hence, from a theoretical point of view, the model (5.1) is an extension of (2.1) with a general Lévy process as driving noise. However, we can also consider (5.1) as a model of the stock price where L_t is a pure-jump Lévy process accounting for the "big" jumps in the price. The Brownian motion part, on the other hand, models the "small" or "normal" variations in the price movements. This is the modelling perspective of Honoré [17], although he considers a slightly different price process, see Section 8.

It is worth mentioning that in Rydberg [33] an approximation procedure for the normal inverse Gaussian Lévy process L_t is suggested where the process is decomposed into a Brownian motion part and a jump part, i.e., $L_t = \sigma B_t + \tilde{L}_t$. For a given ε, the jump process $\tilde{L}_t$ is assumed to be a Lévy process with Lévy measure $\tilde{\nu}(dz) = \mathbf{1}_{(-\varepsilon,\varepsilon)}\,\nu(dz)$, where $\nu(dz)$ is the Lévy measure of L_t and $\sigma^2 = \int_{-\varepsilon}^{\varepsilon} z^2\,\nu(dz)$. We remark that this approximation procedure is not restricted to the normal inverse Gaussian process alone.

Under the condition (2.4) the differential form of S_t reads

$$dS_t = \hat{\mu} S_t\,dt + \sigma S_t\,dB_t + S_{t-}\int_{I\!R\setminus\{0\}} \left(e^z - 1\right)\tilde{N}(dt,dz),$$

where $\hat{\mu} = \mu + \xi + \frac{\sigma^2}{2} + \int_{I\!R\setminus\{0\}}\left(e^z - 1 - z\mathbf{1}_{|z|<1}\right)\nu(dz)$. The wealth process is therefore

$$\begin{aligned} X_t^{\pi,C} &= x - C_t + \int_0^t \left(r + (\hat{\mu} - r)\pi_s\right) X_s^{\pi,C}\,ds + \sigma\int_0^t \pi_s X_s^{\pi,C}\,dB_s \\ &\quad + \int_0^t \pi_{s-} X_{s-}^{\pi,C}\int_{I\!R\setminus\{0\}} \left(e^z - 1\right)\tilde{N}(ds,dz). \end{aligned}$$

The Hamilton-Jacobi-Bellman equation associated to the control problem reads

$$\begin{aligned} &\max\Big\{\beta v_y - v_x; U(y) - \delta v - \beta y v_y + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu} - r)\pi)x v_x + \tfrac{\sigma^2}{2}\pi^2 x^2 v_{xx} + \\ &\int_{I\!R\setminus\{0\}}\Big(v(x + \pi x(e^z - 1), y) - v(x,y) - \pi x v_x(x,y)(e^z - 1)\Big)\,\nu(dz)\Big]\Big\} = 0 \text{ in } \mathcal{D}. \end{aligned} \tag{5.2}$$

In the present context, we see that the Hamilton-Jacobi-Bellman equation is a degenerate elliptic integro-differential variational inequality. We assume $\delta > k(\gamma)$, where

$$\begin{aligned} k(\gamma) = \max_{\pi\in[0,1]}\Big[&\gamma(r + (\hat{\mu} - r)\pi) + \gamma(\gamma - 1)\tfrac{\sigma^2}{2}\pi^2 \\ &+ \int_{I\!R\setminus\{0\}}\Big(\left(1 + \pi(e^z - 1)\right)^\gamma - 1 - \gamma\pi(e^z - 1)\Big)\,\nu(dz)\Big] \end{aligned}$$

and $\gamma \in (0,1)$ is as before the growth exponent of the utility function. Our main result in this section is the following theorem:

Theorem 5.1. *The value function $V(x,y)$ defined in (2.8) is the unique constrained viscosity solution of the integro-differential variational inequality (5.2). Moreover, the value function satisfies $0 \le V(x,y) \le K(1 + x + y)^\gamma$ for all $x, y \in \overline{\mathcal{D}}$. The value function V is uniformly continuous in $\overline{\mathcal{D}}$. Furthermore, if for some $\alpha \in (0,1]$, $\delta > k(\alpha)$, and $U \in C^{0,\alpha}([0,\infty))$, then $V \in C^{0,\alpha}(\overline{\mathcal{D}})$. If $\delta > k(1 + \alpha)$ and $U \in C^{1,\alpha}([0,\infty))$, then $V \in C^{1,\alpha}(\overline{\mathcal{D}})$.*

As in the first order case (see (4.1)), we simplify the presentation by writing (5.2) as

$$\max\Big(G(D_X v); F(X, v, D_X v, D_X^2 v, \mathcal{B}^\pi(X, v))\Big) = 0 \text{ in } \mathcal{D}, \tag{5.3}$$

where $D_X^2 = (\partial^2_{x_i x_j})_{i,j=1,2}$, $\mathcal{B}^\pi$ is exactly as before, and

$$\begin{aligned} F(X, v, D_X v, D_X^2 v, \mathcal{B}^\pi(X, v)) &= U(x_2) - \delta v - \beta x_2 v_{x_2} \\ &+ \max_{\pi \in [0,1]} \Big[(r + (\hat{\mu} - r)\pi) x_1 v_{x_1} + \tfrac{\sigma^2}{2}\pi^2 x_1^2 v_{x_1 x_1} + \mathcal{B}^\pi(X, v)\Big]. \end{aligned}$$

Note that (5.2) is well defined for all $v \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$. To deal with value functions that are not C^2 in the present context, we extend Definition 4.1 as follows:

Definition 5.1. (i) Let $\mathcal{O} \subset \overline{\mathcal{D}}$. Any $v \in C(\overline{\mathcal{D}})$ is a viscosity subsolution (supersolution) of (5.3) in $\mathcal{O}$ if and only if we have, for every $X \in \mathcal{O}$ and $\phi \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$ such that X is a global maximum (minimum) relative to $\mathcal{O}$ of $v - \phi$,

$$\max\Big(G(D_X \phi); F(X, v, D_X \phi, D_X^2 \phi, \mathcal{B}^\pi(X, \phi))\Big) \geq 0 \, (\leq 0).$$

(ii) Any $v \in C(\overline{\mathcal{D}})$ is a constrained viscosity solution of (4.1) if and only if v is a supersolution of (5.3) in $\mathcal{D}$ and v is a subsolution of (5.3) in $\overline{\mathcal{D}}$.

For $\kappa > 0$, $\phi \in C^{(}\overline{\mathcal{D}})$, $v \in C_1$, let us define

$$\begin{aligned} F(X, v, D_X \phi, D_X^2 \phi, \mathcal{B}^{\pi,\kappa}(X, v, D_X\phi), \mathcal{B}^\pi_\kappa(X, \phi)) &= U(x_2) - \delta v - \beta x_2 \phi_{x_2} \\ + \max_{\pi \in [0,1]} \Big[(r + (\hat{\mu} - r)\pi) x_1 \phi_{x_1} + \tfrac{\sigma^2}{2}\pi^2 x_1^2 \phi_{x_1 x_1} &+ \mathcal{B}^{\pi,\kappa}(X, v, D_X \phi) + \mathcal{B}^\pi_\kappa(X, \phi)\Big], \end{aligned}$$

where $\mathcal{B}^{\pi,\kappa}$ and $\mathcal{B}^\pi_\kappa$ are exactly as in the first order case, see Section 4. Then we obtain an equivalent formulation of viscosity solutions in $C_1(\overline{\mathcal{D}})$.

Lemma 5.2. *Let $v \in C_1(\overline{\mathcal{D}})$ and $\mathcal{O} \subset \overline{\mathcal{D}}$. Then v is a viscosity subsolution (supersolution) of (4.1) in $\mathcal{O}$ if and only if we have, for every $\phi \in C^2(\overline{\mathcal{D}})$ and $\kappa > 0$,*

$$\max\Big(G(D_X \phi); F(X, v, D_X \phi, D_X^2 \phi, \mathcal{B}^{\pi,\kappa}(X, v, D_X \phi), \mathcal{B}^\pi_\kappa(X, \phi))\Big) \geq 0 \tag{5.4}$$

whenever $X \in \mathcal{O}$ is a global maximum (minimum) relative to $\mathcal{O}$ of $v - \phi$.

This lemma is a straightforward extension of Lemma 4.1 and the proof is omitted.

By more or less repeating the arguments from Section 3 and Section 4, we can prove that the value function $V(x, y)$ defined in (2.8) has the regularity stated in Theorem 5.1, is sublinearly growing, and is a constrained viscosity solution of (5.2). To prove that the value function is the only solution of (5.2), we need a comparison principle similar to Theorem 4.5. To prove comparison results for viscosity solutions of second order equations, it is convenient to use a formulation of a viscosity solution based on the notions of subjet and superjet.

Definition 5.2. Let $\mathcal{S}^N$ denotes the set of $N \times N$ symmetric matrices, $\mathcal{O} \subset \overline{\mathcal{D}}$, $v \in C(\mathcal{O})$, and $X \in \mathcal{O}$. The second order superjet (subjet) $J^{2,+(-)}_{\mathcal{O}} v(X)$ is the set of $(P, A) \in I\!R^2 \times \mathcal{S}^2$ such that

$$v(Y) \leq (\geq 0) \, v(X) + \langle P, Y - X\rangle + \tfrac{1}{2}\langle A(Y - X), Y - X\rangle + o(|X - Y|^2) \text{ as } \mathcal{O} \ni Y \to X.$$

Its closure $\overline{J}^{2,+(-)}_{\mathcal{O}} v(X)$ is the set of (P, A) for which there is a sequence $(P_n, A_n) \in J^{2,+(-)}_{\mathcal{O}} v(X_n)$ such that $(X_n, v(X_n), P_n, A_n) \to (X, v(X), P, A)$.

Let $v \in C(\overline{\mathcal{D}})$ and $\mathcal{O} \subset \overline{\mathcal{D}}$. Then using the arguments in, e.g., [14] one can easily prove that $(P, A) \in \overline{J}^{2,-(+)}_{\mathcal{O}} v(X)$ if and only if there exists $\phi \in C^2(\overline{\mathcal{D}})$ such that $D_X \phi(X) = P$, $D_X^2 \phi(X) = A$, and $v - \phi$ has a global minimum (maximum) relative to $\mathcal{O}$ at X. In view of the above discussion, the following formulation of viscosity solutions in C_1 is now immediate.

Lemma 5.3. *Let $v \in C_1(\overline{\mathcal{D}})$ be a subsolution (supersolution) and $\mathcal{O} \subset \overline{\mathcal{D}}$. Then, for all $\kappa \in (0,1)$, $X \in \mathcal{O}$, $(P,A) \in \overline{J}_{\mathcal{O}}^{2,-(+)} v(X)$, there exists $\phi \in C^2(\overline{\mathcal{D}})$ such that*

$$\max\Big(G(P); F(X,v,P,A,\mathcal{B}^{\pi,\kappa}(X,v,P),\mathcal{B}^{\pi}_{\kappa}(X,\phi))\Big) \geq 0\,(\leq 0).$$

The test function ϕ is such that $v-\phi$ has a global maximum relative to $\mathcal{O}$ at X_n with $X_n \to X$.

A similar formulation is also used in Pham [32]. To prove a comparison principle for (5.3), we shall need the following lemma from Crandall, Ishii, and Lions [10]:

Lemma 5.4 ([10]). *Let $\mathcal{O} \subset I\!\!R^N$ be locally compact. Let $u_1, -u_2$ be upper semicontinuous and ϕ twice continuously differentiable in a neighbourhood of $\mathcal{O}\times\mathcal{O}$. Suppose $(\hat{X},\hat{Y}) \in \mathcal{O}\times\mathcal{O}$ is a local maximum of $u_1(X)-u_2(Y)-\varphi(X,Y)$ relative to $\mathcal{O}\times\mathcal{O}$. Then there exists $A,B \in S^N$ such that*

$$(D_X\varphi(\hat{X},\hat{Y}),A) \in \overline{J}_{\mathcal{O}}^{2,+}u_1(\hat{X}), \qquad (-D_Y\varphi(\hat{X},\hat{Y}),B) \in \overline{J}_{\mathcal{O}}^{2,-}u_2(\hat{Y}),$$

and for any $\varsigma > 0$,

$$\Big(-\|D^2\varphi(\hat{X},\hat{Y})\| + \frac{1}{\varsigma}\Big)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\varphi(\hat{X},\hat{Y}) + \varsigma\big(D^2\varphi(\hat{X},\hat{Y})\big)^2. \tag{5.5}$$

A slight refinement of this lemma can be found as the "Theorem on Sums" in Crandall [7].

Let $\underline{v} \in C(\overline{\mathcal{D}})$ be a subsolution of (5.3) in $\overline{\mathcal{D}}$ and $\overline{v} \in C(\overline{\mathcal{D}})$ a supersolution of (5.3) in $\mathcal{D}$. Choosing $\tilde{K}$ and $\overline{\gamma}$ properly, one can, following closely the proof of Proposition 4.4, show that $w = \tilde{K} + \big(1 + x_1 + \frac{x_2}{2\beta}\big)^{\overline{\gamma}}$ and thus $\overline{v}^{\theta} = (1-\theta)\overline{v} + \theta w$ ($\theta \in (0,1]$) are strict supersolutions of (5.3) in $\mathcal{D}$. We claim that $\underline{v} \leq \overline{v}^{\theta}$ in $\overline{\mathcal{D}}$, which immediately implies that the comparison principle holds between $\underline{v}$ and $\overline{v}$. Except for the treatment of the second order term, which relies in an essential way on Lemma 5.4, the proof of our claim is very similar to the proof of Theorem 4.5, which we also refer to for the details that are not found below.

As in the first order case, it is sufficient to prove that $\underline{v} \leq \overline{v}^{\theta}$ in $\overline{\mathcal{K}}$, where $\mathcal{K}$ defined in (4.10). Assume to the contrary that (4.11) holds for some $Z \in \overline{\mathcal{K}}$. Then either $Z \in (0,R)\times(0,R)$ or $Z \in \Gamma_{SC}$, where R is used in (4.10) and Γ_{SC} is defined in (4.12). Here we consider only the latter case, the case $Z \in \mathcal{K}$ is treated similarly (consult Case II in the proof of Theorem 4.5). Let (X_α, Y_α) be a maximizer of the function $\Phi(X,Y) : \overline{\mathcal{K}}\times\overline{\mathcal{K}} \to I\!\!R$ defined in (4.14). Using Lemma 5.4 with $\varphi(X,Y) = |\alpha(X-Y)+\varepsilon\eta(Z)|^2 + \varepsilon|X-Z|^2$, $u_1 = \underline{v}$, $u_2 = \overline{v}^{\theta}$, and $\mathcal{O} = \overline{\mathcal{K}}$, we conclude that there exist $A = (a_{ij})_{i,j=1,2}$, $B = (b_{ij})_{i,j=1,2} \in \mathcal{S}^2$ such that

$$\begin{aligned}
&(P,A) \in \overline{J}_{\overline{\mathcal{K}}}^{2,+}\underline{v}(X_\alpha), && P = D_X\varphi(X_\alpha,Y_\alpha) = 2\alpha[\alpha(X_\alpha-Y_\alpha)+\varepsilon\eta(Z)] + 2\varepsilon(X_\alpha - Z),\\
&(Q,B) \in \overline{J}_{\overline{\mathcal{K}}}^{2,-}\overline{v}^{\theta}(Y_\alpha), && Q = -D_Y\varphi(X_\alpha,Y_\alpha) = 2\alpha[\alpha(X_\alpha-Y_\alpha)+\varepsilon\eta(Z)].
\end{aligned}$$

Furthermore, as easy calculation reveals that

$$D^2\varphi(X_\alpha,Y_\alpha) = 2\alpha^2\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\varepsilon\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

and then following, e.g., [10] it is not difficult to show that (5.5) implies

$$\limsup_{\alpha\to\infty}\Big(\tfrac{\sigma^2}{2}\pi x_{\alpha 1}^2 a_{11} - \tfrac{\sigma^2}{2}\pi y_{\alpha 1}^2 b_{11}\Big) \leq 0. \tag{5.6}$$

Since $\overline{v}^{\theta}$ is a strict supersolution of (5.3) in $\mathcal{D}$ there exists, thanks to Lemma 5.2, $\psi \in C^2(\overline{\mathcal{D}})$ such that

$$F(Y_\alpha, \overline{v}^{\theta}, Q, B, \mathcal{B}^{\pi,\kappa}(Y_\alpha,\overline{v}^{\theta},Q), \mathcal{B}^{\pi}_{\kappa}(Y_\alpha,\psi)) < 0. \tag{5.7}$$

Similarly, since $\underline{v}$ is a subsolution of (5.3) in $\overline{\mathcal{D}}$, there exists $\phi \in C^2(\overline{\mathcal{D}})$ such that

$$F(X_\alpha, \underline{v}, P, A, \mathcal{B}^{\pi,\kappa}(X_\alpha, \underline{v}, P), \mathcal{B}^{\pi}_{\kappa}(X_\alpha, \phi)) \geq 0. \tag{5.8}$$

Having (5.6) in mind, we now subtract (5.7) from (5.8) and send $\alpha \to \infty$, which eventually leads to the contradiction (4.21) (consult Case I in the proof of Theorem 4.5).

Summing up, we have proven the following comparison theorem:

Theorem 5.5. *Let $\gamma' > 0$ be such that $\delta > k(\gamma')$. Assume $\underline{v} \in C_{\gamma'}(\overline{\mathcal{D}})$ is a subsolution of* (5.3) *in $\overline{\mathcal{D}}$ and $\overline{v} \in C_{\gamma'}(\overline{\mathcal{D}})$ is a supersolution of* (5.3) *in $\mathcal{D}$. Then $\underline{v} \leq \overline{v}$ in $\overline{\mathcal{D}}$.*

6 An example with HARA utility

In this section we study an example where we can construct an explicit solution to the control problem. Our example is taken from Hindy and Huang [19]. They construct an explicit solution to the optimization problem when the utility function is of HARA (Hyperbolic Absolute Risk Aversion) type and the price of the stock follows a geometric Brownian motion. We show in this section that a more realistic price model with a Lévy process instead of Brownian motion leads to a similar solution. We consider a pure-jump Lévy process which leads to the first order integro-differential variational inequality (2.11). We are able to solve this equation, and construct optimal consumption and portfolio allocation strategies by closely following the arguments in [19]. Note, however, that our results are not as explicit as those in [19]. For instance, the optimal allocation strategy π^* is the solution of an integral equation involving the Lévy measure of the noise process. We remark that a Brownian motion term in the price process (see Section 5) can easily be included in the calculations below.

For $\gamma \in (0,1)$, consider the utility function $U(y) = \frac{y^\gamma}{\gamma}$. Here the parameter γ indicates the degree of risk aversion. Motivated by Hindy and Huang [19], we guess that the optimization problem has a constrained viscosity solution of the form

$$V(x,y) = \begin{cases} k_1 y^\gamma + k_2 y^\gamma \left[\dfrac{x}{ky}\right]^\rho, & 0 \leq x < ky, \\ k_3 \Big(\dfrac{y + \beta x}{1 + \beta k}\Big)^\gamma, & x \geq ky > 0, \end{cases} \tag{6.1}$$

for some constants k_1, k_2, k_3, k, and $\rho > \gamma$. This solution is constructed from the assumption that we can split the state space into two parts, on which each of the terms in the variational inequality (2.11) is effective. Hence, for $0 \leq x < ky$, we construct the solution from the assumption that

$$\begin{aligned} &\tfrac{y^\gamma}{\gamma} - \delta V - \beta y V_y + \max_{\pi \in [0,1]} \Big[(r + (\hat{\mu} - r)\pi) x V_x + \\ &\int_{\mathbb{R}\setminus\{0\}} \Big(V(x + \pi x(e^z - 1), y) - V(x,y) - \pi x V_x(x,y)(e^z - 1) \Big) \nu(dz) \Big] = 0 \end{aligned} \tag{6.2}$$

and, when $x \geq ky > 0$,

$$\beta V_y - V_x = 0. \tag{6.3}$$

We see that the integral in (6.2) is well defined by the conditions in (2.4). In what follows, all the displayed integrals are convergent by the same conditions. In the rest of this section we derive expression for the different constants in the solution, and find the optimal allocation and consumption processes. Optimize the kernel of (6.2) with respect to π to find the first order condition for an optimum

$$(\hat{\mu} - r) x V_x + \int_{\mathbb{R}\setminus\{0\}} \Big(V_x(x + \pi x(e^z - 1), y) x(e^z - 1) - x V_x(x,y)(e^z - 1) \Big) \nu(dz) = 0.$$

Inserting the guessed solution (6.1) for $x < ky$, we get the expression

$$(6.4) \qquad (\hat{\mu} - r) + \int_{I\!R\setminus\{0\}} \Big(\big(1 + \pi(e^z - 1)\big)^{\rho-1}(e^z - 1) - (e^z - 1) \Big)\, \nu(dz) = 0.$$

Assume from now on that π^* is a solution of (6.4)). Note that π^* is constant with respect to time which gives that the investment rule is to hold a constant fraction of the wealth in the stock. With this π^*, we can find equations for the unknown constants k_1 and ρ. Inserting (6.1) into (6.2), we obtain

$$\begin{aligned} y^\gamma \left(\tfrac{1}{\gamma} - \delta k_1 - \beta\gamma k_1\right) + k_2 y^\gamma \left[\frac{x}{ky}\right]^\rho \Big\{ -\delta - \beta(\gamma - \rho) + (r + (\hat{\mu} - r)\pi^*)\rho \\ + \int_{I\!R\setminus\{0\}} \Big(\big(1 + \pi^*(e^z - 1)\big)^\rho - 1 - \rho\pi^*(e^z - 1) \Big)\, \nu(dz) \Big\} = 0. \end{aligned}$$

The only way this can be zero is when

$$(6.5) \qquad (r + (\hat{\mu} - r)\pi^* + \beta)\rho = \delta + \beta\gamma - \int_{I\!R\setminus\{0\}} \Big(\big(1 + \pi^*(e^z - 1)\big)^\rho - 1 - \rho\pi^*(e^z - 1) \Big)\, \nu(dz)$$

and $k_1 = \dfrac{1}{\gamma(\delta + \beta\gamma)}$. The first equation is an expression for ρ.

From now on we assume that (6.4) and (6.5) have a solution $(\pi^*, \rho) \in [0, 1] \times (\gamma, 1)$. We can find expressions for k_2 and k_3 by imposing a *smooth fit* condition along the boundary $x = ky$. From continuity we easily get $k_1 + k_2 = k_3$. Moreover, if the derivatives of V are to be continuous as well, we need to have $V_x = \beta V_y$ when $x = ky$ for the solution (6.1) ($x \le ky$). But differentiating and equating give

$$k_2 = \frac{\beta k_1 \gamma}{\rho/k - \beta(\gamma - \rho)} = \frac{\beta k}{(\delta + \beta\gamma)(\rho(1 + \beta k) - \beta k\gamma)}.$$

We complete the proof that V is a constrained viscosity solution of the Hamilton-Jacobi-Belmann equation (2.11). For $x \le ky$, we need to show that $\beta V_y - V_x \le 0$. Direct differentiation gives

$$\begin{aligned} V_x &= k_2 y^\gamma \left[\frac{x}{ky}\right]^{\rho-1} \frac{\rho}{ky} = k_2 \frac{\rho}{k} y^{\gamma-1} \left[\frac{x}{ky}\right]^{\rho-1} \\ V_y &= k_1\gamma y^{\gamma-1} + k_2(\gamma - \rho) y^{\gamma-\rho-1} \left[\frac{x}{k}\right]^\rho = k_1\gamma y^{\gamma-1} + k_2(\gamma - \rho) y^{\gamma-1} \left[\frac{x}{ky}\right]^\rho. \end{aligned}$$

Hence

$$\beta V_y - V_x = y^{\gamma-1} \Big(k_1\beta\gamma + \beta k_2(\gamma - \rho) \left[\frac{x}{ky}\right]^\rho - k_2 \frac{\rho}{k} \left[\frac{x}{ky}\right]^{\rho-1} \Big).$$

Inserting the expressions for k_1 and k_2 yields

$$\beta V_y - V_x = \frac{\beta y^{\gamma-1}}{\delta + \beta\gamma} \Big(1 - (1 - \rho) \left[\frac{x}{ky}\right]^\rho - \rho \left[\frac{x}{ky}\right]^{\rho-1} \Big).$$

We see that $\beta V_y - V_x \le 0$ if and only if $h(z) := 1 - (1 - \rho)z^\rho - \rho z^{\rho-1} \le 0$ for all $z \in [0, 1]$. But $h(1) = 0$ and $h'(z) = \rho(1 - \rho)z^{\rho-2}(1 - z) \ge 0$. Hence $h(z)$ is an increasing function on $[0, 1]$ with maximum $h(1) = 0$, which implies $h(z) \le 0$. This completes the proof of $\beta V_y - V_x \le 0$ for $x \le ky$.

For the second case we specify the value of k to be the same as in [19] and show that this gives the desired inequality. Let $k = \dfrac{1 - \rho}{\beta(\rho - \gamma)}$. This gives $k_3 = \dfrac{\rho(1 - \gamma)}{\gamma(\rho - \gamma)(\delta + \beta\gamma)}$ and thus $V(x, y) = c(y + \beta x)^\gamma$ for $x \ge ky$, where

$$c = \frac{\rho}{\gamma(\delta + \beta\gamma)} \Big(\frac{1 - \gamma}{\rho - \gamma}\Big)^{1-\gamma}.$$

We show next that

$$\frac{y^\gamma}{\gamma} - \delta V - \beta y V_y + \max_{\pi\in[0,1]}\Big[(r + (\hat\mu - r)\pi)xV_x + \\ \int_{I\!R\setminus\{0\}}\Big(V(x+\pi x(e^z-1),y) - V(x,y) - \pi x V_x(e^z-1)\Big)\,\nu(dz)\Big] \le 0,$$

whenever $x \ge ky$. Inserting the expression for $V(x,y)$ in the left-hand side of the above inequality and using $\beta x/(y+\beta x) \in (0,1)$, we get

$$\frac{y^\gamma}{\gamma} - \delta c(y+\beta x)^\gamma + c(y+\beta x)^\gamma \max_{\pi\in[0,1]}\Big[(r+(\hat\mu - r)\pi)\gamma\tfrac{\beta x}{y+\beta x} - \beta\gamma\tfrac{y}{y+\beta x} + \\ \int_{I\!R\setminus\{0\}}\Big(\big(1+\tfrac{\beta x}{y+\beta x}\pi(e^z-1)\big)^\gamma - 1 - \gamma\tfrac{\beta x}{y+\beta x}\pi(e^z-1)\Big)\,\nu(dz)\Big] \\ \le \frac{y^\gamma}{\gamma} - c(y+\beta x)^\gamma\big(\delta - k(\gamma)\big).$$

But since $x \ge ky$ and $\delta - k(\gamma)$ and c are both positive, we have

$$\frac{y^\gamma}{\gamma} - c(\delta - k(\gamma))(y+\beta x)^\gamma \le \frac{y^\gamma}{\gamma} - c(\delta-k(\gamma))(1+\beta k)^\gamma y^\gamma \\ = y^\gamma\big(\tfrac{1}{\gamma} - c(\delta - k(\gamma))(1+\beta k)^\gamma\big),$$

which is less than or equal to zero if and only if $\frac{1}{\gamma} - c(\delta - k(\gamma))(1+\beta k)^\gamma \le 0$. But this happens if and only if

$$\frac{\rho(1-\gamma)}{\rho-\gamma} \ge \frac{\delta+\beta\gamma}{\delta - k(\gamma)}. \tag{6.6}$$

By construction V is a constrained viscosity solution in $\{x \ge 0, y > 0\}$. Note that a subsolution in $\{x \ge 0, y > 0\}$ is also a subsolution in $\overline{\mathcal{D}}$. We refer to the first remark in Section 3 in [1] for a proof of this. Thanks to Theorem 4.5, V is thus the unique constrained viscosity solution of (2.11). Summing up, we have proven the following theorem:

Theorem 6.1. *For $\gamma \in (0,1)$, let $U(y) = \frac{y^\gamma}{\gamma}$ and assume (6.6) holds. Then the value function $V(x,y)$ associated with our optimization problem is explicitly given by (6.1), where*

$$k_1 = \frac{1}{\gamma(\delta+\beta\gamma)}, \quad k_2 = \frac{1-\rho}{(\rho-\gamma)(\delta+\beta\gamma)}, \quad k_3 = \frac{\rho(1-\gamma)}{\gamma(\rho-\gamma)(\delta+\beta\gamma)}, \quad k = \frac{1-\rho}{\beta(\rho-\gamma)}.$$

The optimal allocation of money in the stock is given by π^* where $\pi^* \in [0,1]$ and $\rho \in (\gamma, 1]$ are solutions (when such exist) to the system of equations

$$(\hat\mu - r) + \int_{I\!R\setminus\{0\}}\big(1+\pi(e^z-1)\big)^{\rho-1}(e^z-1) - (e^z-1)\,\nu(dz) = 0,$$
$$(r + (\hat\mu - r)\pi + \beta)\rho = \delta + \beta\gamma - \int_{I\!R\setminus\{0\}}\Big(\big(1+\pi(e^z-1)\big)^\rho - 1 - \rho\pi(e^z-1)\Big)\,\nu(dz).$$

Note that k_1, k_2, and k_3 are equal to the constants found by Hindy and Huang [19]. However, our expressions for ρ and π^* are quite different. Furthermore, π^* is independent of time and thus gives a constant fraction of wealth to be invested in the stock.

An optimal consumption process C_t^* (not necessarily unique) is provided by the following theorem:

Theorem 6.2. *An optimal consumption process C_t^* is given as*

$$C_t^* = \Delta C_0^* + \int_0^t \frac{X_t^*}{1+\beta k}\,dZ_s, \qquad k = \frac{1-\rho}{\beta(\rho-\gamma)},$$

$$\Delta C_0^* = \max\Big\{0, \frac{x - kY_{0-}}{1+\beta k}\Big\}, \quad Z_t = \sup_{0\le s\le t}\Big[\ln\frac{\hat{X}_t}{\hat{Y}_t} - \ln k\Big]^+, \qquad \hat{Y}_t = (Y_0 + \beta\Delta C_0^*)e^{-\beta t},$$

and

$$\hat{X}_t = (x - \Delta C_0^*) + \int_0^t \big(r + (\hat{\mu} - r)\pi^*\big)\hat{X}_s\, ds + \int_0^t \pi^* \hat{X}_{s-} \int_{I\!R\setminus\{0\}} \big(e^z - 1\big)\, \tilde{N}(ds, dz).$$

The processes X^* and Y^* are the state variables associated with C^*.

Proof. This argument follows closely the proof in [19, Prop. 5]. From the results in [19], we need to find a k ratio barrier policy which ensure that $X_t^*/Y_t^* \le k$, $P-$a.s. at every time instant t. This leads to an initial jump of C_t^* if $x/Y_{0-} > k$, from where we get the expression of ΔC_0^*. Now define

$$Z_t = \sup_{0\le s\le t}\Big[\ln\frac{\hat{X}_t}{\hat{Y}_t} - \ln k\Big]^+$$

and let $\ln(X_t^*/Y_t^*)$ be the "regulated" process defined by

$$\ln\frac{X_t^*}{Y_t^*} = \ln\frac{\hat{X}_t}{\hat{Y}_t} - Z_t. \tag{6.7}$$

Note that the processes $\hat{X}_t$ and $\hat{Y}_t$ are unregulated in the sense that we do not apply any consumption process except for the initial jump. The process Z_t is easily seen to be nondecreasing, $Z_0(\omega) = 0$, and increasing only when $\ln(X_t^*/Y_T^*) = \ln k$. Applying Itô's formula, we find that

$$\begin{aligned}
d\ln\frac{X_t^*}{Y_t^*} &= d\ln X_t^* - d\ln Y_t^* - \left(\frac{1}{X_t^*} + \frac{\beta}{Y_t^*}\right) dC_t^* \\
&= (r + \beta + (\hat{\mu} - r)\pi^*)\, dt - \left(\frac{1}{X_t^*} + \frac{\beta}{Y_t^*}\right) dC_t^* \\
&\quad + \int_{I\!R\setminus\{0\}} \ln\Big(1 + \pi^*(e^z - 1)\Big)\, \tilde{N}(dt, dz) \\
&\quad + \int_{I\!R\setminus\{0\}} \Big(\ln\Big(1 + \pi^*(e^z - 1)\Big) - \pi^*(e^z - 1)\Big)\, \nu(dz)
\end{aligned}$$

and

$$\begin{aligned}
d\ln\frac{\hat{X}_t}{\hat{Y}_t} = d\ln\hat{X}_t - d\ln\hat{Y}_t &= (r + \beta + (\hat{\mu} - r)\pi^*)\, dt \\
&\quad + \int_{I\!R\setminus\{0\}} \ln\Big(1 + \pi^*(e^z - 1)\Big)\, \tilde{N}(dt, dz) \\
&\quad + \int_{I\!R\setminus\{0\}} \Big(\ln\Big(1 + \pi^*(e^z - 1)\Big) - \pi^*(e^z - 1)\Big)\, \nu(dz).
\end{aligned}$$

Thus, relation (6.7) is fulfilled exactly when

$$Z_t = \int_0^t \left(\frac{Y_s^* + \beta X_s^*}{X_s^* Y_s^*}\right) dC_s^* \quad \text{or} \quad C_t^* = \int_0^t \frac{X_s^* Y_s^*}{Y_s^* + \beta X_s^*}\, dZ_s = \int_0^t \frac{X_s^*}{1+\beta k}\, dZ_s.$$

Here the relation for C_t^* follows since Z_t only increases when $X_t^*/Y_t^* = k$. This completes the proof of the theorem. □

7 Merton's problem with consumption and HARA utility

In this section we consider Merton's problem with consumption when the stock price is modelled as (2.1). Merton's problem can be thought of as the case when $\beta \to \infty$ in the particular model considered in Section 6. In this problem we thus optimize the expected utility of the consumption directly. The consumption process is assumed to be absolute continuous with respect to the Lebesgue measure on the real positive half-line, and can thus be specified on the form $C_t = \int_0^t c_s\, ds$, where c_s is the consumption rate at time s. The value function will only be dependent on one variable, namely the intimal fortune x. We note that this problem has been treated by Framstad *et al.* [15] when the price process S_t is modelled as the solution of a stochastic differential equation with jumps, see also the paper [16] where they take into account transaction costs. Their model also include a Brownian motion term. However, they have a more restrictive condition on the Lévy measure in a neighbourhood of zero. For example, the normal inverse Gaussian Lévy process of Barndorff-Nielsen [6] does not fit into the framework of [15, 16]. Even though we concentrate our calculations to the pure-jump case, one can easily incorporate a Brownian motion term in the stock price (as in Section 5) and derive analogous expressions to those found below.

In the present context, the wealth process is given as

$$dX_t = \big(r + (\hat{\mu} - r)\pi_t\big) X_t\, dt - c_t\, dt + X_{t-}\pi_{t-} \int_{I\!R\setminus\{0\}} \left(e^z - 1\right) \tilde{N}(dt, dz)$$

with initial wealth $X_0 = x$. We consider the optimal control problem

$$V(x) = \sup_{c,\pi\in\mathcal{A}_x} \mathrm{E}^x \left[\int_0^T e^{-\delta t} \left[\tfrac{c_t^\gamma}{\gamma}\right] dt\right], \qquad \text{for } \gamma \in (0,1),$$

where the set of admissible controls $\mathcal{A}_x$ is defined as follows: $\pi, c \in \mathcal{A}_x$ if

- (cm_i) c_t is a positive and adapted process such that $\int_0^t \mathrm{E}[c_s]\, ds < \infty$ for all $t \geq 0$.
- (cm_{ii}) π_t is progressively measurable with values in $[0,1]$.
- (cm_{iii}) c_t is such that $X_t^{\pi,c} \geq 0$ almost everywhere for all $t \geq 0$.

Note that condition (cm_{iii}) introduces a state space constraint into our control problem. The Hamilton-Jacobi-Bellman equation for this problem is

$$\begin{aligned}\max_{c\geq 0,\pi\in[0,1]} \Big[&(r + (\hat{\mu} - r)\pi) x v'(x) - c v'(x) - \delta v(x) + \tfrac{c^\gamma}{\gamma} + \\ &\int_{I\!R\setminus\{0\}} \Big(v(x + \pi x(e^z - 1)) - v(x) - \pi x v'(x)(e^z - 1)\Big)\, \nu(dz)\Big] = 0 \text{ in } \{x > 0\}.\end{aligned} \tag{7.1}$$

Note that the integral in (7.1) as well as the other integrals displayed in this section are convergent by the conditions in (2.4). We now construct an explicit (unique) constrained viscosity solution to this problem. First maximize with respect to c to obtain

$$-V'(x) + c^{\gamma-1} = 0 \implies c = [V'(x)]^{\frac{1}{\gamma-1}}.$$

Maximizing with respect to π gives the expression

$$(\hat{\mu} - r) x V'(x) + \int_{I\!R\setminus\{0\}} \Big(V'(x + \pi x(e^z - 1)) x(e^z - 1) - x V'(x)(e^z - 1)\Big)\, \nu(dz) = 0.$$

We guess a solution on the form $V(x) = Kx^\gamma$. Then a straightforward calculation gives the following integral equation for π:

$$\int_{I\!R\setminus\{0\}} \Big(\big(1 + \pi(e^z - 1)\big)^{\gamma-1}(e^z - 1) - (e^z - 1)\Big)\, \nu(dz) = r - \hat{\mu}. \tag{7.2}$$

Note that a π solving this equation will be independent on t. Using the guessed solution, we can obtain an expression for c as well:

$$c = x \cdot (K\gamma)^{\frac{1}{\gamma-1}}. \tag{7.3}$$

This expression gives us an explicit consumption rule, that is, consume the fraction $(K\gamma)^{1/\gamma-1}$ of your total wealth. We now set out to find the constant K. Inserting (7.3) into the Hamilton-Jacobi-Bellman equation (7.1), we get

$$\max_{\pi\in[0,1]}\Big[(r+(\hat{\mu}-r)\pi)\gamma - (K\gamma)^{\frac{1}{\gamma-1}}\gamma - \delta + \tfrac{1}{\gamma}(K\gamma)^{\frac{\gamma}{\gamma-1}}K^{-1}$$
$$+\int_{I\!R\setminus\{0\}}\Big(\big(1+\pi(e^z-1)\big)^{\gamma} - 1 - \gamma\pi(e^z-1)\,\nu(dz)\Big)\Big]Kx^{\gamma} = 0.$$

We thus conclude that $K = \frac{1}{\gamma}\left[\frac{1-\gamma}{\delta-k(\gamma)}\right]^{1-\gamma}$, where $k(\gamma)$ is defined in (2.9). Note that the condition $\delta > k(\gamma)$ imposed in Section 2 implies that K is positive.

We state a condition ensuring the existence of a unique solution $\pi \in [0,1]$ to (7.2). To this end, define the function

$$f(\pi) = \int_{I\!R\setminus\{0\}}\Big(\big(1+\pi(e^z-1)\big)^{\gamma-1}(e^z-1) - (e^z-1)\Big)\,\nu(dz) + (\hat{\mu}-r).$$

Inserting $\pi = 0$ and $\pi = 1$, we obtain

$$f(0) = \hat{\mu} - r > 0, \qquad f(1) = (\hat{\mu}-r) + \int_{I\!R\setminus\{0\}}\Big(e^{(\gamma-1)z}(e^z-1) - (e^z-1)\Big)\,\nu(dz)$$
$$= (\hat{\mu}-r) - \int_{I\!R\setminus\{0\}}(1-e^{-(1-\gamma)z})(e^z-1)\,\nu(dz).$$

In order to have a solution in $[0,1]$, we need $f(1) < 0$, i.e.,

$$\int_{I\!R\setminus\{0\}}(1-e^{-(1-\gamma)z})(e^z-1)\,\nu(dz) > (\hat{\mu}-r).$$

This solution is unique since

$$f'(\pi) = (\gamma-1)\int_{I\!R\setminus\{0\}}\big(1+\pi(e^z-1)\big)^{\gamma-2}(e^z-1)^2\,\nu(dz) < 0.$$

It is well known that in the classical case of $S_t = S_0\exp(\mu t + \sigma B_t)$, a geometric Brownian motion, the optimal allocation of money in the portfolio is also independent of time; namely, $\pi^*_{\mathrm{GBM}} = \frac{\mu+\sigma^2/2-r}{(1-\gamma)\sigma^2}$. On the other hand, we have seen that S_t given as in (2.1) also gives a constant fraction, denoted by π^*_J, which solves (7.2). It is of practical interest to see how the two optimal investment strategies relate. This depends on the sign of $f(\pi^*_{\mathrm{GBM}})$. However, for any practical purposes, this sign can only be checked when we have fitted both price models to the same set of logreturn data. We remark that the conclusion in [15] is not correct from the point of view of a practitioner with two pricing models for the same asset. Framstad *et al.* [15] observe that in the jump diffusion pricing model you invest less in the stock than in the standard geometric model. However, this conclusion is based on the fact that their jump model is simply geometric Brownian motion with additional multiplicative jump-noise. The parameters of the drift and diffusion terms are the same in both models. Thus, since you simply add noise (e.g., volatility) in the jump model, their conclusion is obvious from a practical investment point of view. To reach a useful conclusion you must adjust the parameters in the jump model accordingly to the data series when fitting. Where to put the most of your fortune is no longer clear.

8 Other models and concluding remarks

Instead of modelling the price process S_t directly as in (2.1) or (5.1), one can let S_t be the solution of a stochastic differential equation with jumps

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t + S_{t-}\int_{-1}^{\infty} z\,\tilde{N}(dt,dz). \tag{8.1}$$

Note that S_t is positive due to the restriction of the jump size to be greater than -1. Condition (2.4) must be substituted by

$$\int_{1}^{\infty} z\,\nu(dz) < \infty. \tag{8.2}$$

Under this restriction on the Lévy measure we can show, by arguing as before, that the value function $V(x,y)$ is the unique constrained viscosity solution of the Hamilton-Jacobi-Belman equation

$$\begin{aligned} &\max\Big\{\beta v_y - v_x; U(y) - \delta v - \beta y v_y + \max_{\pi\in[0,1]}\Big[(r+(\mu-r)\pi)x v_x + \tfrac{1}{2}\sigma^2\pi^2x^2 v_{xx} + \\ &\int_{-1}^{\infty}\Big(v(x+\pi x z, y) - v(x,y) - \pi x z v_x(x,y)\Big)\,\nu(dz)\Big]\Big\} = 0 \text{ in } \mathcal{D}. \end{aligned} \tag{8.3}$$

The condition (8.2), which ensures that (8.3) is well defined for all sublinearly growing $v \in C^2$, is satisfied for the normal inverse Gaussian Lévy process discussed in Section 2 and for α-stable Lévy processes with $\alpha > 1$.

In [15], the price model (8.1) is chosen for the analysis of Merton's problem with consumption. Using verification theorems, they show that the value function in Merton's problem with consumption (see Section 7) is a unique classical solution of (8.3) under the condition (8.2) and $\nu(\{(-1,\infty)\}) < \infty$. Honoré [17] has developed estimation techniques for price processes of the kind (8.1). This opens for a numerical comparison of the different stock price models. In future work we will investigate the relation between the models discussed in the present paper when they are fitted to market data.

Finally, except for a few special cases such as those considered in Sections 6 and 7, the Hamilton-Jacobi-Bellman equation (2.11) cannot be solved explicitly and one has to consider numerical approximations. The construction and analysis (within the viscosity solution framework) of numerical schemes for integro-differential variational inequalities will be reported in future work.

Acknowledgements: K. H. Karlsen acknowledges support from the University of Bergen (Computational Science). K. Reikvam acknowledges support from the Norwegian Research Council under the grant NFR 118868/410.

References

[1] O. Alvarez, A singular stochastic control problem in an unbounded domain, *Comm. Partial Differential Equations* **19** (1994), no. 11-12, 2075–2089.

[2] O. Alvarez and A. Tourin, Viscosity solutions of nonlinear integro-differential equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13** (1996), no. 3, 293–317.

[3] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Birkhäuser Boston, Boston, MA, 1997.

[4] M. Bardi et al., *Viscosity solutions and applications*, Lecture Notes in Math., 1660, Springer, Berlin, 1997.

[5] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, Springer, Paris, 1994.

[6] O. E. Barndorff-Nielsen, Processes of Normal inverse Gaussian type, *Finance and Stochastics* **2** (1998), 41-68.

[7] M. G. Crandall, Viscosity solutions: a primer, in *Viscosity solutions and applications (Montecatini Terme, 1995)*, 1–43, Lecture Notes in Math., 1660, Springer, Berlin, 1997.

[8] M. G. Crandall, L. C. Evans and P.-L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **282** (1984), no. 2, 487–502.

[9] M. G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1983), no. 1, 1–42

[10] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67.

[11] M. H. A. Davis, V. G. Panas and T. Zariphopoulou, European option pricing with transaction costs, *SIAM J. Control Optim.* **31** (1993), no. 2, 470–493.

[12] I. Capuzzo-Dolcetta and P.-L. Lions, Hamilton-Jacobi equations with state constraints, *Trans. Amer. Math. Soc.* **318** (1990), no. 2, 643–683.

[13] E. Eberlein and U. Keller, Hyperbolic distributions in finance, *Bernoulli* **1**(3) (1995), 281-299.

[14] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Applications of Mathematics 25, New York: Springer Verlag, (1993).

[15] N. C. Framstad, B. Øksendal and A. Sulem, Optimal consumption and portfolio in a jump diffusion market, Preprint 1999/5 (1999), Norwegian School of Economics.

[16] N. C. Framstad, B. Øksendal and A. Sulem, Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs, Working paper.

[17] P. Honoré, Pitfalls in Estimating Jump-Diffusion Models, Preprint, Centre for Analytical Finance, Aarhus, No. 18, 1998.

[18] A. Hindy and C. Huang, On intertemporal preferences for uncertain consumption: A continuous time approach, *Econometrica* **60** (1992), 781-801.

[19] A. Hindy and C. Huang, Optimal consumption and portfolio rules with durability and local substitution, *Econometrica* **61** (1993), 85-122.

[20] A. Hindy, C. Huang and H. Zhu, Numerical analysis of a free-boundary singular control problem in finacial economics, *J. of Economics Dynamics and Control* **21** (1997), 297-327.

[21] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes, 2nd Edition*, North-Holland/Kodansha, (1989).

[22] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs, *Comm. Pure Appl. Math.* **42** (1989), no. 1, 15–45.

[23] H. Ishii and P.-L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, *J. Differential Equations* **83** (1990), no. 1, 26–78.

[24] R. Jensen, The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations, *Arch. Rational Mech. Anal.* **101** (1988), no. 1, 1–27.

[25] R. Jensen, Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations, *Indiana Univ. Math. J.* **38** (1989), no. 3, 629–667.

[26] M. A. Katsoulakis, Viscosity solutions of second order fully nonlinear elliptic equations with state constraints, *Indiana Univ. Math. J.* **43** (1994), no. 2, 493–519.

[27] J.-M. Lasry and P.-L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem, *Math. Ann.* **283** (1989), no. 4, 583–630.

[28] R. Jensen, P.-L. Lions and P. E. Souganidis, A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations, *Proc. Amer. Math. Soc.* **102** (1988), no. 4, 975–978.

[29] P. L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. I. The dynamic programming principle and applications, *Comm. Partial Differential Equations* **8** (1983), no. 10, 1101–1174.

[30] P. L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness, *Comm. Partial Differential Equations* **8** (1983), no. 11, 1229–1276.

[31] P.-L. Lions and P. E. Souganidis, Viscosity solutions of second-order equations, stochastic control and stochastic differential games, in *Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986)*, 293–309, Springer, New York, 1988.

[32] H. Pham, Optimal stopping of controlled jump diffusion processes: A viscosity solution approach. *J. Math. Syst. Estim. Control* **8** (1998), No.1, 27 pp.

[33] T. H. Rydberg, The normal inverse Gaussian Lévy process: Simulation and approximation. *Commun. Statist.-Stochastic Models* **13** (4) (1997), 887-910.

[34] A. Sayah, Équations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. I. Unicité des solutions de viscosité, *Comm. Partial Differential Equations* **16** (1991), no. 6-7, 1057–1074.

[35] A. Sayah, Équations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. II. Existence de solutions de viscosité, *Comm. Partial Differential Equations* **16** (1991), no. 6-7, 1075–1093.

[36] H. M. Soner, Optimal control with state-space constraint. I, *SIAM J. Control Optim.* **24** (1986), no. 3, 552–561.

[37] H. M. Soner, Optimal control with state-space constraint. II, *SIAM J. Control Optim.* **24** (1986), no. 6, 1110–1122.

[38] H. M. Soner, Optimal control of jump-Markov processes and viscosity solutions, in *Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986)*, 501–511, Springer, New York, 1988.

[39] H. M. Soner, Controlled Markov processes, viscosity solutions and applications to mathematical finance, in *Viscosity solutions and applications (Montecatini Terme, 1995)*, 134–185, Lecture Notes in Math., 1660, Springer, Berlin, 1997.

[40] S. Zhu, Dynamic programming and variational inequalities in singular stochastic control, Doctoral Dissertation, Division of Applied Mathematics, Brown University (1991).