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Superposition of Ornstein - Uhlenbeck Type Processes

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Abstract

A class of superpositionsof Ornstein-Uhlenbeck type processes is constructed, in terms of integrals with respect to independently scattered random measures. Under specified conditions the resulting processes exhibit long range dependence. By integration the superpositions yield cumulative processes with stationary increments, and integration with respect to processes of the latter type is defined. A limiting procedure results in processes that, in the case of square integrability, are second order selfsimilar with stationary increments. Certain other of the limiting processes are stable and selfsimilar with stationary increments.

Contents

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1. Introduction

In studying observational processes that show significant dependence over long time periods a possible approach is to try to model the process or processes at hand by means of superposition of independent processes with short range dependence.

Cox (1984), in a review of the roles of long range dependence and selfsimilarity in statistics, introduced, on a heuristic basis, a method for construction of processes with long range dependence by weighted integration of processes with short range dependence. In Cox (1991) this was applied in a study of the relations of nonlinearity and time irreversibility to long range dependence. A somewhat similar, rigorously based, method was proposed in Barndorff-Nielsen, Jensen and Sørensen (1990) and there applied to the modelling of velocity fields in stationary turbulence, cf. also Barndorff-Nielsen, Jensen and Sørensen (1993, 1998).

Recent work on modelling observational series of financial assets have described log price processes as following a diffusion type model where the squared diffusion coefficient itself obeys a stochastic differential equation and constitutes a stationary process. In extension of this, weighted sums of such stationary processes were used in order to capture the timewise dependencies in the price developments that are an essential feature of the financial markets. More specifically, superposition of independent Ornstein-Uhlenbeck type processes have provided flexible and analytically tractable parametric models. The integrated squared volatility process equals the quadratic variation of the log price process and plays an essential role in the analysis and applications of the models. See Barndorff-Nielsen (1998b) and Barndorff-Nielsen and Shephard (1998a,b,c).

It should also be noted that questions of moduli of continuity and large increments of infinite sums of classical, i.e. Gaussian, Ornstein-Uhlenbeck processes have been discussed in papers by Csáki, Csörgõ, Lin and Révesz (1991) and Lin (1995). See also Walsh (1981).

These developments have motivated the present study of superposition of Ornstein-Uhlenbeck type processes and their integrals, based on the theory of independently scattered random measures. An overall aim is to develop flexible classes of processes that incorporate long range dependence and selfsimilarity-like properties and are capable, furthermore, of describing some of the other key distributional features of typical data in finance, turbulence and other fields. We note that, in several respects, the class of strictly selfsimilar processes is too limited in scope for such modelling purposes. In particular, they cannot simultaneously show semiheavy tailed behaviour for short time lags and close to Gaussian behaviour for large time lags, such as do typical observational series from both finance and turbulence.

We recall that a stationary process $x = \{x(t)\}_{t \in \mathbf{R}}$ is said to exhibit long range dependence if the correlation function r of x behaves as

$$r(u) \sim L(u) u^{-2\bar{H}}$$

for $u \to \infty$ and where L is a slowly varying function and $\bar{H} \in (0, \frac{1}{2})$. Throughout we shall write $H = 1 - \bar{H}$ and we assume that $H \in (0, 1]$. When x is long range dependent the cumulative process, x^* say, derived from x is approximately second order selfsimilar, see for instance Cox (1984) or Barndorff-Nielsen, Jensen and Sørensen (1990).

A process $x^* = \{x^*(t)\}_{0 \le t}$ is selfsimilar with exponent H if

$$\{x^*(ct)\}_{t \in \mathbf{R}_+} \stackrel{\mathcal{L}}{=} c^H \{x^*(t)\}_{t \in \mathbf{R}_+}$$

for all $c > 0$. In that case one says that x^* is H-ss, and if, moreover, x^* has stationary increments we write H-sssi. For a comprehensive discussion of selfsimilarity, see Samorodnitsky and Taqqu (1994).

An H-ss process whose increments are stationary to second order (at least) will be referred to as an H-sssi$_2$ process. A class of such processes, driven by bivariate Lévy processes, is discussed in Barndorff-Nielsen and Pérez-Abreu (1998).

Further, if a process has stationary increments and is square integrable with the same type of covariance function as if it was selfsimilar we write H-ss$_2$si.

In the sequel we shall use the following notation for cumulant and Laplace transforms of a random variate x:

$$\mathrm{C}\{\zeta \ddagger x\} = \log \mathrm{E}\{e^{i\zeta x}\}$$

$$\bar{\mathrm{K}}\{u \ddagger x\} = \log \mathrm{E}\{e^{-ux}\}$$

For instance, if x is a random variable of the form $x = \sigma\varepsilon$ where σ and ε are independent with ε standard normal and σ positive (a form of key importance in finance) then

$$\mathrm{C}\{\zeta \ddagger x\} = \bar{\mathrm{K}}\{\zeta^2/2 \ddagger \sigma^2\}$$

Section 2 summarizes results on Lévy processes, selfdecomposability, Ornstein-Uhlenbeck type processes, and independently scattered random measures, needed in the subsequent sections. In Section 3 a class of superpositions, in terms of integrals, of Ornstein-Uhlenbeck type processes is introduced; under certain conditions the resulting processes will exhibit long range dependence. By integration the superpositions yield cumulative processes with stationary increments and these are investigated in Section 4. Integration of real functions with respect to the cumulative processes is considered in Section 5. A limiting procedure, discussed in Section 6, results in processes that, in the case of square integrability, are second order selfsimilar with stationary increments, i.e. H-ss$_2$si. Certain other of the limiting processes are stable and (strictly) selfsimilar with stationary increments.

2. Prerequisites

This section reviews a number of, mostly wellknown, results on Lévy processes, selfdecomposability, Ornstein-Uhlenbeck type processes, and independently scattered random measures.

2.1. Lévy processes

Recall that a random variable x is infinitely divisible if its cumulant function has the Lévy-Khintchine form

$$\mathrm{C}\{\zeta \ddagger x\} = a\zeta + \frac{b}{2}\zeta^2 + \int_{\mathbf{R}} (e^{i\zeta u} - 1 - i\zeta\tau(u))Q(\mathrm{d}u) \tag{2.1}$$

where $a \in \mathbf{R}$, $b > 0$ and

$$\tau(u) = \begin{cases} u & \text{if} \quad |u| \le 1 \\ \frac{u}{|u|} & \text{if} \quad |u| > 1 \end{cases}$$

and where the Lévy measure Q is a Radon measure on $\mathbf{R}$ such that $Q(\{0\}) = 0$ and

$$\int_{\mathbf{R}} \min(1, u^2) Q(\mathrm{d}u) < \infty$$

For any Lévy measure Q we shall use the notation

$$Q^+(x) = Q([x, \infty)) \quad \text{and} \quad Q^-(x) = Q((-\infty, x])$$

for the tail masses of Q.

A stochastic process $\{z(t)\}_{0 \le t}$ is said to be a Lévy process if it has independent increments and cadlag sample paths and is continuous in probability. If the increments are stationary z is said to be homogeneous. In the following, unless otherwise stated, we take the term Lévy process to mean a homogeneous Lévy process z such that $z(t) \xrightarrow{p} 0$ as $t \downarrow 0$.

If z is a Lévy process then the cumulant function of z satisfies $\mathrm{C}\{\zeta \ddagger z(t)\} = t\mathrm{C}\{\zeta \ddagger z(1)\}$. Note that to any infinitely divisible random variable x there corresponds a Lévy process z such that $x \overset{\mathcal{L}}{=} z(1)$; we speak of z as the Lévy process generated by x.

More generally, a stochastic process $x = \{x(t) : t \in T\}$, T an arbitrary index set, is said to be infinitely divisible if all its finite dimensional distributions are infinitely divisible. Any such process generates a generalized Lévy process $z = \{z(s, t) : s \ge 0,\, t \in T\}$ by the prescription

$$\mathrm{C}\{\zeta_1, ..., \zeta_m \ddagger z(s, t_1), ..., z(s, t_m)\} = s\mathrm{C}\{\zeta_1, ..., \zeta_m \ddagger x(t_1), ..., x(t_m)\}$$

for all the finite dimensional laws.

2.2. Selfdecomposability

An infinitely divisible random variable x is selfdecomposable if its characteristic function ϕ has the property that for every $c \in (0, 1)$ there exists a characteristic function ϕ_c such that $\phi(t) = \phi(ct)\phi_c(t)$ for all $t \in \mathbf{R}$. Equivalently, x is selfdecomposable if its Lévy measure U is of the form $U(\mathrm{d}x) = u(x)\mathrm{d}x$ with $u(x) = |x|^{-1}\,\bar{u}(x)$ where $\bar{u}(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

A further important characterization is that a random variable x is selfdecomposable if and only if x is representable as

$$x = \int_0^\infty e^{-s}\mathrm{d}\grave{z}(s) \tag{2.2}$$

where $\grave{z}$ is a Lévy process whose law is determined uniquely by that of x (cf., for instance, Jurek and Mason (1993; Theorem 3.6.6)). The Lévy measure W of $\grave{z}(1)$ is related to the Lévy density u of x by the formula

$$W^+(x) = xu(x) \tag{2.3}$$

for $x > 0$ and

$$W^-(x) = |x|\,u(x) \tag{2.4}$$

for $x < 0$. Furthermore, if the Lévy density u of x is differentiable then W has a density w with respect to Lebesgue measure and u and w are related by

$$w(x) = -u(x) - xu'(x) \tag{2.5}$$

see Barndorff-Nielsen (1998b). The process $\grave{z} = \{\grave{z}(t)\}_{t\geq 0}$ is termed the *background driving Lévy process* or BDLP corresponding to x.

In the following x will stand for a selfdecomposable random variable whose Lévy density u is differentiable and $\acute{z} = \{\acute{z}(t)\}_{t\geq 0}$ will denote the Lévy process generated by x, i.e. the Lévy process such that $\acute{z}(1) = x$. Furthermore, the BDLP determined by x is denoted by $\grave{z}$.

Lemma 2.1 The BDLP of $\acute{z}(t)$ is equivalent in law to $\{\grave{z}(ts)\}_{s\geq 0}$. □

Proof This follows immediately from the fact that the Lévy densities u of $\acute{z}(1)$ and w of $\grave{z}(1)$ are related by (2.5). □

For brevity we shall use the notation

$$\acute{\kappa}(\zeta) = \mathrm{C}\{\zeta \ddagger x\}$$

and

$$\grave{\kappa}(\zeta) = \mathrm{C}\{\zeta \ddagger \grave{z}(1)\}$$

and we write $\acute{\kappa}_m$ and $\grave{\kappa}_m$ for the corresponding m-th order cumulants and $\acute{\mu}_m$ for the m-th order central moment of $\acute{z}(1)$.

In consequence of (2.2) we have

$$\acute{\kappa}(\zeta) = \int_0^\infty \grave{\kappa}\left\{e^{-s}\zeta\right\}\mathrm{d}s \tag{2.6}$$

The validity of this known result follows essentially from the formal calculation below, which uses $\mathrm{C}\{\zeta \ddagger \mathrm{d}z(s)\} = \mathrm{C}\{\zeta \ddagger z(1)\}\mathrm{d}s$ and product integration:

$$\begin{aligned}
\acute{\kappa}(\zeta) &= \log \mathrm{E}\{e^{i\zeta x}\} \\
&= \log \mathrm{E}\left\{\exp\left[i\zeta \int_0^\infty e^{-s}\mathrm{d}\grave{z}(s)\right]\right\} \\
&= \log \prod_0^\infty \mathrm{E}\left\{\exp\left[i\zeta e^{-s}\mathrm{d}\grave{z}(s)\right]\right\} \\
&= \log \prod_0^\infty \exp\left\{C\{\zeta e^{-s} \ddagger \grave{z}(1)\}\mathrm{d}s\right\} \\
&= \log \prod_0^\infty \exp\left\{\grave{\kappa}\left\{e^{-s}\zeta\right\}\mathrm{d}s\right\} \\
&= \log\left[\exp\left\{\int_0^\infty \grave{\kappa}\left\{e^{-s}\zeta\right\}\mathrm{d}s\right\}\right] \\
&= \int_0^\infty \grave{\kappa}\left\{e^{-s}\zeta\right\}\mathrm{d}s
\end{aligned}$$

Formal differentiation of (2.6) followed by partial integration yields

$$\grave{\kappa}(\zeta) = \zeta\acute{\kappa}'(\zeta) \tag{2.7}$$

This relation holds in fact under the assumption that $\acute{\kappa}$ is differentiable for $\zeta \neq 0$ and provided $\zeta\acute{\kappa}'(\zeta) \to 0$ for $0 \neq \zeta \to 0$, as may be shown by a limiting argument.

In consequence of (2.7),

$$\acute{\kappa}_m = \grave{\kappa}_m/m \tag{2.8}$$

provided the cumulants exist.

2.3. Ornstein-Uhlenbeck type processes

For any $t > 0$ and $\lambda > 0$ we may rewrite the representation (2.2) in the following way

$$\begin{aligned}
x &= \int_0^\infty e^{-\lambda s}\mathrm{d}\grave{z}(\lambda s) \\
&= \int_t^\infty e^{-\lambda s}\mathrm{d}\grave{z}(\lambda s) + \int_0^t e^{-\lambda s}\mathrm{d}\grave{z}(\lambda s) \\
&= e^{-\lambda t}\int_0^\infty e^{-\lambda s}\mathrm{d}\grave{z}(\lambda(s+t)) + e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}\grave{z}(\lambda(t-s))
\end{aligned}$$

and here, due to the continuity in probability of $\grave{z}$,

$$\int_0^\infty e^{-\lambda s}\mathrm{d}\grave{z}(\lambda(s+t)) \overset{\mathcal{L}}{=} x$$

and

$$\int_0^t e^{\lambda s}\mathrm{d}\grave{z}(\lambda(t-s)) \overset{\mathcal{L}}{=} \int_0^t e^{\lambda s}\mathrm{d}\grave{z}(\lambda s)$$

Consequently, x is representable as

$$x = e^{-\lambda t}x_0 + w_t$$

where x_0 and w_t are independent and $x_0 \overset{\mathcal{L}}{=} x$ and

$$w_t = e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}\grave{z}(\lambda s)$$

In fact, a stronger statement is true provided the cumulant function $\acute{\kappa}$ of x has the property that $\zeta\acute{\kappa}'(\zeta)$ is continuous at 0. In that case (see, for instance, Barndorff-Nielsen, Jensen and Sørensen (1998)), for any $\lambda > 0$, the stochastic differential equation

$$\mathrm{d}x(t) = -\lambda x(t)\mathrm{d}t + \mathrm{d}\grave{z}(\lambda t),$$

has a stationary solution $x(t)$ such that $x(t) \overset{\mathcal{L}}{=} x$. A stationary process $x(t)$ of this kind is said to be an Ornstein - Uhlenbeck type process, or an OU *process* for short. When $x(t)$ is square integrable with $E\{x(t)\} = 0$ it has correlation function $r(u) = \exp\{-\lambda u\}$.

2.4. Independently scattered random measures

The present subsection briefly reviews the definition and some main properties of independently scattered random measures. Comprehensive accounts of the theory of independently scattered random measures are available in Rajput and Rosinski (1989) and Kwapień and Woyczyński (1992).

Let Ω be a Borel subset of $\mathbf{R}^d$ and let $\mathcal{S}$ be a $\sigma-$ring of Ω (i.e. countable unions of sets in $\mathcal{S}$ belong to $\mathcal{S}$ and if A and B are sets in $\mathcal{S}$ with $A \subset B$ then $B\backslash A$ is also in $\mathcal{S}$). The σ-algebra generated by $\mathcal{S}$ will be denoted by $\sigma(\mathcal{S})$. A collection of random variables $z = \{z(A); A \in \mathcal{S}\}$ defined on a probability space is said to be an *independently scattered random measure* (i.s.r.m.) if for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$, the random variables $z(A_n), n = 1, 2, ..,$ are independent and if

$$z(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty z(A_n) \text{ a.s.}$$

whenever $\cup_{n=1}^{\infty} A_n \in \mathcal{S}$.

We shall be interested in the case when z is infinitely divisible, that is, for each $A \in \mathcal{S}$, $z(A)$ is an infinitely divisible random variable whose cumulant function can be written as

$$\mathrm{C}\{\zeta \ddagger z(A)\} = i\zeta m_0(A) - \frac{1}{2}\zeta^2 m_1(A) + \int_{\mathbf{R}} (e^{i\zeta x} - 1 - i\zeta\tau(x))Q(A, \mathrm{d}x) \qquad (2.9)$$

where m_0 is a signed measure, m_1 is a positive measure, $Q(A, \mathrm{d}x)$ is (for fixed A) a measure in $\mathcal{B}(\mathbf{R})$ without atoms at 0 such that $\int_{\mathbf{R}} \min(1, |x|^2)Q(A, \mathrm{d}x) < \infty$ and where

$$\tau(x) = \begin{cases} x \text{ if } |x| \leq 1 \\ \frac{x}{|x|} \text{ if } |x| > 1 \end{cases} .$$

In this case z is said to have the Lévy characteristics (m_0, m_1, Q). There is a one to one correspondence between infinitely divisible independently scattered random measures and the class of parameters m_0, m_1 and Q. We shall refer to Q as a *generalized Lévy measure*.

In the present paper we restrict the discussion to the case where the

$$m_0 = m_1 = 0 \qquad (2.10)$$

and where Q factorizes as

$$Q(A, \mathrm{d}x) = M(A)W(\mathrm{d}x) \qquad (2.11)$$

for some measure M on Ω and some Lévy measure W on $\mathbf{R}$. We denote the cumulant function associated with W by κ, i.e.

$$\kappa(\zeta) = \int_{\mathbf{R}} (e^{i\zeta x} - 1 - i\zeta\tau(x))W(\mathrm{d}x)$$

Formally, then

$$\mathrm{C}\{\zeta \ddagger z(\mathrm{d}\omega)\} = \kappa(\zeta)M(\mathrm{d}\omega) \qquad (2.12)$$

Integration of functions f on T with respect to z is defined first for real simple functions $f = \sum_{j=1}^{n} x_j 1_{A_j}$, where $A_j \in \mathcal{S}$, by

$$\int_A f \mathrm{d}z = \sum_{j=1}^{n} x_j z(A \cap A_j). \qquad (2.13)$$

where A is any subset of T for which $A \in \sigma(\mathcal{S})$ and $A \cap A_j \in \mathcal{S}$, $j = 1, ..., n$. In general, a function $f : (\Omega, \sigma(\mathcal{S})) \to (R, \mathcal{B}(\mathbf{R}))$ is said to be $z-$integrable if there exists a sequence $\{f_n\}$ of simple functions as above, such that $f_n \to f$ a.e.

$[M]$ and for every $A \in \sigma(\mathcal{S})$, the sequence $\{\int_A f_n dz\}$ converges in probability as $n \to \infty$. If f is $z-$integrable, we write

$$\int_A f \mathrm{d}z = \mathrm{p-} \lim_{n\to\infty} \int_A f_n \mathrm{d}z. \tag{2.14}$$

The integral $\int_A f \mathrm{d}z$ is well defined (does not depend on the approximating sequence).

A key result for many calculations is embodied in the next formula.

Proposition 2.1

$$\mathrm{C}\{\zeta \ddagger \int_A f dz\} = \int_A \kappa(\zeta f(\omega)) M(\mathrm{d}\omega) \tag{2.15}$$

□

Proof This more or less well known result follows essentially from the following formal calculation, using product integration, the independence scattering property of z, and formula (2.12):

$$\begin{aligned}
\exp\left\{\mathrm{C}\{\zeta \ddagger \int_A f dz\}\right\} &= \mathrm{E}\left\{\{\exp\{i\zeta \int_A f dz\}\right\} \\
&= \mathrm{E}\left\{\prod_{\omega\in A} \exp\{i\zeta f(\omega)\mathrm{d}z(\omega)\}\right\} \\
&= \prod_{\omega\in A} \mathrm{E}\left\{\exp\{i\zeta f(\omega)\mathrm{d}z(\omega)\}\right\} \\
&= \prod_{\omega\in A} \exp\left\{\mathrm{C}\{\zeta f(\omega) \ddagger \mathrm{d}z(\omega)\}\right\} \\
&= \prod_{\omega\in A} \exp\{\kappa(\zeta f(\omega)) M(\mathrm{d}\omega)\} \\
&= \exp\left\{\int_A \kappa(\zeta f(\omega)) M(\mathrm{d}\omega)\right\}
\end{aligned}$$

□

Proposition 2.2 A function f on Ω is $z-$integrable if and only if the following two conditions hold:

(i) $\int_\Omega V_0(f(\omega)) M(\mathrm{d}\omega) < \infty$
(ii) $\int_\Omega |V(f(\omega))| M(\mathrm{d}\omega) < \infty$

where

$$V_0(y) = \int_{\mathbf{R}} \min\{1, (yx)^2\} W(\mathrm{d}x) \tag{2.16}$$

$$V(y) = \int_{\mathbf{R}} (\tau(yx) - y\tau(x))\, W(\mathrm{d}x) \tag{2.17}$$

□

For a proof, see Rajput and Rosinski (1989).

For later use we note that for $y > 0$ we have

$$\begin{aligned} V_0(y) &= \int_{|x|\le y^{-1}} (yx)^2 W(\mathrm{d}x) + \int_{|x|>y^{-1}} W(\mathrm{d}x) \\ &= y^2 \int_{|x|\le y^{-1}} x^2 W(\mathrm{d}x) + W^+(y^{-1}) + W^-(-y^{-1}) \end{aligned} \tag{2.18}$$

and

$$V(y) = I_{1+}(y) + I_{1-}(y) + I_{2+}(y) + I_{2-}(y) + I_3(y) \tag{2.19}$$

where

$$I_{1+}(y) = y \int \mathbf{1}_{\{1<x\le y^{-1}\}}(x-1)W(\mathrm{d}x)$$

$$I_{1-}(y) = y \int \mathbf{1}_{\{-y^{-1}<x\le -1\}}(x+1)W(\mathrm{d}x)$$

$$I_{2+}(y) = \int \mathbf{1}_{\{0<x\le 1\}} \mathbf{1}_{\{y^{-1}<x\}}(1-yx)W(\mathrm{d}x)$$

$$I_{2-}(y) = \int \mathbf{1}_{\{-1\le x<0\}} \mathbf{1}_{\{x<-y^{-1}\}}(1-yx)W(\mathrm{d}x)$$

$$I_3(y) = (1-y) \int \mathbf{1}_{\{\max\{1,y^{-1}\}<x\}} W(\mathrm{d}x)$$

For the equality (2.19) we have used that

$$\tau(yx) - y\tau(x) = \begin{cases} 0 & \text{if} \quad |x| \le 1 \wedge |yx| \le 1 \\ yx - y\mathrm{sign}x & \text{if} \quad |x| > 1 \wedge |yx| \le 1 \\ \mathrm{sign}(yx) - yx & \text{if} \quad |x| \le 1 \wedge |yx| > 1 \\ \mathrm{sign}(yx) - y\mathrm{sign}x & \text{if} \quad |x| > 1 \wedge |yx| > 1 \end{cases}$$

3. Superposition of Ornstein-Uhlenbeck type processes

Let $\Omega = \mathbf{R} \times \mathbf{R}_+$, with points $\omega = (s, \xi)$, and let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbf{R} \times \mathbf{R}_+$. Furthermore, let z be an independently scattered random measure on $(\Omega, \mathcal{B})$ with characteristics $(0, 0, Q)$ and with Q of the form $Q(A, \mathrm{d}u) = M(A)W(\mathrm{d}u)$. Henceforth, let

$$\grave{\kappa}(\zeta) = \int_{\mathbf{R}} (e^{i\zeta x} - 1 - i\zeta\tau(x))W(\mathrm{d}x)$$

Theorem 3.1 Suppose that the measure M factorizes as

$$M(\mathrm{d}\omega) = \mathrm{d}s\nu(\mathrm{d}\xi) \tag{3.1}$$

where ν is a probability measure on $\mathbf{R}_+$. Assume furthermore that the Lévy measure W is such that W^- and W^+ are of the form

$$W^-(x) = |x|\, u(x) \quad \text{and} \quad W^+(x) = xu(x) \tag{3.2}$$

u being the Lévy density of a selfdecomposable distribution on $\mathbf{R}$.

Define the family $x(\cdot, \mathrm{d}\xi) = \{x(t, \mathrm{d}\xi) : t \in \mathbf{R}\}$ of random measures on $\mathbf{R}_+$ by

$$x(t, B) = \int_B e^{-\xi t} \int_{-\infty}^{\xi t} e^s z(\mathrm{d}s, \mathrm{d}\xi) \tag{3.3}$$

and let

$$x(t) = x(t, \mathbf{R}_+) \tag{3.4}$$

Then $x = \{x(t) : t \in \mathbf{R}\}$ is a welldefined, infinitely divisible and stationary process, and the cumulant transforms of the finite dimensional distributions of x are given by

$$\mathrm{C}\{\zeta_1, ..., \zeta_m \ddagger x(t_1), ..., x(t_m)\} = \int_{\mathbf{R}_+} \int_{\mathbf{R}} \grave{\kappa}(\sum_{j=1}^{m} \mathbf{1}_{[0,\infty)}(t_j - s)\zeta_j e^{-\xi(t_j - s)})\xi \mathrm{d}s\nu(\mathrm{d}\xi) \tag{3.5}$$

where $\grave{\kappa}$ is the cumulant function corresponding to the Lévy measure W and $t_1 < ... < t_m$. $\square$

Remark Note that condition (3.2) implies that W is the Lévy measure of the BDLP corresponding to the selfdecomposable law whose Lévy density is u, cf. (2.3) and (2.4).

Remark Formal calculation from the formulae (3.4) and (3.3) gives

$$\mathrm{d}x(t) = \int_{\mathbf{R}_+} \{-\xi x(t, \mathrm{d}\xi)\mathrm{d}t + z(\mathrm{d}t, \mathrm{d}\xi)\} \tag{3.6}$$

showing that x is a superposition of, perhaps infinitesimally determined, Ornstein-Uhlenbeck type processes. We shall refer to any such process as a supOU *process.*

Proof As the first step we need to verify that the random measure $x(t, \cdot)$ is welldefined. For this we rewrite $x(t, B)$ as

$$x(t, B) = \int_{\Omega} h(s, \xi; t, B) z(\mathrm{d}s, \mathrm{d}\xi)$$

where

$$h(s, \xi; t, B) = e^{-\xi t} e^{s} \mathbf{1}_{(-\infty, \xi t) \times B}(s, \xi) \tag{3.7}$$

We must establish that conditions (i) and (ii) of Proposition 2.2 are satisfied for $f(\omega)$ equal to $h(s, \xi; t, B)$. For this it suffices to consider the case $B = \mathbf{R}_+$. The two conditions then take the form

$$\int_{\mathbf{R}_+} \int_{-\infty}^{\xi t} V_0(e^{-\xi t + s}) \mathrm{d}s \nu(\mathrm{d}\xi) < \infty$$

$$\int_{\mathbf{R}_+} \int_{-\infty}^{\xi t} \left| V(e^{-\xi t + s}) \right| \mathrm{d}s \nu(\mathrm{d}\xi) < \infty$$

A change of variables transforms the integrals to

$$\int_{\mathbf{R}_+} \int_{0+}^{1} V_0(r) r^{-1} \mathrm{d}r \nu(\mathrm{d}\xi) \tag{3.8}$$

$$\int_{\mathbf{R}_+} \int_{0+}^{1} |V(r)| \, r^{-1} \mathrm{d}r \nu(\mathrm{d}\xi) \tag{3.9}$$

and since ν is a probability measure it only remains to show that $\int_{0+}^{1} V_0(r) r^{-1} \mathrm{d}r$ and $\int_{0+}^{1} |V(r)| \, r^{-1} \mathrm{d}r$ are finite.

By (2.18),

$$\int_{0+}^{1} V_0(r) r^{-1} \mathrm{d}r = J_0 + J_+ + J_-$$

where the three terms on the right corresponds to three terms in the expression (2.18). Now,

$$\begin{aligned} J_0 &= \int_{0+}^{1} r \int_{|x| \le r^{-1}} x^2 W(\mathrm{d}x) \\ &= \int_{0+}^{1} r \int_{|x| \le 1} x^2 W(\mathrm{d}x) + \int_{0+}^{1} r \int_{1 < |x| \le r^{-1}} x^2 W(\mathrm{d}x) \\ &= \frac{1}{2} \int_{|x| \le 1} x^2 W(\mathrm{d}x) + \int_{1 < |x| \le r^{-1}} x^2 W(\mathrm{d}x) \int_{0+}^{|x|^{-1}} r \mathrm{d}r \\ &= \frac{1}{2} \int_{\mathbf{R}} \min\{1, x^2\} W(\mathrm{d}x) < \infty \end{aligned}$$

since W is a Lévy measure. Furthermore,

$$
\begin{aligned}
J_+ &= \int_{0+}^{1} W^+(r^{-1})r^{-1}\mathrm{d}r \\
&= \int_{1}^{\infty} W^+(x)x^{-1}\mathrm{d}x \\
&= \int_{1}^{\infty} u(x)\mathrm{d}x < \infty
\end{aligned}
$$

by (3.2). A similar calculation yields the finiteness of J_-.

Turning to the second condition we find

$$\int_{0+}^{1} |V(r)|\, r^{-1}\mathrm{d}r \le K_{1+} + K_{1-} + K_{2+} + K_{2-} + K_3$$

where the K-s correspond to the five terms in (2.19) and

$$
\begin{aligned}
K_{1+} &= \int_{0+}^{1} r \int_{1}^{r^{-1}} (x-1)W(\mathrm{d}x)r^{-1}\mathrm{d}r \\
&= \int_{1}^{\infty} (1-x^{-1})W(\mathrm{d}x) < \infty
\end{aligned}
$$

$$K_{1-} = \int_{-\infty}^{-1} (|x|^{-1}-1)W(\mathrm{d}x) < \infty$$

$$K_{2+} = K_{2-} = 0$$

$$
\begin{aligned}
K_3 &= \int_{0+}^{1} (r^{-1}-1) \int_{r^{-1}}^{\infty} W(\mathrm{d}x)\mathrm{d}r \\
&= \int_{1}^{\infty} (1-x^{-1})W(\mathrm{d}x) < \infty
\end{aligned}
$$

Thus both conditions for integrability have been verified.

To show (3.5) we write

$$
\begin{aligned}
\sum_{j=1}^{m} \zeta_j x(t_j) &= \int_{\mathbf{R}_+} \sum_{j=1}^{m} \zeta_j e^{-\xi t_j} \int_{-\infty}^{\xi t_j} e^{s} z(\mathrm{d}s, \mathrm{d}\xi) \\
&= \int_{\Omega} g(s,\xi) z(\mathrm{d}s, \mathrm{d}\xi)
\end{aligned}
$$

where

$$g(s,\xi) = \sum_{j=1}^{m} \zeta_j e^{-\xi t_j} \mathbf{1}_{(-\infty, \xi t_j]}(s) e^{s}$$

Hence, by formula (2.15),

$$\begin{aligned}\mathrm{C}\{\zeta_1,...,\zeta_m \ddagger x(t_1),...,x(t_m)\} &= \int_{\mathbf{R}_+}\int_{\mathbf{R}} \grave{\kappa}(\sum_{j=1}^{m}\zeta_j e^{-\xi t_j}\mathbf{1}_{(-\infty,\xi t_j]}(s)e^{s})\mathrm{d}s\nu(\mathrm{d}\xi) \\ &= \int_{\mathbf{R}_+}\int_{\mathbf{R}} \grave{\kappa}(\sum_{j=1}^{m}\zeta_j e^{-\xi(t_j-s)}\mathbf{1}_{[0,\infty)}(t_j-s))\mathrm{d}s\xi\nu(\mathrm{d}\xi)\end{aligned}$$

The stationarity and infinite divisibility of the process x follow immediately from this expression and the infinite divisibility of $\grave{\kappa}$. $\square$

Corollary 3.1 We have $\mathrm{C}\{\zeta \ddagger x(t)\} = \acute{\kappa}(\zeta)$ where $\acute{\kappa}$ is the cumulant function of the selfdecomposable law with Lévy density u. $\square$

Proof Formula (3.5) implies, in particular, that

$$\mathrm{C}\{\zeta \ddagger x(t)\} = \int_0^\infty \grave{\kappa}(\zeta e^{-s})\mathrm{d}s$$

and the result now follows from formula (2.6). $\square$

Corollary 3.2 Assuming that x is square integrable, the autocorrelation function r of x is given by

$$r(\tau) = \int_0^\infty e^{-\tau\xi}\nu(\mathrm{d}\xi) = \exp\bar{\mathrm{K}}\{\tau \ddagger \xi\} \tag{3.10}$$

for $\tau \geq 0$ and where for the last expression we interpret ξ as a random variable with distribution ν. $\square$

Example 3.1 Suppose that ν is the gamma law $\Gamma(2\bar{H},1)$ where $\bar{H} > 0$. Then

$$r(\tau) = (1+u)^{-2\bar{H}} \tag{3.11}$$

In particular, then, the process x exhibits second order long range dependence if $H \in (\frac{1}{2},1)$ where $H = 1-\bar{H}$. $\square$

Note Corollaries 3.1 and 3.2 together show that to any selfdecomposable distribution D with finite second moment and to any cumulant transform $\bar{\mathrm{K}}$ of a distribution ν on $\mathbf{R}_+$ there exists a stationary process x on $\mathbf{R}$ whose one-dimensional marginal law is D and whose autocorrelation function satisfies (3.10).

The inverse Gaussian and the normal inverse Gaussian laws referred to in the the following examples have some special interest in the context of finance, see for instance Barndorff-Nielsen (1997, 1998a,b) and Barndorff-Nielsen and Shephard (1998a,b).

Example 3.2 IG-supOU *processes.* The inverse Gaussian law $IG(\delta, \gamma)$ is the distribution on the positive halfline having density

$$\frac{\delta}{\sqrt{(2\pi)}} e^{\delta\gamma} x^{-3/2} \exp\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\}$$

where $\delta > 0$ and $\gamma \geq 0$. The distribution is selfdecomposable and for $\gamma > 0$ there exists a supOU process x with one-dimensional marginal distribution $IG(\delta, \gamma)$ and autocorrelation function (3.11). □

Example 3.3 NIG-supOU *processes.* The normal inverse Gaussian law $NIG(\delta, \gamma)$ is the symmetric distribution on the real line having density

$$\frac{\delta\gamma}{\pi} e^{\gamma} (\delta^2 + x^2)^{-1/2} K_1(\gamma(\delta^2 + x^2)) \tag{3.12}$$

where $\delta > 0$ and $\gamma \geq 0$. The distribution is selfdecomposable and for $\gamma > 0$ there exists a supOU process x with one-dimensional marginal distribution $IG(\delta, \gamma)$ and autocorrelation function (3.11). □

4. The integrated processes

As mentioned in the Introduction, the properties of integrals of the superposition processes x considered in the previous section are of interest for instance in mathematical finance.

Thus, let x be a process of the type considered in Theorem 3.1 and define the integrated process $x^* = \{x^*(t) : t \geq 0\}$ by

$$x^*(t) = \int_0^t x(s)\mathrm{d}s \tag{4.1}$$

Then x^* is a process with stationary increments whose cumulant function we shall denote by κ^*, i.e.

$$\kappa^*(\zeta, t) = \mathrm{C}\{\zeta \ddagger x^*(t)\}$$

Define

$$\varepsilon(t; \lambda) = \lambda^{-1}(1 - e^{-\lambda t}) \tag{4.2}$$

and recall that $x(t)$ has cumulant function $\acute{\kappa}$ (cf. Corollary 3.1).

Theorem 4.1 Assume that the cumulant function $\acute{\kappa}(\zeta)$ of $x(t)$ is differentiable for $\zeta \neq 0$ and that $\zeta\acute{\kappa}'(\zeta) \to 0$ for $0 \neq \zeta \to 0$.

Then the cumulant function κ^* of $x^*(t)$ satisfies

$$\kappa^*(\zeta, t) = \zeta \int_0^\infty \int_0^t \acute{\kappa}'(\varepsilon(u;\xi)\zeta)\mathrm{d}u\nu(\mathrm{d}\xi) \tag{4.3}$$

□

Proof By (3.3) and (3.4) we have

$$\begin{aligned} x^*(t) &= \int_0^t \int_0^\infty \int_{\mathbf{R}} e^{-\xi s+u}\mathbf{1}_{(u/\xi,\infty)}(s)z(\mathrm{d}u,\mathrm{d}\xi)\mathrm{d}s \\ &= \int_0^\infty \int_{-\infty}^0 \left\{\int_0^t e^{-\xi s+u}\mathrm{d}s\right\} z(\mathrm{d}u,\mathrm{d}\xi) \\ &\quad + \int_0^\infty \int_0^{\xi t} \left\{\int_{u/\xi}^t e^{-\xi s+u}\mathrm{d}s\right\} z(\mathrm{d}u,\mathrm{d}\xi) \\ &= \int_0^\infty \int_{-\infty}^0 \varepsilon(t;\xi)e^u z(\mathrm{d}u,\mathrm{d}\xi) \\ &\quad + \int_0^\infty \int_0^{\xi t} \xi^{-1}(1-e^{-\xi t+u})z(\mathrm{d}u,\mathrm{d}\xi) \end{aligned} \tag{4.4}$$

whence, using (2.15), (2.6) and (2.7),

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger x^*(t)\} &= \int_0^\infty \int_{-\infty}^0 \grave{\kappa}(\varepsilon(t;\xi)e^u\zeta)\mathrm{d}u\nu(\mathrm{d}\xi) \\ &\quad + \int_0^\infty \int_0^{\xi t} \grave{\kappa}(\xi^{-1}(1-e^{-\xi t+u})\zeta)\mathrm{d}u\nu(\mathrm{d}\xi) \\ &= \int_0^\infty \int_0^\infty \grave{\kappa}(\varepsilon(t;\xi)\zeta e^{-u})\mathrm{d}u\nu(\mathrm{d}\xi) \\ &\quad + \int_0^\infty \int_0^t \xi\grave{\kappa}(\varepsilon(u;\xi)\zeta)\mathrm{d}u\nu(\mathrm{d}\xi) \\ &= \int_0^\infty \acute{\kappa}(\varepsilon(t;\xi)\zeta)\nu(\mathrm{d}\xi) \\ &\quad + \zeta \int_0^\infty \int_0^t \xi\varepsilon(u;\xi)\acute{\kappa}'(\varepsilon(u;\xi)\zeta)\mathrm{d}u\nu(\mathrm{d}\xi) \end{aligned} \tag{4.5}$$

Further, noting that

$$\varepsilon(u;\xi)_{/u} = 1 - \xi\varepsilon(u;\xi) \tag{4.6}$$

(where / indicates differentiation), we find

$$\begin{aligned} \zeta \int_0^t \xi\varepsilon(u;\xi)\acute{\kappa}'\{\varepsilon(u;\xi)\zeta\}\mathrm{d}u &= \zeta \int_0^t \acute{\kappa}'\{\varepsilon(u;\xi)\zeta\}\mathrm{d}u - \int_0^t \acute{\kappa}\{\varepsilon(u;\xi)\zeta\}_{/u}\mathrm{d}u \\ &= \zeta \int_0^t \acute{\kappa}'\{\varepsilon(u;\xi)\zeta\}\mathrm{d}u - \acute{\kappa}\{\varepsilon(u;\xi)\zeta\} \end{aligned}$$

and this combined with (4.5) yields (4.3). $\square$

Theorem 4.2 Assume that $\acute{\kappa}(\zeta)$ is analytic in a neighbourhood of the origin. The cumulants of $x^*(t)$ are then given by

$$\kappa_m^*(t) = \acute{\kappa}_m m \mathbf{I}_{m-1}(t) \tag{4.7}$$

where the $\acute{\kappa}_m$ are the cumulants of $x(t)$ and where

$$\mathbf{I}_{m-1}(t) = \int_0^\infty \left\{ a_{m-1} + t\xi + \sum_{k=1}^{m-1} (-1)^{k-1} \binom{m-1}{k} k^{-1} e^{-kt\xi} \right\} \xi^{-m} \nu(\mathrm{d}\xi) \tag{4.8}$$

with

$$a_{m-1} = \sum_{k=1}^{m-1} (-1)^k \binom{m-1}{k} k^{-1} \ .$$

$\square$

Proof From (4.3) we find

$$\kappa_m^*(t) = \acute{\kappa}_m m \mathbf{I}_{m-1}(t)$$

where

$$\begin{aligned}
\mathbf{I}_{m-1}(t) &= \int_0^\infty \int_0^t \varepsilon(u;\xi)^{m-1} \mathrm{d}u \nu(\mathrm{d}\xi) \\
&= \int_0^\infty \xi^{-m+1} \int_0^t (1 - e^{-\xi u})^{m-1} \mathrm{d}u \nu(\mathrm{d}\xi) \\
&= \int_0^\infty \xi^{-m} \int_0^{\xi t} (1 - e^{-w})^{m-1} \mathrm{d}w \nu(\mathrm{d}\xi) \\
&= \int_0^\infty \int_0^{\xi t} \left\{ 1 + \sum_{k=1}^{m-1} (-1)^k \binom{m-1}{k} e^{-kw} \right\} \mathrm{d}w \xi^{-m} \nu(\mathrm{d}\xi) \\
&= \int_0^\infty \left\{ \xi t + a_{m-1} + \sum_{k=1}^{m-1} (-1)^{k-1} \binom{m-1}{k} k^{-1} e^{-k\xi t} \right\} \xi^{-m} \nu(\mathrm{d}\xi)
\end{aligned}$$

$\square$

Example 4.1 As in Example 3.1, suppose that ν is the gamma law $\Gamma(2\bar{H}, 1)$. To calculate $\mathbf{I}_{m-1}(t)$ we recall that the incomplete gamma function

$$\Gamma(\alpha, \delta) = \int_\delta^\infty x^{\alpha-1} e^{-x} \mathrm{d}x$$

satisfies

$$\Gamma(\alpha,\delta)=\Gamma(\alpha)-\sum_{n=0}^{\infty}\frac{(-1)^n\delta^{\alpha+n}}{n!(\alpha+n)} \tag{4.9}$$

provided $\alpha\neq 0,-1,-2,\ldots$. Furthermore, $\Gamma(\alpha)$ may be expressed as

$$\Gamma(\alpha)=\int_1^\infty x^{\alpha-1}e^{-x}\mathrm{d}x+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha+n)^{-1}\ .$$

Consequently, writing (4.8) as

$$\mathbf{I}_{m-1}(t)=\lim_{\delta\downarrow 0}\int_\delta^\infty\left\{a_{m-1}+t\xi+\sum_{k=1}^{m-1}(-1)^{k-1}\tbinom{m-1}{k}k^{-1}e^{-kt\xi}\right\}\xi^{-m}\nu(\mathrm{d}\xi)$$

we find that for $\bar{H}\neq 1/2$

$$\begin{aligned}\mathbf{I}_{m-1}(t) \;=\; & \Gamma(2\bar{H})^{-1}\{\Gamma(-m+2\bar{H})a_{m-1}+\Gamma(-m+1+2\bar{H})t\\ & +\Gamma(-m+2\bar{H})\sum_{k=1}^{m-1}(-1)^{k-1}\tbinom{m-1}{k}k^{-1}(1+2kt)^{m-2\bar{H}}\}\end{aligned} \tag{4.10}$$

Here we have used the fact that, since $\mathbf{I}_{m-1}(t)$ is known to exist and is finite, the singular terms arising from (4.8) must cancel. $\square$

The technique used to derive the relation (4.3) can be extended to yield formulae for the cumulant functions of the finite dimensional distributions of x^*. In particular, for the joint law of $x^*(s)$ and $x^*(t)$, where $0<s<t$, we have

$$\mathrm{C}\{(\eta,\zeta)\ddagger(x^*(s),x^*(t))\}=\int_0^\infty G(\xi)\nu(\mathrm{d}\xi) \tag{4.11}$$

where

$$G(\xi)=\int_0^s\acute{\kappa}'(\varepsilon(u;\xi)\eta+\varepsilon(t-s+u;\xi)\zeta)\mathrm{d}u+\zeta\int_s^t\acute{\kappa}'(\varepsilon(t-u;\xi)\zeta)\mathrm{d}u \tag{4.12}$$

We omit the derivation. A more general approach is discussed in Section 5 below.

If the stationary process x is square integrable and has mean 0 then, by (4.1),

$$\begin{aligned}\mathrm{Cov}\{x^*(s)x^*(t)\} &= \int_0^s\int_0^t\mathrm{E}\{x(\sigma)x(\tau)\}\mathrm{d}\sigma\mathrm{d}\tau\\ &= \kappa_2\int_0^s\int_0^t r(\tau-\sigma)\mathrm{d}\sigma\mathrm{d}\tau\\ &= \kappa_2\int_0^s\int_0^t\bar{\mathrm{L}}_\nu(|\tau-\sigma|)\mathrm{d}\sigma\mathrm{d}\tau\\ &= \kappa_2\left(2\int_0^s\int_0^\sigma\bar{\mathrm{L}}_\nu(\tau)\mathrm{d}\tau\mathrm{d}\sigma+\int_0^s\int_s^t\bar{\mathrm{L}}_\nu(\tau-\sigma)\mathrm{d}\tau\mathrm{d}\sigma\right)\end{aligned} \tag{4.13}$$

where $\bar{\mathrm{L}}_\nu(\tau) = \int_0^\infty e^{-\tau\xi}\nu(\mathrm{d}x)$.

Example 4.2 Suppose again that ν is the gamma law $\Gamma(2\bar{H},1)$. Then the expression (4.13) reduces to

$$\mathrm{Cov}\{x^*(s)x^*(t)\} = \kappa_2 c_2[\{1+s\}^{2H} + \{1+t\}^{2H} - \{1+t-s\}^{2H} - 1 - 4Hs] \quad (4.14)$$

with

$$c_2 = \{(2H-1)2H\}^{-1}$$

□

5. Integration

We proceed to define integration of real functions f with respect to the cumulative process x^* and thereafter to calculate the cumulant functionals of such integrals. In other words, we shall determine the characteristic functional of x^*, as a means to study the law of that process.

To arrive at a suitable definition of integrals

$$\int_0^\infty f(t)\mathrm{d}x^*(t) \quad (5.1)$$

we first argue formally as if the integral had been defined. In view of (4.1), (3.3) and (3.4) we write

$$\begin{aligned} \int_0^\infty f(t)\mathrm{d}x^*(t) &= \int_0^\infty f(t)x(t)\mathrm{d}t \\ &= \int_0^\infty \int_0^\infty \int_{-\infty}^{\xi t} f(t)e^{-\xi t}e^s z(\mathrm{d}s,\mathrm{d}\xi)\mathrm{d}t \\ &= \int_0^\infty \int_{-\infty}^\infty F(s,\xi)z(\mathrm{d}s,\mathrm{d}\xi) \end{aligned}$$

where, for $s \in \mathbf{R}$ and $\xi \in \mathbf{R}_+$,

$$F(s,\xi) = e^s \int_0^\infty f(t)e^{-\xi t}\mathbf{1}_{[\xi^{-1}s,\infty)}(t)\mathrm{d}t \quad (5.2)$$

Now, let $\mathcal{F}_z$ be the class of functions f on $[0,\infty)$ such that $F(s,\xi)$ is integrable with respect to the independently scattered random measure z, on the basis of which the process x^* is defined (cf. Theorem 3.1). For any $f \in \mathcal{F}_z$ we then define the integral of f with respect to x^* by

$$\int_0^\infty f(t)\mathrm{d}x^*(t) = \int_0^\infty \int_{-\infty}^\infty F(s,\xi)z(\mathrm{d}s,\mathrm{d}\xi) \quad (5.3)$$

Theorem 5.1 In the setting of Theorem 3.1, assume that the cumulant function $\acute{\kappa}(\zeta)$ of $x(t)$ is differentiable for $\zeta \neq 0$ and that $\zeta\acute{\kappa}'(\zeta) \to 0$ for $0 \neq \zeta \to 0$.

Then, for any function $f \in \mathcal{F}_z$,

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger \int_0^\infty f(t)\mathrm{d}x^*(t)\} &= \int_0^\infty \int_{-\infty}^\infty \grave{\kappa}(\zeta \int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}s\xi\nu(\mathrm{d}\xi) \\ &= \zeta \int_0^\infty \int_0^\infty f(s)\acute{\kappa}'(\zeta \int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}s\nu(\mathrm{d}\xi) \end{aligned} \tag{5.4}$$

□

Proof Note first that, extending the definition of f to all of $\mathbf{R}$ by letting $f(t) = 0$ for $t < 0$, we may rewrite $F(s,\xi)$ as

$$F(s,\xi) = e^s \int_{s/\xi}^\infty f(t)e^{-\xi t}\mathrm{d}t$$

Hence, by Proposition 2.1 and with the assumptions in Theorem 3.1, we find

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger \int_0^\infty f(t)\mathrm{d}x^*(t)\} &= \int_0^\infty \int_{-\infty}^\infty \grave{\kappa}(\zeta F(s,\xi))\mathrm{d}s\nu(\mathrm{d}\xi) \\ &= \int_0^\infty \int_{-\infty}^\infty \grave{\kappa}(\zeta F(\xi s,\xi))\xi\mathrm{d}s\nu(\mathrm{d}\xi) \\ &= \int_0^\infty \int_{-\infty}^\infty \grave{\kappa}(\zeta \int_s^\infty f(t)e^{-\xi(t-s)}\mathrm{d}t)\mathrm{d}s\xi\nu(\mathrm{d}\xi) \\ &= \int_0^\infty \int_{-\infty}^\infty \grave{\kappa}(\zeta \int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}s\xi\nu(\mathrm{d}\xi) \end{aligned}$$

the first form in (5.4). By the relation (2.7) this yields

$$\begin{aligned} &\mathrm{C}\{\zeta \ddagger \int_0^\infty f(t)\mathrm{d}x^*(t)\} \\ &= \zeta \int_0^\infty \int_{-\infty}^\infty \int_0^\infty f(\tau+s)e^{-\xi\tau}\mathrm{d}\tau\acute{\kappa}'(\zeta \int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}s\xi\nu(\mathrm{d}\xi) \end{aligned} \tag{5.5}$$

Now, suppose for the time being that f is continuously differentiable, that the integrals

$$\int_0^\infty f'(\tau+s)e^{-\xi\tau}\mathrm{d}\tau \quad \text{and} \quad \int_0^\infty f(\tau+s)e^{-\xi\tau}\mathrm{d}\tau$$

exist for every $s \in \mathbf{R}$ and that the second integral tends to 0 as $s \to \infty$. Note that it also tends to 0 for $s \to -\infty$, because $f(t) = 0$ for $t < 0$. That is

$$\acute{\kappa}(\zeta \int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t) \to 0 \quad \text{for} \quad \mathrm{s} \to \pm\infty \tag{5.6}$$

Since

$$\int_0^\infty f'(\tau+s)e^{-\xi\tau}\mathrm{d}\tau = f(s)+\xi\int_0^\infty f(\tau+s)e^{-\xi\tau}\mathrm{d}\tau$$

we have

$$\acute{\kappa}(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}t)_{/s} = \acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\{f(s)+\xi\int_0^\infty f(\tau+s)e^{-\xi\tau}\mathrm{d}\tau\}$$

or, equivalently,

$$\begin{aligned}\int_0^\infty f(\tau+s)e^{-\xi\tau}\mathrm{d}\tau\acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t) &= \xi^{-1}f(s)\acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\\ &\quad -\xi^{-1}\acute{\kappa}(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}t)_{/s}\end{aligned}$$

where $/s$ indicates differentiation with respect to s. Insertion into (5.5) and a splitting of the second integral gives

$$\begin{aligned}&\mathrm{C}\{\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x^*(t)\}\\ &= \zeta\int_0^\infty\int_{-\infty}^\infty f(s)\acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}s\nu(\mathrm{d}\xi))\\ &\quad -\zeta\int_0^\infty\int_{-\infty}^\infty \acute{\kappa}(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)_{/s}\mathrm{d}s\nu(\mathrm{d}\xi))\end{aligned}$$

and, in view of (5.6), this reduces to

$$\mathrm{C}\{\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x^*(t)\} = \zeta\int_0^\infty\int_0^\infty f(s)\acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\mathrm{d}s\nu(\mathrm{d}\xi))$$

which is the second form (5.4). Finally, a limit argument shows that the restrictions imposed on f in the course of the proof can be lifted. $\square$

The formulae (4.3) and (4.11) are easily seen to be special cases of formula (5.4).

6. A class of limiting processes

Let $x^*_\lambda(t) = \lambda^{-1}x^*(\lambda t)$ and let the independently scattered random measure $z(\mathrm{d}u,\mathrm{d}\xi)$ be, in fact, dependent on λ, which we shall indicate by writing $z_\lambda(\mathrm{d}u,\mathrm{d}\xi)$. Then, since

$$\varepsilon(\lambda t;\xi) = \lambda\varepsilon(t;\lambda\xi) \tag{6.1}$$

we find, from (4.4),

$$\begin{aligned}
x_\lambda^*(t) &= \lambda^{-1}\int_0^\infty\int_{-\infty}^0 \varepsilon(\lambda t;\xi)e^u z_\lambda(\mathrm{d}u,\mathrm{d}\xi)\\
&\quad+\lambda^{-1}\int_0^\infty\int_0^{\lambda\xi t}\xi^{-1}(1-e^{-\lambda\xi t+u})z_\lambda(\mathrm{d}u,\mathrm{d}\xi)\\
&= \int_0^\infty\int_{-\infty}^0 \varepsilon(t;\xi)e^u z_\lambda(\mathrm{d}u,\lambda^{-1}\mathrm{d}\xi)\\
&\quad+\int_0^\infty\int_0^{\xi t}\xi^{-1}(1-e^{-\xi t+u})z_\lambda(\mathrm{d}u,\lambda^{-1}\mathrm{d}\xi) \qquad (6.2)
\end{aligned}$$

In particular, suppose that for $\lambda\to\infty$ the measure $z_\lambda(\mathrm{d}u,\lambda^{-1}\mathrm{d}\xi)$ converges, at least in law, to a limiting independently scattered random measure $z_0(\mathrm{d}u,\mathrm{d}\xi)$. Then the process $x_\lambda^*=\{x_\lambda^*(t)\}_{0\le t}$ will converge, at least in law, to a limiting process $x_0^*=\{x_0^*(t)\}_{0\le t}$.With sufficiently strong assumptions on the type of convergence, the integrals in (6.2) will converge too. However, it may happen that x_λ^* converges in law without the integrals converging.

As a particular setting, assume that ν is the gamma law $\Gamma(2\bar{H},1)$, with $\bar{H}=1-H$ and $\frac{1}{2}<H<1$. Then the generalized Lévy measure of $z(\mathrm{d}u,\lambda^{-1}\mathrm{d}\xi)$ is

$$\lambda^{-2\bar{H}}\Gamma(2\bar{H})^{-1}\xi^{2\bar{H}-1}e^{-\xi/\lambda}W(\mathrm{d}x)\mathrm{d}s\mathrm{d}\xi \qquad (6.3)$$

Letting W depend on λ, more specifically substituting $\lambda^{2\bar{H}}W(\mathrm{d}x)$ for $W(\mathrm{d}x)$ in (6.3), we obtain a generalized Lévy measure

$$Q_\lambda(\mathrm{d}\omega,\mathrm{d}x)=\Gamma(2\bar{H})^{-1}\xi^{2\bar{H}-1}e^{-\xi/\lambda}W(\mathrm{d}x)\mathrm{d}s\mathrm{d}\xi \qquad (6.4)$$

and we may then define $z_\lambda(\mathrm{d}u,\mathrm{d}\xi)$ as the independently scattered random measure whose generalized Lévy measure is $Q_\lambda(\mathrm{d}\omega,\mathrm{d}x)$. With this setup we have, by Theorem 5.1,

$$\mathrm{C}\{\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x_\lambda^*(t)\}=\Gamma(2\bar{H})^{-1}\zeta\int_0^\infty\int_0^\infty f(s)\acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\xi^{2\bar{H}-1}e^{-\xi/\lambda}\mathrm{d}s\mathrm{d}\xi$$

and hence, provided the integral

$$I(f)=\int_0^\infty\int_0^\infty f(s)\acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\xi^{2\bar{H}-1}\mathrm{d}s\mathrm{d}\xi \qquad (6.5)$$

exists for a sufficiently broad class of functions f we obtain that, for $\lambda\to\infty$, the process x_λ^* converges in law to a process x_0^* the law of which is determined by the cumulant functional

$$\mathrm{C}\{\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x_0^*(t)\}=\Gamma(2\bar{H})^{-1}\zeta\int_0^\infty\int_0^\infty f(s)\acute{\kappa}'(\zeta\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t)\xi^{2\bar{H}-1}\mathrm{d}s\mathrm{d}\xi \qquad (6.6)$$

Now, for $\theta \neq 0$, let

$$k(\theta) = \acute{\kappa}'(\theta)/\theta \tag{6.7}$$

Theorem 6.1 Let $z_\lambda(\mathrm{d}u, \mathrm{d}\xi)$ be the independently scattered random measure having generalized Lévy measure (6.4), with $\bar{H} = 1 - H \in (0, \frac{1}{2})$, and let $x^*_\lambda(t)$ be defined by (6.2). Assume that the function k is defined for all $\theta \neq 0$ and bounded.

Then $x^*_\lambda = \{x^*_\lambda(t)\}_{0 \leq t}$ converges in law to a process $x^*_0 = \{x^*_0(t)\}_{0 \leq t}$ with cumulant functional (6.6). □

Proof It suffices to show that $I(f)$ exists for all functions f in the class

$$\bigcup_{B \in \mathbf{R}_+} \bigcup_{T \in \mathbf{R}_+} \{f : 0 < f(t) \leq B \text{ for } t \in (0,T],\ f(t) = 0 \text{ for } t \notin (0,T]\} \tag{6.8}$$

Thus let f be such a function and let $K = \sup_{\theta \neq 0} |k(\theta)|$. Invoking the definition of k we find

$$I(f) = \zeta \int_0^T f(s) \int_0^\infty k(\zeta \int_0^{T-s} f(t{+}s) e^{-\xi t} \mathrm{d}t) \int_0^{T-s} f(t{+}s) e^{-\xi t} \mathrm{d}t \xi^{2\bar{H}-1} \mathrm{d}\xi \mathrm{d}s \tag{6.9}$$

Hence

$$\begin{aligned}
|I(f)| &\leq |\zeta| B^2 K \int_0^\infty \int_0^T \int_0^{T-s} e^{-\xi t} \mathrm{d}t \xi^{2\bar{H}-1} \mathrm{d}s \mathrm{d}\xi \\
&= |\zeta| B^2 K \int_0^\infty \int_0^T (1 - e^{(T-s)\xi}) \xi^{2\bar{H}-2} \mathrm{d}s \mathrm{d}\xi \\
&= |\zeta| B^2 K \int_0^\infty \{e^{-T\xi} - 1 + T\xi\} \xi^{2\bar{H}-3} \mathrm{d}s \mathrm{d}\xi \\
&< \infty
\end{aligned}$$

□

Example 6.1 Let W be the Lévy measure of the BDLP for the symmetric NIG-supOU process discussed in Example 3.3. Then the conditions of Theorem 6.1 are fulfilled. In particular, the cumulant function of $NIG(\delta, \gamma)$ is (Barndorff-Nielsen, 1997)

$$\acute{\kappa}(\zeta) = \delta\gamma - \delta\{\gamma^2 + \zeta^2\}^{1/2} \tag{6.10}$$

and hence

$$\acute{\kappa}'(\zeta) = -\delta\zeta\{\gamma^2 + \zeta^2\}^{-1/2}$$

and

$$k(\theta) = -\delta\{\gamma^2 + \theta^2\}^{-1/2}$$

which is bounded. $\square$

Recall, from Section 1, that a process x^* is said to be H-ss$_2$si if its covariance function has the same form as if the process was exactly selfsimilar and provided x^* has stationary increments.

Example 6.2 *A class of* H-ss$_2$si *processes.* Consider the class of limiting processes x_0^* determined by Theorem 6.1. If x_0^* is square integrable with mean 0 and variance κ_{02} (as is the case, for instance, in Example 6.1) then, from (4.14),

$$\mathrm{Cov}\{x_0^*(s)x_0^*(t)\} = \kappa_{02}\{(2H-1)2H\}^{-1}\{s^{2H} + t^{2H} - (t-s)^{2H}\} \tag{6.11}$$

for $0 < s \le t$. Hence x_0^* is an H-ss$_2$si process. $\square$

Example 6.3 The limiting form of (4.7) and (4.10) corresponding to (6.6) is

$$\kappa_{0m}^*(t) = \acute{\kappa}_m m \mathbf{I}_{0m-1}(t)$$

where

$$\begin{aligned}\mathbf{I}_{0m-1}(t) &= 2^{m-2\bar{H}}\{(m-2\bar{H})...(1-2\bar{H})\}^{-1} \\ &\cdot \left\{\sum_{k=1}^{m-1}(-1)^{m+k-1}\binom{m-1}{k}k^{m-2\bar{H}-1}\right\}t^{m-2\bar{H}}\end{aligned}$$

and we have used that $\Gamma(x+1) = x\Gamma(x)$. $\square$

Boundedness of $k(\theta)$, as assumed in Theorem 6.1, is not a necessary condition for the existence of the integral $I(f)$.

Example 6.4 *A class of stable* H'-sssi *processes.* Choosing $W(\mathrm{d}x) = |x|^{-\alpha-1}$, $\alpha \in (0,2)$, we obtain that the stationary process x has the symmetric α-stable law as one-dimensional marginal and

$$\acute{\kappa}(\zeta) = -|\zeta|^{\alpha}$$

It follows that

$$k(\theta) = -\alpha \mathrm{sign}\theta\, |\theta|^{\alpha-2}$$

which is not bounded. However, arguing as in the proof of Theorem 6.1, we find from (6.9)

$$|I(f)| \le \alpha|\zeta|^{\alpha-1}B^{\alpha}\int_0^T\int_0^{\infty}|\int_0^{T-s}e^{-\xi t}\mathrm{d}t|^{\alpha-1}\xi^{2\bar{H}-1}\mathrm{d}\xi\mathrm{d}s$$

$$
\begin{aligned}
&= \alpha\,|\zeta|^{\alpha-1}\,B^\alpha \int_0^T \int_0^\infty |\xi^{-1}(1-e^{-(T-s)\xi})|^{\alpha-1}\xi^{2\bar{H}-1}\mathrm{d}\xi\mathrm{d}s \\
&= \alpha\,|\zeta|^{\alpha-1}\,B^\alpha \int_0^T \int_0^\infty (1-e^{-\xi s})^{\alpha-1}\xi^{2\bar{H}-\alpha}\mathrm{d}\xi\mathrm{d}s \\
&= \alpha\,|\zeta|^{\alpha-1}\,B^\alpha \int_0^\infty M_\alpha(\xi)\xi^{-(\alpha-2\bar{H})}\mathrm{d}\xi
\end{aligned}
$$

where

$$M_\alpha(\xi) = \int_0^T (1-e^{-\xi s})^{\alpha-1}\mathrm{d}s$$

We have

$$
M_\alpha(\xi) \to \begin{array}{lll} \alpha^{-1}T^\alpha\xi^{\alpha-1} & \text{for} & \xi\to 0 \\ T & \text{for} & \xi\to\infty \end{array}
$$

Hence (recall that we have assumed $0 < 2\bar{H} < 1$) $I(f)$ exists provided

$$\alpha - 2\bar{H} > 1 \tag{6.12}$$

In other words, under the condition (6.12), we have established the existence of a process x_0^* with cumulant functional

$$\mathrm{C}\{\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x_0^*(t)\} = -\Gamma(2\bar{H})^{-1}\alpha|\zeta|^\alpha\int_0^\infty\int_0^\infty f(s)|\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t|^{\alpha-1}\xi^{2\bar{H}-1}\mathrm{d}s\mathrm{d}\xi \tag{6.13}$$

This shows that $x_0^*(t)$ is symmetric α-stable. Furthermore, by construction, x_0^* has stationary increments and, as we shall now verify, it is also selfsimilar.

For this we note that x_0^* is H'-selfsimilar if

$$\mathrm{C}\{\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x_0^*(ct)\} = \mathrm{C}\{c^{H'}\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x_0^*(t)\}$$

for every $c\in(0,1)$. Moreover we have

$$\int_0^\infty f(t)\mathrm{d}x_0^*(ct) = \int_0^\infty f(c^{-1}t)\mathrm{d}x_0^*(t)$$

Hence, by (6.12) and by substituting s by cs, t by ct and ξ by $c^{-1}\xi$,

$$
\begin{aligned}
\mathrm{C}\{\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x_0^*(ct)\} &= -\Gamma(2\bar{H})^{-1}\alpha|\zeta|^\alpha \\
&\quad\cdot\int_0^\infty\int_0^\infty f(c^{-1}s)|\int_0^\infty f(c^{-1}(t+s))e^{-\xi t}\mathrm{d}t|^{\alpha-1}\xi^{2\bar{H}-1}\mathrm{d}s\mathrm{d}\xi \\
&= -\Gamma(2\bar{H})^{-1}\alpha c^{\alpha-2\bar{H}}|\zeta|^\alpha \\
&\quad\cdot\int_0^\infty\int_0^\infty f(s)|\int_0^\infty f(t+s)e^{-\xi t}\mathrm{d}t|^{\alpha-1}\xi^{2\bar{H}-1}\mathrm{d}s\mathrm{d}\xi \\
&= \mathrm{C}\{c^{1-2\bar{H}/\alpha}\zeta\ddagger\int_0^\infty f(t)\mathrm{d}x_0^*(t)\}
\end{aligned}
$$

Consequently, for $1 < \alpha < 2$ and $0 < \bar{H} < (\alpha - 1)/2$ the process x_0^* is H'-sssi where $H' = 1 - 2\bar{H}/\alpha \in (\alpha^{-1}, 1)$. $\square$

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