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HIGHER SKEIN MODULES, II

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Abstract. In our previous paper [1] we introduced a notion of higher Conway skein modules of links. In this paper we introduce higher Homfly skein modules of links in an oriented 3-manifold and partially compute them in terms of the first skein module.

1. Introduction

Let R be a commutative ring with unity. Fix three elements $x, y, h \in R$ such that x, h are invertible in R. For an oriented 3-manifold M, denote by $\tilde{A} = \tilde{A}(M)$ the free R-module generated by the isotopy classes of oriented (non-empty) links in M. By a singular link in M, we mean an immersion of a finite system of oriented circles in M with only double transversal intersections. Using the formula

$$\tilde{r}(X_\bullet) = xX_+ - yX_- - hX_0,$$

(cf. Figure 1) we resolve each singular link $L \subset M$ with n double points into a formal sum $\tilde{r}(L) \in \tilde{A}$ of 3^n terms. Denote by $\tilde{A}_n$ the R-submodule of $\tilde{A}$ generated by $\tilde{r}(L)$ where L runs over all singular links with n double points. Clearly,

$$\tilde{A} = \tilde{A}_0 \supset \tilde{A}_1 \supset \tilde{A}_2 \supset \dots$$

The R-module $\tilde{A}/\tilde{A}_1$ is a version of the Homfly skein module of M. We shall denote this module by $Q = Q(M)$. We call the R-modules $\tilde{A}_n/\tilde{A}_{n+1}$ with $n = 1, 2, \dots$ the *higher Homfly skein modules* of M. Of course, all these modules depend on the choice of x, y, h. In this paper we shall partially compute these modules in terms of the first skein module Q. From now on, we fix an oriented 3-manifold M.

Each R-module H admits a completion

$$H_+ = \operatorname{proj}\lim_N (H/(x-y)^N H)$$

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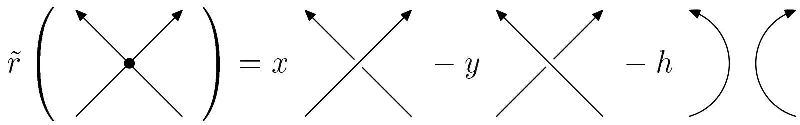

FIGURE 1. The resolution $\tilde{r}$ of a double point.

where $N = 0, 1, 2, \ldots$. It is clear that H_+ is a module over the ring $R_+ = \operatorname{proj}\lim_N (R/(x-y)^N R)$. The transformation $H \mapsto H_+$ extends in the obvious way to a functor from the category of R-modules into the category of R_+-modules.

In Section 2 we shall construct for each pair of non-negative integers p and q an R-homomorphism

$$t^{p,q} : Q \to \tilde{A}_{p+q}/\tilde{A}_{p+q+1}.$$

Let $t^n : Q^{n+1} \to \tilde{A}_n/\tilde{A}_{n+1}$ be the direct sum of the R-homomorphisms $t^{p,n-p}$:

$$t^n = \oplus_{p=0}^n t^{p,n-p} : Q^{n+1} \to \tilde{A}_n/\tilde{A}_{n+1}$$

where Q^{n+1} is the direct sum of $n+1$ copies of Q. By functoriality, t^n induces a R_+-homomorphism $t_+^n : Q_+^{n+1} \to (\tilde{A}_n/\tilde{A}_{n+1})_+$.

Theorem 1.1. *For every $n \geq 0$, the homomorphism $t_+^n : Q_+^{n+1} \to (\tilde{A}_n/\tilde{A}_{n+1})_+$ is surjective.*

Now, we shall specify algebraic conditions on R, x, y, h which will ensure that the homomorphism in this theorem is an isomorphism.

Recall that a differential in R is an additive homomorphism $d : R \to R$ such that $d(ab) = ad(b) + d(a)b$ for any $a, b \in R$. We shall impose the following condition:

$(*)$ There exist an invertible element r of R and differentials $d_1, d_2 : R \to R$ such that

$$d_1(x-y)\, d_2(xh^{-1}) - d_2(x-y)\, d_1(xh^{-1}) = r \mod (x-y). \tag{1.1}$$

Below we give examples of tuples (R, x, y, h) satisfying this condition.

Now we can state our main theorem which computes the $+$-completions of the higher skein modules of M in terms of the first skein module Q.

Theorem 1.2. *Under condition $(*)$, the homomorphism $t_+^n : Q_+^{n+1} \to (\tilde{A}_n/\tilde{A}_{n+1})_+$ is an isomorphism for all $n \geq 0$.*

Examples. Let $R = K[x^{\pm 1}, h^{\pm 1}]$ be the ring of Laurent polynomials on variables x, h with coefficients in a commutative ring with unity K. The condition $(*)$ is satisfied for any monomial $y = kx^p h^q$ with $k \in K, p, q \in \mathbb{Z}$ such that $p+q-1$ is invertible in K. Indeed, it suffices to take $d_1 = \frac{\partial}{\partial x}, d_2 = \frac{\partial}{\partial h}$ and $r = (p+q-1)xh^{-2}$. For instance, in the case $K = \mathbb{Z}$ the condition $(*)$ is satisfied for $y = kx^p h^{-p}$ and $y = kx^p h^{2-p}$ with $k, p \in \mathbb{Z}$. To cover the standard choice $y = x^{-1}$ it suffices to assume that $1/2 \in K$.

The next theorem is a step towards determining the structure of the modules $(\tilde{A}/\tilde{A}_n)_+$.

Theorem 1.3. *Under condition $(*)$, for every $n \geq 0$, the short exact sequence*

$$0 \to \tilde{A}_n/\tilde{A}_{n+1} \to \tilde{A}/\tilde{A}_{n+1} \to \tilde{A}/\tilde{A}_n \to 0$$

induces a short exact sequence

$$0 \to (\tilde{A}_n/\tilde{A}_{n+1})_+ \to (\tilde{A}/\tilde{A}_{n+1})_+ \to (\tilde{A}/\tilde{A}_n)_+ \to 0. \tag{1.2}$$

An easy induction yields the following corollary.

Corollary 1.4. *If condition $(*)$ is satisfied and Q_+ is a projective R_+-module, then for each $n \geq 0$, the exact sequence (1.2) splits and*

$$(\tilde{A}/\tilde{A}_n)_+ = \bigoplus_{i=0}^{n-1} (\tilde{A}_i/\tilde{A}_{i+1})_+ = Q_+^{n(n+1)/2}.$$

The assumption of this corollary holds for instance for $M = S^3$, $R = \mathbb{Q}[x^{\pm 1}, h^{\pm 1}]$ and $y = kx^p h^q$ with $k \in \mathbb{Q}, p, q \in \mathbb{Z}$ such that $k \neq 0, p+q \neq 1$. Indeed, in this case Q is a free module of rank 1 generated by the trivial knot. This computes the modules $(\tilde{A}/\tilde{A}_n)_+$ associated with S^3. Choosing $y = 1$ and quotienting by $x - 1$ we recover the results of [1].

The paper is organized as follows. In Section 2 we define several useful transformations of links and prove Theorem 1.1. In Section 3 we prove Theorems 1.2 and 1.3. In Section 4 we discuss in more detail the case $M = S^3$.

2. Transformations of links

2.1. **Transformations u and t_1.** For any oriented link $L \subset M$ we can consider the union $L \amalg O$ of L with an oriented trivial knot O in an embedded ball in $M \backslash L$. The mapping $L \mapsto L \amalg O$ extends by R-linearity to a homomorphism $\tilde{A} \to \tilde{A}$ denoted u.

A more interesting R-homomorphism $t_1 : \tilde{A} \to \tilde{A}$ is defined on the link generators of $\tilde{A}$ by inserting the singular tangle T_1 drawn in Figure 2. More precisely, for an oriented link $L \subset M$, we choose a small subarc of L, replace it with T_1 and apply the resolution $\tilde{r}$ to this singular link with one double point. The resulting element $t_1(L)$ of $\tilde{A}$ does not depend on the choice of the subarc on L: by definition of $\tilde{r}$, we have

$t_1(L) = (x-y)L - hu(L)$. The mapping $L \mapsto t_1(L)$ extends to a R-linear endomorphism, t_1, of $\tilde{A}$. Clearly, $t_1 = (x-y) - hu$ where $x-y$ is multiplication by $x-y \in R$. The definition of t_1 implies that $t_1(\tilde{A}_n) \subset \tilde{A}_{n+1}$ for all $n \geq 0$. Hence t_1 induces an R-homomorphism $\tilde{A}_n/\tilde{A}_{n+1} \to \tilde{A}_{n+1}/\tilde{A}_{n+2}$ denoted also t_1.

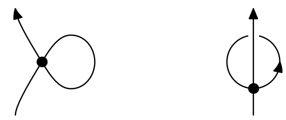

FIGURE 2. Singular tangles T_1 and T_2.

Iterating t_1 we obtain for each non-negative integer q an endomorphism t_1^q of $\tilde{A}$. It is clear that t_1^q acts on the generator represented by an oriented link L by inserting q copies of T_1 at q disjoint small subarcs of L and applying $\tilde{r}$.

2.2. **Transformation t_2^q.** The transformations t_2^q with $q = 0, 1, 2, ...$ are defined similarly to t_1^q except that instead of T_1 we use the singular tangle T_2, drawn in Figure 2. Thus, t_2^q acts on an oriented link L by inserting q copies of T_2 at disjoint small subarcs of L and applying $\tilde{r}$. In contrast to t_1, this transformation does not give a well defined endomorphism of $\tilde{A}$ (unless $q = 0$). We have a weaker result as follows.

Lemma 2.1. *For each $q \geq 0$, the mapping $L \mapsto t_2^q(L)$ extends by R-linearity to a well defined R-homomorphism $t_2^q : \tilde{A}/\tilde{A}_1 \to \tilde{A}_q/\tilde{A}_{q+1}$.*

Proof. The proof is based on the identity shown in Figure 3. To prove this identity we observe that both singular tangles on the right-hand side contain one double point which is a self-crossing of a strand. We apply the resolution $\tilde{r}$ to these double points. This transforms the right-hand side into an algebraic sum of six terms. Four of them cancel and the remaining two terms give exactly the expression on the left-hand side.

The identity in Figure 3 shows that inserting q copies of T_2 at disjoint small subarcs of an oriented link and applying $\tilde{r}$ we obtain an element of $\tilde{A}_q/\tilde{A}_{q+1}$ independent of the choice of the subarcs. This implies our claim.

□

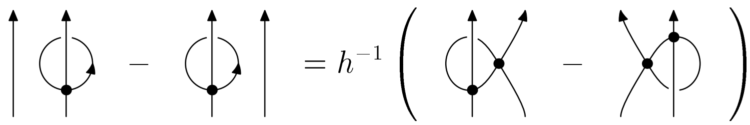

FIGURE 3. Inserting T_2 at different arcs.

2.3. **Transformation $t^{p,q}$.** Recall the notation $Q = \tilde{A}/\tilde{A}_1$. For each pair of non-negative integers p, q, we consider an R-homomorphism

$$t^{p,q} = t_1^p t_2^q : Q \to \tilde{A}_{p+q}/\tilde{A}_{p+q+1}$$

defined as the composition of $t_2^q : Q \to \tilde{A}_q/\tilde{A}_{q+1}$ and $t_1^p : \tilde{A}_q/\tilde{A}_{q+1} \to \tilde{A}_{p+q}/\tilde{A}_{p+q+1}$. The following Proposition directly implies Theorem 1.1.

Proposition 2.2. *For each $n \geq 0$, the module $(\tilde{A}_n/\tilde{A}_{n+1})_+$ is generated by the images of the homomorphisms $t_+^{p,n-p} : Q_+ \to (\tilde{A}_n/\tilde{A}_{n+1})_+$ with $p = 0, 1, ..., n$.*

The proof goes by adjusting the arguments of [1], Section 2 to our present setting. We begin with the following fundamental 8T-relation.

Lemma 2.3. *We have the identity in Figure 4, once all double points are resolved as in Figure 1.*

It is understood that all eight local pictures in Figure 4 represent singular tangles in a ball $B \subset M$. They are completed by one and the same singular tangle in $M \setminus \mathrm{Int} B$ to form eight singular links in M. Alternatively, one can view the identity in Figure 4 as a formal relation between singular tangles which lies in the kernel of the resolution map $\tilde{r}$.

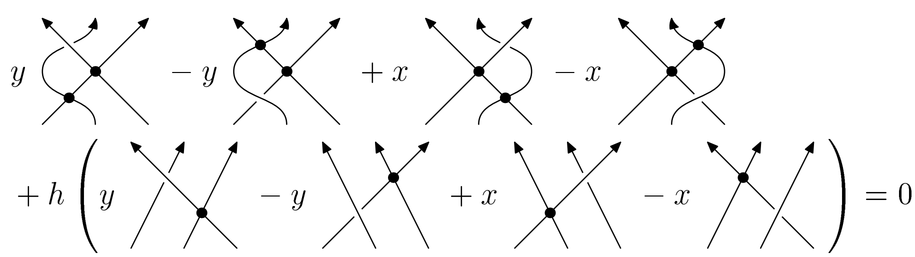

FIGURE 4. The 8T-relation.

Proof. Consider the strand leading from the second input to the second output in the first four pictures. This strand contains one double point and one over/under-crossing. Resolve this double point in each of these four pictures. This yields an algebraic sum of eight terms with coefficient containing no power of h and of four terms linear in h. The eight terms cancel while the sum of four terms is exactly the opposite of the sum in the second row in Figure 4.

□

2.4. **Proof of Proposition 2.2.** It suffices to prove that the images of $t^{p,n-p} : Q \to \tilde{A}_n/\tilde{A}_{n+1}$ with $p = 0, 1, ..., n$ span the quotient $(\tilde{A}_n/\tilde{A}_{n+1})/(x-y)$. Observe that

$$(\tilde{A}_n/\tilde{A}_{n+1})/(x-y) = \tilde{A}_n/((x-y)\tilde{A}_n + \tilde{A}_{n+1}).$$

We begin by deriving a few consequences of the 8T-relation. Let B be a closed 3-ball in M. For any singular $(3,3)$-tangle in $M\backslash \mathrm{Int}B$ with $n-1$ double points we complete the eight singular tangles in B drawn in Figure 4 with that tangle so as to obtain eight singular links in M. The first four singular tangles in Figure 4 yield after resolution of double points elements of $\tilde{A}_{n+1}$, which shall be ignored in the following calculations proceeding in $\tilde{A}_n/\tilde{A}_{n+1}$. Thus we can complete the second row in Figure 4 by any singular $(3,3)$-tangle with $n-1$ double points and obtain a 4-term relation in $\tilde{A}_n/\tilde{A}_{n+1}$. In particular, let us connect the middle top strand to the bottom left strand and add a negative crossing at the bottom in the four pictures in the second row of Figure 4. By the argument above, we obtain an identity in $\tilde{A}_n/\tilde{A}_{n+1}$ shown in Figure 5.

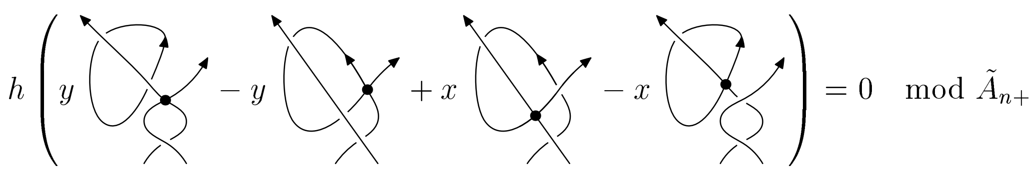

FIGURE 5. An identity in $\tilde{A}_n/\tilde{A}_{n+1}$.

Observe that the left-most term in Figure 5 is in

$$u\tilde{A}_n \subset (x-y)\tilde{A}_n - t_1(\tilde{A}_n) \subset (x-y)\tilde{A}_n + \tilde{A}_{n+1}.$$

Dividing by hx, we obtain a *basic relation* in $\tilde{A}_n/((x-y)\tilde{A}_n + \tilde{A}_{n+1})$, see Figure 6.

Let $L \subset M$ be a singular link with n double points. Applying the basic relation to all double points of L we obtain an expansion of $\tilde{r}(L)$ mod $((x-y)\tilde{A}_n + \tilde{A}_{n+1})$ as a sum of 2^n terms. Each of these terms has the form $\tilde{r}(K)$ where K is obtained from a certain non-singular link by inserting p copies of T_1 and $n-p$ copies of T_2 with $0 \le p \le n$. Therefore

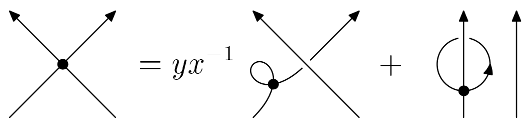

FIGURE 6. The basic relation in $\tilde{A}_n/((x-y)\tilde{A}_n + \tilde{A}_{n+1})$.

$\tilde{r}(L) \mod ((x-y)\tilde{A}_n + \tilde{A}_{n+1})$ belongs to the submodule generated by the images of $t^{p,n-p}$. This completes the proof of the proposition.

□

3. Proof of Theorems 1.2 and 1.3

3.1. **Preliminaries on differentials.** Each differential $d : R \to R$ induces an additive homomorphism $d : \tilde{A} \to \tilde{A}$ by

$$d(\sum_i k_i L_i) = \sum_i d(k_i) L_i, \tag{3.1}$$

where $k_i \in R$ and $\{L_i\}_i$ are oriented links in M. Clearly,

$$d(ka) = kd(a) + d(k)a$$

for any $k \in R$ and $a \in \tilde{A}$.

The following lemma is established in [1], Lemma 5.2.

Lemma 3.1. *For each $n \geq 0$, $d(\tilde{A}_{n+1}) \subset \tilde{A}_n$.*

The obvious formula $d((x-y)^N) = 0 \mod (x-y)^{N-1}$ implies that the differential d in R induces a differential in R_+.

Let d_1, d_2 be differentials in R satisfying $(*)$. By Lemma 3.1, both d_1 and d_2 induce R-homomorphisms $\tilde{A}_n/\tilde{A}_{n+1} \to \tilde{A}_{n-1}/\tilde{A}_n$ for any $n \geq 0$. Therefore the composition $(d_1)^p(d_2)^{n-p}$ induces an R-linear homomorphism $\tilde{A}_n/\tilde{A}_{n+1} \to \tilde{A}/\tilde{A}_1 = Q$ for any $n \geq p \geq 0$. We denote the latter homomorphism by $d^{p,n-p}$.

Proposition 3.2. *Let $d^n : \tilde{A}_n/\tilde{A}_{n+1} \to Q^{n+1}$ be the direct sum of the homomorphisms $d^{p,n-p}$ where $p = 0, ..., n$. Then the induced R_+-homomorphism $d^n_+ : (\tilde{A}_n/\tilde{A}_{n+1})_+ \to Q^{n+1}_+$ is an isomorphism.*

Proof. We can extend the differential d_j with $j = 1, 2$ to linear combinations of tangles with coefficients in R (or R_+): it suffices to use Formula 3.1 where L_i are tangles. For a singular tangle T, set $d_j(T) = d_j(\tilde{r}(T))$. Clearly, $d_j(kT) = kd_j(T) + d_j(k)T$ for any $k \in R$. Observe also that the usual gluing of tangles extends by linearity to their linear combinations. It is clear that if TT' is the result of gluing of two tangles (or their linear combinations) T, T' then

$$d_j(TT') = d_j(T)\,T' + T d_j(T'). \tag{3.2}$$

Now we shall compute the action of d_1, d_2 on the singular tangles T_1, T_2 drawn in Figure 2. Denote by I the unknotted vertical strand oriented upwards. The formula $\tilde{r}(T_1) = (x-y)I - huI$ implies that $uI = h^{-1}(x-y)I \mod \tilde{r}(T_1)$. It follows from definitions that for $j = 1, 2$,

$$d_j(T_1) = d_j(x-y)I - d_j(h)uI = d_j(x-y)I - d_j(h)h^{-1}(x-y)I \mod \tilde{r}(T_1).$$

Set

$$\alpha_j = d_j(x-y) - d_j(h)h^{-1}(x-y) \in R.$$

Then

$$d_j(T_1) = \alpha_j I \mod \tilde{r}(T_1).$$

To compute the derivatives of T_2, observe that $\tilde{r}(T_2) = xH - yuI - hI$ where H is the tangle drawn in Figure 7. Therefore,

$$H = x^{-1}(yu+h)I \mod \tilde{r}(T_2) = x^{-1}(yh^{-1}(x-y)+h)I \mod \tilde{r}(T_1), \tilde{r}(T_2).$$

It follows from definitions that

$$d_j(T_2) = d_j(x)H - d_j(y)uI - d_j(h)I = \beta_j I \mod \tilde{r}(T_1), \tilde{r}(T_2)$$

where

$$\beta_j = d_j(x)x^{-1}(yh^{-1}(x-y)+h) - d_j(y)h^{-1}(x-y) - d_j(h) \in R.$$

Set

$$\Delta = \det \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}.$$

It is easy to compute that

$$\Delta = xh^{-2}(d_1(x-y)d_2(xh^{-1}) - d_2(x-y)d_1(xh^{-1})) = xh^{-2}r \mod (x-y)$$

where r is the invertible element of R provided by the condition $(*)$. Now, a standard algebraic argument shows that Δ is inverible in R_+. This allows us to introduce two formal linear combinations of T_1, T_2 over R_+ by

$$E_1 = \Delta^{-1}\beta_2 T_1 - \Delta^{-1}\alpha_2 T_2 \quad \text{and} \quad E_2 = -\Delta^{-1}\beta_1 T_1 + \Delta^{-1}\alpha_1 T_2.$$

By definition, $\tilde{r}(E_1) = \Delta^{-1}\beta_2\tilde{r}(T_1) - \Delta^{-1}\alpha_2\tilde{r}(T_2)$ and $\tilde{r}(E_2) = -\Delta^{-1}\beta_1\tilde{r}(T_1) + \Delta^{-1}\alpha_1\tilde{r}(T_2)$. We can easily compute the derivative d_j of E_1, E_2 modulo $\tilde{r}(T_1), \tilde{r}(T_2)$. Indeed, for any linear combination $aT_1 + bT_2$ with $a, b \in R_+$ we have

$$d_j(aT_1 + bT_2) = a d_j(T_1) + b d_j(T_2) \mod \tilde{r}(T_1), \tilde{r}(T_2).$$

This implies

$$d_j(E_i) = \delta_i^j I \mod \tilde{r}(T_1), \tilde{r}(T_2)$$

where δ_i^j is the Kronecker delta.

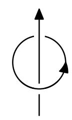

FIGURE 7. The tangle H.

For each $p = 0, ..., n$ we define a R_+-linear homomorphism $e^{p,n-p}$: $Q_+ \to (\tilde{A}_n/\tilde{A}_{n+1})_+$ in the same way as $t^{p,n-p}$ but using E_1, E_2 instead of T_1, T_2. Thus, $e^{p,n-p}$ acts on an oriented link L by inserting p copies of E_1 and $n-p$ copies of E_2 at n disjoint small subarcs of L and applying the resolution $\tilde{r}$. It is clear that each $e^{p,n-p}$ is a linear combination of the homomorphisms $t_+^{0,n}, t_+^{1,n-1}, ..., t_+^{n,0}$. Therefore $e^{p,n-p}$ is a well defined homomorphism. Note that we can express both T_1 and T_2 as linear combinations of E_1, E_2 with coefficients in R_+. Thus, each $t_+^{p,n-p}$ is a linear combination of the homomorphisms $e^{0,n}, e^{1,n-1}, ..., e^{n,0}$. Proposition 2.2 implies that the module $(\tilde{A}_n/\tilde{A}_{n+1})_+$ is generated by the images of the homomorphisms $e^{0,n}, e^{1,n-1}, ..., e^{n,0}$. Denote by e^n : $Q_+^{n+1} \to (\tilde{A}_n/\tilde{A}_{n+1})_+$ the direct sum of the homomorphisms $(p!\,(n-p)!)^{-1}e^{p,n-p} : Q_+ \to (\tilde{A}_n/\tilde{A}_{n+1})_+$ where $p = 0, 1, ..., n$. It is clear that e^n is surjective.

The computations above and Formula 3.2 imply that

$$d_1 \circ e^{p,n-p} = pe^{p-1,n-p} : Q_+ \to (\tilde{A}_{n-1}/\tilde{A}_n)_+$$

and

$$d_2 \circ e^{p,n-p} = (n-p)e^{p,n-p-1} : Q_+ \to (\tilde{A}_{n-1}/\tilde{A}_n)_+.$$

Therefore

$$d^{p',n-p'} \circ e^{p,n-p} = p!(n-p)!\delta_{p'}^p\ e^{0,0} = p!(n-p)!\delta_{p'}^p\ \mathrm{id}_{Q_+}.$$

We can rewrite these equalities in the form $d_+^n \circ e^n = \mathrm{id}$. The surjectivity of e^n implies that e^n and d_+^n are mutually inverse isomorphisms.

□

3.2. **Proof of Theorem 1.2.** Theorem 1.1 implies that t_+^n is surjective. It is injective because $\operatorname{Ker} t_+^n \subset \operatorname{Ker} e^n = 0$ where e^n is the homomorphism introduced in the proof of Proposition 3.2.

□

3.3. **Proof of Theorem 1.3.** The general properties of projective limits imply that the sequence (1.2) is exact except possibly in the term $(\tilde{A}_n/\tilde{A}_{n+1})_+$. We need to prove that the homomorphism $(\tilde{A}_n/\tilde{A}_{n+1})_+ \to (\tilde{A}/\tilde{A}_{n+1})_+$ induced by the inclusion $\tilde{A}_n/\tilde{A}_{n+1} \to \tilde{A}/\tilde{A}_{n+1}$ is injective.

By Lemma 3.1, the differentials d_1, d_2 induce additive (but not R-linear) homomorphisms $\tilde{A}/\tilde{A}_{n+1} \to \tilde{A}/\tilde{A}_n$ for any $n \geq 0$. Therefore the composition $(d_1)^p(d_2)^{n-p}$ induces an additive homomorphism $\tilde{A}/\tilde{A}_{n+1} \to Q$ for any $n \geq p \geq 0$. Denote the latter homomorphism by $D^{p,n-p}$.

An easy induction shows that $D^{p,n-p}((x-y)^N a) = 0 \mod (x-y)^{N-n}$ for any $a \in \tilde{A}/\tilde{A}_{n+1}$ and all $N \geq n$. Therefore $D^{p,n-p}$ induces an additive homomorphism $(\tilde{A}/\tilde{A}_{n+1})_+ \to Q_+$ denoted $D_+^{p,n-p}$.

It follows from definitions that $D^{p,n-p} \circ i^n = d^{p,n-p}$ where i^n is the inclusion $\tilde{A}_n/\tilde{A}_{n+1} \hookrightarrow \tilde{A}/\tilde{A}_{n+1}$. Hence

$$D_+^{p,n-p} \circ i_+^n = d_+^{p,n-p} : (\tilde{A}_n/\tilde{A}_{n+1})_+ \to Q_+.$$

Thus, the kernel of $i_+^n : (\tilde{A}_n/\tilde{A}_{n+1})_+ \to (\tilde{A}/\tilde{A}_{n+1})_+$ is annihilated by the homomorphisms $d^{0,n}, d^{1,n-1}, ..., d^{n,0}$. Proposition 3.2 implies that this kernel is zero so that i_+^n is injective.

4. Case $M = S^3$

Throughout this section, we assume that $M = S^3$. If y is invertible in R then the standard arguments show that the skein module Q of the 3-sphere is a free module of rank 1 generated by an unknot. In the sequel we assume that $y = x^{-1} \in R = \mathbb{Q}[x^{\pm 1}, h^{\pm 1}]$. Note that the condition $(*)$ is satisfied for $d_1 = \frac{\partial}{\partial x}, d_2 = \frac{\partial}{\partial h}$. In the following we will write $d_x = \frac{\partial}{\partial x}, d_h = \frac{\partial}{\partial h}$. We can explicitly describe the module $(\tilde{A}/\tilde{A}_n)_+$ as follows.

Theorem 4.1. *For each $n \geq 0$,*

$$(\tilde{A}/\tilde{A}_n)_+ = \bigoplus_{\substack{l,m \geq 0 \\ l+m<n}} R_+ \, \tilde{r}(G^l_{l+m})$$

where G^l_{l+m} is the singular link in S^3 shown in Figure 8.

Proof. This theorem directly follows from Theorem 1.2, Corollary 1.4 and the definition of t^n given in Sections 1 and 2.

□

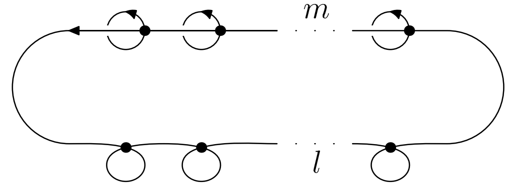

FIGURE 8. The singular link G^l_{l+m}.

Corollary 4.2. *There are unique R-linear homomorphisms $\tilde{\nabla}_{l,m} : \tilde{A} \to R_+$ numerated by pairs of non-negative integers (l,m) such that for any $a \in \tilde{A}$,*

$$a = \sum_{l,m} \tilde{\nabla}_{l,m}(a)\, \tilde{r}(G^l_{l+m}) \in \operatorname{proj\,lim}_n (\tilde{A}/\tilde{A}_n)_+.$$

Applying this to any oriented link $L \subset S^3$ we obtain an expansion

$$L = \sum_{l,m} \tilde{\nabla}_{l,m}(L)\, \tilde{r}(G^l_{l+m}) \in \operatorname{proj\,lim}_n (\tilde{A}/\tilde{A}_n)_+.$$

Substituting $x = 1$ in $\tilde{\nabla}_{l,m}(L)$, we obtain the link polynomial $\nabla_{l,m}(L)$ introduced in [1].

Clearly, $\tilde{\nabla} = \tilde{\nabla}_{0,0}$ is the Homfly polynomial which can be described as the (unique) mapping from the set of isotopy classes of oriented links in S^3 into the ring R such that

(i) the value of $\tilde{\nabla}$ on an unknot is equal to 1;

(ii) for any three oriented links X_+, X_-, X_0 coinciding outside a 3-ball and looking as in Figure 9 inside this ball, we have that

$$x\tilde{\nabla}(X_+) - x^{-1}\tilde{\nabla}(X_-) = h\tilde{\nabla}(X_0).$$

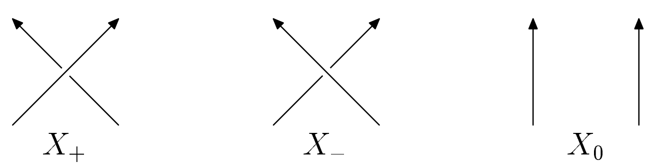

FIGURE 9. X_+, X_-, X_0.

It follows from Theorem 4.1 that for each $n \geq 0$, the set $\{\tilde{\nabla}_{l,m} \mid l + m < n\}$ is a basis of the free R_+-module $(\tilde{A}/\tilde{A}_n)^*_+ = \operatorname{Hom}_R((\tilde{A}/\tilde{A}_n)_+, R_+)$. Another basis of the same module can be derived from the Homfly polynomial as follows. For $p, q \geq 0$, denote by $(d^*_x)^p (d^*_h)^q (\tilde{\nabla})$ the R-linear mapping $\tilde{A} \to R$ sending each oriented link L into $\partial^{p+q}\tilde{\nabla}(L)/\partial^p x \partial^q h \in R$. Below we prove the following theorem.

Theorem 4.3. *For $p, q \geq 0$, the homomorphism $(d_x^*)^p(d_h^*)^q(\tilde{\nabla})$ annihilates $\tilde{A}_{p+q+1}$. For every $n \geq 0$, the set $\{(d_x^*)^p(d_h^*)^q\tilde{\nabla} \mid p+q<n\}$ is a basis of the free R_+-module $(\tilde{A}/\tilde{A}_n)_+^*$.*

Using the transformation matrix relating the two bases of $(\tilde{A}/\tilde{A}_n)_+^*$ constructed above we can express each polynomial $\tilde{\nabla}_{l,m}$ as a linear combination of the derivatives of $\tilde{\nabla}$. More precisely, we have the following corollary.

Corollary 4.4. *There are unique $c_{p,q}^{l,m} \in R_+$ where $l, m, p, q \geq 0$ and $p+q \leq l+m$ such that for any l, m and any oriented link $L \subset S^3$,*

$$\tilde{\nabla}_{l,m}(L) = \sum_{p,q\geq 0, p+q \leq l+m} c_{p,q}^{l,m}((d_x^*)^p(d_h^*)^q(\tilde{\nabla}))(L).$$

For instance, $c_{0,0}^{0,0} = 1$. A direct comparison on the generators $\tilde{r}(G_0^0), \tilde{r}(G_1^0), \tilde{r}(G_1^1)$ of $(\tilde{A}/\tilde{A}_2)_+$ shows that

$$\tilde{\nabla}_{0,1} = z^{-1}(-(1+x^{-2})d_h^*(\tilde{\nabla}) - h^{-1}(x-x^{-1})d_x^*(\tilde{\nabla})),$$

$$\tilde{\nabla}_{1,0} = z^{-1}((x^{-1}h + 2x^{-1}h^{-1} - 2x^{-3}h^{-1})d_h^*(\tilde{\nabla}) + d_x^*(\tilde{\nabla}))$$

where

$$z = \det \begin{bmatrix} -(1+x^{-2}) & -h^{-1}(x-x^{-1}) \\ (x^{-1}h + 2x^{-1}h^{-1} - 2x^{-3}h^{-1}) & 1 \end{bmatrix}$$

is invertible in R_+. In particular, substituting $x = 1$ in $\tilde{\nabla}_{l,m}(L)$, we obtain $\nabla_{0,1} = d_h^*(\nabla)$ and $\nabla_{1,0} = -(h/2)d_h^*(\nabla) - (1/2)d_x^*(\tilde{\nabla})|_{x=1}$ where $\nabla = \tilde{\nabla}|_{x=1}$ is the Conway polynomial. The first of these formulas was already obtained in [1].

The following more general formula computes $c_{p,q}^{l,m}$ in the case $l+m = p+q$ in terms of $\alpha_1, \alpha_2, \beta_1, \beta_2, \Delta$ introduced in the proof of Proposition 3.2:

$$c_{p,q}^{l,m} = (-1)^{m+p}(p!q!)^{-1}\Delta^{-p-q} \sum_{r=\max(0,l-q)}^{\min(p,l)} \binom{p}{r}\binom{q}{l-r} \alpha_1^{q+r-l}\alpha_2^{p-r}\beta_1^{l-r}\beta_2^r.$$

This expression can be deduced from the computations in the proof of Proposition 3.2 (cf. the argument in the next subsection).

Corollary 4.5. *If K_1 and K_2 are two links which have the same Homfly polynomial, then we have that $K_1 - K_2$ projects to zero in all $(\tilde{A}/\tilde{A}_n)_+$.*

This follows directly from Theorem 4.3.

4.1. **Proof of Theorem 4.3.** The first claim is obtained following the lines of [1], Section 5. The second claim is deduced by induction from Proposition 3.2 using the formula

$$((d_x^*)^p(d_h^*)^{n-p}(\tilde{\nabla}))(a) = (-1)^n\tilde{\nabla}(d^{p,n-p}(a))$$

for any $a \in \tilde{A}_n/\tilde{A}_{n+1}$.

References

[1] J.E. Andersen and V. Turaev, Higher skein modules. Preprint 1998

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