A note on the multi-dimensional monotone follower problem and its connection to optimal stopping

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Abstract

We generalize a result by Karatzas and Shreve, [15] to the multi-dimensional case. A viscosity solution approach is taken to show that the value function of the multi-dimensional monotone follower problem coincides with the integral of the value function of associated stopping problems. The connection holds under a strong factorization property of the running cost function.

Keywords: Monotone follower problem, singular stochastic control, optimal stopping, variational inequalities, viscosity solutions

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1 Introduction

We give a multi-dimensional generalization to a result by Karatzas and Shreve [15] on the equivalence between the monotone follower problem and optimal stopping. The monotone follower problem consists of optimally controlling a Brownian motion with a control which may be singular with respect to the Lebesgue measure. Such control problems, and generalizations of it, are also known as singular stochastic control. Karatzas and Shreve [15] show that the value function of the monotone follower problem coincides with the integral of the value function of an associated stopping problem. Such a connection was first noticed by Bather and Chernoff [1], but later proved rigourosly by [15]. We would also like to mention the paper by El Karoui and Karatzas, [7], where a connection to the Skorohod problem is applied in proving the connection. For a special class of diffusions, we prove that a similar relation holds for the multi-dimensional singular stochastic control problem. Our approach is based on the viscosity solution method in [3], where the connection was established for general one-dimensional diffusions.

In a recent paper by Boetius and Kohlman, [2], a one-dimensional generalization of the results of [15] is proved using comparison results. They also consider a multi-dimensional problem where the control of the diffusion takes place in one of the variables. We note that our case is more general since all the coordinates of the diffusion can be controlled.

There is a lot of interest on singular stochastic control problems, both from the theoretical and the applied point of view. We would like to mention a few works on the subject; Benes et al. [4], Karatzas [13, 14], Karatzas and Shreve [16], Zhu [24], Fleming and Soner [8], Haussmann and Suo [10, 11], Lungu and Øksendal [18, 19] and Myhre [22].

The multi-dimensional diffusion process we consider must have a special dependency with respect to its initial condition. This is valid in the case of a correlated Wiener process with drift, which consitute a direct extension of the results in [15]. The running and final cost functions must, however, factorize in their space variables, which means that no cross terms between space variables are allowed. Under some additional conditions, we prove that the value function of the multi-dimensional monotone follower problem coincides with the sum of the integrals of associated stopping problems. A corollary of this is that the multi-dimensional singular control problem factorizes into n one-dimensional control problems. The representation is quite natural in light of the factorization of the cost functions and the structure of the diffusion. The proof highly relies on results by Haussman and Suo [11], which proves that the value function of the follower problem is a viscosity solution of a variational inequality.

The paper is organized as follows: In Section 2 we formulate the multidimensional singular stochastic control problem and state the associated stopping problems. The basic assumptions on the parameter functions and some preliminary results are stated in Section 3. The viscosity property of the value function is discussed in Section 4, where we prove the main result of this paper, namely the connection to the sums of the integral of associated stopping problems. We end the section with a discussion of the connection between the optimal control and the optimal stopping times.

2 Formulation of the problems

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{F}_t$ for $0 \leq t \leq T$ be the σ-algebra generated by the m-dimensional (standard) Brownian motion $B(s)$, $0 \leq s \leq t$. $T < \infty$ is a fixed time horizon. We assume $\mathcal{F}_t$ to satisfy the standard conditions with $\mathcal{F}_T = \mathcal{F}$. Let $X^{t,\mathbf{x}}(s)$ be the n-dimensional diffusion process starting in $\mathbf{x} = (x_1, .., x_n) \in I\!R^n$ at time t defined as

$$X^{t,\mathbf{x}}(s) = \mathbf{x} + \int_t^s \mu(u, X^{t,\mathbf{x}}(u))\, du + \int_t^s \sigma(u, X^{t,\mathbf{x}}(u))\, dB(u) \tag{2.1}$$

where $\mu : [0,T] \times I\!R^n \to I\!R^n$ and $\sigma : [0,T] \times I\!R^n \to I\!R^{n\times m}$ are bounded continuous functions, continuously differentiable in the space variables and with a restricted dependency structure on the different arguments. We assume that $X_i^{t,\mathbf{x}}(s)$ is only dependent on the i'th coordinate of the initial point $\mathbf{x}$:

$$X_i^{t,\mathbf{x}}(s) = x_i + \int_t^s \mu_i(u, X_i^{t,\mathbf{x}}(u))\, du + \sum_{j=1}^m \int_t^s \sigma_{ij}(u, X_i^{t,\mathbf{x}}(u))\, dB_j(u)$$

for $i = 1, 2, .., n$. For this special class of diffusions we thus have $X_i^{t,\mathbf{x}}(s) = X_i^{t,x_i}(s)$. Note that our diffusion process really is a slight generalization of a correlated multidimensional Wiener process with drift. For example, Let $\mu \in I\!R^n$ and $\sigma \in I\!R^{n\times m}$. Then

$$X^{t,\mathbf{x}}(s) = \mathbf{x} + \mu(s-t) + \sigma(B(s) - B(t))$$

satisfies the structural assumptions above.

The problem consists in optimally controlling the diffusion process under some cost criterion. Denote by $\mathcal{A}_n(t)$ the class of $I\!R^n$-valued $\mathcal{F}_s$-adapted processes $\xi = \{(\xi_1(s), \ldots, \xi_n(s)) : t \leq s \leq T\}$ such that a.s. ω and for every $i = 1, \ldots, n$

(i) $\xi_i(t, \omega) = 0$,

(ii) $s \to \xi_i(s,\omega)$ is nondecreasing and left-continuous with right limits.

Denote the controlled process $X^{t,\mathbf{x},\xi}(s)$ i.e.

$$X^{t,\mathbf{x},\xi}(s) = \mathbf{x} + \int_t^s \mu(u, X^{t,\mathbf{x},\xi}(u))\, du + \int_t^s \sigma(u, X^{t,\mathbf{x},\xi}(u))\, dB_u - \xi(s) \tag{2.2}$$

The value function for the follower problem is given by

(2.3)

$$V(t,\mathbf{x}) = \inf_{\xi \in \mathcal{A}_n(t)} \mathrm{E}^{t,\mathbf{x}} \left[\int_t^T h(s, X(s))\, ds + \int_{[t,T)} f(s)\, d\xi(s) + g(X(T)) \right]$$

where $h : [0,T] \times I\!R^n \to I\!R, f : [0,T] \times I\!R^n$ and $g : I\!R^n \to I\!R$ are Borel measurable functions. See next Section section for the precise assumptions on h, f and g. The integral with respect to ξ is interpreted as

$$\int_{[t,T)} f(s)\, d\xi(s) = \sum_{i=1}^n \int_{[t,T)} f_i(s)\, d\xi_i(s) \tag{2.4}$$

and $\int_{[t,T)} f_i(s)\, d\xi_i(s)$ is understood in the Lebesgue-Stiltjes sense;

$$\int_{[t,T)} f_i(s)\, d\xi_i(s) = \int_t^T f_i(s)\, d\xi_i^c(s) + \sum_{t \le s < T} f_i(s) \triangle \xi_i(s) \tag{2.5}$$

where $\xi_i^c(s)$ is the continuous part of $\xi_i(s)$ and $\triangle\xi_i(s) = \xi_i(s+) - \xi_i(s)$. Note that the (possible) last jump of $\xi_i(s)$ at time T is not accounted for in the integral. The function h represents the running cost, while using the control will lead to costs according to the function f. After reaching T we have to pay g.

The controls ξ are not necessarily absolute continuous with respect to the Lebesgue measure on $I\!R^n$. In fact, it will in many cases behave like a local time on the boundary of some open region in $[0,T) \times I\!R^n$ (see e.g. [15, 16]). Thus, Problem (2.3) is usually refered to as a *singular* stochastic control problem.

From [12] we know that $X^{t,\mathbf{x}}(s)$ is differentiable with respect to the intital condition $\mathbf{x}$. Since $X_i^{t,\mathbf{x}}(s)$ is only dependent on x_i we have $\partial_j X_i^{t,\mathbf{x}}(s) = 0$ for $j \neq i$, where we use the notation $\partial_j \psi(\mathbf{x}) = \frac{\partial \psi(\mathbf{x})}{\partial x_j}$. Denote the partial derivatives of $X_i^{t,\mathbf{x}}(s)$ with respect to x_i by $Y_i^{t,1}(s)$, i.e.

$$\partial_i X_i^{t,x}(s) = Y_i^{t,1}(s)$$

Hence,

(2.6)

$$Y_i^{t,1}(s) = 1 + \int_t^s \partial_i\mu(s, X^{t,\mathbf{x}}(s))Y_i^{t,1}(s)ds + \int_t^s \sum_{j=1}^m \partial_i\sigma_{ij}(s, X^{t,\mathbf{x}}(s))Y_i^{t,1}(s)dB_j(s)$$

With the notation $Y_i^{t,y}(s)$ we mean the process that starts in y at time t. We now state the corresponding stopping problems. For each $i = 1, \ldots, n$, define the optimal stopping problems,

$$\begin{aligned}(2.7)\qquad U_i(t,x,y) = \inf_{t\le\tau\le T} \mathrm{E}^{t,x,y}\Big[\int_t^\tau \partial_x h_i(s, X_i^{t,x}(s))Y_i^{t,y}(s)\,ds \\ + f_i(\tau)Y_i^{t,y}(\tau)\mathbf{1}_{\tau<T} + g_i'(X_i^{t,x}(\tau))Y_i^{t,y}(\tau)\mathbf{1}_{\tau=T}\Big]\end{aligned}$$

where the τ's are stopping times with respect to $\mathcal{F}_s$ and $X_i^{t,x}(s)$ is the i'th coordinate of the diffusion $X^{t,\mathbf{x}}(s)$ defined in the section above. Each $U_i(t,x,y)$ is a measurable, positive-valued function defined on $[0,T]\times I\!R^2$. Observe that $U_i(T,x,y) = g_i'(x,y)$ and $U_i(t,x,y) \le f_i(t)y$ for $t < T$.

3 Assumptions and and some preliminary results

A basic assumption throughout the paper is a factorization property of the cost functions h and g:

$$(3.1)\qquad h(t, x_1, \ldots, x_n) = h_1(t, x_1) + \ldots + h_n(t, x_n)$$

and

$$(3.2)\qquad g(x_1, \ldots, x_n) = g_1(x_1) + \ldots + g_n(x_n)$$

Under appropriate regularity conditions on the parameters of the monotone follower problem, we will show that for $\mathbf{x} = (x_1, \ldots, x_n)$

$$V(t,\mathbf{x}) = \sum_{i=1}^n \int_{-\infty}^{x_i} U_i(t,z,1)\,dz$$

This relation actually says that the multi-dimensional monotone follower problem factorizes into n one-dimensional follower problems. Thus, if $\xi^*(s)$ is an optimal control, it will consist of coordinates which optimally controls each of the $X_i(s)$ separatly. This factorization of the control problem is natural in view of the simple nature of $X^{t,\mathbf{x}}(s)$ and the factorization of h and g.

We make the following regularity assumptions on the functions $h_i : [0,T]\times I\!R \to [0,\infty)$, $f_i : [0,T] \to [0,\infty)$ and $g_i : I\!R \to [0,\infty)$, for $i = 1,\dots,n$:

(3.3) h_i is bounded and Lipschitz continuous in both variables, continuously differentiable in x where $\partial_x h_i$ is non-negative,

(3.4) g_i is Lipschitz continuous of polynomial growth, continuously differentiable, and g_i' is non-negative,

(3.5) $f_i(\cdot) > 0$ and Lipschitz continuous.

The following relation between $f_i(\cdot)$ and $g_i'(\cdot)$ holds:

$$\sup_{I\!R} g_i'(x) \le \inf_{t\in[0,T]} f_i(t) \tag{3.6}$$

In addition, both $h_i(\cdot,\cdot)$ and $g_i(\cdot)$ vanish at $-\infty$, i.e.

$$\lim_{x\to-\infty} h_i(t,x) = 0 = \lim_{x\to-\infty} g_i(x) \tag{3.7}$$

for every $t \in [0,T]$, $i = 1,\dots,n$. From now on we assume that all the conditions stated above holds for the parameters in the problem. From [3] we have the following continuity and integrability result for $U_i(t,x,1)$:

Proposition 3.1. *For every $i = 1,\dots,n$, $U_i(t,x,1)$ is uniformly continuous on $[0,T]\times I\!R$. Moreover, $U(t,x,1)$ is integrable in x on $(-\infty,\hat{x})$ for every $\hat{x} \in I\!R$.*

Proof. The continuity results in [3], Section 3.2, are strictly speaking only valid for one-dimensional Brownian motion. However, by straightforward modifications of the argument given there, we see that the results can be extended to multidimensional Brownian motion. $\square$

We state a "dynamic programming principle". First, introduce the continuation region for U_i:

$$D_i = \Big\{(t,x,y) \in [0,T) \times I\!R^2\,;\, U_i(t,x,y) < f_i(t)y\Big\} \tag{3.8}$$

From Shiryayev [23] the following holds: Let τ_i be the first exit time from D_i. Then, for any stopping time $\tau \le \tau_i$,

$$U_i(t,x,y) = \mathrm{E}^{t,x,y}\Big[\int_t^\tau \partial_x h_i(s,X_i(s))Y_i(s)\,ds + U_i(\tau,X_i(\tau))Y_i(\tau)\Big] \tag{3.9}$$

Otherwise, for a general stopping time τ,

$$U_i(t,x,y) \le \mathrm{E}^{t,x,y}\Big[\int_t^\tau \partial_x h_i(s,X_i(s))Y_i(s)\,ds + U_i(\tau,X_i(\tau))Y_i(\tau)\Big] \tag{3.10}$$

(3.9) and (3.10) are exactly the kind of "dynamic programming principle" we need to prove the connection between U_i and V.

Direct calculation gives the following natural factorization result for U_i

Lemma 3.2. *The value function $U_i(t,x,y)$ satisfies:*

$$U_i(t,x,y) = yU_i(t,x,1) \tag{3.11}$$

From now on we will only consider the stopping problem with $y=1$, i.e. the value function $U_i(t,x) := U_i(t,x,1)$.

Introduce for $i=1,..,n$ the continuation regions for the stopping problem when $y=1$;

$$D_i^1 = \Big\{(t,x)\in[0,T)\times I\!R;\, U_i(t,x) < f(t)\Big\} \tag{3.12}$$

Like in [3] we make the following structural condition on the continuation regions D_i^1: For $i=1,..,n$ the following holds

(C) The region D_i^1 defined in (3.12) is connected in the sense that if $(t,\hat{x})\in D_i^1$ then $(t,x)\in D_i^1$ for any $x\in(-\infty,\hat{z})$

4 The connection

The variational inequality connected to the singular stochastic control problem is

$$\min\left(\mathcal{L}V(t,\mathbf{x}) + h(t,\mathbf{x});\, f_i(t) - \frac{\partial V}{\partial x_i}(t,\mathbf{x}), i=1,\dots,n\right) = 0 \tag{4.1}$$

$$V(T,\mathbf{x}) = g(\mathbf{x}) \tag{4.2}$$

where $\mathcal{L}$ is the generator for the diffusion $X^{t,\mathbf{x}}(s)$ known to be

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^n \mu(t,x_i)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(t,x_i)\frac{\partial^2}{\partial x_i \partial x_j}$$

The notion of viscosity solution was introduced by Crandall and Lions [6] for first-order equations and Lions [17] for second-order equations. For a general overview of the theory we refer to [5]. We recall the definition of viscosity solution: Denote by $C^{1,2}([0,T]\times I\!R^n)$ the space of functions $\phi : [0,T]\times I\!R^n \to I\!R$ which are once continuously differentiable in t and twice continuously differentiable in $\mathbf{x}$.

Definition 4.1. Assume $V(t, \mathbf{x})$ is continuous on $[0, T] \times I\!R^n$ and $V(T, \mathbf{x}) = g(\mathbf{x})$:

(i) $V(t, \mathbf{x})$ is a *viscosity subsolution* of (4.1) if for every $\phi \in C^{1,2}([0, T] \times I\!R^n)$

$$\min\left(\mathcal{L}\phi(\bar{t}, \bar{\mathbf{x}}) + h(\bar{t}, \bar{\mathbf{x}}), f_i(\bar{t}) - \frac{\partial \phi}{\partial x_i}(\bar{t}, \bar{\mathbf{x}}), i = 1, \ldots, n\right) \geq 0$$

where $(\bar{t}, \bar{\mathbf{x}}) \in [0, T) \times I\!R^n$ is the maximizer of $V(t, \mathbf{x}) - \phi(t, \mathbf{x})$.

(ii) $V(t, \mathbf{x})$ is a *viscosity supersolution* of (4.1) if for every $\phi \in C^{1,2}([0, T] \times I\!R^n)$

$$\min\left(\mathcal{L}\phi(\bar{t}, \bar{\mathbf{x}}) + h(\bar{t}, \bar{\mathbf{x}}), f_i(\bar{t}) - \frac{\partial \phi}{\partial x_i}(\bar{t}, \bar{\mathbf{x}}), i = 1, \ldots, n\right) \leq 0$$

where $(\bar{t}, \bar{\mathbf{x}}) \in [0, T) \times I\!R^n$ is the minimizer of $V(t, \mathbf{x}) - \phi(t, \mathbf{x})$.

(iii) $V(t, \mathbf{x})$ is a *viscosity solution* of (4.1) and (4.2) if it is both a viscosity subsolution and supersolution.

From [11, Th. 5.5] we have the following result for the connection between the value function $V(t, \mathbf{x})$ defined in (2.3) and the variational inequality (4.1) and (4.2):

Theorem 4.1. *If $g(\cdot) \equiv 0$ then $V(t, \mathbf{x})$ is the unique viscosity solution of (4.1) for which $V(T, \mathbf{x}) = 0$.*

Note that in the paper of [11] they consider a singular stochastic control problem involving a continuous control in addition to the control $\xi(t)$. Our problem is a special case of their formulation.

Let $V(t, \mathbf{x})$ be defined as in (2.3) and $U_i(t, x)$ as in (2.7). We first investigate the relation between the stopping problem $U_i(t, x)$ and a one-dimensional singular control problem. Define

$$V_i(t, x) = \int_{-\infty}^{x} U_i(t, z)\, dz \tag{4.3}$$

By extending the results in [3] in an obvious way, we get the following proposition,

Proposition 4.2. *$V_i(t, x)$ is a viscosity solution of the variational inequality*

$$\min\left(\mathcal{L}_i V_i(t, x) + h_i(t, x); f_i(t) - \frac{\partial V_i}{\partial x_i}(t, x)\right) = 0 \tag{4.4}$$

(4.5) $$V_i(T,x) = g_i(x)$$

where

$$\mathcal{L}_i = \frac{\partial}{\partial t} + \mu_i(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sum_{j=1}^{m} \sigma_{ij}^2(t,x)\frac{\partial^2}{\partial x^2}$$

If $g_i(\cdot) \equiv 0$, $i = 1, \dots, n$, we can conclude by Th. 4.1. that $V_i(t,x)$ is the value function of a singular control problem:

(4.6) $$V_i(t,x) = \inf_{\xi \in \mathcal{A}_1(t)} \mathrm{E}^{t,x}\Big[\int_t^T h_i(s, X_i(s))\, ds + \int_{[t,T)} f_i(s)\, d\xi(s)\Big]$$

We proceed with the n-dimensional case: For $\mathbf{x} = (x_1, \dots, x_n)$, define the function

(4.7) $$W(t,\mathbf{x}) = \sum_{i=1}^{n} \int_{-\infty}^{x_i} U_i(t,z)\, dz$$

We show that $W(t,\mathbf{x})$ is a viscosity solution of (4.1):

Theorem 4.3. *Assume condition (C) is satisfied. Then the function $W(t,\mathbf{x})$ is a viscosity solution of (4.1) satisfying the boundary condition $W(T,\mathbf{x}) = g(\mathbf{x})$.*

Proof. This argument follows the lines of the proof of Th. 4.1 in [3]:

The boundary condition $g(\mathbf{x})$ is obviously satisfied since,

$$\int_{-\infty}^{x_i} U_i(T,z)\, dz = \int_{-\infty}^{x} g_i'(z)\, dz = g_i(x_i)$$

where we used the assumption $\lim_{z \to -\infty} g_i(z) = 0$.

We first treat the viscosity supersolution case: Let $\phi \in C^{1,2}([0,T] \times I\!R^n)$ and $(\bar{t}, \bar{\mathbf{x}})$ be a minimizer of $W(t,\mathbf{x}) - \phi(t,\mathbf{x})$. Without any loss of generality we may assume that

$$\phi(\bar{t},\bar{\mathbf{x}}) = W(\bar{t},\bar{\mathbf{x}}) \quad \text{and} \quad \phi(t,\mathbf{x}) \le W(t,\mathbf{x})$$

Suppose $(\bar{t}, \bar{x}_i) \notin D_i^1$ for at least one $i \in \{1, \dots, n\}$. Since $(\bar{t}, \bar{\mathbf{x}})$ is an optimum and both $\phi(t,\mathbf{x})$ and $W(t,\mathbf{x})$ are differentiable at $(\bar{t}, \bar{\mathbf{x}})$, we have

$$\frac{\partial \phi}{\partial x_i}(\bar{t},\bar{\mathbf{x}}) = U_i(\bar{t}, \bar{x}_i) = f_i(\bar{t})$$

Thus we see that the supersolution property is satisfied in this case independent of what $\mathcal{L}\phi + h$ is. Suppose now that the minimum is inside all the regions D_i^1, $i = 1, \dots, n$. Since D_i^1 is open and assumed to have property (C), there exists a stopping time τ_i which is smaller than or equal to $\tau_{D_i^1}(\bar{t}, z)$ for all z and not being equal to $\bar{t}$. Namely, $\tau_i = \inf_z\{\tau_{D_i^1}(\bar{t}, z)\}$. (We use the notation $\tau_{D_i^1}$ for the first exit time out of $D_i^1(t, z)$ when the process starts at (t, z)). Let $\tau = \inf_i \tau_i$. The Fubini-Tonelli theorem and the dynamical programming principle (3.9) in each D_i^1 give

$$
\begin{aligned}
\phi(\bar{t}, \bar{\mathbf{x}}) &= W(\bar{t}, \bar{\mathbf{x}}) \\
&= \sum_{i=1}^{n} \int_{-\infty}^{\bar{x}_i} \left(\mathrm{E}^{\bar{t},z} \left[\int_{\bar{t}}^{\tau} \partial_x h_i(s, X_i(s)) Y_i(s)\, ds + U_i(\tau, X_i(\tau)) Y_i(\tau) \right] \right) dz \\
&= \sum_{i=1}^{n} \mathrm{E} \left[\int_{\bar{t}}^{\tau} \left(\int_{-\infty}^{\bar{x}_i} \partial_x h_i(s, X_i^{\bar{t},z}(s)) Y_i^{\bar{t},z}(s)) \right) dt \right] \\
&+ \sum_{i=1}^{n} \mathrm{E} \left[\int_{-\infty}^{\bar{x}_i} U_i(\tau, X_i^{\bar{t},z}(\tau)) Y_i^{\bar{t},z}(\tau))\, dz \right]
\end{aligned}
$$

For each i, do the substitution $u = X_i^{\bar{t},z}(s)$ in the integrals above. This yields,

$$
\int_{-\infty}^{\bar{x}_i} \partial_x h_i(s, X_i^{\bar{t},z}(s)) Y_i^{\bar{t},z}(s)\, ds = \int_{-\infty}^{X_i^{\bar{t},\bar{x}_i}(s)} \partial_x h_i(s, u)\, du = h_i(s, X_i^{\bar{t},\bar{x}_i}(s))
$$

In the last equality we have used that $\lim_{x \to -\infty} X_i^{t,x}(s) = -\infty$ a.s. and the assumption $\lim_{x \to -\infty} h_i(t, x) = 0$. Equivalently, we have

$$
\int_{-\infty}^{\bar{x}_i} U_i(\tau, X_i^{\bar{t},z}(\tau)) Y_i^{\bar{t},z}(\tau)\, dz = \int_{-\infty}^{X_i^{\bar{t},\bar{x}_i}(\tau)} U_i(\tau, u)\, du
$$

Hence, by summing up and using $\phi(t, \mathbf{x}) \leq W(t, \mathbf{x})$ we get

$$
\begin{aligned}
\phi(\bar{t}, \bar{\mathbf{x}}) &= \mathrm{E}^{\bar{t},\bar{\mathbf{x}}} \left[\int_{\bar{t}}^{\tau} h(s, X(s))\, ds \right] + \mathrm{E}^{\bar{t},\bar{\mathbf{x}}} \left[\sum_{i=1}^{n} \int_{-\infty}^{X_i(\tau)} U_i(\tau, u)\, du \right] \\
&\geq \mathrm{E}^{\bar{t},\bar{\mathbf{x}}} \left[\int_{\bar{t}}^{\tau} h(s, X(s))\, ds + \phi(\tau, X(\tau)) \right]
\end{aligned}
$$

Dynkin's formula yields

$$
0 \geq \mathrm{E}^{\bar{t},\bar{\mathbf{x}}} \left[\int_{\bar{t}}^{\tau} \mathcal{L}\phi(s, X(s)) + h(s, X(s))\, ds \right]
$$

A limiting argument when $\tau \to \bar{t}$ gives that $\mathcal{L}\phi + h \leq 0$. Hence, $W(t, \mathbf{x})$ is a viscosity supersolution.

Consider now the viscosity subsolution case: Let $\phi \in C^{1,2}([0,T] \times I\!R^n)$ and $(\bar{t}, \bar{\mathbf{x}})$ be a maxmizer of $W(t, \mathbf{x}) - \phi(t, \mathbf{x})$. Without any loss of generality we may assume that

$$\phi(\bar{t}, \bar{\mathbf{x}}) = W(\bar{t}, \bar{\mathbf{x}}) \quad \text{and} \quad \phi(t, \mathbf{x}) \geq W(t, \mathbf{x})$$

Since $U_i(t,x) \leq f_i(t)\mathbf{1}_{t<T} + g_i'(x)\mathbf{1}_{t=T}$ we know that $f_i(\bar{t}) - \frac{\partial \phi}{\partial x_i}(\bar{t}, \bar{\mathbf{x}}) \geq 0$. It remains to show that $\mathcal{L}\phi(\bar{t}, \bar{\mathbf{x}}) + h(\bar{t}, \bar{\mathbf{x}}) \geq 0$ in order to have the viscosity supersolution property. Use the dynamic programming principle (3.10) for each of the U_i's and argue as above,

$$\begin{aligned}
\phi(\bar{t}, \bar{\mathbf{x}}) &= \sum_{i=1}^{n} U_i(t, x_i)\, dz \\
&\leq \sum_{i=1}^{n} \int_{-\infty}^{\bar{x}_i} \left(\mathrm{E}^{\bar{t},z} \left[\int_{\bar{t}}^{\tau} \partial_x h_i(s, X_i(s)) Y_i(s)\, ds + U_i(\tau, X_i(\tau)) Y_i(\tau) \right] \right) dz \\
&= \mathrm{E}^{\bar{t},\bar{\mathbf{x}}} \left[\int_{\bar{t}}^{\tau} h(s, X(s))\, ds + \sum_{i=1}^{n} \int_{-\infty}^{X_i(\tau)} U_i(\tau, u)\, du \right] \\
&\leq \mathrm{E}^{\bar{t},\bar{\mathbf{x}}} \left[\int_{\bar{t}}^{\tau} h(s, X(s))\, ds + \phi(\tau, X(\tau)) \right]
\end{aligned}$$

Dynkin's formula now yields,

$$0 \leq \mathrm{E}^{\bar{t},\bar{\mathbf{x}}} \left[\int_{\bar{t}}^{\tau} \mathcal{L}\phi(s, X(s)) + h(s, X(s))\, ds \right]$$

A limiting argument when $\tau \to \bar{t}$ gives that $\mathcal{L}\phi + h \geq 0$. Hence, $W(t, \mathbf{x})$ is a viscosity subsolution. This completes the proof. □

It is now obvious that if $g_i(\cdot) \equiv 0$, for $i = 1, \dots, n$,

$$V(t, \mathbf{x}) = \sum_{i=1}^{n} V_i(t, x_i) \tag{4.8}$$

where the V_i's are the singular control problems defined in (4.6). Hence, we see that the n-dimensional singular stochastic control problem factorizes into n one-dimensional singular control problems.

For singular control problems one can - as already mentioned - split the domain of definition into an in-action and action region (see e.g. [15, 16]). If we denote the in-action region by $\tilde{D}$, an optimal control ξ^* can in many cases be constructed as follows: Push the process immediately out of $\tilde{D}^c$ if it starts inside the action region. Thereafter the control will behave like a local time on the boundary $\partial\tilde{D}$, pushing the process just enough to keep it inside the in-action region. As long as the process is in the interior of $\tilde{D}$ the control does nothing. The region of in-action is defined as

$$\tilde{D} := \left\{ (t, \mathbf{x}) \in [0, T) \times I\!R^n \, ; \, \frac{\partial V}{\partial x_i}(t, \mathbf{x}) < f_i(t), i = 1, \ldots, n \right\} \tag{4.9}$$

When the singular control problem coincides with the sums of the integrals of the stopping problems as shown above, we have

$$\tilde{D} = \bigcap_{i=1}^{n} D_i^1 \tag{4.10}$$

From the theory of optimal stopping the optimal time to stop is - under certain conditions - when the process reaches the boundary of the continuation region. This relates the optimal stopping time and the optimal control process,

$$\tau_i^* = \inf\{ s \geq t \, ; \, \xi_i^*(s) > 0\}$$

I.e., the optimal stopping time is the first time one intervenes with the optimal control. This connection was proven by [15, 16] in the one-dimensional case.

Karatzas and Shreve [15] also proved the existence of ξ^* under regularity conditions on the parameters. One of these conditions says that $\partial_{x_i} h$ should be non-decreasing. From Haussmann and Suo [11], on the other hand, we need h to be bounded in order to have the viscosity property of the value function.

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