Phillips-Sarnak's conjecture for $\Gamma_0(8)$ with primitive character

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Abstract

We prove a conjecture of Phillips and Sarnak about the disappearance of embedded eigenvalues for the Laplacian $A(\Gamma_0(8);\chi_8)$, where χ_8 is the primitive character mod 8, under an analytic family of character perturbations $\chi^{(\alpha)}$ with $\chi^{(\frac{1}{2})} = \chi_8$. Eigenvalues λ with odd eigenfunctions give rise to resonances, and at least one eigenfunction from the corresending eigenspace is transformed into a resonance function. Here $\lambda = s(1-s)$, $s = \frac{1}{2} + it$, $t \neq \frac{\pi i n}{\log 2}$, $n \in Z$. This indicates that the Weyl law is violated by the operators $A(\Gamma_0(8);\chi^{(\alpha)})$ for $\frac{1}{2} - \varepsilon < \alpha < \frac{1}{2} + \varepsilon$, $\alpha \neq \frac{1}{2}$, and some $\varepsilon > 0$.

Introduction

It was proved by Selberg [Sel] that the Laplacian $A(\Gamma)$ for congruence subgroups Γ of the modular group $\Gamma_{\mathbb{Z}}$ has an infinite sequence of embedded eigenvalues $\{\lambda_i\}$ satisfying a Weyl law $\#\{\lambda_i \leqq \lambda\} \sim \frac{|F|}{4\pi}\lambda$ for $\lambda \to \infty$. Here $|F|$ is the area of the fundamental domain F of the group Γ, and the eigenvalues λ_i are counted according to multiplicity. The same holds true for the Laplacian $A(\Gamma;\chi)$, where χ is a character on Γ and $A(\Gamma;\chi)$ is associated with a congruence subgroup Γ_1 of Γ. It is an important question whether this is a characteristic of congruence groups or it may hold also for some non-congruence subgroups of $\Gamma_{\mathbb{Z}}$. To investigate this problem Phillips and Sarnak studied perturbation theory for Laplacians $A(\Gamma)$ with regular perturbations derived from modular forms of weight 4 [P-Sal] and singular perturbations by characters derived from modular forms of weight 2 [P-Sa2]. Their work on singular perturbations was inspired by Wolpert [W1], [W2]. See also [DIPS] for a short version of these ideas and related conjectures. Central to their approach was the application of perturbation theory and in that connection the evaluation of the integral of the product of the Eisenstein series $E(s_j)$ at an eigenvalue $\lambda_j = s_j(1-s_j)$ with the first order perturbation M applied to the eigenfunction v_j. If this integral $I(s_j)$, which we call the Phillips-Sarnak integral, is non-zero, the corresponding eigenfunction ϕ_j becomes a resonance function with resonance $\lambda_j(\alpha)$ under the perturbation $\alpha M+ \alpha^2 N$ for small $\alpha \neq 0$ and a resonance. This follows from the fact that $|I(s_j)|^2$ is a constant times $\operatorname{Im} \lambda_j''(0)$, a fact known as Fermi's Golden rule. The strategy of Phillips and Sarnak is on the one hand to prove Fermi's Golden Rule for Laplacians $A(\Gamma)$ and on the other hand to prove that $I(s_j) \neq 0$ under certain conditions. For congruence groups with singular character perturbation closing two or more cusps a fundamental difficulty presents itself due to the appearance of new resonances of $A(\Gamma, \alpha)$ for $\alpha \neq 0$, which condense at every point of the continuous spectrum of A as $\alpha \to 0$. These resonances (poles of the S-matrix) were discovered by Selberg [Se2] for the group $\Gamma(2)$ with singular character perturbation closing 2 cusps, so we call them the Selberg resonances. Any method of proving that eigenvalues become resonances or remain eigenvalues has to deal with these resonances, which arise from the continuous spectrum of the cusps, which are closed by the perturbation. This makes the problem very difficult in that case. In the case of regular perturbations derived from cusp forms it is not difficult to prove Fermi's Golden Rule, but it is very hard to prove that the integral is not zero.

We consider instead as our basic operator $A(\Gamma_0(N), \chi)$ with χ a real primitive character mod N. The perturbation of this operator through the characters $\chi^{(\alpha)}$ $\frac{1}{2}$ α near is regular, and there is no Selberg phenomenon. Then we can study the question whether embedded eigenvalues become resonances under this perturbation. Specifically, we choose as the basic groups $\Gamma_0(N)$ with $N = 4n$ and $n = 3 \bmod 4$ or $n = 2 \bmod 4$. This group has 4 cusps and genus 0. The reason for this choice is that $\chi^{\left(\frac{1}{2}\right)}$ for $\Gamma_0(8)$ is a primitive character and hence all cusp forms are new and we have the simplest complete Hecke theory.

In Section 1 we give an exposition of all primitive characters on the groups $\Gamma_0(N)$ in the general case with explicit description of parabolic generators and closed and open cusps. Moreover, these characters are fundamental in number theory, since they are related to quadratic fields. The perturbation can also be expressed in the form $\alpha M + \alpha^2 N$, where $M = -4\pi i y^2 \left(\omega_1 \frac{\partial}{\partial x} - \omega_2 \frac{\partial}{\partial y}\right)$, $N = 4\pi y^2 \left|\omega\right|^2$. This explicit form is important for perturbation theory of weight 2.

In Section 2 we discuss the Eisenstein series, and in Section 3 we prove the Weyl law for eigenvalues of $\Gamma_0(N, \chi)$ using the factorization formula for the Selberg zeta function [F] and Huxley explicit formula for the scattering matrix for $\Gamma_1(N)$ [Hu] is off-diagonal and symmetric, the off-diagonal elements being expressed by the Dirichlet series associated with the primitive character (Lemma 3). From this follows that $A(\Gamma, \chi^{\left(\frac{1}{2}\right)})$ has an infinite sequence of eigenvalues following a Weyl law. We further discuss the Hecke theory for $A(\Gamma, \chi^{\left(\frac{1}{2}\right)})$, which is basic for the understanding of embedded eigenvalues. We show that all forms are new forms and derive the important property, that all even Fourier coefficients of these forms are 0.

The fact that $I(s_j) \neq 0$ for all eigenvalues $s_j(1 - s_j)$ of L with odd eigenfunctions v_j is based on the non-vanishing of the L-series $L(s; v_j)$ on the line $\{s \,|\mathrm{Re}\, s = 1\}$. In Section 3, Theorem 1, we establish this property using a general property about zeros on $\{\mathrm{Re}\, s = 1\}$ of L-functions, which are holomorphic in $\{\mathrm{Re}\, s \geq 1\}$ except for a pole at $s = 1$ (Theorem 1).

In Section 4 we prove that the Phillips-Sarnak integral $I(s_j) \neq 0$ for all odd eigenfunctions v_j corresponding to embedded eigenvalues $s_j(1 - s_j)$, $s_j = \frac{1}{2} + it_j$, $t_j \neq \frac{\pi i n}{\log 2}$, $n \in Z$ (Theorem 3). We do this by unfolding the integral and obtain an expression for $I(s_j)$ apart from a non-zero factor as a Dirichlet series with coefficients equal to the products of the Fourier coefficients of the modular form ω defining the perturbation and those of the eigenfunction v_j. We then express this Dirichlet series by the Dirichlet L-series of v_j and apply Theorem 1 to obtain $I(s_j) \neq 0$ for $s_j = \frac{1}{2} + it_j$, $t_j \neq \frac{\pi i n}{\log 2}, n \in Z$.

In Section 5 we prove Fermi's Golden Rule, which together with Theorem 2 implies that embedded eigenvalues with odd eigenfunctions give rise to resonances under these perturbations. The method used by Phillips and Sarnak [P-Sa3] and Petridis [Pe] for proving Fermi's Golden Rule utilized the Lax-Phillips scattering theory, obtaining the eigenvalues and resonances of $A(\Gamma)$ as discrete spectrum of a certain non-selfadjoint operator. The proof given here uses analytic continuation of the resolvent between certain weighted Banach spaces introduced by Faddeev [F]. For a given eigenvalue λ_j this gives rise to a quasi-projection $P(\alpha)$, obtained by integration of the analytically continued resolvent around λ_j. Then we can expand the eigenfunction to second order and obtain the explicit expression for $\operatorname{Im} \lambda_2$ known as Fermi's Golden rule, where λ_2 is the second order coefficient of $\lambda_j(\alpha)$. This is obtained for all odd eigenfunctions associated with a given eigenvalue, and it follows that at least one eigenfunction from each eigenspace containing odd eigenfunctions becomes a resonance function (Theorem 4).

Although we can prove that the Phillips-Sarnak integrals are non-zero only in the case of congruence groups $\Gamma_0(8)$ with character $\chi^{\left(\frac{1}{2}\right)}$, this result makes it possible to draw some conclusions about whether embedded eigenvalues of $L(\Gamma_0(8), \chi^{(\alpha)})$ for other values of α, if such eigenvalues exist, leave or stay under this perturbation (Remark 1).

Replacing $\operatorname{Re} \int_{z_0}^{\gamma z_0} \omega(t)dt$ by $\operatorname{Im} \int_{z_0}^{\gamma z_0} \omega(t)dt$ in the definition of characters $\omega^{(\alpha)}$, we obtain a perturbation $\alpha \tilde{M} + \alpha^2 N$, which keeps the parity. Now the Phillips-Sarnak integral for even eigenfunction is non-zero, but in this case it does not imply that such eigenvalues turn into resonances. On the contrary, all eigenvalues are constant, since the operators $L(\alpha)$ are unitarily equivalent due to the periods being real (Remark 2).

To make this paper self-contained we have included all details of the necessary calculations.

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1 The basic operators

We consider the Hecke group $\Gamma = \Gamma_0(8)$. The index of Γ in the modular group $\Gamma_{\mathbb{Z}}$ is 12, the number of cusps is 4, it is of genus $g = 0$, and it contains no elliptic elements.

We now construct the canonical generators of Γ. It is clear that $\Gamma_0(8) \subset \Gamma_0(4)$ with index 2, and $\Gamma_0(4)$ is generated by 3 parabolic elements with one relation,

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \ \tilde{S} = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}, \ \tilde{A}\tilde{B}\tilde{S} = 1.$$

As usual we identify matrices which differ by a factor -1.

In [B-V1] we studied the normal subgroups Γ_d of the main congruence subgroup $\Gamma(2)$, generated by parabolic generators only. The group $\Gamma(2)$ is isomorphic to $\Gamma_0(4)$ by the map

$$\begin{pmatrix} a & 2b \\ 2c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ 4c & d \end{pmatrix}.$$

Using the definition of $\Gamma(2)$ we construct the group $\tilde{\Gamma}_2 \subset \Gamma_0(4)$ of index 2 by the parabolic generators S_1, S_2, S_3, S_4 with the relation

$$S_1 S_2 S_3 S_4 = 1$$

where

$$S_1 = \tilde{B}^2, \ S_2 = \tilde{B}^{\cdot -1}\tilde{A}\tilde{B}, \ S_3 = \tilde{A}, \ S_4 = \left(\tilde{B}\tilde{A}\right)^{-2}$$

$$S_1 = \begin{pmatrix} 1 & 0 \\ -8 & 1 \end{pmatrix}, \ S_2 = \begin{pmatrix} -3 & 1 \\ -16 & 5 \end{pmatrix}, \ S_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ S_4 = \begin{pmatrix} 5 & 2 \\ -8 & -3 \end{pmatrix}. \qquad (1)$$

So this group $\tilde{\Gamma}_2$ is a subgroup of $\Gamma_0(8)$, because $S_1, S_2, S_3 \in \Gamma_0(8)$. But also $\tilde{\Gamma}_2, \Gamma_0(8)$ both have the same index 2 in $\Gamma_0(4)$. Thus $\Gamma_0(8) = \tilde{\Gamma}_2$.

We introduce now the one-dimensional unitary representation (character) χ of $\Gamma = \Gamma_0(8)$ just using the standard construction

$$\chi \begin{pmatrix} a & b \\ 8c & d \end{pmatrix} = \hat{\chi}(d),\ \gamma = \begin{pmatrix} a & b \\ 8c & d \end{pmatrix} \in \Gamma_0(8) \tag{2}$$

where $\hat{\chi}(d)$ is one of the arithmetical characters mod 8. The definition becomes clear from

$$\begin{pmatrix} a & b \\ 8c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ 8c' & d' \end{pmatrix} = \begin{pmatrix} * & * \\ * & 8b'c + dd' \end{pmatrix}.$$

We take as $\hat{\chi}(n)$ the primitive character

$$\chi_8(n) = \begin{cases} 1 & n = \pm 1 \pmod 8 \\ -1 & n = \pm 3 \pmod 8 \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

This character has the important property $\chi_8(n) = \chi_8(-n)$, which makes the definition (2) correct. We denote by χ_8 the particular character of the group $\Gamma_0(8)$ defined by $\chi_8(\gamma) = \chi_8(d)$.

It is easy to see that

$$\chi_8(S_1) = 1,\ \chi_8(S_2) = -1,\ \chi_8(S_3) = 1,\ \chi_8(S_4) = -1. \tag{4}$$

So this character closes two cusps of the fundamental domain F of Γ, which correspond to S_2, S_4, and keep open the other two cusps.

We will now construct the modular form $\omega(z)$, which defines our perturbation. For z in the upper half-plane H, let

$$P(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z} \tag{5}$$

be the holomorophic Eisenstein series of weight 2 for the modular group $\Gamma_{\mathbb{Z}}$. We have

$$P\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 P(z) - \frac{6i}{\pi} c(cz+d),\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathbb{Z}} \tag{6}$$

in particular

$$P\left(-\frac{1}{z}\right) = z^2 P(z) - \frac{6i}{\pi} z$$
$$P(z+1) = P(z).$$

We are looking for the holomorphic modular form $\omega(z)$ of weight 2 for the group $\Gamma_0(8)$ with the property that the form $\omega(z)$ is small in the two open cusps of F, which correspond to S_1, S_3.

Lemma 1 *The function $\omega(z)$ defined for $z \in H$ by*

$$\omega(z) = P(z) - 7P(2z) + 14P(4z) - 8P(8z)$$

is a holomorphic form of weight 2, which is exponentially small in the open cusps defined by S_1 and S_3.

Proof. Let us consider a linear combination

$$\omega_\alpha(z) = P(z) + \alpha_1 P(2z) + \alpha_2 P(4z) + \alpha_3 P(8z) \tag{7}$$

with real parameters α_i, $i = 1, 2, 3$. We determine the coefficients α_i in order to get the form with the desired properties. First we will find the cusps z_j of a fundamental domain F for the group Γ, which correspond to the system of generators, S_1, S_2, S_3, S_4.

We have

$$\left.\begin{array}{ll} S_1 \cdot 0 = 0 & z_1 = 0 \\ S_2 z_2 = z_2, & \frac{3z-1}{16z-5} = z, \quad 16z^2 - 8z + 1 = 0, \quad z = z_2 = 1/4 \\ S_3 \infty = \infty & z_3 = \infty \\ S_4(-1/2) = -1/2 & z_4 = -1/2 \end{array}\right\} \tag{8}$$

Then in order to calculate the asymptotics of $\omega(z)$ in the cusps z_i it is convenient to find the transformations of $\Gamma_{\mathbb{Z}}$ which map these cusps to ∞.

Let us introduce the generators of the group $\Gamma_{\mathbb{Z}}$

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = S_3, \; V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

Then it is not difficult to find representations of S_1, S_2, S_3, S_4 by U and V. We have

$$\begin{cases} S_1 = (VUV)^8 = VU^8V \\ S_2 = (VU^{-1}V)^4U(VUV)^4 = (VU^{-4}V)U(VU^4V) \\ S_3 = U \\ S_4 = (VU^2V)U^2(VU^{-2}V) \end{cases} . \tag{10}$$

Let us determine now the values of the parameters α_i. First we want $\omega(z)$ to be a holomorphic modular form for $\Gamma_0(8)$. We obtain, using (6),

$$\begin{aligned} \omega_\alpha\left(\frac{az+b}{8cz+d}\right) &= P\left(\frac{az+b}{8cz+d}\right) + \alpha_1 P\left(\frac{az+b}{8cz+d}\cdot 2\right) \\ &+\alpha_2 P\left(\frac{az+b}{8cz+d}\right) + \alpha_3 P\left(\frac{az+b}{8cz+d}\cdot 8\right) \end{aligned} \tag{11}$$

$$\frac{2az+2b}{8cz+d} = \frac{a(2z)+2b}{4c(2z)+d}, \quad \frac{4az+4b}{8cz+d} = \frac{a(4z)+4b}{2c(4z)+d}$$
$$\frac{8az+8b}{8cz+d} = \frac{a(8z)+8b}{c(8z)+d}$$

For $\begin{pmatrix} a & b \\ 8c & d \end{pmatrix} \in \Gamma_0(8)$ we have $\begin{pmatrix} a & 2b \\ 4c & d \end{pmatrix}$, $\begin{pmatrix} a & 4b \\ 2c & d \end{pmatrix} \in \Gamma_{\mathbb{Z}}$, $\begin{pmatrix} a & 8b \\ c & d \end{pmatrix} \in \Gamma_{\mathbb{Z}}$.

We continue equality (11)

$$\begin{aligned} \omega_\alpha\left(\frac{az+b}{8cz+d}\right) &= (8cz+d)^2P(z) - \frac{8c(8cz+d)}{\pi}6i \\ &+\alpha_1\left[(8cz+d)^2P(2z) - \frac{4c(8cz+d)}{\pi}6i\right] \\ &+\alpha_2\left[(8cz+d)^2P(4z) - \frac{2c(8cz+d)6i}{\pi}\right] \\ &+\alpha_3\left[(8cz+d)^2P(8z) - \frac{c(8cz+d)}{\pi}6i\right]. \end{aligned} \tag{12}$$

From (12) follows

$$\omega_\alpha\left(\frac{az+b}{8cz+d}\right) = (8cz+d)^2\omega_\alpha(z) + \frac{6ic(8cz+d)}{\pi}\left[-8-4\alpha_1-2\alpha_2-\alpha_3\right]$$

and we obtain the first condition

$$\alpha_3 + 2\alpha_2 + 4\alpha_1 + 8 = 0. \tag{13}$$

In order to get the second condition let us consider the asymptotics of $\omega_\alpha(z)$ when $z = x + iy$, $y > 0$, $y \to \infty$ and $-1 \leqq x \leqq 1$

$$\omega_\alpha(z) = P(z) + \alpha_1 P(2z) + \alpha_2 P(4z) + \alpha_3 P(8z) \sim_{y\to\infty} 1 + \alpha_1 + \alpha_2 + \alpha_3$$

(see (5)). Thus we obtain the second condition

$$\alpha_3 + \alpha_2 + \alpha_1 + 1 = 0 \tag{14}$$

because we like to have $\omega(z)$ small at infinity. From (13), (14) we get

$$\alpha_3 = -(8 + 4\alpha_1 + 2\alpha_2),\ \alpha_3 = -(1 + \alpha_1 + \alpha_2)$$

and

$$a_2 = 1 + \alpha_1 - 8 - 4\alpha_1 = -7 - 3\alpha_1$$

$$\alpha_3 = 6 + 2\alpha_1.$$

So we have only a one-parameter family of forms ω_α

$$\omega_\alpha = P(z) + \alpha_1 P(2z) - (7 + 3\alpha_1)P(4z) + (6 + 2\alpha_1)P(8z) \tag{15}$$

which are holomorphic modular forms of weight 2 for $\Gamma_0(8)$ and are small in the cusp $z_3 = \infty$. We now determine the last parameter α_1 by the condition for ω_α to be small also in the cusp z_1. That means we want $\omega_\alpha(-1/z)$ to be small when $y \to \infty$, $z = x+iy$, $-1 \leqq x \leqq 1$. We take $\omega_\alpha(z)$ from (15) and we obtain (see (6))

$$\omega_\alpha(-1/z) = z^2P(z) - \frac{6iz}{\pi} + \alpha_1\left[\frac{z^2}{4}P\left(\frac{z}{2}\right) - \frac{3iz}{\pi}\right] \tag{16}$$

$$- (7+3\alpha_1)\left[\frac{z^2}{16}P\left(\frac{z}{4}\right) - \frac{3iz}{2\pi}\right] + (6+2\alpha_1)\left[\frac{z^2}{64}P\left(\frac{z}{8}\right) - \frac{3iz}{4\pi}\right]$$

$$= z^2\left[P(z) + \frac{\alpha_1}{4}P\left(\frac{z}{2}\right) - \frac{7+3\alpha_1}{16}P\left(\frac{z}{4}\right) + \frac{6+2\alpha_1}{64}P\left(\frac{z}{8}\right)\right]$$

$$+\frac{iz}{\pi}\left(-6 - 3\alpha_1 + \frac{(7+3\alpha_1)3}{2} - \frac{(6+2\alpha_1)3}{4}\right)$$

$$= z^2\left[P(z) + \frac{\alpha_1}{4}P\left(\frac{z}{2}\right) - \frac{7+3\alpha_1}{16}P\left(\frac{z}{4}\right) + \frac{6+2\alpha_1}{64}P\left(\frac{z}{8}\right)\right].$$

We obtain from (5) and (16)

$$\omega_\alpha(-1/z) = \left(1 + \frac{\alpha_1}{4} - \frac{7+3\alpha_1}{16} + \frac{6+2\alpha_1}{64}\right)z^2 + O\left(e^{-\varepsilon y}\right)_{y\to\infty} \tag{17}$$

for some $\varepsilon > 0$. So we like the first term on the right hand side of (17) to disappear. Thus we obtain the final condition

$$1 + \frac{\alpha_1}{4} - \frac{7+3\alpha_1}{16} + \frac{6+2\alpha_1}{64} = 0. \tag{18}$$

We get $32 + 8\alpha_1 - 14 - 6\alpha_1 + 3 + \alpha_1 = 0$, $\alpha_1 = -7$. From (15) and (18) follows the explicit expression for the desired form $\omega(z)$

$$\omega(z) = P(z) - 7P(2z) + 14P(4z) - 8P(8z). \tag{19}$$

This completes the proof of the Lemma. ■

We shall now study the integrals

$$\int_{z_0}^{\gamma z_0} \omega(t)dt \tag{20}$$

where z_0 is any fixed point from the upper half plane and $\gamma \in \Gamma_0(8)$. We are especially interested in the values of these integrals for $\gamma = S_i$, $i = 1,2,3,4$ (see (1)). We will see that in these cases the integrals are just the constant terms in the Fourier decompositions of the form $\omega(t)$ relative to the parabolic subgroups of $\Gamma_0(8)$ generated by S_j $j = 1,2,3,4$. And we will calculate these constant terms explicitly.

Lemma 2 *We have*

$$\int_{z_0}^{S_1 z_0} \omega(t)dt = 0 \quad \int_{z_0}^{S_2 z_0} \omega(t)dt = 6 \quad \int_{z_0}^{S_3 z_0} \omega(t)dt = 0 \quad \int_{z_0}^{S_4 z_0} \omega(t)dt = -6$$

Proof. We start with S_3 because it is the simplest case.

$$\begin{aligned} &\int_{z_0}^{S_3 z_0} \omega(t)dt \qquad (21)\\ &= \int_{z_0}^{z_0+1} \left[P\left(x+iy_0\right) - 7P\left(2x+2iy_0\right) + 14P\left(4x+4iy_0\right) - 8P\left(8x+8iy_0\right)\right]dx. \end{aligned}$$

We apply now the Fourier decomposition (5) for $P(z)$ to (21) and obtain

$$\int_{z_0}^{S_3 z_0} \omega(t)dt = 1 - 7 + 14 - 8 = 0.$$

Let us consider the integral with S_1. We have (see (10))

$$\int_{z_0}^{S_1 z_0} \omega(t)dt = \int_{z_0}^{VU^8Vz_0} \omega(t)dt \qquad (22)$$

We make the change of variable in (22) $t = Vz$,

$$\int_{z_0}^{VU^8Vz_0} \omega(t)dt = \int_{Vz_0}^{U^8Vz_0} \omega(Vz)d(Vz) \qquad (23)$$

because $V^2 = 1$. Then we have $d(Vz) = d\left(-1/z\right) = \frac{dz}{z^2}$

$$\begin{aligned} \frac{1}{z^2}\omega(Vz) &= \frac{1}{z^2}\left[P(-1/z) - 7P(-2/z) + 14P(-4/z) - 8P(-8/z)\right] \qquad (24)\\ &= \frac{1}{z^2}\left\{z^2P(z) - \frac{6i}{\pi}z - 7\left[\frac{z^2}{4}P\left(\frac{z}{2}\right) - \frac{6i}{\pi}\cdot\frac{z}{2}\right]\right.\\ &\left.+14\left[\frac{z^2}{16}P\left(\frac{z}{4}\right) - \frac{6i}{\pi}\cdot\frac{z}{4}\right] - 8\left[\frac{z^2}{64}P\left(\frac{z}{8}\right) - \frac{6i}{\pi}\cdot\frac{z}{8}\right]\right\}\\ &= P(z) - \frac{7}{4}P(z/2) + \frac{7}{8}P(z/4) - \frac{1}{8}P(z/8) - \frac{6i}{\pi}z^2\left(z - \frac{7}{2}z + \frac{7}{2}z - z\right)\\ &= P(z) - \frac{7}{4}P(z/2) + \frac{7}{8}P(z/4) - 1/8P(z/8). \end{aligned}$$

We have used in (24) the transformation formula (6). We now put $Vz_0 = w_0 = \tilde{x}_0 + i\tilde{y}_0$. From (23), (24) we obtain

$$\begin{aligned}
&\int_{Vz_0}^{U^8 Vz_0} \omega(Vz)d(Vz) \qquad (25)\\
&= \int_{w_0}^{w_0+8} \left[P\left(x + i\tilde{y}_0\right) - \frac{7}{4}P\left(\frac{x+i\tilde{y}_0}{2}\right) + \frac{7}{8}P\left(\frac{x+i\tilde{y}_0}{4}\right) - 1/8P\left(\frac{x+i\tilde{y}_0}{8}\right)\right] dx \\
&= 8\left[1 - \frac{7}{4} + \frac{7}{8} - \frac{1}{8}\right] = 0
\end{aligned}$$

We used the Fourier decomposition (5) in (25), and we have

$$\int_{z_0}^{S_1 z_0} \omega(t)dt = 0.$$

We will calculate now the integral with S_2. We have

$$S_2 = (VU^{-4}V)U(VU^4V),\ t = VU^{-4}Vz = \frac{z}{4z+1},\ dt = \frac{dz}{(4z+1)^2}$$

$$\begin{aligned}
\int_{z_0}^{S_2 z_0} \omega(t)dt &= \int_{VU^4Vz_0}^{U(VU^4V)z_0} \omega(VU^{-4}Vz)d(VU^{-4}Vz) \qquad (26)\\
&= \int_{w_1}^{w_1+1} \omega\left(\frac{z}{4z+1}\right)\frac{dz}{(4z+1)}
\end{aligned}$$

where $w_1 = VU^4Vz_0$. We then obtain

$$\begin{aligned}
&\frac{1}{(4z+1)^2}\omega\left(\frac{z}{4z+1}\right) \qquad (27)\\
&= \frac{1}{(4z+1)^2}\left[P\left(\frac{z}{4z+1}\right) - 7P\left(\frac{2z}{4z+1}\right) + 14P\left(\frac{4z}{4z+1}\right) - 8P\left(\frac{8z}{4z+1}\right)\right].
\end{aligned}$$

The three first terms on the right hand side of (27) are related to the following transformations from $\Gamma_{\mathbb{Z}}$,

$$\frac{z}{4z+1} \to \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},\ \frac{2z}{4z+1} = \frac{(2z)}{2(2z)+1} \to \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

$$\frac{4z}{4z+1} = \frac{(4z)}{(4z)+1} \to \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so we can apply directly the transformation formula (6) to them. For the last term we use the following transformation first and then apply (6)

$$P\left(\frac{8z}{4z+1}\right) = P\left(\frac{8z}{4z+1} - 2\right) = P\left(-\frac{2}{4z+1}\right) = P\left(-\frac{1}{\left(\frac{4z+1}{2}\right)}\right).$$

We continue (27),

$$\begin{aligned}
&\frac{1}{(4z+1)^2}\omega\left(\frac{z}{4z+1}\right) \qquad (28)\\
&= \frac{1}{(4z+1)^2}\left[(4z+1)^2P(z) - \frac{6i}{\pi}4(4z+1) - 7\left((4z+1)^2P(2z) - \frac{6i}{\pi}2(4z+1)\right)\right.\\
&+ 14\left((4z+1)^2P(4z) - \frac{6i}{\pi}(4z+1)\right) - 8\left(\left(\frac{4z+1}{2}\right)^2 P\left(\frac{4z+1}{2}\right)\right.\\
&\left.\left.- \frac{6i}{\pi}\cdot\frac{4z+1}{2}\right)\right]\\
&= P(z) - 7P(2z) + 14P(4z) - 2P\left(\frac{4z+1}{2}\right)\\
&- \frac{6i}{\pi(4z+1)^2}\left(16z + 4 - 56z - 14 + 56z + 14 - 16z - 4\right)\\
&= P(z) - 7P(2z) + 14P(4z) - 2P\left(\frac{4z+1}{2}\right).
\end{aligned}$$

Now we use (28) together with (26) and obtain

$$\int_{w_1}^{w_1+1} \omega\left(\frac{z}{4z+1}\right)\frac{dz}{(4z+1)^2} = \int_{w_1}^{w_1+1}(1-7+14-2)dx = 6.$$

For the last integral we have

$$\begin{aligned}
\int_{z_0}^{S_4 z_0} \omega(t)dt &= \int_{VU^{-2}Vz_0}^{U^2(VU^{-2}V)z_0} \omega(VU^2Vz)d(VU^2Vz) \qquad (29)\\
&= \int_{w_2}^{w_2+2} \omega\left(\frac{z}{-2z+1}\right)\frac{dz}{(2z-1)^2}
\end{aligned}$$

where $w_2 = VU^{-2}Vz_0$. Then we have

$$P\left(\frac{4z}{-2z+1}\right) = P\left(\frac{4z}{-2z+1}+2\right) = P\left(\frac{2}{-2z+1}\right) = P\left(-\frac{1}{\left(\frac{2z-1}{2}\right)}\right)$$

and

$$P\left(\frac{8z}{-2z+1}\right) = P\left(-\frac{1}{\left(\frac{2z-1}{4}\right)}\right) \tag{30}$$

$$\begin{aligned}
\omega\left(\frac{z}{-2z+1}\right)\frac{1}{(2z-1)^2} &= \frac{1}{(2z-1)^2}\left[P\left(\frac{z}{-2z+1}\right) - 7P\left(\frac{2z}{-2z+1}\right)\right.\\
&\left.+14P\left(\frac{4z}{-2z+1}\right) - 8P\left(\frac{8z}{-2z+1}\right)\right]\\
&= \frac{1}{(2z-1)^2}\left[(2z-1)^2P(z) - \frac{6i}{\pi}(-2)(-2z+1)\right.\\
&- 7\left((2z-1)^2 P(2z) - \frac{6i}{\pi}(-1)(-2z+1)\right)\\
&+ 14\left(\left(\frac{2z-1}{2}\right)^2 P\left(\frac{2z-1}{2}\right) - \frac{6i}{\pi}\cdot\frac{2z-1}{2}\right)\\
&\left.-8\left(\left(\frac{2z-1}{4}\right)^2 P\left(\frac{2z-1}{4}\right) - \frac{6i}{\pi}\cdot\frac{2z-1}{4}\right)\right]\\
&= P(z) - 7P(2z) + \frac{7}{2}P\left(\frac{2z-1}{2}\right) - \frac{1}{2}P\left(\frac{2z-1}{4}\right)\\
&- \frac{6i}{\pi}(4z-2-14z+7+14z-7-4z+2)\frac{1}{(2z-1)^2}\\
&= P(z) - 7P(2z) + \frac{7}{2}P\left(\frac{2z-1}{2}\right) - \frac{1}{2}P\left(\frac{2z-1}{4}\right).
\end{aligned}$$

We continue (29) using (30)

$$\int_{z_0}^{S_4 z_0} \omega(t)dt = \int_{w_2}^{w_2+2}(1-7+\frac{7}{2}-1/2)dx = -6$$

and Lemma 2 is proved. ■

We will introduce now the one-parameter family of one-dimensional unitary representations $\chi^{(\alpha)}$ of the group $\Gamma_0(8)$. For that reason we define

$$\tilde{\omega} = \frac{\omega}{6}. \tag{31}$$

Then by definition

$$\chi^{(\alpha)}(\gamma) = \exp\left\{2\pi i\alpha \int_{z_0}^{\gamma z_0} \tilde{\omega}(t)dt\right\},\ \gamma \in \Gamma, \alpha \in [0,1]\,. \tag{32}$$

It is not difficult to see that $\chi^{(\alpha)}$ is a group of characters,

$$\chi^{(1/2)} = \chi_8 \tag{33}$$

(see (4)), and $\chi^{(0)}$ is the trivial character.

Now we will describe the perturbation problem (see [B-V2]). We consider a family of selfadjoint operators $A(\Gamma;\chi^{(\alpha)})$ defined by the Laplacian of H acting on functions $f(z)$ satisfying

$$f(\gamma z) = \chi^{(\alpha)}(\gamma)f(z),\ \gamma \in \Gamma,\ z \in H.$$

The domain of definition of $A(\Gamma;\chi^{(\alpha)})$ is a dense subspace of $L_2(F)$ (F is a fundamental domain of Γ), varying with α. Using an idea of Phillips and Sarnak [P-Sa2] we can transform these operators to operators acting on functions which are purely Γ-automorphic. But instead of this we take as our basic operator the operator with the congruence character χ_8, $A(\Gamma;\chi^{(1/2)})$, and bring other operators to its domain of definition. That makes the perturbation problem regular.

We define for a function f with the property $f(\gamma z) = \chi^{(1/2)}(\gamma)(z)$ another function $g(z)$ by

$$g(z) = f(z) \cdot \exp 2\pi i\alpha\, \mathrm{Re} \int_{z_0}^{z} \tilde{\omega}(t)dt = f(z)\Omega(z,\alpha) \tag{34}$$

Then we have

$$g(\gamma z) = \chi^{(1/2+\alpha)}(\gamma)g(z),\ \gamma \in \Gamma \tag{35}$$

with $\chi^{(\alpha)}$ given by (32). Applying the negative Laplacian

$$-\Delta = -4y^2 \frac{\partial^2}{\partial z \partial \bar{z}}$$

to the function $g(z)$ we obtain that the operator $A(\Gamma; \chi^{(\alpha+1/2)})$ is unitarily equivalent to the operator on $\left(\Gamma, \chi^{\left(\frac{1}{2}\right)}\right)$-automorphic functions defined by

$$L(\alpha + 1/2) = L + \alpha M + \alpha^2 N \tag{36}$$

where $L = A\left(\Gamma; \chi^{\left(\frac{1}{2}\right)}\right)$

$$\left.\begin{array}{l} M = -4\pi i y^2 \left(\tilde{\omega}_1 \frac{\partial}{\partial x} - \tilde{\omega}_2 \frac{\partial}{\partial y}\right) = -4\pi i y^2 \left(\tilde{\omega} \frac{\partial}{\partial \bar{z}} + \tilde{\omega} \frac{\partial}{\partial z}\right) \\ N = 4\pi^2 y^2 \left|\tilde{\omega}(z)\right|^2 = 4\pi^2 y^2 \left(\tilde{\omega}_1^2 + \tilde{\omega}_2^2\right) \end{array}\right\} \tag{37}$$

and $\tilde{\omega} = \tilde{\omega}_1 + i\tilde{\omega}_2$, $\bar{\omega} = \tilde{\omega}_1 - i\tilde{\omega}_2$. The domain of definition $D(L(\alpha + 1/2))$ equals $\Omega(z, \alpha)^{-1} D(A(\Gamma; \chi^{(\alpha+1/2)}))$ and

$$L(\alpha + 1/2) = \Omega(\cdot, \alpha)^{-1} A(\Gamma; \chi^{(\alpha+1/2)}) \Omega(\cdot, \alpha) \tag{38}$$

Note that M maps odd functions to even and even to odd. Recall that functions satisfying $f(-x+iy) = -f(x+iy)$ are odd and functions satisfying $f(-x+iy) = f(x+iy)$ are even by definition. Note also that a function f satisfying $f(\gamma z) = \chi^{(1/2)}(\gamma) f(z)$ is allowed to be odd or even. It is also true for the trivial character.

It is not difficult to see that the differential operators M, N map $(\Gamma, \chi^{(\alpha)})$-automorphic functions to $(\Gamma, \chi^{(\alpha)})$-automorphic functions for any $\alpha \in [0, 1]$. We shall use this for $\chi^{\left(\frac{1}{2}\right)}$.

2 Scattering matrix, eigenvalues and Hecke theory

The next step consists in studying embedded eigenvalues for the operator $A(\Gamma; \chi^{\left(\frac{1}{2}\right)})$. We will prove now that there are infinitely many such eigenvalues. In fact, we will prove the Weyl law for the distribution function $N(\lambda; \Gamma; \chi^{\left(\frac{1}{2}\right)})$. In order to do this we will find the automorphic scattering matrix $\phi(s;\Gamma;\chi^{(\frac{1}{2})})$ [S]. From the general theory (see [V]) we know that in this case we have a continuous spectrum of $A(\Gamma;\chi^{(\frac{1}{2})})$ of multiplicity 2, because only 2 cusps are open, namely z_1, z_3. That means that the matrix ϕ is of order 2.

Lemma 3 *The scattering matrix $\phi(\Gamma;\chi^{(\frac{1}{2})})$ is given by*

$$\phi(s;\Gamma;\chi^{(\frac{1}{2})}) = \begin{pmatrix} 0 & \varphi_{13} \\ \varphi_{31} & 0 \end{pmatrix}, \tag{39}$$

where

$$\varphi_{13} = \varphi_{31} = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \cdot \frac{1}{8^s} \cdot \frac{L_8(2s-1)}{L_8(2s)}$$

and $L_8(s)$ is the Dirichlet series $\sum_{n=1}^{\infty} \frac{\chi_8(n)}{n^s}$ for $\operatorname{Re} s > 1$. The function $L_8(s)$ has an analytic continuation to the complex plane and satisfies the functional equation

$$\pi^{-\frac{1}{2}(1-s)} 8^{\frac{1}{2}(1-s)} \Gamma\left(\frac{1-s}{2}\right) L_8(1-s) = \pi^{-\frac{1}{2}s} 8^{\frac{s}{2}} \frac{\sqrt{8}}{\tau(\chi_8)} \Gamma\left(\frac{s}{2}\right) L_8(s) \tag{40}$$

where $\tau(\chi_8)$ is the Gauss sum, equal to $\sqrt{8}$. $A(\Gamma;\chi^{(\frac{1}{2})})$ has an infinite sequence of eigenvalues $\{\lambda_j\}$ satisfying a Weyl law,

$$\#\{\lambda_j \leqq \lambda\} \sim \frac{|F|}{4\pi} \cdot \lambda = \lambda \text{ for } \lambda \to \infty. \tag{41}$$

Proof. We have two Eisenstein series related to the cusps z_1, z_3,

$$E_1(z;s;\Gamma;\chi^{(\frac{1}{2})}) = \sum_{\gamma \in \Gamma_1 \backslash \Gamma} y^s(g_1^{-1}\gamma z)\chi^{(\frac{1}{2})}(\gamma), \ \operatorname{Re} s > 1 \tag{42}$$

$$E_3(z;s;\Gamma;\chi^{(\frac{1}{2})}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s(\gamma z)\chi^{(\frac{1}{2})}(\gamma), \quad \operatorname{Re} s > 1$$

where $y(z) = \operatorname{Im} z$, $\Gamma_\infty \subset \Gamma$ is the cyclic subgroup of Γ generated by $S_3, \Gamma_1 \subset \Gamma$ is the cyclic subgroup of Γ generated by S_1. Then

$$g_1^{-1} S_1 g_1 = S_3, \ g_1 = \begin{pmatrix} 0 & \frac{1}{2\sqrt{2}} \\ -2\sqrt{2} & 0 \end{pmatrix}.$$

These series are absolutely convergent for $\mathrm{Re}\, s > 1$. We have a Fourier decomposition for both E_1 and E_3, which we can write

$$E_\alpha(g_\beta z; s; \Gamma; \chi^{(\frac{1}{2})}) = \delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s; \Gamma; \chi^{(\frac{1}{2})}) y^{1-s} + \sum_{n\neq 0} a_n(s; y; \Gamma; \chi^{(\frac{1}{2})}) e^{2\pi i n x}$$

with certain coefficients a_n. We take $\alpha = 1, 3;\ \beta = 1, 3;\ g_3 = 1$.

From general theory (see [Se1]) we find the following expressions for the matrix elements $\varphi_{11}, \varphi_{13}, \varphi_{31}, \varphi_{33}$.

$$\varphi_{11}(s; \Gamma; \chi^{(\frac{1}{2})}) = \varphi_{33}(s; \Gamma; \chi^{(\frac{1}{2})})$$

$$= \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{(8n)^{2s}} \sum_{d (\mathrm{mod}\ 8n)_{(d,8n)=1}} \chi_8(d)$$

$$\varphi_{13}(s; \Gamma; \chi^{(1/2)}) = \varphi_{31}(s; \Gamma; \chi^{(1/2)})$$

$$= \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{\chi_8(n)}{(2\sqrt{2n})^{2s}} \sum_{d (\mathrm{mod}\ n)_{(d,n)=1}} 1.$$

Then it is not difficult to see that $\varphi_{11} = \varphi_{33} = 0$ and

$$\varphi_{13} = \varphi_{31} = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \frac{L_8(2s - 1)}{L_8(2s)} \cdot \frac{1}{8^s} \tag{43}$$

where $L_8(s)$ is the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\chi_8(n)}{n^s},\ \mathrm{Re}\, s > 1.$$

The Dirichlet series $L_8(s)$ has analytic continuation to the complex plane and satisfies the functional equation (40). It follows that the function

$$\xi_8(s) = \left(\frac{8}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L_8(s), \qquad \xi_8(1 - s) = \xi_8(s),$$

is an entire function of order 1. Then from general theory (see [V]) it follows that there exists an infinite sequence of embedded eigenvalues of $A(\Gamma; \chi^{(\frac{1}{2})})$ and their distribution function $N(\lambda; \Gamma; \chi^{(\frac{1}{2})}) = \#\{\lambda_i \leqq \lambda\}$ follows the Weyl law (41).

This concludes the proof of Lemma 3. $\blacksquare$

Next we recall the Hecke theory in application to the group $\Gamma_0(8)$ and its character $\chi^{(1/2)}$ (for a survey for the general theory see [I]).

Let f be a continuous $(\Gamma;\chi^{(1/2)})$-automorphic function, i.e.

$$f(\gamma z)=\chi^{(1/2)}(\gamma)f(z),\ \forall\gamma\in\Gamma,\ z\in H$$

and let $n\in\mathbb{Z}_+$ be an odd number. Then the Hecke operator is defined by

$$T_nf(z)=\frac{1}{\sqrt{n}}\sum_{ad=n}\chi_8(d)\sum_{b(\mathrm{mod}\ d)}f\left(\frac{az+b}{d}\right)$$

and acts in the subspace of cusp forms $\mathcal{H}_0(\Gamma;\chi^{(1/2)})\subset\mathcal{H}(\Gamma;\chi^{(1/2)})=L_2(F)$. It is bounded and $\chi_8(n)$-hermitian, i.e.

$$\langle T_nf,g\rangle=\chi_8(n)\,\langle f,T_ng\rangle\,,$$

m,n odd. For m and n odd we have

$$T_m\cdot T_n=\sum_{d|(m,n)}\chi_8(d)T_{mn/d^2}. \tag{44}$$

All T_m commute with each other and commute with the automorphic Laplacian $A(\Gamma;\chi^{(1/2)})$. So we can take in the space $\mathcal{H}_0(\Gamma;\chi^{(1/2)})$ a common basis of eigenfunctions for all T_m and $A(\Gamma;\chi^{(1/2)})$.

$$\left\{v_j(z)=v_j(z;\Gamma;\chi^{(1/2)})\right\}_{j=0}^{\infty} \tag{45}$$

$A(\Gamma;\chi^{(1/2)})v_j=\lambda_jv_j,\ j=0,1,...\ T_mv_j=\rho_j(m)v_j,\ m=1,3,...\ .$

There is a delicate question about the normalization of these eigenfunctions. Clearly for the L_2 theory of $A(\Gamma;\chi)$ the normalization

$$\langle v_j,v_k\rangle=\delta_{jk}$$

is important, where $\langle , \rangle$ is the inner product in $L_2(F)$

$$\langle v_j, v_k \rangle = \int_F v_j(z)\overline{v_k(z)}\frac{dxdy}{y^2},$$

For Hecke theory there is another normalization which is more natural and which is related to Fourier decomposition of the functions v_j. We explain this after discussing old and new forms for $\Gamma_0(N)$. We recall briefly the definition of old and new forms for $\Gamma_0(N)$ and χ generated by a Dirichlet character mod N.

If χ is mod M and $v(z) \in \mathcal{H}_0(\Gamma_0(M); \chi)$ then $v(dz) \in \mathcal{H}_0(\Gamma_0(N);\chi)$ whenever $dM \mid N$. By definition $\mathcal{H}_0^{old}(\Gamma_0(N); \chi)$ is the subspace of $\mathcal{H}_0(\Gamma_0(N);\chi)$ spanned by forms $v(dz)$, where $v(z)$ is defined for $\Gamma_0(M)$ with character χ (mod M), $M < N$, $dM \mid N$ and v is a common eigenfunction for all Hecke operators T_m with $(m, M) = 1$. Let the space $\mathcal{H}_0^{new}$ be the orthogonal complement,

$$\mathcal{H}_0(\Gamma_0(N); \chi) = \mathcal{H}_0^{old}(\Gamma_0(N); \chi) \oplus \mathcal{H}_0^{new}(\Gamma_0(N); \chi).$$

From this definition it is clear that there is no old form for the pair $\Gamma_0(8)$ and $\chi^{(1/2)}$, because χ_8 is a primitive character mod 8 [D]. For any function $v_j(z)$ from (45) we have a Fourier decomposition

$$v_j(z; \Gamma; \chi^{(1/2)}) = \sqrt{y} \sum_{n \neq 0} \tilde{\rho}_j(n) K_{s_j - \frac{1}{2}}(2\pi \left| n \right| y) e^{2\pi i n x} \tag{46}$$

where $K_s(z)$ is the modified Bessel function. For non-trivial new forms we know that $\tilde{\rho}_j(1) \neq 0$. So the second normalization of functions (45) is by the condition

$$\tilde{\rho}_j(1) = 1. \tag{47}$$

After this normalization we obtain $\rho_j(m) = \tilde{\rho}_j(m)$ for all j, m.

There is another Hecke operator U_2 which is important in this connection, defined by

$$U_2 f(z) = \frac{1}{\sqrt{2}} \left(f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right)\right).$$

In our situation it is not difficult to prove that U_2 is a unitary operator in $\mathcal{H}$ which commutes with all regular Hecke operators and with the automorphic Laplacian. Because of the multiplicity one thorem the basis (45) is automatically the basis of eigenfunction for U_2 and for all $T(n), n \in \mathbb{Z}_+$. We have $T(n)v_j = \rho_j(n)v_j$. The remarkable property of U_2 is

$$U_2 v_j = \rho_j(2) v_j, \rho_j(2) = \pm 1. \tag{48}$$

For the Fourier coefficients $\rho_j(m), \rho_j(n)$ we have the relation of multiplicativity.

$$\rho_j(m)\rho_j(n) = \sum_{d|(m,n)} \chi_8(d)\rho_j(mn/d^2). \tag{49}$$

We proceed to the construction of a Dirichlet series corresponding to (46), (47). Because of the specific properties of the perturbation (37) we have to consider odd eigenfunctions (46), i.e.

$$\rho_j(n) = -\rho_j(-n). \tag{50}$$

For the Hecke theory of Euler products, we will consider next, it is not an important restriction. The difference between odd and even functions becomes visible, when we will derive the functional equation for the corresponding Dirichlet series.

3 Non-vanishing of Hecke L-functions

For each function $v_j(z)$ from (46) with (47), we define the Dirichlet series

$$L(s; v_j) = \sum_{n=1}^{\infty} \frac{\rho_j(n)}{n^s}. \tag{51}$$

From studying the Rankin-Selberg convolution we can see that the series (51) is absolutely convergent for $\operatorname{Re} s > 1$.

From (44), (49) follows

Theorem 1 *Let $L(s, v_j)$ be the series (51) and the function $v_j(z)$ be as in (45). Then for* $\operatorname{Re} s > 1$ *we have an Euler product representation for $L(s; v_j)$*

$$L(s; v_j) = \prod_p (1 - \rho_j(p)p^{-s} + \chi(p)p^{-2s})^{-1}.$$

The product is taken over all primes.

We can also write it in the form

$$L(s; v_j) = (1 - \rho_j(2)2^{-s})^{-1} \prod_{p \neq 2} (1 - \rho_j(p)p^{-s} + \chi(p)p^{-2s})^{-1}$$

since $\chi(2) = 0$. From Theorem (48) we know $\rho_j(2) = \pm 1, j = 1, 2, 3, \ldots.$

We derive now the functional equation for the pair of Dirichlet series

$$\begin{cases} L(s; v_j) = \sum_{n=1}^{\infty} \frac{\rho_j(n)}{n^s} \\ L(s; \hat{v}_j) = \sum_{n=1}^{\infty} \frac{\overline{\rho_j(n)}}{n^s}, \operatorname{Re} s > 1. \end{cases}$$

We only consider the case of odd eigenfunctions since that is important for this paper.

We have together with (46) by definition

$$\hat{v}_j(z) = \sum_{n \neq 1}^{\infty} \overline{\rho_j(n)} \sqrt{y} K_{s_j - 1/2}(2\pi |n| y) e^{2\pi i n x}.$$

We define two involutions $Kf(z) = \overline{f(z)}, Hf(z) = f(-1/8z)$.

If $s_j - 1/2 \in i\mathbb{R}$ or $s_j \in (\frac{1}{2}, 1)$ then $K_{s_j - \frac{1}{2}}(2\pi |n| y)$ is a real-valued function and for odd v_j we have $\overline{v}_j(z) = -\hat{v}_j(z)$. We will write $v_j(z) = v_j(x, y)$, where $z = x + iy$. We have $v_j(-x, y) = -.v_j(x, y)$. The action of the involutions H, K can be written as follows

$$\begin{cases} \overline{v_j(u, v)} = \pm v_j(x, y) \\ u = -\frac{x}{8(x^2 + y^2)}, v = \frac{y}{8(x^2 + y^2)}. \end{cases}$$

We apply the partial derivative $\frac{\partial}{\partial x}$ and obtain

$$-\frac{1}{8y^2}\frac{\partial\overline{v_j}}{\partial u}\mid_{x=0}=\pm\frac{\partial v_j}{\partial x}\mid_{x=0}.$$

This is equivalent to

$$\pm B(y)=\pm 8^{3/2}y^3\sum_{n=1}^{\infty}\overline{\rho_j(n)}nK_{s_j-\frac{1}{2}}(2\pi ny)=\sum_{n=1}^{\infty}\rho_j(n)nK_{s_j-1/2}(2\pi n/8y). \tag{52}$$

We multiply the left hand side of (52) by $4\pi 8^{s/2-3/2}y^{s-3}$ and integrate it from 0 to ∞ in y. We obtain

$$\begin{aligned}&\int_0^\infty 4\pi^{s/2-3/2}y^{s-3}B(y)dy\\&=\pi^{-s}8^{s/2}\Gamma\left(\frac{s+s_j}{2}+1/4\right)\Gamma\left(\frac{s-s_j}{2}+3/4\right)\cdot L(s;\hat{v}_j)\\&=\Omega(s;\hat{v}_j).\end{aligned}$$

That is because

$$\int_0^\infty y^sK_{s_j-1/2}(y)dy=2^{s-1}\Gamma\left(\frac{s+s_j}{2}+1/4\right)\Gamma\left(\frac{s-s_j}{2}+3/4\right).$$

We can now write the integral obtained as a sum of two integrals

$$4\pi^{s/2-3/2}\int_0^\infty B(y)y^{s-3}dy=4\pi 8^{s/2-3/2}\left(\int_0^{1/\sqrt{8}}+\int_{1/\sqrt{8}}^\infty\right). \tag{53}$$

In the first integral we use (52) for $B(y)$ and then map $y\to 1/8y$.

Then we obtain that (53) is equal to

$$4\pi\left(8^{3/2}\int_{1\sqrt{8}}^{\infty}y^s\sum_{n=1}^{\infty}\overline{\rho_j(n)}nK_{s_j-1/2}(2\pi ny)dy\right.$$
$$\left.\pm N^{\frac{1-s}{2}}\int_{1/\sqrt{8}}^{\infty}y^{1-s}\sum_{n=1}^{\infty}\rho_j(n)nK_{s_j-1/2}(2\pi ny)dy\right)=C(s;\hat{v}_j)\pm C(1-s;v_j).$$

It is clear that $C(s;v_j)$, $C(s;\hat{v}_j)$ are the entire functions of s. Then we have

$$\Omega(s;\hat{v}_j)=C(s;\hat{v}_j)\pm C(1-s;v_j).$$

The analogous calculation shows

$$\Omega(s;v_j)=C(s;v_j)\pm C(1-s;\hat{v}_j)$$

$$\Omega(1-s;v_j)=C(1-s;v_j)\pm C(s;\hat{v}_j)$$

and we finally obtain

$$\pm\Omega(1-s;v_j)=\Omega(s;\hat{v}_j).$$

We shall prove that the functions $L(s;v_j)$ and $L(s,\hat{v}_j)$ are regular and non-vanishing on the boundary of the critical strip.

We start with the Rankin-Selberg convolution. For each eigenfunction $v_j(z)$ from (46) we define the series

$$\sum_{n=1}^{\infty}\frac{|\rho_j(n)|^2}{n^s} \tag{54}$$

which is absolutely convergent for $\operatorname{Re}s>1$.

For $\operatorname{Re}s>1$ we consider the following Selberg integral

$$\int_{F_0(N)}|v_j(z)|^2E_\infty(z;s;\Gamma_0(8);1)d\mu(z)=A(s)$$

where

$$E_\infty(z;s) = E_\infty(z;s;\Gamma_0(8);1) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(8)} y^s(\gamma z)$$

Using the unfolding of the Eisenstein series we obtain

$$A(s) = \int_0^\infty y^{s-1}\sum_{n\neq 0}|\rho_j(n)|^2 K_{ir_j}^2(2\pi|n|y)dy =$$
$$= \frac{\Gamma^2(s/2)\Gamma(\frac{s}{2}+ir_j)\Gamma(s/2-ir_j)}{4\pi^s\Gamma(s)}\sum_{n=1}^\infty \frac{|\rho_j(n)|^2}{n^s}.$$

It is well known that $E(z,s;\Gamma_0(8);1)$ has analytic continuation to the whole s-plane, and at $\mathrm{Re}\, s > 1/2$ it has only a simple pole at $s = 1$ with residue equal to $\mu(F_0(8))^1$ (inerse μ-area of the fundamental domain of $\Gamma_0(8)$). From that follows that the Rankin-Selberg convolution (54) is a regular function in $\mathrm{Re}\, s > 1/2$ except for a simple pole at $s = 1$.

We want to see now the Euler product for the Rankin-Selberg convolution (54). The method is due to Rankin (see [R]). The main difference from Rankin's case is that our coefficients ρ_j may be complex numbers, and that we hve also the exceptional prime $p = 2$.

First consider the main case $(n, 2) = 1$. We have

$$\rho_j(n) = \chi(n)\bar{\rho}_j(n), j = 1, 2, ...$$

and for $\chi(n) = -1$, $\rho_j(n)$ is purely imaginary (it can not be zero). In both the cases $\chi(n) = \pm 1$ we have

$$|\rho_j(n)|^2 = \chi(n)\rho_j^2(n). \tag{55}$$

From (49) follows

$$\left\{ \begin{array}{l} \rho_j^2(p^n) = (\rho_j(p)\rho_j(p^{n-1} - \chi(p)\rho_j(p^{n-2}))^2 \\ (\chi(p)\rho_j(p^{n-3}))^2 = (-\rho_j(p^{n-1} + \rho_j(p)\rho_j(p^{n-2}))^2 \end{array} \right. .$$

Then multiplying the second line by $\chi(p)$ and taking the difference, we obtain

$$\rho_j^2(p^n) - \rho_j^2(p)\rho_j^2(p^{n-1}) + \chi(p)\rho_j^2(p^{n-1}) + \chi(p)\rho_j^2(p)\rho_j^2(p^{n-2}) - \rho_j^2(p^{n-2}) - \chi(p)\rho_j^2(p^{n-3}) = 0.$$

Multiplying this by $\chi(p^n)$ and using (55) we obtain

$$|\rho_j(p^n)|^2 - |\rho_j(p^{n-1})|^2|\rho_j(p)|^2 + |\rho_j(p^{n-1})|^2 + |\rho_j(p)|^2\rho_j(p^{n-2})|^2 - |\rho_j(p^{n-2})|^2 - |\rho_j(p^{n-3})|^2 = 0.$$

In the case $p = 2$ we have

$$|\rho_j(2^n)| = 1, n = 1, 2, \dots.$$

Then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^{2s}} &= \prod_{p \neq 2} \left(1 + \frac{|\rho_j(p)|^2}{p^{2s}} + \frac{|\rho_j(p^2)|^2}{p^{4s}} + \frac{|\rho_j(p^3)|^2}{p^{6s}} + \dots\right) \cdot \left(1 + \frac{1}{2^{2s}} + \frac{1}{2^{4s}} + \dots\right) \\
&= (1 - 2^{-2s})^{-1} \cdot \prod_{p \neq 2} \frac{1 + p^{-2s}}{1 - |\rho_j(p)|^2 p^{-2s} + p^{-2s} + |\rho_j(p)|^2 p^{-4s} - p^{-4s} - p^{-6s}} \\
&= (1 - 2^{-2s})^{-1} \cdot \prod_{p \neq 2} (1 + p^{-2s})(1 - p^{-2s})^{-1}(1 + (2 - |\rho_j(p)|^2)p^{-2s} + p^{-4s})^{-1} \\
&= (1 - 2^{-2s})^{-1} \cdot \prod_{p \neq 2} (1 - p^{-4s})(1 - p^{-2s})^{-2}(1 + (2 - |\rho_j(p)|^2)p^{-2s} + p^{-4s})^{-1} \\
&= \zeta(2s)L(2s; \hat{\chi})L^{-1}(4s; \hat{\chi}) \prod_{p \neq 2} (1 + (2 - |\rho_j(p)|^2)p^{-2s} + p^{-4s})^{-1}, \qquad (56)
\end{aligned}$$

where $L(s; \hat{\chi}$ is the Dirichlet L-series with principal character mod 8

$$L(s;\hat{\chi}) = \prod_p (1-\hat{\chi}(p)p^{-s})^{-1} = \zeta(s)(1-2^{-s}).$$

The products in (56) are taken over all primes $p \neq 2$. For $p \neq 2$ we now introduce new functions $\alpha_j(p)$, $\beta_j(p)$, which are important to define symmetric power L-series, by

$$\begin{cases} \alpha_j(p)+\beta_j(p) = \rho_j(p) \\ (\alpha_j(p)\beta_j(p) = \chi(p) \end{cases}. \tag{57}$$

We have $(\alpha_j(p)+\beta_j(p))^2 = \rho_j^2(p) = \alpha_j^2(p) + 2\chi(p) + \beta_j^2(p)$, and

$$\begin{cases} \chi(p)\alpha_j^2(p) + \chi(p)\beta_j^2(p) = |\rho_j(p)|^2 - 2 \\ (\alpha_j^2(p)\beta_j^2(p) = 1 \end{cases}. \tag{58}$$

Applying (58) to (56) we obtain by definition

$$\begin{aligned} &\prod_{p\neq 2}(1+(2-|\rho_j(p)|^2)p^{-2s}+p^{-4s})^{-1} \\ &= \prod_{p\neq 2}\left(1-\frac{\chi(p)\alpha_j^2(p)}{p^{2s}}\right)^{-1}\left(1-\frac{\chi(p)\beta_j^2(p)}{p^{2s}}\right) = L_2(2s;v_j). \end{aligned}$$

Combining with (56) we finally obtain

$$L(s;v_j\times\overline{v}_j) = \frac{L(s;\hat{\chi})}{L(2s;\hat{\chi})}L_2(s;v_j)\zeta(s)$$

where $L(s;v_j\times\overline{v}_j)$ is the Rankin-Selberg convolution (54).

We can also write for $\operatorname{Re} s > 1$

$$\begin{cases} L(s;v_j) = (1\pm 2^{-s})^{-1}\prod\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_j(p)}{p^s}\right)^{-1} \\ L(s;\hat{v}_j) = (1\pm 2^{-s})^{-1}\prod_{p\neq 2}\left(1-\frac{\overline{\alpha_j(p)}}{p^s}\right)^{-1}\left(1-\frac{\overline{\beta_j(p)}}{p^s}\right)^{-1}. \end{cases}. \tag{59}$$

For the proof of the next theorem we will make use of the following general criterion proved in [M-M] (Theorem 1.2).

Theorem 2 *Let $f(s)$ be a function satisfying*

1. *f is holomorphic and $f(s) \neq 0$ in $\{s | \operatorname{Re} s = \sigma > 1\}$*

2. *f is holomorphic on the line $\sigma = 1$ except for a pole of order $e \geq 1$ at $s = 1$*

3. *$\log f(s)$ can be written as a Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

   with $b_n \geq 0$ for $\sigma > 1$.

Then if f has a zero on the line $\sigma = 1$, the order of the zero is bounded by $e/2$.

We now want to prove the following

Theorem 3 *$L(s; v_j)$ and $L(s, \hat{v}_j)$ are regular for $s = 1 + it$, $s = it$, $t \in \mathbb{R}$, and*

$$L(1 + it; v_j) \neq 0, L(it; v_j) \neq 0, L(1 + it, \hat{v}_j) \neq 0, L(it; \hat{v}_j) \neq 0, j = 1, 2, \ldots. \tag{60}$$

Proof. Clearly, (60) is analogous to the prime number theorem, $\zeta(1 + it) \neq 0$, for the Riemann zeta function. This kind of property for different zeta-functions is very important in number theory (see, for example, [M-M]).

From the functional equation follows that it is enough to prove the inequalities

$$L(1 + it; v_j) \neq 0, L(1 + it; \hat{v}_j) \neq 0$$

because we know all singular points of the Euler Γ-function.

Consider the following product

$$f(s) = L(s; v_j \times \hat{v}_j) L(2s; \hat{\chi}) L(s, \hat{v}_j)(1 - 2^{-s}(1 - \rho_j(2)2^{-s})(1 - \overline{\rho}_j(2)2^{-s}).$$

Let $\mathrm{Re}\, s > 1$, then from (59) follows

$$\log f(s) = -\sum_{p\neq 2}\{2\log(1-p^{-s}) + \log(1-\chi(p)\alpha_j^2(p)p^{-s}) + \log(1-\chi(p)\beta_j^2(p)p^{-s}) \quad (61)$$
$$+\log(1-\alpha_j(p)p^{-s}) + \log(1-\beta_j(p)p^{-s}) + \log(1-\overline{\alpha}_j(p)p^{-s})$$
$$+\log(1-\overline{\beta}_j(p)\cdot p^{-s})\}.$$

For $|x| < 1$ we have $\log(1-x) = -\sum_{n=1}^{\infty} x^n/n$. Using this we continue (61)

$$\log f(s) = \sum_{p\neq 2}\sum_{n=1}^{\infty}\frac{1}{np^{ns}}(2+\chi(p)^n\alpha_j^{2n}(p) + \chi(p)^n\beta_j^{2n}(p) + \alpha_j^n(p)$$
$$+\beta_j^n(p) + \bar{\alpha}_j^n(p) + \bar{\beta}_j^n(p))$$
$$= \sum_{p\neq 2}\sum_{n=1}^{\infty}\frac{a_{n,p}}{np^{ns}}.$$

We will show now $a_{n,p} \geq 0$.

We consider two cases: $\chi(p) = 1, \chi(p) = -1$. In the first case

$$a_{n,p} = 2 + 2\alpha_j^n(p) + 2\beta_j^n(p) + \alpha_j^{2n}(p) + \beta_j^{2n}(p) = (1+\alpha_j^n(p))^2 + (1+\beta_j^n(p)) \geq 0$$

because in that case $\alpha_j(p), \beta_j(p)$ are real numbers. In the second case we have that $\alpha_j(p) = i\tilde{\alpha}_j(p) = i\tilde{\beta}_j(p)$, and $\tilde{\alpha}_j(p), \tilde{\beta}_j(p)$ are real numbers. We have

$$a_{n,p} = 2 + \tilde{\alpha}_j^{2n} + \tilde{\beta}_j^{2n} + \tilde{\alpha}_j^n(i)^n((-1)^n+1) + \tilde{\beta}_j^n(i)^n((-1)^n+1) \quad (62)$$

and this is real and ≥ 0 if $n = 2m-1, m = 1, 2, \ldots$. We consider $n = 2m, m = 1, 2, \ldots$

$$a_{n,p} = 2+\tilde{\alpha}_j^{4m}+\tilde{\beta}_j^{4m}+(-1)^m\cdot 2\tilde{\alpha}_j^{2m}+(-1)^m\cdot\tilde{\beta}_j^{2m} = (1+(-1)^m\tilde{\alpha}_j^{2m})^2+(1+(-1)^m\tilde{\beta}_j^{2m})^2 \geq 0$$

and we have proved that $a_{n,p} \geq 0$ for all $p \neq N, n = 1, 2, \ldots$.

Let us assume first that $L(s; v_j) = 0$ at $s = 1$. That means also $L(s; \hat{v}_j) = 0$ at $s = 1$. Since $L(s; v_j \times \hat{v}_j)$ has only a simple pole at $s = 1$, we see tht the function $f(s)$ has a zero at $s = 1$. On the other hand, singe $\log f(s)$ has the property 3, $\log f(s) > 0$ for $s > 1$, so $f(s) > 1$ for $s > 1$, a contradiction. So we have $L(1; v_j) \neq 0$, $L(1, \hat{v}_j) \neq 0$.

Suppose now that $L(1 + it; v_j) = 0$ for some $t \neq 0$. Then $f(s)$ has a zero of order ≥ 0 on the line $\mathrm{Re}\, s > 1$, $s \neq 1$, since $L(s; \hat{v}_j)$ and $L(s; v_j \times \hat{v}_j)$ are regular at $s = 1 + it$. Also, $f(s)$ has a pole of order 1 at $s = 1$, since $L(s; v_j \times \hat{v}_j)$ has a simple pole at $s = 1$ and $L(1; v_j) \neq 0$, $L(1; \hat{v}_j) \neq 0$. This is in contradiction with Theorem 2, and Theorem 3 is proved. ■

4 The Phillips-Sarnak integral

In this section we study the Phillips-Sarnak integral, adapted to our perturbation (36). For any odd eigenfunction (45), which corresponds to an embedded eigenvalue $\lambda_j > \frac{1}{4}$ (actually, according to the Selberg eigenvalue conjecture, extended to the case of congruence character, all $\lambda_j \geq \frac{1}{4}$) we define the integral over the fundamental domain F of $\Gamma_0(8)$

$$I_j(s) = \int_F (Mv_j)(z) E_3(z, s) d\mu(z) \tag{63}$$

where

$$E_3(z, s) = E_3(z, s; \Gamma; \chi^{(\frac{1}{2})})$$

is the Eisenstein series from (42).

Theorem 4 *$I_j(s)$ is well defined and $I_j(s) \neq 0$ for $s = \frac{1}{2} + it$, $t \in \mathbb{R}$, except for $t = \frac{\pi i n}{\log 2}, n \in \mathbb{Z}$.*

Proof. We take first $\mathrm{Re}\, s > 1$. It is not difficult to see that $v_j(z)$ is a function of exponential decay in all cusps z_1, z_2, z_3, z_4 of F. This follows from the fact that $v_j(z)$ is an eigenfunction of the Laplacian, which is $(\Gamma, \chi^{\left(\frac{1}{2}\right)})$-automorphic.

In open cusps z_1, z_3, v_j is a cusp form and has Fourier decompositions (46) for z_3 and similarly for z_1

$$\sqrt{y_1}\sum_{n\neq 0}\rho_j^{(1)}(n)K_{s_j-\frac{1}{2}}(2\pi\left|n\right|y_1)e^{2\pi i n x_1}$$

where x_1 and y_1 are local coordinates related to z_1.

In the closed cusps it has Fourier decompositions

$$\sqrt{y_2}\sum_{n=-\infty}^{\infty}\rho_j^{(2)}(n)K_{s_j-\frac{1}{2}}(2\pi\left|n+\frac{1}{2}\right|y_2)e^{2\pi i(n+\frac{1}{2})x_2} \text{ in } z_2$$

$$\sqrt{y_4}\sum_{n=-\infty}^{\infty}\rho_j^{(4)}(n)K_{s_j-\frac{1}{2}}(2\pi\left|n+\frac{1}{2}\right|y_4)e^{2\pi i(n+\frac{1}{2})x_4} \text{ in } z_4$$

in corresponding parabolic coordinates.

Then it is obvious that

$$v_j(\gamma z) = \chi^{\left(\frac{1}{2}\right)}(\gamma)v_j(z), (Mv_j)(\gamma z) = \chi^{\left(\frac{1}{2}\right)}(\gamma)(Mv_j)(z)$$

$$E_3(\gamma z, s) = \chi^{\left(\frac{1}{2}\right)}(\gamma)E_3(z, s).$$

We have also

$$(\chi^{\left(\frac{1}{2}\right)}(\gamma))^2 = 1 \text{ for } \gamma \in \Gamma.$$

This means that the integral (62) is well defined and we can unfold the Eisenstein series $E_3(z, s)$, obtaining

$$I(s) = \int_0^{\infty}\frac{dy}{y^2}\int_{-1/2}^{1/2}dx(Mv_j(z))y^s \tag{64}$$

where

$$y^{-2}Mv_j(z) = -\frac{2\pi i}{3}\left(\tilde{\omega}_1 v_{jx} - \tilde{\omega}_2 v_{jy}\right). \tag{65}$$

In (64) we remember the definitions (31), (37). Also we denote

$$v_{jx} = \frac{\partial v_j}{\partial x},\ v_{jy} = \frac{\partial v_j}{\partial y}.$$

Then we have

$$\int_{-1/2}^{1/2} \tilde{\omega}_1(x,y) v_{jx}(x,y)dx = \omega_1 v_j \Big|_{-1/2}^{1/2} - \int_{-1/2}^{1/2} \tilde{\omega}_{1x}(x,y)) v_j(x,y)dx \quad (66)$$
$$= - \int_{-1/2}^{1/2} \tilde{\omega}_{1x}(x,y)) v_j(x,y)dx$$

because ω and v_j are periodic in x with period 1. Similarly

$$\int_0^\infty y^s \tilde{\omega}_2 v_{jy} dy = y^s \tilde{\omega}_2 v_j \Big|_0^\infty - s \int_0^\infty y^{s-1} \tilde{\omega}_2 v_j dy - \int_0^\infty y^s \tilde{\omega}_{2y} v_j dy \quad (67)$$
$$= -s \int_0^\infty y^{s-1} \tilde{\omega}_2 v_j dy - \int_0^\infty y^s \tilde{\omega}_{2y} v_j dy.$$

Also we have

$$\left. \begin{array}{l} \tilde{\omega}_1(x,y) = \sum_{n=1}^\infty a_n e^{-2\pi ny} \cos 2\pi nx \\ \tilde{\omega}_2(x,y) = \sum_{n=1}^\infty a_n e^{-2\pi ny} \sin 2\pi nx \end{array} \right\}. \quad (68)$$

Using (63)-(68) we obtain

$$I_j(s) = \quad (69)$$
$$= \frac{2\pi i}{3} \int_0^\infty y^s dy \int_{-1/2}^{1/2} dx (\tilde{\omega}_{1x} - \tilde{\omega}_{2y}) v_j - \frac{2\pi i}{3} s \int_0^\infty y^{s-1} dy \int_{-1/2}^{1/2} \tilde{\omega}_2 v_j dx$$
$$= -\frac{2\pi i}{3} s \int_0^\infty y^{s-1} dy \int_{-1/2}^{1/2} \tilde{\omega}_2(x,y) v_j(x,y) dx.$$

Then we apply to (69) the Fourier decomposition (46) with (47), (50) and (68). We obtain

$$I(s) = \frac{2\pi}{3} s \int_0^\infty y^{s-1/2} \sum_{n=1}^\infty a_n \rho_j(n) e^{-2\pi ny} K_{s_j-1/2}(2\pi ny) dy \quad (70)$$
$$= \frac{2\pi}{3} s \cdot \frac{1}{(2\pi)^{s+1/2}} \left(\int_0^\infty t^{s-1/2} e^{-t} K_{s_j-1/2}(t) dt \right) \sum_{n=1}^\infty \frac{a_n \rho_j(n)}{n^{s+1/2}}.$$

The standard integral in brackets is equal to

$$\sqrt{\pi} \cdot 2^{-s-1/2} s \frac{\Gamma(s+s_j)\Gamma(s-s_j+1)}{\Gamma(s+1)}, \quad (71)$$

and we finally obtain

$$I(s) = \frac{\pi^{1-s}s}{3 \cdot 2^{2s}} \cdot \frac{\Gamma(s+s_j)\Gamma(s-s_j+1)}{\Gamma(s+1)} \sum_{n=1}^{\infty} \frac{a_n \rho_j(n)}{n^{s+1/2}}. \tag{72}$$

We will study the Dirichlet series in (72) in more detail. We take $b(n) = (-1/24)a_n$. We have

$$\sum_{n=1}^{\infty} \frac{b(n)\rho_j(n)}{n^{s+1/2}} = \sum_{\beta=0}^{3} \alpha_\beta \cdot \frac{\rho_j^\beta(2)}{2^\beta(s+1/2)} \sum_{n=0}^{\infty} \frac{\sigma(2^n)\rho_j^n(2)}{2^{n(s+1/2}} \prod_{p\neq 2} \frac{\sigma(p^n)\rho_j(p^n)}{p^{n(s+1/2)}} = \theta_1\theta_2\theta_3 \tag{73}$$

where the form $\omega(z)$ from Lemma 1 is written $\omega(z) = \sum_{d|8} \alpha_d P(dz)$, $d = 2^\beta$, $\alpha_d = \alpha_\beta$. We have

$$\theta_2 = \sum_{n=0}^{\infty} \frac{2^{n+1}-1}{2-1} \cdot \frac{\rho_j^n(2)}{2^{n(s+1/2)}} = (1-\rho_j(2)2^{-s+1/2})^{-1}(1-\rho_j(2)2^{-s-1/2})^{-1}$$

$$\begin{aligned}\theta_3 &= \prod_{p=neq2} \sum_{n=0}^{\infty} \frac{1}{p-1} \left(\frac{p\rho_j(p^n)}{p^{n(s-1/2)}} - \frac{\rho_j(p^n)}{p^{n(s+1/2)}} \right) \\ &= \prod_{p\neq 2} (1-\chi(p)p^{-2s})(1-\rho_j(p)p^{-(s-1/2)} + \chi(p)p^{-2s+1})^{-1} \cdot (1-\rho_j(p)^{-(s+1/2)} + \chi(p)p^{-2s-1})^{-1}.\end{aligned}$$

Then we have

$$\theta_2 \cdot \theta_3 = L^{-1}(2s;\chi)L(s+1/2;v_j)L(s-1/2;v_j) \tag{74}$$

From Theorem 2 follows that for $s = 1/2 + ir$, $r \in R$, (74) is not zero. The sum θ_1 can produce zeros if

$$s = 1/2 + \frac{\pi i n}{\log 2}, n \in \mathbb{Z} \tag{75}$$

and that proves Theorem 3.

■

5 Perturbation of embedded eigenvalues and Fermi's Golden Rule

We have proved in Theorem 3 that the Phillips-Sarnak integral $I_j(s) \neq 0$ for $s = \frac{1}{2} + it$, $t \in \mathbb{R}\backslash\{0\}$, provided the eigenfunction v_j of $A = A(\Gamma_0(8); \chi^{\left(\frac{1}{2}\right)})$ is odd. For even eigenfunctions $I_j(s) = 0$ because of symmetry. We shall now prove Fermi's Golden Rule, which says that in both cases $|I_j(s)|^2$ is a constant times $\operatorname{Im} I_{j2}$, where $\lambda_j(\alpha) = \lambda_j + \alpha^2\lambda_{j2} + o(\alpha^2)$.

For this purpose we study the resolvent $R(s, \alpha) = (A - s(1-s))^{-1}$ acting as a bounded operator between weighted Banach spaces $C_{\mu,\nu} = C_{\mu,\nu}(F)$ defined as follows. $C_{\mu,\nu}(F)$ is the space of continuous functions on F such that

$$|f(\gamma_i z)| \leq K(\operatorname{Im}\gamma_i z)^{\mu} \text{ for } i = 1, 3$$

$$|f(\gamma_i z)| \leq K(\operatorname{Im}\gamma_i z)^{\nu} \text{ for } i = 2, 4$$

with the norm $\|f\|_{\mu,\nu} = \max\{\max\limits_{i=1,3} \sup\limits_{\operatorname{Im}\gamma_i z \geq 1} |f(\gamma_i z)| (\operatorname{Im}\gamma_i z)^{-\mu}, \max\limits_{i=2,4} \sup\limits_{\operatorname{Im}\gamma_i z \geq 1} |f(\gamma_i z)| (\operatorname{Im}\gamma_i z)^{-\nu}\}$. We recall that γ_1,γ_3 correspond to the open cusps z_1, z_3 while γ_2 and γ_4 correspond to the closed cusps z_2, z_4, $\gamma_i = g_i^{-1}$, see 2.

The analysis of the resolvent using weighted Banach spaces goes back to D. Faddeev [F], who proved that the resolvent kernel $r(z, z'; s)$ has an analytic continuation to $\{s | \operatorname{Re} s > 0\}$ giving rise to an analytic continuation of the resolvent $R(s)$ as an operator in $B(C_{-1}, C_1)$. In the λ-plane the resolvent $R(\lambda)$ continues analytically from the upper (lower) half-plane across $\left(\frac{1}{4}, \infty\right)$ to the second sheet in the lower (upper) half-plane. This was extended to Laplacians with character by A. Venkov [V]. From the results of [F] and [V] we obtain

Lemma 4 *For any $\alpha \in \left(\frac{1}{2}, \frac{1}{2}\right)$ the resolvent $R(s, \alpha)$ of $L(\alpha) = L + \alpha M + \alpha^2 N$ has an analytic continuation $\tilde{R}(s, \alpha)$ to $\{s | 0 < \operatorname{Re} s < 2\}$ as an operator in $B(C_{-1,1}, C_{1,-1})$.*

One-dimensional eigenvalues λ_j continue analytically as functions $\lambda_j(\alpha)$. They are analyzed in 1) below. Multi-dimensional eigenvalues in general split up in Puiseux cycles. Those of order 1 again give rise to analytic functions $\lambda_j(\alpha)$. The cycles of order $p \geq 2$ give rise to branch points, but due to the special property that an embedded eigenvalue λ_j can only move to the second sheet,

$$\lambda(\alpha) = \lambda + \mu_1 \alpha + \cdots + \mu_{2np} \alpha^{2np} + o(\alpha^{2np})$$

where $\mu_{p1}, \cdots, \mu_{(2n-1)p}$ are real and $\operatorname{Im} \mu_{2np} < 0$.

For this theory we refer to Howland [H]. The analysis of [H] utilizes a symmetrized version of the resolvent equation based on a factorization of the perturbation. The operator $V(\alpha) = \alpha M + \alpha^2 N$ is not suited for factorization, but we can replace $\mathcal{H} = L_2(F; d\mu)$ by the Banach space $C_0(F)$ and develop the theory of [H] in an analogous way. For this it is important that $M\tilde{R}(\alpha)$ and $N\tilde{R}(\alpha)$ are bounded on $C_{-1,1}(F)$. This follows from the fact that M and N are small in the two open cusps, while the resolvent kernel is small in the closed cusps. We obtain

Lemma 5 *$V(\alpha)\tilde{R}(s) \in B(C_{-1,1})$ for $0 < \operatorname{Re} s < 2$*

$$\left\| V(\alpha)\tilde{R}(s) \right\|_{B(C_{-1,1})} \underset{\alpha \to 0}{\to} 0,$$

and for $|\alpha| < \varepsilon$ in $B(C_{-1,1}, C_{1,-1})$

$$\tilde{R}(s, \alpha) = \tilde{R}(s)(1 + V(\alpha)\tilde{R}(s))^{-1}.$$

Theorem 5 *Let $L = L(\Gamma_0(8), \chi^{(\frac{1}{2})})$ and $L(\alpha) = L + \alpha M + \alpha^2 N$.*

Let $\lambda_0 = s_0(1 - s_0) > \frac{1}{4}$ be an embedded eigenvalue of L with eigenspace J and $\dim J = l$. *Let K be the subspace of J consisting of all odd eigenfunction in J and let* $\dim K = m$. *There exists an orthonormal basis $\phi_1 ... \phi_m$ of K and positive numbers* $1, ..., m$, *such that*

$$\operatorname{Im} \lambda_i(\alpha) = -\lambda_i \alpha^2 + o(\alpha^2) \text{ for } \alpha \to 0,\ i = 1, ..., m.$$

The coefficients λ_i are given by Fermi's Golden Rule

$$\lambda_i = \frac{1}{8t_0} \sum_{j=1}^{2} \left| \left\langle E_j \left(\frac{1}{2} + it_0 \right), M\phi_i \right\rangle \right|^2, \quad i = 1, ..., m,$$

where $s_0 = \frac{1}{2} + it_0$.

There exists at least one eigenfunction $\phi \in K$, such that $\phi(\alpha)$ is a resonance function with resonance $\lambda(\alpha)$ for $0 < |\alpha| < \varepsilon$. The function ϕ can be taken as one of the basis vectors ϕ_i with $\lambda_i > 0$.

For every even eigenfunction $\phi \in J \ominus K$ we have

$$\operatorname{Im} \lambda_i(\alpha) = 0(\alpha^4).$$

Here the resonances $\lambda_i(\alpha)$ are poles of the analytic continuation of the resolvent from the upper half-plane across $\left(\frac{1}{4}, \infty\right)$ to the second sheet. In the s-plane this corresponds to $\left\{ s = \sigma + i\tau \,\middle|\, \sigma < \frac{1}{2}, t < 0 \right\}$.

Proof. 1) The case $l = m = 1$. /Mail/Diverse

Here $\lambda_0 > \frac{1}{4}$ is a simple eigenvalue with an odd eigenfunction ϕ_0, $\|\phi_0\| = 1$. Let $\tilde{R}(\alpha, \lambda)$ be the analytic continuation of the resolvent $R(\alpha, \lambda) = (L(\alpha) - \lambda)^{-1}$ as an operator in $\mathcal{B}(C_{-1,1}, C_{1,-1})$ from the upper half-plane across $\left(\frac{1}{4}, \infty\right)$ to the second sheet in the lower half-plane.

Since $\tilde{R}(\alpha, \lambda)$ has a simple pole at $\lambda = \lambda_0$, for $|\alpha| < \varepsilon$, $\tilde{R}(\alpha, \lambda)$ has a simple pole at $\lambda(\alpha)$ near $\lambda_0 = \lambda(0)$. A "quasi-projection" on the null space $\mathcal{N}(L(\alpha) - \lambda(\alpha)) \subset C_{-1,1}$ is given by

$$\tilde{P}(\alpha) = -\frac{1}{2\pi i} \int_C \tilde{R}(\alpha, \lambda) d\lambda \in \mathcal{B}(L_2^{\delta}, L_2^{-\delta}),$$

where $C = \{\lambda \,||\lambda - \lambda_0| = \rho_0\}$ is a circle containing for each α with $|\alpha| < \varepsilon$ the point λ_α and no other pole of $\tilde{R}(\alpha, \lambda)$.

Let $\phi(\alpha) = \tilde{P}(\alpha)\phi_0$. The operator-valued function $\tilde{P}(\alpha)$ is analytic for $|\alpha| < \varepsilon$. Since $\varphi_0 \in C_{-1,1}$, $\phi(\alpha)$ is analytic with values in $C_{1,-1}$.

Consider $L(\alpha)$ as an unbounded, closed, densely defined operator, denoted by $L_{1,-1}(\alpha)$, on its maximal domain in $C_{1,-1}$, and let M and N act in $C_{1,-1}$. Then $L_{1,-1}(\alpha) = L + \alpha M + \alpha^2 N$ is analytic for $|\alpha| < \varepsilon$ with constant domain of its sesqui-linear form. It has continuous spectrum equal to $\left\{\lambda = s(1-s) \,\middle|\, \frac{1}{2} - \delta \leqq Res \leqq \frac{1}{2}\right\}$, since for each such s with $Res > \frac{1}{2} - \delta$ the Eisenstein series $E(\alpha, s)$ belongs to $\mathcal{D}(L_{1,-1}(\alpha))$, provided $s \neq s(\alpha),\ s(\alpha)(1 - s(\alpha)) = \lambda(\alpha),\ s(\alpha)$ being a pole of the Eisenstein series $E(\alpha, s)$.

We have for $|\alpha| < \varepsilon$

$$L_\delta(\alpha)\phi(\alpha) = \lambda(\alpha)\phi(\alpha)$$

and hence

$$\langle L_\delta(\alpha)\phi(\alpha), \phi\rangle = \lambda(\alpha)\langle \phi(\alpha), \phi\rangle$$

where the duality $\langle\cdot,\cdot\rangle$ between $C_{-1,1} = C_{-1,1}(F, d\mu)$ and $C_{1,-1}(F, d\mu)$ is defined by

$$\langle f, g\rangle = \int_F f\bar{g}\,d\mu(z) \text{ for } f \in C_{-1,1},\ g \in C_{1,-1}.$$

It follows that for α small $\lambda(\alpha)$ is real-analytic. We choose ε such that this holds for $|\alpha| < \varepsilon$. Expanding $\phi(\alpha)$ and $\lambda(\alpha)$ to second order around $\alpha = 0$, we get

$$L_\delta(\alpha) = L + \alpha M + \alpha^2 N, \text{ where } L, M, N \text{ act in } C_{1,-1}.$$

$$\phi(\alpha) = \phi_0 + \alpha\phi_1 + \alpha^2\phi_2 + O(\alpha^3) \text{ in } C_{1,-1}.$$

$$\lambda(\alpha) = \lambda_0 + \alpha\lambda_1 + \alpha^2\lambda_2 + O(\alpha^3).$$

The first order equation is

$$L\phi_1 + M\phi_0 = \lambda_0\phi_1 + \lambda_1\phi_0.$$

Taking inner product with ϕ_0, we get

$$\langle L\phi_1, \phi_0\rangle + \langle M\phi_0, \phi_0\rangle = \lambda_0\langle\phi_1, \phi_0\rangle + \lambda_1.$$

Since ϕ_0 is exponentially decreasing, we have $\langle L\phi_1, \phi_0 \rangle = \langle \phi_1, L\phi_0 \rangle = \lambda_1 \langle \phi_1, \phi_0 \rangle$. Moreover, ϕ_0 is odd and hence $M\phi_0$ is even, so $\langle M\phi_0, \phi_0 \rangle = 0$, and we obtain $\lambda_1 = 0$. The second order equation is

$$L\phi_2 + M\phi_1 + N\phi_0 = \lambda_0\phi_2 + \lambda_1\phi_1 + \lambda_2\phi_0.$$

Taking inner product with ϕ_0, we get

$$\langle L\phi_2, \phi_0 \rangle + \langle M\phi_1, \phi_0 \rangle + \langle N\phi_0, \phi_0 \rangle = \lambda_0 \langle \phi_2, \phi_0 \rangle + \lambda_1 \langle \phi_1, \phi_0 \rangle + \lambda_2.$$

Since $\langle L\phi_2, \phi_0 \rangle = \langle \phi_2, L\phi_0 \rangle = \lambda_0 \langle \phi_2, \phi_0 \rangle$ and $\lambda_1 = 0$, we obtain

$$\langle M\phi_1, \phi_0 \rangle + \langle N\phi_0, \phi_0 \rangle = \lambda_2.$$

Since $\langle N\phi_0, \phi_0 \rangle$ is real, this implies

$$\operatorname{Im} \lambda_2 = \operatorname{Im} \langle M\phi_1, \phi_0 \rangle. \tag{76}$$

We shall now derive an expression for ϕ_1. We have

$$\begin{aligned}
\phi_1 = \lim_{\alpha \to 0} \frac{1}{\alpha}(\phi(\alpha) - \phi_0) &= \lim_{\alpha \to 0} \left\{ \frac{-1}{2\pi i} \int_C \left\{ \tilde{R}(\alpha, \lambda) - \tilde{R}(0, \lambda) \right\} \right\} \phi_0 d\lambda \qquad (77) \\
&= \lim_{\alpha \to 0} \left\{ \frac{1}{2\pi i} \int_C \tilde{R}(\alpha)(M + \alpha N)\tilde{R}(0, \lambda)\phi_0 d\lambda \right\} \\
&= \frac{1}{2\pi i} \int_C \tilde{R}(0, \lambda) M \tilde{R}(0, \lambda)\phi_0 d\lambda
\end{aligned}$$

$$= \frac{-1}{2\pi i} \int_C \frac{1}{\lambda - \lambda_0} \tilde{R}(0, \lambda) M\phi_0 d\lambda = -\lim_{\lambda \to \lambda_0} \tilde{R}'(0, \lambda) M\phi_0 = -\tilde{R}'(0, \lambda_0) M\phi_0,$$

where

$$\tilde{R}'(0, \lambda) = \tilde{R}(0, \lambda) - \frac{\tilde{P}(0)}{\lambda_0 - \lambda} \text{ for } |\lambda - \lambda_0| < \delta$$

$$\tilde{R}'(0, \lambda_0) = \lim_{\lambda \to \lambda_0} \tilde{R}'(0, \lambda)$$

and for $\operatorname{Im} \lambda > 0$

$$R'(0, \lambda) = R(0, \lambda) - \frac{P}{\lambda_0 - \lambda} = R(0, \lambda)(1 - P)$$

is the reduced resolvent of L at $\lambda = \lambda_0$. Thus

$$\tilde{R}'(0, \lambda_0) = \lim_{\varepsilon \downarrow 0} R'(0, \lambda_0 + i\varepsilon) \in \mathcal{B}(C_{-1,1}, C_{1,-1}). \tag{78}$$

Here the identity in $\mathcal{B}(C_{-1,1}, C_{1,-1})$

$$\left\{ \tilde{R}(\alpha, \lambda) - \tilde{R}(0, \lambda) \right\} \phi_0 = \tilde{R}(\alpha, \lambda)(\alpha M + \alpha^2 N)\tilde{R}(0, \lambda)\phi_0$$

can be obtained as follows. We have

$$[(L_{1,-1}(\alpha) - \lambda) - (L_{1,-1}(0) - \lambda] \, \tilde{R}(0, \lambda)\phi_0 = (\alpha M + \alpha^2 N)\tilde{R}(0, \lambda)\phi_0.$$

Since $\tilde{R}(0, \lambda)\phi_0 = (\lambda_0 - \lambda)^{-1}\phi_0$ is exponentially decreasing, both terms on the l.h.s. as well as the r.h.s. are in $C_{-1,1}$, and we obtain

$$\tilde{R}(0, \lambda)\phi_0 - \tilde{R}(\alpha, \lambda)\phi_0 = \tilde{R}(\alpha, \lambda)(\alpha M + \alpha^2 N)\tilde{R}(0, \lambda)\phi_0.$$

From (76), (77) and (78) we obtain

$$\operatorname{Im} \lambda_2 = - \operatorname{Im} \lim_{\varepsilon \downarrow 0} (R'(\lambda_0 + i\varepsilon) M\phi_0, M\phi_0).$$

By the spectral theorem applied to the operator $L(1 - P)$ with spectral measure $E(\mu)$ we have, setting $\psi_0 = M\phi_0$,

$$\begin{aligned} \lambda_2 &= - \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} (\mu - \lambda_0 - i\varepsilon)^{-1} d(E(\mu)\psi_0, \psi_0) \\ &= - \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{\mu - \lambda_0 + i\varepsilon}{(\mu - \lambda_0)^2 + \varepsilon^2} d(E(\mu)\psi_0, \psi_0). \end{aligned}$$

Thus, $\operatorname{Im} \lambda_2$ is given by

$$\operatorname{Im} \lambda_2 = - \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{\varepsilon}{(\mu - \lambda_0)^2 + \varepsilon^2} d(E(\mu)\psi_0, \psi_0).$$

Choose $\delta_0 > 0$ such that $[\lambda_0 - \delta_0, \lambda_0 + \delta_0]$ contains no eigenvalues of L different from λ_0. Then we have, using that $E\left([\lambda_0 - \delta, \lambda_0 + \delta]\right)\psi_0 \in \mathcal{H}_{ac}(L(1 - P))$, for $0 < \delta < \delta_0$

$$\operatorname{Im} \lambda_2 = - \lim_{\varepsilon \downarrow 0} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{\varepsilon}{(\mu - \lambda_0)^2 + \varepsilon^2} \frac{d}{d\mu} (E(\mu)\psi_0, \psi_0) d\mu.$$

From this we obtain, setting $f(\mu) = (E(\mu)\psi_0, \psi_0)$, for any $\delta < \delta_0$

$$\operatorname{Im}\lambda_2 = -\lim_{\varepsilon\downarrow 0}\left[\operatorname{Arctan}\frac{\mu-\lambda_0}{\varepsilon}f'(\mu)\right]_{\lambda_0-\delta}^{\lambda_0+\delta} + \lim_{\varepsilon\downarrow 0}\int_{\lambda_0-\delta}^{\lambda_0+\delta}\operatorname{Arctan}\frac{\mu-\lambda_0}{\varepsilon}f''(\mu)d\mu.$$

Since $\int_{\lambda_0-\delta}^{\lambda_0+\delta}\operatorname{Arctan}\frac{\mu-\lambda_0}{\varepsilon}f''(\mu)d\mu \to 0$ for $\delta \to 0$, uniformly in ε, it suffices to determine the limit as $\varepsilon \downarrow 0$ of the first term. For any $\delta < \delta_0$ this equals

$$-\lim_{\varepsilon\downarrow 0}\left\{\operatorname{Arctan}\frac{\delta}{\varepsilon}f'(\lambda_0+\delta) + \operatorname{Arctan}\frac{\delta}{\varepsilon}f'(\lambda_0-\delta)\right\}$$
$$= -\pi\left\{f'(\lambda_0+\delta) + f'(\lambda_0-\delta)\right\}.$$

Letting $\delta \to 0$, the second term goes to 0, and we obtain

$$\operatorname{Im}\lambda_2 = \lim_{\delta\to 0}(-\pi\left\{f'(\lambda_0+\delta) + f'(\lambda_0-\delta)\right\}) = -\pi f'(\lambda_0).$$

We have proved that

$$\operatorname{Im}\lambda_2 = -\pi\frac{d}{d\mu}(E(\mu)M\phi_0, M\phi_0)\Big|_{\mu\,=\,\lambda_0}. \tag{79}$$

In the s-variable we have, setting $s = \frac{1}{2} + it,\ \mu = s(1-s) = \frac{1}{4} + t^2$, so if $\lambda_0 = \frac{1}{4} + t_0^2$, we get

$$\operatorname{Im}\lambda_2 = -\frac{\pi}{2t_0}\frac{d}{dt}\left(E\left(\frac{1}{4}+t^2\right)\psi_0, \psi_0\right)\Big|_{t\,=\,t_0}.$$

We calculate this expression in terms of the Eisenstein series as follows. For any smooth function h with support in the interval $(\lambda_0-\delta_0, \lambda_0+\delta_0)$, $h(L)$ is given by the kernel

$$h(L)(z,z') = \frac{1}{4\pi}\int_{-\infty}^{\infty} h\left(\frac{1}{4}+r^2\right)\sum_{i=1}^{2} E_i\left(z, \frac{1}{2}+ir\right)\overline{E_i}\left(z', \frac{1}{2}+ir\right).$$

Thus

$$(h(L)\psi_0, \psi_0) = \frac{1}{4\pi}\int_{-\infty}^{\infty} h\left(\frac{1}{4}+r^2\right)\sum_{i=1}^{2}\left|\left\langle E_i\left(\frac{1}{2}+ir\right), \psi_0\right\rangle\right|^2.$$

On the other hand,

$$(h(L)\psi_0,\psi_0)=\int_{-\infty}^{\infty}h(\lambda)\frac{d}{d\lambda}(E(\lambda)\psi_0,\psi_0)d\lambda$$
$$=\int_{-\infty}^{\infty}h\left(\frac{1}{4}+r^2\right)\frac{d}{dr}\left(E\left(\frac{1}{4}+r^2\right)\psi_0,\psi_0\right)dr.$$

Thus for all smooth functions h with support in $(\lambda_0-\delta_0,\lambda_0+\delta_0)$

$$\int_{-\infty}^{\infty}h\left(\frac{1}{4}+r^2\right)\frac{1}{4\pi}\sum_{i=1}^{2}\left|\left\langle E_i\left(\frac{1}{2}+ir\right),\psi_0\right\rangle\right|^2$$
$$=\int_{-\infty}^{\infty}h\left(\frac{1}{4}+r^2\right)\frac{d}{dr}\left(E\left(\frac{1}{4}+r^2\right)\psi_0,\psi_0\right)dr.$$

This implies

$$\frac{d}{dt}\left(E\left(\frac{1}{4}+t^2\right)\psi_0,\psi_0\right)|_{t=t_0}=\frac{1}{4\pi}\sum_{i=1}^{2}\left|\left\langle E_i\left(\frac{1}{2}+ir\right),\psi_0\right\rangle\right|^2$$

and hence

$$\operatorname{Im}\lambda_2=-\frac{1}{8t_0}\sum_{i=1}^{2}\left|\left\langle E_i\left(\frac{1}{2}+it_0\right),M\phi_0\right\rangle\right|^2 \tag{80}$$

2) Assume that $\dim K=\dim J=m>1$. Then the eigenvalue λ_0 of L may split up into m resonances or eigenvalues for $\alpha\neq 0$. As above we define the analytic continuation $\tilde{R}(\alpha,\lambda)\in\mathcal{B}(C_{-1,1},C_{1,-1})$ of the resolvent $R(\alpha,\lambda)$ and the quasi-projection $\tilde{P}(\alpha)=\frac{-1}{2\pi i}\int_C\tilde{R}(\alpha,\lambda)d\lambda$, where $|\alpha|<\varepsilon$ and C is a circle with center λ_0 containing in its interior precisely the poles into which λ_0 splits up. These are simple poles of $\tilde{R}(\alpha,\lambda)$ for $|\alpha|<\varepsilon$, since λ_0 is a simple pole of $\tilde{R}(0,\lambda)$, and the total dimension of the eigenspaces of $L_{1,-1}(\alpha)$ corresponding to this λ_0-group of poles equals m.

Consider the sesquilinear form $\langle(L_\delta(\alpha)-\lambda_0)P(\alpha)\phi,\psi\rangle$ on $K\times K$. For $\phi\in K$

$$(L(\alpha)-\lambda_0)P(\alpha)\phi=\left\{(L-\lambda_0)+\alpha M+\alpha^2N\right\}(\phi+\alpha\phi_1+\alpha^2\phi_2+o(\alpha^2))$$
$$=\alpha\left[(L-\lambda_0)\phi_1+M\phi\right]+\alpha^2\left[(L-\lambda_0)\phi_2+M\phi+N\phi\right]+o(\alpha^2).$$

For $\psi \in K$ this yields

$$\langle (L(\alpha) - \lambda_0)P(\alpha)\phi, \psi\rangle = \alpha(M\phi, \psi) + \alpha^2 \left\{\langle M\phi_1, \psi\rangle + (N\phi, \psi)\right\} + o(\alpha^2)$$

since $\langle (L - \lambda_0)\phi_1, \psi\rangle = \langle (L - \lambda_0)\phi_2, \psi\rangle = 0$. Also $(M\phi, \psi) = 0$, since $M\phi$ is even and ψ is odd. Thus

$$\langle (L(\alpha) - \lambda_0)P(\alpha)\phi, \psi\rangle = \alpha^2 \left[\langle M\phi_1, \psi\rangle + (N\phi, \psi)\right] + o(\alpha^2).$$

The derivative ϕ_1 is calculated as above. We have

$$\phi_1 = -R'(0, \lambda_0)M\phi$$

where $R'(0, \lambda_0) = \lim_{\varepsilon\downarrow 0} R'(0, \lambda_0 + i\varepsilon)$ is the boundary value of the reduced resolvent of L at λ_0.

From this we derive as above the following expression for $\operatorname{Im}\langle M\phi_1, \phi\rangle$,

$$\operatorname{Im}\langle M\phi_1, \phi\rangle = -\frac{1}{8t_0}\sum_{i=1}^{2}\left|\left\langle E_i\left(\frac{1}{2} + it_0\right), M\phi\right\rangle\right|^2 \tag{81}$$

where $\lambda_0 = \frac{1}{4} + t_0^2$. Since $(N\phi, \phi)$ is real, we obtain

$$\operatorname{Im}\langle (L(\alpha) - \lambda_0)P(\alpha)\phi, \phi\rangle = -\alpha^2\frac{1}{8t_0}\sum_{i=1}^{2}\left|\left\langle E_i\left(\frac{1}{2} + it_0\right), M\phi\right\rangle\right|^2 + o(\alpha^2).$$

By Theorem 3, the sesquilinear form

$$\mathcal{F}(\phi, \psi) = \frac{1}{8t_0}\sum_{i=1}^{2}\left\langle E_i\left(\frac{1}{2} + it_0\right), M\phi\right\rangle \cdot \overline{\left\langle E_i\left(\frac{1}{2} + it_0\right), M\psi\right\rangle}$$

on $K \times K$ as positive definite.

Let $\phi_1, ..., \phi_m$ be an orthonormal basis of K, which diagonalizes $\mathcal{F}(\cdot, \cdot)$, with eigenvalues $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_m$,

$$F\phi_i = \mu_i\phi_i,\ i = 1, ..., m,\ \mathcal{F}(\phi_i, \phi_i) = \mu_i$$

where F is the operator on K defined by

$$\mathcal{F}(\phi,\psi) = (F\phi,\psi).$$

Some of the eigenvalues may be repeated according to multiplicity. In any case, for $\phi = \sum_{i=1}^m \beta_i\phi_i$

$$\mathcal{F}(\phi,\phi) = \sum_{i=1}^m \mu_i \left|\beta_i\right|^2$$

so

$$\operatorname{Im}\langle (L(\alpha)-\lambda_0)P(\alpha)\phi,\phi\rangle = -\alpha^2\sum_{i=1}^m \mu_i\left|\beta_i\right|^2 + o(\alpha^2),$$

in particular

$$\operatorname{Im}\langle (L(\alpha)-\lambda_0)P(\alpha)\phi_i,\phi_i\rangle = -\mu_i\alpha^2 + o(\alpha^2).$$

Thus, the eigenvalue λ_0 does not split to first order, but may split to second order, and the second order terms $\lambda_i(\alpha)$ will have negative imaginary parts for at least one i, $i = 1...m$, for $0 < |\alpha| < \varepsilon$. Since the basis $\phi_i, ..., \phi_m$ of K need not be a Hecke basis, this is what can be inferred from Theorem 3. This means that λ_0 gives rise to at least one resonance $\lambda(\alpha)$ near λ_0 for $0 < |\alpha| < \varepsilon$.

3) Assume that $\dim J > \dim K$. Then $J = D \oplus K$, where D consists of even functions and K of odd functions. For $\phi \in K$ the form $\mathcal{F}(\phi,\phi)$ is negative definite. For $\phi \in D$ we have $\langle E_i, M\phi\rangle = 0$, since $M\phi$ is odd, so $\mathcal{F}(\phi,\phi) = 0$. Diagonalizing the form $\mathcal{F}$ on J, we obtain a basis $\phi_1, ..., \phi_m, \phi_{m+1}, ..., \phi_l$ of J, such that $\phi_1 ... \phi_m$ is a basis of K and $\phi_{m+1}, ..., \phi_l$ is a basis of D. Let $\mu_1, ..., \mu_m$ be the eigenvalues associated to $\phi_1, ..., \phi_m$. Since $\phi_{m+1}, ..., \phi_l$ have eigenvalue 0, we obtain a basis $\phi_1, ..., \phi_{m_1}, \phi_{m_1+1}, ..., \phi_l$ of J, such that the corresponding eigenvalues μ_i satisfy

$$0 < \mu_1 \leqq ... \leqq \mu_m, \mu_{m+1} = ... = \mu_l = 0.$$

Thus,

$$\mathcal{F}(\phi_i,\phi_i) = \mu_i \text{ for } i = 1, ..., m$$

$$\mathcal{F}(\phi_i, \phi_i) = 0 \text{ for } i = m+1, ..., l$$

and

$$\mathcal{F}\left(\sum_{i=1}^{l} \beta_i \phi_i, \sum_{i=1}^{l} \beta_i \phi_i\right) = \sum_{i=1}^{m} \mu_i \beta_i^2.$$

It follows that for $i = 1, ..., m$

$$\operatorname{Im} \langle (L(\alpha) - \lambda_0) P(\alpha) \phi_i, \phi_i \rangle = -\mu_i \alpha^2 + o(\alpha^2)$$

while for $i = m+1, ..., l$

$$\operatorname{Im} \langle (L(\alpha) - \lambda_0) P(\alpha) \phi_i, \phi_i \rangle = o(\alpha^4).$$

Thus, on the subspace K the eigenvalue λ_0 gives rise to at least one resonance $\lambda(\alpha)$ for $0 < |\alpha| < \varepsilon$, while on the subspace D the eigenvalue does not change to second order, and it remains an open question, whether on this space the eigenvalue gives rise to a resonance of $L(\alpha)$.

This completes the proof of the Theorem. ∎

Remark 1 *The general perturbation theory for embedded eigenvalues outlined in this section is equally valid for each of the operators $A(\alpha_0) = A(\Gamma_0(8), \chi^{(\alpha_0)})$, $\alpha_0 \neq \frac{1}{2}$. The arithmetical proof of the fact that $I(s_j) \neq 0$ for odd eigenfunctions v_j is only possible in the case of congruence groups. The points α_i on the character circle, which correspond to congruence groups, have not been identified. However, we can draw the following conclusions about embedded eigenvalues of $L(\Gamma_0(8), \chi^{(\alpha)})$ based on the general perturbation theory. Due to the analyticity in α, each embedded eigenvalue $\lambda(\alpha_0)$ of $A(\alpha_0)$ under character perturbation either stays as an embedded eigenvalue for $\alpha \neq \alpha_0$ or leaves as a resonance. When α_0 is not a congruence point, it is not known whether there exist embedded eigenvalues, but for each congruence point there are embedded eigenvalues obeying a Weyl law. If $\lambda(\alpha)$ remains an eigenvalue and does not go to ∞ as $\alpha \to \alpha_1$ for* *some* α_1*, then it becomes an eigenvalue* $\lambda\left(\frac{1}{2}\right)$*, which stays embedded, hence the corresponding eigenfunction* $\Phi\left(\frac{1}{2}\right)$ *is even. It is also a priori possible that* $\lambda(\alpha_1) = \frac{1}{4}$ *for some* α_1 *and that* $\lambda(\alpha)$ *then becomes a small eigenvalue. This cannot happen at* $\alpha = \frac{1}{2}$ *due to symmetry. If no eigenvalue goes to* ∞ *for finite* α_1 *or reaches* $\frac{1}{4}$*, this together with the Weyl law has implications about eigenvalues leaving or staying for other congruence points than* $\frac{1}{2}$*.*

Remark 2 *For even eigenfunctions the Phillips-Sarnak integral is zero, since* $M\phi$ *is odd for even* ϕ*. It is therefore not known whether even eigenfunctions leave or stay under this perturbation.*

There is another perturbation obtained by replacing $\operatorname{Re}\int_{z_0}^{z}\omega(t)dt$ *by* $\operatorname{Im}\int_{z_0}^{z}\omega(t)dt$ *in the definition of the characters* $\chi(\alpha)$*,*

$$\tilde{L}(\alpha) = L + \alpha\tilde{M} + \alpha^2 N$$

where

$$L = A(\Gamma_0(8), \chi)$$
$$\tilde{M} = -4\pi i y^2\left(\omega_2\frac{\partial}{\partial x} + \omega_1\frac{\partial}{\partial x}\right)$$
$$N = 4\pi^2 y^2(\omega_1^2 + \omega_2^2).$$

It turns out that the operator $\tilde{M}$ *is not L-bounded, and therefore the perturbation theory developed for* M *does not apply.* $\tilde{M}$ *preserves parity, and the Phillips-Sarnak integrals are in fact given by the same Rankin-Selberg convolution and are proved to the nonzero. This does not imply, however, that eigenvalues with even eigenfunctions become resonances under this perturbation. Indeed,* $\operatorname{Im}\int_{z_0}^{\gamma z_0}\omega(t)dt = 0$ *for* $\gamma \in \Gamma_0(8)$*, which implies that* $\chi\cdot\chi^{(\alpha)} = \chi$ *for all* α*, and the functions* $\Omega(\alpha) = \exp\left\{2\pi i \operatorname{Im}\int_{z_0}^{z}\omega(t)dt\right\}$ *are* $\Gamma_0(8)$*-automorphic. Thus, the operators* $\tilde{L}(\alpha)$ *are unitarily equivalent to L for all* α *via* $\tilde{L}(\alpha) = \Omega^{-1}(\alpha)L\Omega(\alpha($ *with domain* $D(\tilde{L}(\alpha) = \Omega(\alpha)D(L)$*, and all eigenvalues stay.*

Appendix 1

The Rankin-Selberg convolution

For Res >1 we consider the following integral

$$\int_{F_0(8)} \mid v_j(z) \mid^2 E_3(z;s;\Gamma_0(8);1)d\mu(z) = R(s) \tag{82}$$

where

$$E_3(z;s;\Gamma_0(8);1) = \sum_{\gamma\epsilon\Gamma_\infty\backslash\Gamma} y^s(\gamma z)$$

and $v_j(z) = v_j(z;\Gamma;\chi^{(\frac{1}{2})})$ is defined in (45), (46). Using the unfolding of the Eisenstein series we obtain

$$R(s) = \int_0^\infty y^{s-1} \sum_{n\neq 0} \mid \rho_j(n) \mid^2 K^2_{ir_j}(2\pi \mid n \mid y)dy \tag{83}$$

$$\frac{\Gamma^2(s/2)\Gamma\left(\frac{s}{2}+ir_j\right)\Gamma\left(\frac{s}{2}-ir_j\right)}{4\pi^s\Gamma(s)} \sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^s}$$

From (82), (83) follows that

$$\sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^s}$$

has meromorphic continuation to all of C and at s=1 has a pole of order $\leqq$ the order of the pole of $E_3(z;s;\Gamma_0(8);1)$ at $s=1$.

We will show now that the Eisenstein series E_3 has at $s=1$ a simple pole. We will find $E_3(z) = E_3(z;s;\Gamma_0(8);1)$ as explicit linear combination of $E(z,s)$, $E(2z,s)$, $E(4z,s)$, $E(8z,s)$where

$$E(z) = E(z,s) = \sum_{\gamma\epsilon\Gamma_\infty\diagdown\Gamma_z} y^s(\gamma z) = y^s + 0_{y\to\infty}(y^{1-s}) \tag{84}$$

and Γ_Z is the modular group. From general theory we know

$$E_3(z) = y^s + 0_{y\to\infty}(y^{1-\mathrm{Re}\, s}), \qquad E_3(g_k z) = 0_{y\to\infty}(y^{1-\mathrm{Re}\, s}) \qquad k = 1,2,4 \tag{85}$$

Here

$$g_k\infty = z_k \ \ cusps \ \ k = 1,2,4 \quad , g_k^{-1} S_k g_k = S_{3.}$$

We have

$$g_1 z = -\frac{1}{8z}, \qquad g_2 z = \frac{z}{4z+1}, \qquad g_4 z = \frac{2z}{-4z+1},$$

and we take

$$E_3(z) = \alpha_1 E(z) + \alpha_2 E(2z) + \alpha_4 E(4z) + \alpha_8 E(8z) \tag{86}$$

From (84) and first condition in (85) we obtain

$$\alpha_1 + \alpha_2 2^s + \alpha_4 4^s + \alpha_8 8^s = 1 \tag{87}$$

Then we have from (85)

$$\alpha_1 E\left(-\frac{1}{8z}\right) + \alpha_2 E\left(-\frac{1}{4z}\right) + \alpha_4 E\left(-\frac{1}{2z}\right) + \alpha_8 E\left(-1/z\right) = 0_{y\to\infty}(y^{1-s})$$

That is equivalent to

$$\alpha_1 E\left(8z\right) + \alpha_2 E\left(4z\right) + \alpha_4 E\left(2z\right) + \alpha_8 E\left(z\right) = 0_{y\to\infty}(y^{1-\mathrm{Re}\, s})$$

We obtain the second condition

$$\alpha_1 8^s + \alpha_2 4^s + \alpha_4 2^s + \alpha_8 = 0 \tag{88}$$

Analogous calculations for $k = 2,\, 4$ in (85) gives 2 more conditions,

$$\alpha_1 + \alpha_2 2^s + \alpha_4 4^s + \alpha_8 2^s = 0 \tag{89}$$

$$\alpha_1 2^s + \alpha_2 4^s + \alpha_4 2^s + \alpha_8 = 0 \tag{90}$$

We solve the system of linear equations (87) - (90). Finally we get

$$\alpha_4 = -\frac{1}{2^{2s}(2^{2s}-1)} \qquad \alpha_8 = \frac{1}{2^s(2^{2s}-1)} \tag{91}$$

We obtain then

$$E_3(z; s; \Gamma_0(8); 1) = \frac{1}{2^s(2^{2s}-1)} \left[E(8z,s) - 2^{-s}E(4z,s)\right] \tag{92}$$

From the well-known Fourier expansion for E(z,s) follows that it has a simple pole at $s = 1$ with the residue $3/\pi = \text{vol } (F_Z)^{-1}$. From (92) follows that E_3 at $s = 1$ has a simple pole with the residue $\frac{1}{4\pi} = (\text{vol } (F_0(8)))^{-1}$ which agrees with the general theory.

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